2008

Global behavior of graph dynamics with applications to Markov chains

Jose Ayala-Hoffmann
Iowa State University

Global behavior of graph dynamics with applications to Markov Chains

by

Jose Ayala-Hoffmann

A thesis submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Mathematics

Program of Study Committee:
Wolfgang Kliemann, Major Professor
Paul Sacks
Stephen Willson

Iowa State University

Ames, Iowa

2008

DEDICATION

A mi amada Loreto, compañera de viajes definitivos.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to express my gratitude and admiration to my advisor Dr. Wolfgang Kliemann, whose deep and multiple visions inspire me to continue the exciting choice of intellectual life.

A very special thanks goes out to my grandmother and mentor, Fresia Weisse, who always will illuminate every second of my existence.

I would also like to thank my parents Victor, Cristina and Tutti, for the huge amount of love and inspiration they provided me through this path.

I would also like to acknowledge my friends for the gift of sincere friendship: Patricio Galdames, Zsolt Gemesi, Todd Teske, Fernando Miranda-Mendoza, Mehmet Dagli, Adil Kaymaz and Yoon-Sup So.

CHAPTER 1. Introduction

The mathematical theory of dynamical systems analyzes, from an axiomatic point of view, the common features of many models that describe the behavior of systems in time. In its abstract form, a dynamical system is given by a time set $\mathbb{T}$ (with semigroup operation $\circ$), a state space M, and a map $\Phi : \mathbb{T} \times M \to M$ that satisfies (i) $\Phi(0, x) = x$ for all $x \in M$, describing the initial value, and (ii) $\Phi(t \circ s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \in \mathbb{T}$ and $x \in M$. Common examples for the time set $\mathbb{T}$ are the natural numbers $\mathbb{N}$ or the nonnegative reals $\mathbb{R}^+$ as semigroups, and the integers $\mathbb{Z}$ or the reals $\mathbb{R}$ as groups (under addition). If the state space M carries an additional structure, such a being a measurable space, a topological space or a manifold, the map Φ is required to respect this structure, i.e. it is assumed to be measurable, continuous, or differentiable, respectively.

At the heart of the theory of dynamical systems is the study of system behavior as $t \to \infty$ or $t \to \pm\infty$ (qualitative behavior), as well the change in behavior under variation of parameters (bifurcation theory). We refer the reader to [9] for a comprehensive introduction to the theory of dynamical systems. The paper [4] summarizes, for continuous dynamical systems on a compact metric space, the qualitative behavior using the concepts of chain recurrence, attractors, and Morse decompositions.

Over the last ten years, composite systems have been studied from a dynamical systems point of view, such as stochastic systems consisting of a stochastic process that enters into a dynamical system (see [2]), or control systems (see [8]). In the case of composite systems one considers a “skew-product” structure, where an underlying (random or control) system $\Psi : \mathbb{T} \times N \to N$ affects the system dynamics of interest $\Phi : \mathbb{T} \times N \times M \to M$, resulting a skew-product flow of the type $(\Psi, \Phi) : \mathbb{T} \times N \times M \to N \times M$. Arnold’s book deals with measurable systems, while the book by Colonius and Kliemann studies systems that are continuous in the $\Psi-$component and smooth in the $\Phi-$component. Both references deal primarily with systems on the real time axis $\mathbb{T} = \mathbb{R}$, with state space M being a smooth manifold.

Recently, so-called "hybrid" systems have attracted much attention in applications in the sciences and engineering. Hybrid systems are composite systems with different time sets, usually the background component Ψ has a discrete time set ($\mathbb{N}$ or $\mathbb{Z}$), while the system itself has a continuous time set ($\mathbb{R}^+$ or $\mathbb{R}$). Typical examples are mechanical systems perturbed by a Markov chain, event-driven systems (such as contingencies in power systems), switching systems, hidden Markov models in statistics, or dynamical systems with modeled digital information component. An analysis of hybrid structures from a dynamical systems point of view is still missing.

As a first step towards a dynamical systems perspective for hybrid systems, this thesis discusses the qualitative behavior of a class of discrete mathematical models from the point of view of global systems behavior. The discrete models include finite directed graphs, certain linear iterated function systems, and Markov chains. The dynamical description of these models requires a discrete time set $\mathbb{T} = \mathbb{N}$, and a state space M endowed with the discrete topology, leading to a discrete semiflow $\Phi : \mathbb{N} \times \mathfrak{P}(S) \to \mathfrak{P}(S)$, where $\mathfrak{P}(S)$ is the power set of some finite set S. Our guideline is the paper [4], which summarizes the qualitative behavior of continuous dynamical systems on a compact metric space. As it turns out, both the discrete topology and the fact that we are dealing with a system on a one-sided time set lead to interesting complications, when we try to adapt the ideas of chain recurrence, attractors, and Morse decompositions to our setup.

The connections between Markov chains, finite directed graphs, and products of (stochastic) matrices have been studied extensively in the literature, compare e.g. [10], [13], [14], and [15]. Most of the books on this topic confine themselves to the case of irreducible Markov chains, while [13] also gives an overview of the reducible case. This thesis starts by considering general finite directed graphs. We analyze their communication structure, i.e. equivalence classes of vertices that can be reached mutually via sequences of edges, and the associated quotient graphs. This leads to the "communicating classes" $\mathcal{C} = \{C_1, ..., C_k\}$ of a graph and a reachability order $\preceq$ on $\mathcal{C}$. The key concept is that of an $L-$graph, i.e. graphs for which each vertex has out-degree ≥ 1. As it turns out, these are exactly the graphs for which the $\omega-$limit sets of the associated semiflow are nonempty.

To each graph $G = (V, E)$, where V is the set of vertices and $E \subset V \times V$ the set of edges, we associate a semiflow $\Phi_G : \mathbb{N} \times \mathfrak{P}(V) \to \mathfrak{P}(V)$. This semiflow is studied from the point of view of qualitative behavior of dynamical systems, i.e. we adapt the concepts of $\omega-$limit sets, (positive) invariance, recurrence, Morse decompositions, attractors and attractor-repeller pairs to Φ_G and prove characterizations equivalent to those of [4]. As it turns out, the finest Morse decomposition of Φ_G corresponds to the decomposition $\mathcal{C}$ of the graph G into communicating classes, and the order on the communicating classes is equivalent to the order that accompanies a Morse decomposition. Moreover, the connected components of the recurrent set of Φ_G are exactly the (finest) Morse sets of Φ_G, i.e. the communicating classes of G. Most of our new results are contained in Section 2.3.

Graphs $G = (V, E)$ (and certain aspects of Markov chains) are often studied using the adjacency matrix A_G: Let $V = \{1, ..., d\}$ and define $(A_G)_{ij} = 1$ if $(i, j) \in E$, and 0 otherwise. Now products of A_G describe the paths, and hence the communication structure of G. We construct a semiflow $\Psi_A : \mathbb{N} \times \mathbb{Q}^d \to \mathbb{Q}^d$ (where $\mathbb{Q}^d$ is the (vertex set of the) unit cube in $\mathbb{R}^d$) that is equivalent to the semiflow Φ_G defined on $\mathfrak{P}(V)$, using logical matrix multiplication. This point of view is somewhat different from the standard approach that uses regular matrix multiplication and that does not lead to an equivalent semiflow. The equivalence allows us to interpret all results obtained for Φ_G in terms of certain linear iterated function systems.

In Chapter 3 we apply the results obtained for graphs and their semiflows to the study of general finite Markov chains. Our results are presented in the form of a "3-language dictionary" Each key concept for Markov chains is "translated " into graph language and into semiflow language. This dictionary is contained in Facts 1 - 13 and Fact 15. Note that our concepts and results from Chapter 2 only deal with the communication structure of graphs (or the qualitative behavior of semiflows) and hence they do not contain the probabilistic information of a Markov chain.

But it turns out that a simple result on the geometric decay of certain probabilities (compare Lemma 106) is sufficient to recapture all the relevant probabilistic information. Facts 14 and 16 - 18 describe the long term behavior of general Markov chains and introduce the concept of multistable states. While many of the results in Chapter 3 can be found in the literature, our presentation unifies many of the concepts and shows which structural (deterministic, graph theoretic, semiflow) properties and which probabilistic properties are really needed to analyze Markov chains. Moreover, we lay the foundation for a dynamical systems approach to hybrid systems with Markov chain type perturbations.

CHAPTER 2. Decompositions of Graphs

Directed graphs are often used to analyze the behavior of discrete systems in time, such as Markov chains, event trees, information systems, switched systems, or discrete control systems. This chapter presents a general decomposition theory for graphs from a discrete systems point of view. We focus on graphs in their simplest form, i.e. on directed finite graphs without valuations or coloring of the edges. For one specific area of application, Markov chains, we will discuss in Chapter 3 how these graph-theoretic results can be used to understand the structure of more complex discrete systems.

The key idea in the analysis of directed graphs as discrete systems is that of "communication" i.e. the question of which vertices are connected via paths. This idea can be developed from the point of view of orbits in graphs, from an adaptation of Morse decompositions for dynamical systems, or from an angle of linear algebra using adjacency matrices. Three sections of this chapter explore these three points of view. We begin with a short review of concepts from graph theory that are useful for our purposes.

2.1 Some Concepts from Graph Theory

A finite graph is an ordered triple G denoted by (V_G, E_G, I_G) where V_G is a nonempty set of finite cardinality, the elements of V_G are called vertices of G. The second component E_G is a set of edges of G, with $V_G \cap E_G = \phi$ i.e., no element can be a vertex and an edge simultaneously. The last term I_G is a correspondence called incidence map, which assigns to an element of E_G an element of the Cartesian product $V_G \times V_G$

$$\begin{aligned} I_G : E_G &\rightarrow V_G \times V_G \\ e &\mapsto I_G(e) = (i, j). \end{aligned}$$

The vertices i, j are called the adjacent vertices of e, and each element of $\mathrm{Im}(I_G)$ is called an incidence. Note that each edge has a direction, and G is a directed graph.

If there is no ambiguity we denote the tuple as $G = (V, E, I)$, using capital letters. Any subset of either V or E or both will be considered a subgraph of G by eliminating the corresponding vertices (and all associated edges) or edges (and associated incidences). A set of two or more edges it said to be multiple if each of them joins the same vertices. A graph is said to be simple if each pair $(i, j) \in V \times V$ in the range of I is associated exactly with one $e \in E$. In other words, the mapping I is injective. From now we confine ourselves to finite, simple, directed graphs. In this case the initial triple G can be viewed as the pair (V, E), with $E \subset V \times V$. The cardinality of a graph G is the number of elements in V, denoted by $\#G$.

A (finite) sequence of linked incidences in G is called a path. More precisely, let $n \in \mathbb{N}$, $n \geq 1$, then a path γ is given by

$$\langle\ i_0, (i_0, i_1), i_1, (i_1, i_2), ..., i_{n-1}, (i_{n-1}, i_n), i_n\ \rangle.$$

Observe that a path begins and finishes with a vertex, and each edge is incident with the vertices immediately preceding and succeeding it. Since we are dealing with simple directed graphs, the incidences are uniquely determined by their adjacent vertices, and so there is no ambiguity to denote a path as

$$\gamma = \langle\ i_0, i_1, ..., i_n\ \rangle. \tag{2.1}$$

We write $(i, j) \in \gamma$ if there exists an edge joining i with j belonging to the path γ. Moreover, we say $i \in \gamma$ if there exists an incidence in γ with i as adjacent vertex. In many arguments only the first and last vertex of a path are crucial, in which case we often use the notation

$$\gamma = \langle\ i_0, i_1, ..., i_n\ \rangle = \gamma_{i_0 i_n} = \langle\ i_0 ... i_n\ \rangle.$$

The length of a path γ is written as $\ell(\gamma)$, and corresponds to the number of edges in γ. Note that the length of the path γ in (2.1) is given by $\ell(\gamma) = n$. A path of length n can be viewed as an element of V^{n+1}, the $(n+1)^{st}$ Cartesian product of V with itself. Given a graph G, we define the set

$$\Gamma^n := \{\gamma : \ell(\gamma) = n,\ \ n \in \mathbb{N}\} \tag{2.2}$$

as the set of all paths of length n, and we set $\Gamma^0 = V$.

We can specify vertices in a path $\gamma = \langle\ i_0, i_1, ..., i_n\ \rangle$ in terms of the projection mappings π_p for $0 \le p \le n$:

$$\pi_p : \Gamma^n \to V,\ \ \pi_p(\gamma) = i_p$$

where i_p is the p^{th} vertex in γ. In other words

$$\gamma = \langle\ \pi_0(\gamma), ..., \pi_p(\gamma), ..., \pi_n(\gamma)\ \rangle.$$

A subpath γ' of γ is a subsequence of γ of consecutive edges (or vertices) belonging to γ. In particular, any edge of a path is a subpath of length one. Besides subpaths, composition of paths will play a role in many of our proofs.

Definition 1 *For two paths γ_1 and γ_2 with $\ell(\gamma_1) = m$ and $\ell(\gamma_2) = n$ such that $\gamma_1 = \langle\ i...j\ \rangle$ and $\gamma_2 = \langle\ j...k\ \rangle$ we define the concatenation of the paths as*

$$\langle \gamma_1, \gamma_2 \rangle = \langle i...j \rangle * \langle j...k \rangle = \langle i...k \rangle$$

with $\ell\,(\,\langle\ i...k\ \rangle\,) = m + n$ and $\pi_m\,(\,\langle\ i...k\ \rangle\,) = j$.

In a directed graph G, the out-degree of a vertex $i \in V$ is defined as the number of edges "going out of the vertex i" given by

$$O^{\circ}(i) = \#\,\{(i, j) :\ \ (i, j) \in E\ \text{ for some }\ j \in V\}.$$

Similarly, the in-degree of a vertex i corresponds to the number of edges "coming into i" ,given by

$$I^{\circ}(i) = \#\,\{(j, i) :\ \ (j, i) \in E\ \text{ for some }\ j \in V\}.$$

Alternatively we can say that the $O^{\circ}(i)$ and $I^{\circ}(i)$ correspond to the number of incidences having the vertex i as first and second coordinates in $V \times V$, respectively.

Definition 2 *A graph G is called an $L-$graph if for every $i \in V$ we have $O^{\circ}(i) \ge 1$.*

Example 3 *We will use the following graph various times in this work to illustrate concepts and results. Consider the graph $G = (V, E)$ with vertex set*

$$V = \{1, ..., 21\}$$

and edge set

$$E = \{(1,2),(2,1),(2,6),(3,4),(4,5),(5,3),(5,4),(5,6),(6,8),$$
$$(6,9),(7,6),(7,8),(8,7),(8,10),(9,8),(10,10),(10,11),$$
$$(11,12),(12,13),(12,14),(13,11),(13,17),(14,15),(15,16)$$
$$(16,15),(17,19),(18,17),(19,18),(19,19),(20,19),(21,21)\}.$$

This graph is an $L-$graph. But note that $I^\circ(20) = 0$.

In what follows, we mainly concentrate on $L-$graphs since they are related to limit sets in Section 2.3.3 (compare Remark 62) and to Markov chains that will be studied in Chapter 3. However, many of the interesting ideas can be extended to general directed graphs. Basically, $L-$graphs are needed to ensure the existence of various objects, such as communicating classes, compare the next section.

Definition 4 *The positive and negative orbit of a vertex $i \in V$ are defined as*

$$\mathcal{O}^+(i) = \{j \in\ V :\quad \exists\, n \geq 1,\ \ \exists\, \gamma \in \Gamma^n \ \ \textit{such that}\ \ \pi_0(\gamma) = i,\ \ \pi_n(\gamma) = j\},$$

$$\mathcal{O}^-(i) = \{j \in\ V :\quad \exists\, n \geq 1,\ \ \exists\, \gamma \in \Gamma^n\ \textit{such that}\ \ \pi_0(\gamma) = j,\ \ \pi_n(\gamma) = i\},$$

where $\pi_0(\gamma)$ and $\pi_n(\gamma)$ represent the initial and the final vertices in γ.

By Definition 4, in an $L-$graph the positive orbit of every vertex is nonempty. We extend this definition to subsets of V via: For $U \subset V$, the positive and negative orbit of U are given by

$$\mathcal{O}^+(U) = \cup\ \{\ \mathcal{O}^+(i)\ :\ i \in U\ \}$$

$$\mathcal{O}^-(U) = \cup\ \{\ \mathcal{O}^-(i)\ :\ i \in U\ \}.$$

Example 5 *(Continuation of Example 3) We compute, e.g., $\mathcal{O}^+(11) = \mathcal{O}^+(12) = \mathcal{O}^+(13) = \{11,12,13,14,15,16,17,18,19\}$ and $\mathcal{O}^-(6) = \{1,2,3,4,5,6,7,8,9\}$, while $\mathcal{O}^-(20) = \varnothing$ and $\mathcal{O}^-(21) = \{21\}$.*

2.2 Orbits and Communicating Classes

2.2.1 Communicating Sets in Graphs

The communication structure in graphs is one of the key issues of this thesis. In this section we introduce the concepts of communicating sets and communicating classes based on the idea of orbits In many ways, communicating classes are similar to control sets for control systems, compare [8], Chapter 3.

Definition 6 *A vertex $i \in V$ has access to a vertex $j \in V$ if there exists a path of length ≥ 1 from i to j. We say that the vertices i and j communicate, written as $i \sim j$, if they have mutual access. A subset U of V is a communicating set if any two vertices of U communicate.*

Proposition 7 *The vertex communication relation $\sim$ in a graph G is symmetric and transitive but, in general, it lacks the reflexivity property.*

Proof. Symmetry is obvious from the definition of mutual access. To see transitivity, take $i, j, k \in V$ with $i \sim j$ and $j \sim k$. By definition there exist paths $\gamma_1 = \langle\ i...j\ \rangle$, $\gamma_2 = \langle\ j...i\ \rangle$, $\gamma_3 = \langle\ j...k\ \rangle$, and $\gamma_4 = \langle\ k...j\ \rangle$. Now the concatenation of paths $\langle\ \gamma_1, \gamma_3\ \rangle = \langle\ i...k\ \rangle$ links the vertices i and k, and the path $\langle\ \gamma_4, \gamma_2\ \rangle = \langle\ k...i\ \rangle$ links vertices k and i, and therefore $i \sim k$, which completes the proof. Note that the relation $\sim$ is reflexive iff for all $i \in V$ there exists a path $\gamma_{ii} = \langle i...i \rangle$, a property that does not always hold. ∎

The lack of reflexivity of the communication relation $\sim$ means that $V/\sim$ may not determine a partition of V. We therefore define a smaller set on which this property holds: We denote the union of all communicating sets by

$$V_c = \{\ i \in V :\ i \sim j \ \text{ for some } \ j \in V\ \}.$$

Note that $V_c \subsetneqq V$ is possible.

Example 8 *(Continuation of Example 3) For this graph we have $V \backslash V_c = \{14, 20\} \neq \varnothing$.*

For $i \in V_c$ we define $[i] := \{j \in V,\ j \sim i\}$. Then $V_c/\sim := \{[i],\ i \in V_c\}$ is a partition of V_c, i.e. $[i] \cap [j] = \varnothing$ for $j \notin [i]$, and $\cup[i] = V_c$.

Definition 9 *Let G be a graph with communication relation $\sim$. Each set $[i]$ for $i \in V_c$ is called a communicating class of G. We denote the set $V_c/\sim$ of all communicating classes by $\mathcal{C}$.*

Note that by definition, communicating classes are communicating sets. They are characterized by their maximality:

Proposition 10 *Communicating classes are maximal communicating sets (with respect to the set inclusion). Vice versa, maximal communicating sets are communicating classes.*

Proof. Assume that $[j] \in \mathcal{C}$ is not maximal, then there exist $i \in [\ j\ ]$, and $k \notin [\ j\ ]$ with $i \sim k$ i.e., $[\ j\ ]$ is not maximal. Since $i \in [\ j\ ]$ there exist paths $\gamma_1 = \langle\ i...j\ \rangle$ and $\gamma_2 = \langle\ j...i\ \rangle$. Moreover, since i and k communicate we have paths $\gamma_3 = \langle\ i...k\ \rangle$ and $\gamma_4 = \langle\ k...i\ \rangle$. The concatenations $\langle\ \gamma_4, \gamma_1\ \rangle$ and $\langle\ \gamma_2, \gamma_3\ \rangle$ imply $k \sim j$ and therefore $k \in [\ j\ ]$, which leads to a contradiction, proving the first claim of the proposition. The second part follows by definition of communicating classes. ■

Our first main result shows that communicating classes can be characterized using orbits of vertices.

Theorem 11 *Every communicating class $C \in \mathcal{C}$ is of the form*

$$C = \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$$

for some $i \in V$. Vice versa, if $C := \mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \varnothing$ for some $i \in V$, then C is a communicating class.

Proof.

(i) Let C be a communicating class with $i \in C$. Then since $i \sim i$, we have that C contains a path $\gamma = \langle\ i...i\ \rangle$ and hence it follows that

$$i \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i),$$

i.e. $C \subset \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$.

Now consider $j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$ for some $i \in V$. Then there exist a path γ and $n \geq 1$ such that $\pi_0(\gamma) = i$ and $\pi_n(\gamma) = j$, as well as a path γ' such that $\pi_0(\gamma') = j$ and $\pi_m(\gamma') = i$, for some $m \geq 1$. This immediately implies $i \sim j$ for every element $j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$, and therefore $\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \subset [i]$.

(ii) Assume that $\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \varnothing$ for some $i \in V$. We have to show that $\mathcal{O}^+(i) \cap \mathcal{O}^-(i)$ is a communicating class, i.e. $\mathcal{O}^+(i) \cap \mathcal{O}^-(i) = [i]$. Take $j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$, then we argue as before that $j \sim i$ and hence $j \in [i]$. On the other hand, if $j \notin \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$, then $j \notin \mathcal{O}^+(i)$ or $j \notin \mathcal{O}^-(i)$. In the first case there is no path from i to j, in the second case there is no path from j to i. Any of these two statements implies that $i \nsim j$, which completes the proof.

■

Example 12 *(Continuation of Example 3) For this graph consider, e.g., the vertex* 8*:* $\mathcal{O}^+(8) = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$ *and* $\mathcal{O}^-(8) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$*, hence* $\mathcal{O}^+(8) \cap \mathcal{O}^-(8) = \{6, 7, 8, 9\}$ *is a communicating class.*

Definition 13 *A transitory vertex i of a graph G is a vertex that does not belong to a communicating class, i.e. $i \in V \backslash V_c$.*

Remark 14 *We note that by Theorem 11 transitory vertices are exactly those vertices $i \in V$ for which $\mathcal{O}^+(i) \cap \mathcal{O}^-(i) = \varnothing$. This also means that $\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \varnothing$ iff $i \in V_c$, i.e. exactly these vertices "anchor" communicating classes.*

Example 15 *(Continuation of Example 3) For this graph we find that the set of transitory vertices is $V \backslash V_c = \{14, 20\}$.*

Remark 16 *We note that the statements in Theorem 11 take on this simple form because we have defined orbits in Definition 4 as starting with paths of length 1, not 0. If we include paths of length 0 in an orbit, then it always holds that $i \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$. This trivial situation then needs to be excluded in Theorem 11. Similarly, we have defined communicating classes in Definition 9 using mutual access, i.e. a vertex $i \in V$ satisfies $i \in V_c$ if there exists a path* *of length ≥ 1 from i to i. This avoids the triviality that each vertex communicates with itself. Note that for systems on continuous state spaces one needs separate non-triviality conditions, such as the existence of an infinite path within a "communicating class" and a condition on the richness of the orbits, see the discussion in [8], Chapter 3 for control systems. In our context, the existence of communicating classes, however, requires a non-degeneracy condition, compare the next section.*

2.2.2 Communicating Sets in $L-$Graphs

Definition 2 states a non-degeneracy condition on the orbits of a graph that will ensure the existence of communicating classes (with certain additional properties). This non-degeneracy condition plays the same role in our present context that is played by the accessibility condition for continuous control systems, compare [8], Chapter 3 and Appendix A. This section explores communication structures within $L-$graphs, starting with the idea of a loop.

Lemma 17 *Each $L-$graph has paths of arbitrary length.*

Proof. Let G be an $L-$graph and $n \geq 1$. Pick $i_0 \in V$, then by definition $O^\circ(i_0) \geq 1$. This ensures the existence of $i_1 \in V$ such that (i_0, i_1) is an incidence in G. Using the same argument we see that there is a vertex $i_2 \in V$ and an incidence (i_1, i_2). Continuing with this process up to step n we infer the existence of a path

$$\gamma = \langle\ i_0, (i_0, i_1), i_1, (i_1, i_2), ..., i_{n-2}, (i_{n-1}, i_n), i_n\ \rangle$$

or equivalently,

$$\gamma = \langle\ i_0 ... i_n\ \rangle$$

with $\ell(\gamma) = n$. ■

Definition 18 *A path γ of length $\ell(\gamma) = n$, $n \geq 1$, is said to be a loop if there exists a vertex $i \in \gamma$ such that $\pi_0(\gamma) = \pi_n(\gamma) = i$.*

Lemma 19 *In a graph G with $\#G = d$ any path γ of length $\ell(\gamma) = n$, $n \geq d$ contains a loop.*

Proof. We consider a path γ in G such that

$$\gamma = \langle\ i_0, i_1, ..., i_d\ \rangle$$

with $\ell(\gamma) = d$. Assume that the subpath $\langle\ i_0...i_{d-1}\ \rangle$ contains no loop (otherwise we are done). Then all the vertices of $\langle\ i_0...i_{d-1}\ \rangle$ are distinct and hence $\{\ i_0, ..., i_{d-1}\ \} = V$. Now $i_d \in V$ implies that there exists $\alpha \in \{0, ..., d-1\}$ with $i_d = i_\alpha$. Hence the subpath $\langle\ i_\alpha...i_d\ \rangle$ is a loop contained in γ. ■

The next three lemmata explore the relationship between loops and communicating classes, leading to the existence of communicating classes in $L-$graphs.

Lemma 20 *Given a graph G and a loop λ in G, then there exists a communicating class $C \subset G$ such that the vertices in λ are contained in C.*

Proof. Consider a loop $\lambda = \langle\ i_0, i_1, ..., i_n, i_0\ \rangle$. Since the elements in λ have mutual access each other, we have $i_k \in [\ i_0\ ]$ for $0 \leq k \leq n$. Therefore each i_k belongs to the same communicating class $C = [\ i_0\ ]$. ■

Lemma 21 *Let G be a graph and $\gamma = \langle\ i_0...i_n\ \rangle$ a path in G. If there is a communicating class $C \subset V$ with $i_0 \in C$, and if there is $\alpha \in \{1, ..., n\}$ with $i_\alpha \notin C$, then $i_\beta \notin C$ for all $\beta \in \{\alpha, ..., n\}$.*

Proof. Using the notation of the statement of the lemma, assume, to the contrary, that there exists $\beta \geq \alpha$ with $i_\beta \in C$. Then there are paths $\gamma_1 = \langle\ i_0...i_\alpha\ \rangle$ and $\gamma_2 = \langle\ i_\alpha...i_\beta...i_0\ \rangle$, showing that $i_0 \sim i_\alpha$ and hence $i_\alpha \in C$, which is a contradiction. ■

Lemma 22 *Let G be a graph. If $i \in V$ belongs to a communicating class $C \subset G$, then there exists a loop λ in C such that $i \in \lambda$.*

Proof. Let G be a graph and C a communicating class in G. Then there is pairwise communication between the elements in C, i.e., given $i, j \in C$ there exists a path $\gamma_{ij} = \langle\ i...j\ \rangle$. In particular, for $i = j$ we have the path $\gamma_{ii} = \langle\ i...i\ \rangle$ with $\pi_0(\gamma) = \pi_n(\gamma) = i$ for some $n \geq 1$. Hence γ is indeed a loop containing i. By Lemma 21 all components of this loop are in C. ■

Proposition 23 *An $L-$graph G has at least one communicating class.*

Proof. Consider an $L-$graph G with $\#V = d$. By Lemma 17 there exists a path γ such that $\ell(\gamma) = n$ with $n \geq d$. By Lemma 19 the path γ contains a loop λ, and by Lemma 20 there exist a communicating class containing the vertices of λ. ■

Example 24 *(Continuation of Example 3) For this graph we obtain eight communicating classes:*

$$
\begin{aligned}
C_1 &= \{1, 2\} \\
C_2 &= \{3, 4, 5\} \\
C_3 &= \{6, 7, 8, 9\} \\
C_4 &= \{10\} \\
C_5 &= \{11, 12, 13\} \\
C_6 &= \{15, 16\} \\
C_7 &= \{17, 18, 19\} \\
C_8 &= \{21\}.
\end{aligned}
$$

Remark 25 *The proof of Proposition 23 actually shows the stronger statement: Let G be an $L-$graph and $i \in V$. Then there exists at least one communicating class C with $C \subset \mathcal{O}^+(i)$. Note that, in general, $i \in \mathcal{O}^+(i)$ may not hold.*

Example 26 *(Continuation of Example 3) To illustrate the Remark, we consider the vertex* 20*: We have $\mathcal{O}^+(20) = \{17, 18, 19\}$, which is itself a communicating class, but $20 \notin \{17, 18, 19\}$.*

As a final idea of this section we explore an order on the set of communicating classes, which will lead to a characterization of so-called forward invariant classes.

Definition 27 *Let G be a graph with a family $\mathcal{C} = \{C_1, ..., C_k\}$ of communicating classes. We define a relation on $\mathcal{C}$ by*

$$
\begin{aligned}
C_\mu \preceq C_\nu \quad &\text{if} \quad \text{there exists a path } \gamma \in \Gamma^n \text{ with} \\
&\pi_0(\gamma) \in C_\mu \text{ and } \pi_n(\gamma) \in C_\nu.
\end{aligned}
$$

Lemma 28 *Let G be a graph with a family $\mathcal{C} = \{C_1, ..., C_k\}$ of communicating classes. The relation $\preceq$ defines a (partial) order on $\mathcal{C}$.*

Proof.

(i) Reflexivity: Let C_μ be a communicating class in G. By Lemma 22, for any $i \in C_\mu$ there exists a loop λ in C_μ such that $i \in \lambda$. Since C_μ is a communicating class, the loop λ gives us a path form C_μ to itself, and therefore $C_\mu \preceq C_\mu$.

(ii) Antisymmetry: Assume that $C_\mu \preceq C_\nu$ and $C_\nu \preceq C_\mu$. From the first relation we get a path $\gamma_{\mu\nu} \in \Gamma^p$ for some $p \geq 1$ such that

$$\pi_0(\gamma_{\mu\nu}) \in C_\mu \quad \text{and} \quad \pi_p(\gamma_{\mu\nu}) \in C_\nu.$$

By the second relation there exists a path $\gamma_{\nu\mu} \in \Gamma^q$ for some $q \geq 1$ with

$$\pi_0(\gamma_{\nu\mu}) \in \mathcal{C}_\nu \quad \text{and} \quad \pi_q(\gamma_{\nu\mu}) \in \mathcal{C}_\mu.$$

Hence the concatenation $\langle \gamma_{\mu\nu}, \gamma_{\nu\mu} \rangle$ is a path in C_μ of length $p+q$. Since communicating classes are maximal, and any two vertices in a communicating class have mutual access, it holds that $\mathcal{C}_\mu = \mathcal{C}_\nu$.

(iii) Transitivity: Let $\mathcal{C}_\mu$, $\mathcal{C}_\nu$, $\mathcal{C}_\xi$ in $\mathcal{C}$ and suppose that $C_\mu \preceq C_\nu$ and $C_\nu \preceq C_\xi$ hold. Since $C_\mu \preceq C_\nu$, there exists a path γ_1 such that $\pi_0(\gamma_1) \in C_\mu$ and $\pi_{n_1}(\gamma_1) \in C_\nu$ for some $n_1 \in \mathbb{N}$. Moreover, since $C_\nu \preceq C_\xi$, there exists a path γ_3 such that $\pi_0(\gamma_3) \in C_\nu$ and $\pi_{n_3}(\gamma_3) \in C_\xi$ for some $n_3 \in \mathbb{N}$. Since C_ν is a communicating class, there exists a path γ_2 in C_ν such that $\pi_0(\gamma_2) = \pi_{n_1}(\gamma_1)$ and $\pi_{n_2}(\gamma_2) = \pi_0(\gamma_3)$ for some $n_2 \in \mathbb{N}$. The path $\gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ is such that $\pi_0(\gamma) \in C_\mu$ and $\pi_m(\gamma) \in C_\xi$ for $m = n_1 + n_2 + n_3$. Therefore, $C_\mu \preceq C_\xi$.

■

Example 29 *(Continuation of Example 3) For this graph we obtain the following order relations among the communicating classes: $C_1 \preceq C_3$, $C_2 \preceq C_3$, $C_3 \preceq C_4 \preceq C_5$, $C_5 \preceq C_6$, $C_5 \preceq C_7$. The class C_8 cannot be compared with any of the other classes.*

Recall that an order relation on a finite set has maximal elements (compare, e.g. [11]). We now characterize these maximal communicating classes.

Definition 30 *A set of vertices U of G, is called forward invariant if*

$$\mathcal{O}^+(U) \subset U.$$

Similarly, U is called backward invariant if

$$\mathcal{O}^-(U) \subset U,$$

and invariant if

$$\mathcal{O}^+(U) \cup \mathcal{O}^-(U) \subset U.$$

Remark 31 *Note that, by definition, a forward invariant communicating class is maximal with respect to the order $\preceq$ introduced in Definition 27.*

Proposition 32 *Each $L-$graph G contains a forward invariant communicating class.*

Proof. Consider an $L-$graph G with set of communicating classes $\mathcal{C} = \{C_1, ..., C_k\}$. By [11] $(\mathcal{C}, \preceq)$ possesses a maximal element C_μ. We show that C_μ is forward invariant: Assume to the contrary that C_μ is not forward invariant. Since then $\mathcal{O}^+(C_\mu)$ is not contained in C_μ,there exist $i, j_0 \in V$ with $i \in C_\mu$ and $j_0 \notin C_\mu$ such that $(i, j_0) \in E$. Since $O^\circ(j_0) \geq 1$, there exists $j_1 \in V$ such that $(j_0, j_1) \in E$. Note that by Lemma 21 j_1 cannot belong to C_μ. By Remark 25 there exists a communicating class $C \subset \mathcal{O}^+(j_0)$ and $C \cap C_\mu = \varnothing$ by Lemma 21. It follows that, $C_\mu \preceq C$, which contradicts maximality of C_μ. ■

Remark 33 *Note that in the proof of Proposition 32 we actually showed the stronger statement: Let G be an $L-$graph and $i \in V$. Then $\mathcal{O}^+(i)$ contains a forward invariant communicating class.*

Remark 34 *Summarizing Remark 31 and Proposition 32 we see that for an $L-$graph the maximal elements of $(\mathcal{C}, \preceq)$ are exactly the forward invariant communicating classes. It also follows directly from Definition 27 that backward invariant communicating classes are minimal* *in $(\mathcal{C}, \preceq)$. Note, however, that minimal communicating classes in $(\mathcal{C}, \preceq)$ need not be backward invariant. For this fact to hold we would need a backward nondegeneracy condition similar to the $L-$graph property, e.g., using the in-degree of vertices. For an analogue of this issue in the theory of control systems in discrete time compare [1].*

Example 35 *(Continuation of Example 3) For this graph we obtain the following results: The maximal (with respect to $\preceq$) and hence forward invariant communicating classes are C_6, C_7 and C_8. The minimal (with respect to $\preceq$) communicating classes are C_1, C_2 and C_8. In the example, these classes are also backward invariant. However, if we extend the graph $G = (V, E)$ to a graph $G' = (V', E')$ with $V' = V \cup \{22\}$, $E' = E \cup \{22, 1\}$, then $C_1 = \{1, 2\}$ is still a minimal communicating class in G' but it is not backward invariant in G'.*

The concept of connectedness, as used in graph theory, will be useful in Chapter 3.

Definition 36 *A graph $G = (V, E)$ is called strongly connected if for any $i, j \in S$ we have $j \in \mathcal{O}^+(i)$ (i.e. if the graph consists of one communicating class), and connected if its symmetric graph $G^s = (V, E \cup E^T)$ is strongly connected. Here $(i, j) \in E^T$ iff $(j, i) \in E$.*

Example 37 *(Continuation of Example 3) The graph of this example is not connected. It has two connected components, the vertex sets $Z_1 = \{1, ..., 20\}$ and $Z_2 = \{21\}$.*

2.2.3 Quotient Graphs

There are (at least) two quotient structures associated with the idea of communicating sets in a graph. The first idea is to simply take the order graph G_q of $(\mathcal{C}, \preceq)$ as defined in Definition 27. Equivalently, this graph is obtained as the quotient $V_c/\sim$. Hence G_q does not necessarily cover all the vertices of a given graph G, and its edges may not be edges of G.

Example 38 *(Continuation of Example 3) In this example the quotient (or order) graph $G_q = (E_q, V_q)$ is of the form*

$$E_q = \{C_1, ..., C_8\}$$

$$V_q = \{(C_1, C_3), (C_2, C_3), (C_3, C_4), (C_4, C_5), (C_5, C_6), (C_5, C_7)\}.$$

The edge (C_5, C_6) of G_q does not correspond to any edge between vertices of C_5 and C_6 in the graph G. And the vertices 14 and 20 are not represented in G_q.

This leads to the concept of an extended quotient graph G_Q, whose vertices include the communicating classes as well as the transitory vertices of G:
For a given directed graph $G = (V, E)$ we define $V_Q := \mathcal{C} \cup V \backslash V_c = \{C,\, C$ is a communicating class$\} \cup \{i \in V,\, i$ is a transitory vertex$\}$ as a set of vertices. The set of edges is constructed as follows: For $A, B \in V_Q$ we set $(A, B) \in E_Q$ if there exist $i \in A$ and $j \in B$ with $(i, j) \in E$. (Note the abuse of notation: if $A \in V_Q$ is a (transitory) vertex of G then "$i \in A$" is to be interpreted as "$i = A$"). The graph $G_Q = (V_Q, E_Q)$ is called the extended quotient graph of G. It is easily seen that the extended quotient graph of G_Q is G_Q itself. All vertices in V_Q have specific interpretations in the context of Markov chains, see Section 3.3.

Example 39 *(Continuation of Example 3) In this example the extended quotient graph $G_Q = (E_Q, V_Q)$ is of the form*

$$
\begin{aligned}
E_Q &= \{C_1, ..., C_8, 14, 20\} \\
V_Q &= \{(C_1, C_3), (C_2, C_3), (C_3, C_4), (C_4, C_5), (C_5, C_7), \\
&\quad (C_5, 14), (14, C_6), (20, C_7)\}.
\end{aligned}
$$

2.3 Semiflows and Morse Decompositions

Our second approach to study decompositions of graphs is based on an idea from the theory of dynamical systems, namely Morse decompositions. A Morse decomposition describes the global behavior of a dynamical system, i.e. the limit sets of a system and the flow between these sets. A Morse decomposition results in an order among the components (the Morse sets) of the decomposition. It can be constructed from attractors and repellers, and the behavior of the system on the Morse sets is characterized by (chain) recurrence.
In this section we first review briefly the idea of Morse decompositions for (continuous) dynamical systems. We then construct an analogue for (discrete) systems defined by directed

$L-$graphs. Unfortunately, this analogy is not complete since systems defined by directed graphs only lead to semiflows, i.e. systems for which the time set is $\mathbb{N}$, and not all of $\mathbb{Z}$.

2.3.1 Overview of Morse Decompositions for Continuous Dynamical Systems

For more details on the material of this section we refer to [4].
Throughout this section we assume that X is a compact, complete metric space.

Definition 40 *A flow or continuous time dynamical system on a metric space X is given by a continuous map $\Phi : \mathbb{R} \times X \to X$ that satisfies $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $x \in X$ and all $t, s \in \mathbb{R}$.*

The orbit of a point $x \in X$ is the set $\mathcal{O}(x) := \{y \in X$, there is $t \in \mathbb{R}$ with $y = \Phi(t, x)\}$.

Definition 41 *A set $K \subset X$ is called invariant if $\mathcal{O}(x) \subset K$ for all $x \in K$; a compact subset $K \subset X$ is called isolated invariant, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int} N$, such that $\mathcal{O}(x) \subset N$ implies $x \in K$.*

Thus an invariant set K is isolated invariant if every trajectory that remains close to K actually belongs to K.

Definition 42 *The ω-limit set of a subset $Y \subset X$ is defined as*

$$\omega(Y) = \left\{ y \in X, \begin{array}{c} \text{there are } t_k \to \infty \text{ and } y_k \in Y \\ \text{such that } \Phi(t_k, y_k) \to y \end{array} \right\},$$

and similarly

$$\omega^*(Y) = \left\{ y \in X, \begin{array}{c} \text{there are } t_k \to -\infty \text{ and} y_k \in Y \\ \text{such that } \Phi(t_k, y_k) \to y \end{array} \right\}.$$

Definition 43 *A Morse decomposition of a flow on a compact metric space is a finite collection $\{\mathcal{M}_i,\ i = 1, ..., n\}$ of nonvoid, pairwise disjoint, and compact isolated invariant sets such that:*
(i) For all $x \in X$ one has $\omega(x), \omega^*(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$.

(ii) Suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l}$ *and* $x_1, ..., x_l \in X \setminus \bigcup_{i=1}^{n} \mathcal{M}_i$ *with* $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ *and* $\omega(x_i) \subset \mathcal{M}_{j_i}$ *for* $i = 1, ..., l$*; then* $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements of a Morse decomposition are called Morse sets.

Thus the Morse sets contain all limit sets and "cycles" are not allowed. As an easy consequence of this definition we obtain the following equivalent characterization.

Proposition 44 *A finite collection* $\{\mathcal{M}_i,\ i = 1, ..., n\}$ *of nonvoid, pairwise disjoint, and compact isolated invariant sets is a Morse decomposition if and only if*

1. *condition (i) in Definition 43 holds;*

2. $\omega^*(x) \cup \omega(x) \subset \mathcal{M}_i$ *implies* $x \in \mathcal{M}_i$*; and*

3. *the following relation "*$\preceq$*" is a (partial) order*

$$\mathcal{M}_i \preceq \mathcal{M}_k \text{ if } \begin{array}{c} \text{there are } \mathcal{M}_{j_0} = \mathcal{M}_i, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_k \text{ and } x_1, ..., x_l \in X \\ \text{with } \omega^*(x_k) \subset \mathcal{M}_{j_{k-1}} \text{ and } \omega(x_k) \subset \mathcal{M}_{j_k} \text{ for } k = 1, ..., l. \end{array}$$

Definition 45 *A Morse decomposition* $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ *is called finer than a Morse decomposition* $\{\mathcal{M}'_1, ..., \mathcal{M}'_{n'}\}$*, if for all* $j \in \{1, ..., n'\}$ *there is* $i \in \{1, ..., n\}$ *with* $\mathcal{M}_i \subset \mathcal{M}'_j$.

If a flow admits a finest Morse decomposition, this decomposition is unique.

Morse decompositions can be constructed from attractors and their complementary repellers.

Definition 46 *For a flow on a compact metric space* X *a compact invariant set* A *is an attractor if it admits a neighborhood* N *such that* $\omega(N) = A$*. A repeller is a compact invariant set* R *that has a neighborhood* N^* *with* $\omega^*(N^*) = R$.

We also allow the empty set as an attractor. A neighborhood N as in Definition 46 is called an *attractor neighborhood*. Every attractor is compact and invariant, and a repeller is an attractor for the time reversed flow.

Definition 47 *For an attractor* A*, the set* $A^* = \{x \in X,\ \omega(x) \cap A = \varnothing\}$ *is a repeller, called the complementary repeller. Then* (A, A^*) *is called an attractor-repeller pair.*

Theorem 48 *For a flow on a compact metric space X a finite collection of subsets $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors*

$$\varnothing = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n = X,$$

such that

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n-1.$$

Corollary 49 *Let $\{\mathcal{M}_i,\, i = 1, ..., n\}$ be the finest Morse decomposition of a flow on a compact metric space, with order $\preceq$. Then the maximal (with respect to $\preceq$) Morse sets are attractors, and the minimal Morse sets are repellers.*

Remark 50 *Note that any dynamical system of the form described in Definition 40 has at least the trivial attractor-repeller pair $A = X$ and $A^* = \varnothing$. Theorem 48 implies that a system with a finite number of attractors has a (unique) finest Morse decomposition.*

Theorem 48 characterizes the global behavior of continuous dynamical systems with a finite number of attractors, i.e. the behavior outside of the Morse sets. The behavior of a system on a Morse set is given by a certain recurrence property, called chain recurrence.

Definition 51 *For $x,\, y \in X$ and $\varepsilon, T > 0$ an (ε, T)-chain from x to y is given by a natural number $n \in \mathbb{N}$, together with points*

$$x_0 = x, x_1, ..., x_n = y \in X \text{ and times } T_0, ...T_{n-1} \geq T,$$

such that $d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, ..., n-1$. Here $d(\cdot,\cdot)$ denotes the metric on X.

Note that the number n of "jumps" is not a priori bounded. Hence one may introduce "trivial jumps"$\dot{}$Furthermore, as the notation suggests, only small values of $\varepsilon > 0$ are of interest.

Definition 52 *A subset $Y \subset X$ is chain transitive if for all $x, y \in Y$ and all $\varepsilon, T > 0$ there exists an (ε, T)-chain from x to y. A point $x \in X$ is chain recurrent if for all $\varepsilon, T > 0$ there exists an (ε, T)-chain from x to x. The chain recurrent set $\mathcal{R}$ is the set of all chain recurrent points.*

Theorem 53 *The chain recurrent set $\mathcal{R}$ satisfies*

$$\mathcal{R} = \bigcap \{A \cup A^*, \;\; A \textit{ is an attractor}\}.$$

In particular, there exists a finest Morse decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ if and only if the chain recurrent set $\mathcal{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathcal{R}$ and the flow restricted to any Morse set $\mathcal{M}$ is chain transitive and chain recurrent, i.e. all points $x \in \mathcal{M}$ are chain recurrent points.

2.3.2 Semiflows Associated with Graphs

When trying to adapt the idea of Morse decompositions and attractors / repellers to systems induced by directed graphs, one faces two main challenges: The first concerns the topology on discrete spaces that has some interesting consequences for limit sets, isolated invariant sets, etc. The second challenge stems from the fact that the out-degree of vertices can be > 1, resulting in set-valued systems. This is the reason why graphs define semiflows (on $\mathbb{N}$ instead of $\mathbb{Z}$). We first define these semiflows and show in an example, why in general they cannot be extended to flows on sets of vertices.

Let $G = (V, E)$ be a directed graph as introduced in Section 2.1. We denote by $\mathcal{S} := \mathfrak{P}(V)$ the power set of the vertex set, which we endow with the discrete topology. (Recall that the discrete topology defines all subsets $\mathcal{A} \subset \mathfrak{P}(V)$ to be open.) We define the discrete metric (on $\mathfrak{P}(V)$) as

$$\rho(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

for $x, y \in \mathfrak{P}(V)$. Observe that the topology derived from the discrete metric on $\mathcal{S}$ is the same as the discrete topology on $\mathcal{S}$. For a detailed discussion we refer to [12].

The directed graph $G = (V, E)$ gives rise to two semiflows, one with the positive integers $\mathbb{N}$ as time set, and one with the negative integers $\mathbb{N}^- := \{-n, \, n \in \mathbb{N}\}$:

$$\Phi_G : \mathbb{N} \times \mathcal{S} \to \mathcal{S},$$

$$\Phi_G(n,A) = \left\{ \begin{array}{c} j \in V : \quad \exists\, i \in A \text{ and } \gamma \in \Gamma^n \text{ such that} \\ \pi_0(\gamma) = i \text{ and } \pi_n(\gamma) = j \end{array} \right\}. \tag{2.3}$$

Similarly, we define

$$\Phi_G^- : \mathbb{N}^- \times \mathcal{S} \to \mathcal{S},$$

$$\Phi_G^-(n,A) = \left\{ \begin{array}{c} j \in V : \quad \exists\, i \in A \text{ and } \gamma \in \Gamma^{-n} \text{ such that} \\ \pi_0(\gamma) = j \text{ and } \pi_{-n}(\gamma) = i \end{array} \right\}. \tag{2.4}$$

We note that the definition of the semiflows $\Phi_G(n,A)$ and $\Phi_G^-(n,A)$ only requires the basic ingredients of a graph: the sets of vertices and of edges. We have formulated (2.3) and (2.4) in terms of paths for convenience - we could as well have used the idea of iterated function systems, compare Section 2.4.

Example 54 *(Continuation of Example 3) Let* $A = \{5, 12, 13\}$ *and* $n = 5$*, then we compute* $\Phi_G(n,A) = \{3,4,5,6,7,8,9,10,11,12,15,16,17,18,19\} \in \mathcal{S}$ *as an example for the definition of images under the semiflow* Φ_G*.*

The next proposition collects some properties of the maps defined in (2.3) and (2.4).

Proposition 55 *Consider a graph* $G = (V, E)$ *and the associated map* Φ_G *defined in (2.3). This map is a semiflow, i.e. it has the properties*

1. Φ_G *is continuous,*
2. $\Phi_G(0, A) = A$ *for all* $A \in \mathcal{S}$*,*
3. $\Phi_G(n+m, A) = \Phi_G(n, \Phi_G(m, A))$ *for all* $A \in \mathcal{S}$*,* $m, n \in \mathbb{N}$*.*

The same properties hold for the negative semiflow Φ_G^-*.*

Proof. Note that in the discrete topology every function is continuous due to the fact that every element of $\mathfrak{P}(V)$ is an open set, proving statement 1. Property 2. holds by definition of Γ^0 in (2.2): The vertices reached from A under the flow at time zero, are the elements in V that belong to A.

To show that property 3. holds, we first assume that the three sets $\Phi_G(m,A)$, $\Phi_G(n+m,A)$, and $\Phi_G(n,\Phi_G(m,A))$ are nonempty. Consider $j \in \Phi_G(n+m,A)$: There exist $i \in A$ and a path $\alpha \in \Gamma^{n+m}$ such that $\pi_0(\alpha) = i$ and $\pi_{n+m}(\alpha) = j$. We split α as the concatenation of two paths β and γ with $\ell(\beta) = m$ and $\ell(\gamma) = n$, having, $\pi_0(\beta) = i$, $\pi_m(\beta) = k$ and $\pi_0(\gamma) = k$, $\pi_n(\gamma) = j$ for some $k \in V$. Observe that by definition we have $k \in \Phi_G(m,A)$ and hence $j \in \Phi_G(n,\Phi_G(m,A))$. On the other hand, if $j \in \Phi_G(n,\Phi_G(m,A))$ then there exist $k \in \Phi_G(m,A)$ and $\beta \in \Gamma^n$ such that $\pi_0(\beta) = k$ and $\pi_n(\gamma) = j \in \Phi_G(n,\Phi_G(m,A))$. Since $k \in \Phi_G(m,A)$, there are $i \in A$ and $\alpha \in \Gamma^m$ such that $\pi_0(\alpha) = i$ and $\pi_m(\alpha) = k$. The concatenation $\gamma = \langle \alpha, \beta \rangle$ satisfies $\pi_0(\gamma) = i$ and $\pi_{m+n}(\gamma) = j$, hence $j \in \Phi_G(n+m,A)$.

To finish the proof for Φ_G, we consider the case that (at least) one of the three sets $\Phi_G(m,A)$, $\Phi_G(n,\Phi_G(m,A))$, and $\Phi_G(n+m,A)$ is empty: Note first of all that by the definition of Φ_G in (2.3) we have $\Phi_G(n,\varnothing) = \varnothing$ for all $n \in \mathbb{N}$. (i) If $\Phi_G(m,A) = \varnothing$ then we have $\Phi_G(n,\Phi_G(m,A)) = \varnothing$ by the preceding argument. Now if $\Phi_G(n+m,A) \neq \varnothing$, then we can construct, as in the previous paragraph, a point $k \in \Phi_G(m,A)$ by splitting a path $\alpha \in \Gamma^{n+m}$ with $\pi_0(\alpha) \in A$ and $\pi_{n+m}(\alpha) \in \Phi_G(n+m,A)$. This contradicts $\Phi_G(m,A) = \varnothing$ and therefore $\Phi_G(n+m,A) = \varnothing$. (ii) Assume next that $\Phi_G(n,\Phi_G(m,A)) = \varnothing$. Using the same reasoning from the previous paragraph, we see that if $j \in \Phi_G(n+m,A)$ then $j \in \Phi_G(n,\Phi_G(m,A))$. Hence it holds that $\Phi_G(n+m,A) = \varnothing$. (iii) If $\Phi_G(n+m,A) = \varnothing$ then there exists no path $\gamma \in \Gamma^{n+m}$ with $\pi_0(\gamma) \in A$. But by the reasoning in the previous paragraph, if $j \in \Phi_G(n,\Phi_G(m,A))$ then there exist $i \in A$ and $\gamma \in \Gamma^{m+n}$ such that $\pi_0(\gamma) = i$ and $\pi_{m+n}(\gamma) = j$, which cannot be true, and hence we see that $\Phi_G(n,\Phi_G(m,A)) = \varnothing$.

The proof for Φ_G^- follows the same lines. ■

Remark 56 *If $G = (V,E)$ is an $L-$graph, then $\Phi_G(n,A) \neq \varnothing$ for $A \neq \varnothing$ and $n \in \mathbb{N}$. This observation may not hold for the negative semiflow Φ_G^- without additional assumptions.*

One might wonder if the semiflows Φ_G defined in (2.3) and Φ_G^- from (2.4) can be combined to a flow on $\mathbb{Z}$. The following example shows that, in general, this is not possible, even if the graph G has additional properties (such as being an $L-$graph) or if one restricts oneself to graphs that are one communicating class.

Example 57 *Consider the graph $G = (V, E)$ with $V = \{1, 2, 3\}$ and*

$$E = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}.$$

This graph is an $L-$graph and it also satisfies $I^{\circ}(i) \geq 1$ for all $i \in V$. Consider $A = \{2\} \in \mathcal{S} = \mathfrak{P}(V)$. We compute $\Phi_G(n, A) = A$ for all $n \in \mathbb{N}$, $\Phi_G^-(-1, A) = \{1, 2\} =: B$, and $\Phi_G(n, B) = \{1, 2, 3\}$ for all $n \geq 1$. But $\{2\} = A = \Phi_G(1, A) = \Phi_G(2-1, A) \neq \Phi_G(2, \Phi_G^-(-1, A)) = \{1, 2, 3\}$. Note that this graph has three communicating classes.

We extend this example by considering the graph $G' = (V', E')$ with $V' = \{1, 2, 3, 4\}$ and $E' = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 4), (3, 3), (3, 4), (4, 1)\}$. This graph is again an $L-$graph and it also satisfies $I^{\circ}(i) \geq 1$ for all $i \in V'$. Consider again $A = \{2\}$: As before we obtain $\{2, 4\} = \Phi_{G'}(2-1, A) \neq \Phi_{G'}(2, \Phi_{G'}^-(-1, A)) = \{1, 2, 3, 4\}$. Note that this graph has exactly one communicating class $C = V$.

Remark 58 *The semiflow Φ_G^- as defined in (2.4) can be interpreted as the positive semiflow of the graph $G^T = (V, E^T)$, where $(i, j) \in E^T$ iff $(j, i) \in E$. Φ_G^- is sometimes called the time-reverse semiflow of Φ_G. Under the corresponding assumptions, all statements for a positive semiflow also hold for its time-reverse counterpart.*

2.3.3 Morse Decompositions of Semiflows

We next turn to some concepts from the theory of dynamical systems and study their analogues for the semiflows defined in (2.3) (and (2.4)). Since the state space $\mathcal{S} = \mathfrak{P}(V)$ of the semiflow Φ_G is finite with the discrete topology, it suffices to introduce the concepts for points $A \in \mathcal{S}$. To avoid trivial situations where $\Phi_G(n, A) = \varnothing$ for some $n \in \mathbb{N}$ and $A \in \mathcal{S}$, $A \neq \varnothing$, we assume that all graphs are $L-$graphs, compare Remarks 56 above and 62 below.

Invariance: A point $A \in \mathcal{S}$ is said to be (forward) invariant if $\Phi_G(n, A) \subset A$ for all $n \in \mathbb{N}$. Note that $A \in \mathcal{S}$ is invariant under Φ_G iff A is a forward invariant set of the underlying graph $G = (V, E)$, compare Definition 30. Hence invariance under Φ_G is a fairly strong requirement of a set $A \in \mathcal{S}$. As we will see, a meaningful Morse decomposition of the semiflow Φ_G only requires a weak form of invariance.

Definition 59 *A point $A \in \mathcal{S}$ is said to be weakly invariant if for all $n \in \mathbb{N}$ we have $\Phi_G(n, A) \cap A \neq \varnothing$.*

Isolated invariance: For a (forward) invariant set $A \in \mathcal{S}$ one could define "forward isolated invariant" in analogy to Definition 41. But because of the discrete topology, we could choose $N(A) = A$, and any forward invariant set then satisfies this property. As we will see, because of the discrete topology employed, a meaningful Morse decomposition of the semiflow Φ_G does not require the property of isolated invariance.

Limit sets: To adapt the concept of a limit set from Definition 42 to the semiflow Φ_G, note that a sequence converges in the discrete topology iff it is eventually constant. Hence limit sets can be defined in the following way:

Definition 60 *The $\omega-$limit set of a point $A \in \mathcal{S}$ under Φ_G is defined as*

$$\omega(A) = \left\{ y \in V, \begin{array}{c} \textit{there are } t_k \to \infty \\ \textit{such that } y \in \Phi(t_k, A) \end{array} \right\} \in \mathcal{S}.$$

Remark 61 *Note that by definition of the discrete topology we have for $A \in \mathcal{S}$ the fact $\omega(A) = \cup\{\omega(\{i\}),\ i \in A\}$.*

Remark 62 *The existence of $\omega-$limit sets and the $L-$graph property are closely related: Let $G = (V, E)$ be a graph and Φ_G its associated semiflow. Then G is an $L-$graph iff $\omega(A) \neq \varnothing$ for all $A \in \mathcal{S} = \mathfrak{P}(V)$. This observation justifies our concentration on $L-$graphs in this section.*

For continuous dynamical systems Morse decompositions are required to contain all of the $\omega-$ and ω^*-limit sets of the system, compare property (i) of Definition 43 and Proposition 44. For semiflows induced by graphs a weaker condition of recurrence turns out to be appropriate:

Definition 63 *Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. A one-point set $\{i\} \in \mathcal{S}$ is called recurrent, if there exists a sequence n_l in $\mathbb{N}$, $n_l \to \infty$, such that $\{i\} \subset \Phi_G(n_l, \{i\})$. A set $B \in \mathcal{S}$ is called recurrent if for each $i \in B$ the one-point set $\{i\}$ is* *recurrent under* Φ_G. *The set* $\mathcal{R} := \{i \in V,\ \{i\}$ *is recurrent*$\}$ *is called the recurrent set of* Φ_G. *If* $\mathcal{R} = V$ *the semiflow* Φ_G *is called recurrent.*

Note that by Definition 63 it holds that $\{i\} \in \mathcal{S}$ is recurrent iff $i \in \omega(\{i\})$ iff $i \in \lambda$ for some loop λ of G.

No-cycle condition: For continuous dynamical systems the no-cycle property (ii) of Definition 43 is essential for the characterization of a Morse decomposition via an order, compare Proposition 44. For semiflows induced by graphs we can formulate an analogue of the no-cycle condition either using only the (forward) semiflow Φ_G or a combination of Φ_G and Φ_G^-.

Definition 64 *Consider the semiflow* Φ_G *and a finite collection* $\mathcal{A} = (A_1, ..., A_n)$ *of points in* $\mathcal{S}$. $\mathcal{A}$ *is said to satisfy the no-cycle condition for* Φ_G *if for any subcollection* $A_{j_0}, ..., A_{j_l}$ *of* $\mathcal{A}$ *with* $\omega(A_{j_\alpha}) \cap A_{j_{\alpha+1}} \neq \varnothing$ *for* $\alpha = 0, ..., l-1$ *it holds that* $A_{j_0} \neq A_{j_l}$.

Remark 65 *Alternatively, we can define* $A \in \mathcal{S}$ *to be a no-return set if for all one-point sets* $\{i\} \in \mathcal{S}$ *we have: If* $\omega^*(\{i\}) \cap A \neq \varnothing$ *and* $\omega(\{i\}) \cap A \neq \varnothing$ *then* $\{i\} \subset A$, *where* $\omega^*(B)$ *is the* $\omega-$*limit set for* $B \in \mathcal{S}$ *under the negative semiflow* Φ_G^-. *This definition mimics the no-cycle property (ii) of Definition 43, compare also property (ii) in Proposition 44. We have chosen Definition 64 because it uses exclusively the positive semiflow.*

With these preparations we can now introduce our concept of a Morse decomposition of the semiflow Φ_G.

Definition 66 *Let* $G = (V, E)$ *be an* $L-$*graph. A Morse decomposition of the semiflow* Φ_G *on* $\mathcal{S} = \mathfrak{P}(V)$ *is a finite collection of nonempty, pairwise disjoint and weakly invariant sets* $\{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ *such that (i)* $\mathcal{R} \subset \cup_{\mu=1}^k \mathcal{M}_\mu$ *and (ii)* $\{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ *satisfies the no-cycle condition from Definition 64.*
The elements of a Morse decomposition are called Morse sets.

Proposition 67 *Let $G = (V, E)$ be an $L-$graph and let $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ be a finite collection of nonempty, pairwise disjoint and weakly invariant sets of the semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. The collection $\mathcal{M}$ is a Morse decomposition of Φ_G iff the following properties hold: (i) $\mathcal{R} \subset \cup_{\mu=1}^{k} \mathcal{M}_\mu$ and (ii) the relation "$\preceq$" defined by*

$$\mathcal{M}_\alpha \preceq \mathcal{M}_\beta \text{ if } \begin{array}{c} \text{there are } \mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_\beta \text{ in } \mathcal{M} \\ \text{with } \omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \varnothing \text{ for } i = 0, ..., l-1 \end{array}$$

is a (partial) order on $\mathcal{M}$.
We use the indices $\mu = 1, ..., k$ in such a way that they reflect this order, i.e. if $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$ then $\alpha \leq \beta$.

Proof. Assume first that $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ is a Morse decomposition, we need to show that the relation "$\preceq$" is an order. (i) Reflexivity: Let $\mathcal{M}_\mu \in \mathcal{M}$, then $\mathcal{M}_\mu$ is weakly invariant, i.e. $\Phi_G(n, A) \cap A \neq \varnothing$ for all $n \in \mathbb{N}$. Since $\mathcal{M}_\mu$ consists of finitely many elements there is at least one $i \in \mathcal{M}_\mu$ such that $i \in \Phi_G(n_k, \mathcal{M}_\mu) \cap \mathcal{M}_\mu \neq \varnothing$ for infinitely many $n_k \in \mathbb{N}$. Hence $i \in \omega(\mathcal{M}_\mu) \cap \mathcal{M}_\mu$ and therefore $\omega(\mathcal{M}_\mu) \cap \mathcal{M}_\mu \neq \varnothing$, which shows $\mathcal{M}_\mu \preceq \mathcal{M}_\mu$. (ii) Antisymmetry: Assume there are $\mathcal{M}_\alpha$, $\mathcal{M}_\beta \in \mathcal{M}$ with $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$ and $\mathcal{M}_\beta \preceq \mathcal{M}_\alpha$. This means there are $\mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_\beta$ in $\mathcal{M}$ with $\omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \varnothing$ for $i = 0, ..., l-1$ and $\mathcal{M}_{k_0} = \mathcal{M}_\beta, \mathcal{M}_{k_1}, ...\mathcal{M}_{k_m} = \mathcal{M}_\alpha$ in $\mathcal{M}$ with $\omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \varnothing$ for $i = 0, ..., m-1$. If there were two different sets in the collection $\mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ...\mathcal{M}_{j_l} = \mathcal{M}_\beta$, $\mathcal{M}_{k_0} = \mathcal{M}_\beta, \mathcal{M}_{k_1}, ...\mathcal{M}_{k_m} = \mathcal{M}_\alpha$, then $\mathcal{M}_\alpha \neq \mathcal{M}_\alpha$, which cannot hold. Hence all sets in this collection are the same, in particular $\mathcal{M}_\alpha = \mathcal{M}_\beta$. (iii) Transitivity: This follows directly from the definition of "$\preceq$".
Assume now that $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ is a finite collection of nonempty, pairwise disjoint and weakly invariant sets such the relation "$\preceq$" is an order. We need to show that $\mathcal{M}$ satisfies the no-cycle condition: If $\mathcal{M}_{j_0}, ..., \mathcal{M}_{j_l}$ is a subcollection of $\mathcal{A}$ with $\omega(A_{j_\alpha}) \cap A_{j_{\alpha+1}} \neq \varnothing$ for $\alpha = 0, ..., l-1$ then $\mathcal{M}_{j_0} \preceq \mathcal{M}_{j_l}$. If this subcollection is disjoint, then $\mathcal{M}_{j_0} \npreceq \mathcal{M}_{j_l}$, in particular $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$. ■

As in the case of continuous dynamical systems, Morse decompositions for semiflows induced by $L-$graphs need not be unique. For instance, the collection $\{V, \varnothing\}$ always is a Morse decomposition of any Φ_G. As is the case of continuous dynamical systems we can use intersections of Morse decompositions to refine existing ones, compare Definition 45. Since the sets V and $\mathcal{S} = \mathfrak{P}(V)$ are finite, the semiflow Φ_G on $\mathcal{S}$ admits a (unique) finest Morse decomposition for any $L-$graph $G = (V, E)$. The next result characterizes the finest Morse decomposition.

Theorem 68 *Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. For a finite collection of nonempty, pairwise disjoint sets $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ the following statements are equivalent:*

1. *$\mathcal{M}$ is the finest Morse decomposition of Φ_G.*

2. *$\mathcal{M} = \mathcal{C}$, the set of communicating classes of G, compare Lemma 28.*

Proof. Assume first that $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ is the finest Morse decomposition of Φ_G and let $x, y \in \mathcal{M}_\mu$ for some $\mu = 1, ..., k$. We have to show that x and y communicate. Observe first that $x \in \mathcal{M}_\mu$ implies that there exists a loop γ of the graph $G = (V, E)$ with $x \in \gamma$. If there is no such loop then $\{\mathcal{M}_\mu \backslash \{x\}, \mathcal{M}_\alpha : \alpha \neq \mu\}$ is still a Morse decomposition and hence $\mathcal{M}$ cannot be the finest one. If x does not communicate with y, take the communicating classes $[x] \neq \varnothing$ and $[y] \neq \varnothing$, $[x] \cap [y] = \varnothing$ together with set $\mathcal{L} = \{\lambda : \lambda \text{ is a loop not in } [x] \cup [y]\}$, to form the new Morse decomposition $\mathcal{M}' = \{[x], [y], \mathcal{L}, \mathcal{M}_\alpha : \alpha \neq \mu\}$, which is finer than the given $\mathcal{M}$, leading to a contradiction.
To see the converse, let $\mathcal{C} = \{C_1, ..., C_k\}$ be the set of communicating classes of the graph $G = (V, E)$. The C_α are clearly nonempty, pairwise disjoint and weakly invariant for all $\alpha = 1, ..., k$. Recall that $\{i\} \in \mathcal{S}$ is recurrent iff $i \in \lambda$ for some loop λ of G, and therefore we have $\mathcal{R} \subset \cup_{\mu=1}^{k} C_\mu$. Finally, Lemma 28 shows that the relation "$\preceq$" defined in Proposition 67 is indeed an order relation. Hence, the two ordered sets $(\mathcal{C}, \preceq)$ and $(\mathcal{M}, \preceq)$ agree. ■

Example 69 *(Continuation of Example 3) According to Theorem 68 the finest Morse decomposition of this graph is given by $\mathcal{M} = \{\mathcal{M}_\mu,\ \mu = 1, ..., 8\}$ where $\mathcal{M}_\mu = C_\mu$ and the C_μ are the communicating classes from Example 24. The order in Example 29 is exactly the order among the finest Morse sets of Φ_G.*

Remark 70 *The proofs of Proposition 67 and of Theorem 68 show the relationship between $\omega-$limit sets of Φ_G and loops of $G=(V,E)$: For each $A \in \mathcal{S}$ the limit set $\omega(A)$ contains at least one loop. And vice versa, if $i \in \lambda$ is a vertex of a loop λ of G, then $i \in \omega(\{i\})$. This shows that for the finest Morse decomposition $\mathcal{M}=\{\mathcal{M}_\mu,\ \mu=1,...,k\}$ of Φ_G we have*

$$\cup_{\mu=1}^{k}\mathcal{M}_\mu=\{i\in\lambda,\lambda \text{ is a loop of } G\}\subset\cup\{\omega(A),A\in\mathcal{S}\}.$$

This situation is different from the one for continuous dynamical systems, where Morse sets may contain points that are not contained in limit sets, see [4], Example 5.11. Indeed, the next example shows that for discrete semiflows not all points in $\omega-$limit sets need to be elements of a Morse set.

Example 71 *Consider the $L-$graph $G=(V,E)$ given by $V=\{1,2,3,4,5\}$ and*

$$E=\{(1,2),(2,1),(2,3),(3,4),(4,5),(5,4)\},$$

and its associate flow Φ_G. Note that $\omega(\{1,2\})=V$ and hence each Morse decomposition of Φ_G has to include all points in V. Define $\mathcal{M}^1=\{\{1,2,3\},\{4,5\}\}$ and $\mathcal{M}^2=\{\{1,2\},\{3,4,5\}\}$. Both $\mathcal{M}^1$ and $\mathcal{M}^2$ are Morse decompositions of Φ_G, but $\mathcal{M}^1\cap\mathcal{M}^2=\{\{1,2\},\{3\},\{4,5\}\}$ is not a Morse decomposition since the set $\{3\}$ is not weakly invariant. Note that the point 3 is a transitory vertex of the graph G with $3\in\omega(\{1,2\})$.

Remark 72 *Consider an $L-$graph $G=(V,E)$ and its semiflow Φ_G on $\mathcal{S}$. It follows from Theorem 68 that $i\in\cup\{\omega(A),\ A\in\mathcal{S}\}\setminus\cup\{\mathcal{M}_\mu,\ \mathcal{M}_\mu$ is a finest Morse set$\}$ iff i is a transitory vertex and there exists a (finest) Morse set $\mathcal{M}$ with $i\in\Phi_G(n,\mathcal{M})$ for some $n\geq 1$.*

2.3.4 Attractors and Recurrence in Semiflows

Next we will adapt the concept of an attractor to the semiflow of an $L-$graph and analyze the connection with Morse decompositions. We can define attractors for semiflows in complete analogy to Definition 46 for continuous dynamical systems:

Definition 73 *Let $G=(V,E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S}=\mathfrak{P}(V)$. A point $A\in\mathcal{S}$ is called an attractor if there exists a set $N\subset V$ with $A\subset N$ such that $\omega(N)=A$.*

A set N as in Definition 73 is called an *attractor neighborhood.* Note that in the discrete topology A is a neighborhood of itself and hence a point $A \in \mathcal{S}$ is an attractor iff $\omega(A) = A$. We also allow the empty set as an attractor.

A definition of repellers for semiflows of graphs is not obvious, but the idea of complementary repellers from Definition 47 carries over with an obvious modification for semiflows:

Definition 74 *For an attractor $A \in \mathcal{S}$, the set*

$$A^* = \{i \in V,\ \omega(\{i\}) \backslash A \neq \varnothing\} \in \mathcal{S}$$

is called the complementary repeller of A, and (A, A^*) is called an attractor-repeller pair.

Morse decompositions of semiflows can be characterized by attractor-repellers pairs, in analogy to Theorem 48:

Theorem 75 *Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. A finite collection of sets $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ defines a Morse decomposition of Φ_G if and only if there is a strictly increasing sequence of attractors*

$$\varnothing = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n \subset V,$$

such that

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n-1.$$

Proof. Recall the indexing convention for Morse sets from Proposition 67.

(i) Let $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ be a Morse decomposition of Φ_G. In analogy to the continuous time case we define the sets A_k for $k = 1, ..., n$ as follows:

$$A_k = \{x \in V,\ \omega^*(x) \cap (\mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1}) \neq \varnothing\}.$$

Note that for semiflows of graphs we have $A_k = \mathcal{O}^+(\mathcal{M}_n \cup ... \cup \mathcal{M}_{n-k+1})$. We first need to show that each A_k is an attractor. The inclusion $\omega(A_k) \subset A_k$ follows directly from the characterization above of A_k as a positive orbit. To see that $A_k \subset \omega(A_k)$ pick $x \in A_k$. Then there exists $\mu \in \{n-k+1, ..., n\}$ with $x \in \mathcal{O}^+(\mathcal{M}_\mu)$. According to Theorem 68 the set $\mathcal{M}_\mu$ is a communicating class of the graph $G = (V, E)$ and hence every element $z \in \mathcal{M}_\mu$ is in a loop γ that is completely contained in $\mathcal{M}_\mu$, compare Lemma 22. Therefore there is $z \in A_k$ with $x \in \Phi_G(n_l, \{z\})$ for as sequence $n_l \to \infty$, i.e. $A_k \subset \omega(A_k)$. Hence each A_k is an attractor.

Next we show that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ for $0 \le i \le n-1$. To see that $\mathcal{M}_{n-i} \subset A_{i+1}$ pick $x \in \mathcal{M}_{n-i}$. Since $\mathcal{M}_{n-i}$ is a communicating class of the graph G we have $\omega^*(x) \cap \mathcal{M}_{n-i} \neq \varnothing$ and therefore $x \in A_{i+1}$. To see that $\mathcal{M}_{n-i} \subset A_i^*$ assume that there exists $x \in \mathcal{M}_{n-i}$ with $x \notin A_i^*$, i.e. $\omega(x) \backslash A_i = \varnothing$ or $\omega(x) \subset A_i$. But $x \in \mathcal{M}_{n-i}$ means $\omega(x) \cap \mathcal{M}_{n-i} \neq \varnothing$, and by definition we have $\mathcal{M}_{n-i} \cap A_i = \varnothing$, which is a contradiction. This shows $\mathcal{M}_{n-i} \subset A_{i+1} \cap A_i^*$ for $0 \le i \le n-1$. To see the reverse inclusion, let $x \in A_{i+1} \cap A_i^*$, i.e. $x \in \mathcal{O}^+(\mathcal{M}_n \cup ... \cup \mathcal{M}_{n-i})$ and $\omega(x) \backslash A_i \neq \varnothing$. Recall that by Remark 70 $\omega(x)$ contains a loop γ of the graph G, and by Lemma 22 and Theorem 68 each loop is contained in a Morse set. Hence $\omega(x) \cap (\mathcal{M}_n \cup ... \cup \mathcal{M}_{n-i}) \neq \varnothing$, which by definition of a Morse decomposition means that $x \in \mathcal{M}_{n-i}$.

(ii) Let $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ for $0 \le i \le n-1$ be defined as in the statement of the theorem. We have to show that $(\mathcal{M}_1, ..., \mathcal{M}_n)$ form a Morse decomposition. We start by proving that the sets $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ are nonempty. Note first of all that $A_1, ..., A_n \neq \varnothing$ by assumption. We have by definition of attractor-repeller pairs that $V = A_0^* \supset A_1^* \supset ... \supset A_{n-1}^* \supset A_n^*$. Now $A_{n-1}^* \neq \varnothing$ can be seen like this: If $A_{n-1}^* = \varnothing$ then for all $x \in V$ we have $\omega(x) \backslash A_{n-1} = \varnothing$, i.e. $\omega(x) \subset A_{n-1}$. Hence there is $m \in \mathbb{N}$ such that for all $\alpha \ge m$ we have $\Phi_G(\alpha, V \backslash A_{n-1}) \subset A_{n-1}$ and therefore A_n cannot be an attractor. We conclude that $A_0^*, A_1^*, ..., A_{n-1}^* \neq \varnothing$. Now if $A_{i+1} \cap A_i^* = \varnothing$ then we have by the same reasoning as before: For all $x \in A_{i+1}$ it holds that $\omega(x) \backslash A_i = \varnothing$, i.e. $\omega(x) \subset A_i$ and A_{i+1} cannot be an attractor.

The sets $\mathcal{M}_i$ are pairwise disjoint: Let $\alpha < \beta$, then $M_{n-\alpha} \cap M_{n-\beta} = A_{\alpha+1} \cap A_\alpha^* \cap A_{\beta+1} \cap A_\beta^* = A_{\alpha+1} \cap A_\beta^* \subset A_\beta \cap A_\beta^* = \varnothing$.

The sets $\mathcal{M}_i$ are weakly invariant: As above, it suffices to prove that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ contains a loop of the graph G. If there is no loop in $A_{i+1} \cap A_i^*$, then there exists $m \in \mathbb{N}$ such that for all $\alpha \ge m$ we have $\Phi_G(\alpha, A_{i+1} \cap A_i^*) \subset A_i$ and therefore A_{i+1} cannot be an attractor.

The collection $(\mathcal{M}_1, ..., \mathcal{M}_n)$ satisfies the no-cycle condition: This is just a restatement of the assumption that $A_0 \subset A_1 \subset A_2 \subset ... \subset A_n$ is a strictly increasing sequence of attractors.

We have shown so far that $\mathcal{M} = (\mathcal{M}_1, ..., \mathcal{M}_n)$ satisfies the conditions of a Morse decomposition, except for $\mathcal{R} \subset \cup_{\mu=1}^{n} \mathcal{M}_\mu$. Now let $\mathcal{M}' = (\mathcal{M}'_1, ..., \mathcal{M}'_k)$ be the finest Morse decomposition of Φ_G. Since the recurrence condition was not used in the first part of the proof of Theorem 68, and since $\{i\} \in \mathcal{S}$ is recurrent iff $i \in \lambda$ for some loop λ of G, we know by Lemma 20 that $\mathcal{R} \subset \cup_{\mu=1}^{k} \mathcal{M}'_\mu \subset \cup_{\mu=1}^{n} \mathcal{M}_\mu$. Altogether we see that $\mathcal{M} = (\mathcal{M}_1, ..., \mathcal{M}_n)$ is a Morse decomposition. ■

Corollary 76 *Let $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{S} : \mu = 1, ..., k\}$ be the finest Morse decomposition of a semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$, with order $\preceq$. Then the maximal (with respect to $\preceq$) Morse sets are attractors. Furthermore, the smallest (with respect to set inclusion) non-empty attractors are exactly the maximal (with respect to $\preceq$) Morse sets.*

Proof. If M is a maximal Morse set of the semiflow Φ_G, then, according to Theorem 68 and Proposition 67, M is a maximal communicating class of the graph G. Hence M is forward invariant, $\omega(M) = M$ and M does not contain any attractor, except for the empty set.
Vice versa, if A is a smallest (with respect to set inclusion) non-empty attractor, then A is a Morse set according to Theorem 75. If A is not maximal (with respect to $\preceq$), then A is not forward invariant for the graph G and hence there exists a point $x \in \mathcal{O}^+(A) \backslash A$ such that $\mathcal{O}^+(x) \cap A = \varnothing$ (by Lemma 21). According to Remark 33, $\mathcal{O}^+(x)$ contains a maximal communicating class, which is an attractor $A' \subsetneqq A$ and hence A is not a smallest non-empty attractor. ■

Example 77 *(Continuation of Example 3) For this graph, consider the following sequence of* *attractors together with their complementary repellers:*

$$A_0 = \varnothing,\ A_0^* = V$$
$$A_1 = \{21\},\ A_1^* = \{1, ..., 20\}$$
$$A_2 = \{15, 16, 21\},\ A_2^* = \{1, ..., 13, 17, .., 20\}$$
$$A_3 = \{11, ..., 19, 21\},\ A_3^* = \{1, ..., 10\}$$
$$A_4 = \{6, ..., 19, 21\},\ A_4^* = \{1, ..., 5\}$$
$$A_5 = V,\ A_5^* = \varnothing.$$

This sequence of attractor-repeller pairs leads to the Morse decomposition $\mathcal{M}_5 = A_1 \cap A_0^* = \{21\}$, $\mathcal{M}_4 = A_2 \cap A_1^* = \{15, 16\}$, $\mathcal{M}_3 = A_3 \cap A_2^* = \{11, 12, 13, 17, 18, 19\}$, $\mathcal{M}_2 = A_4 \cap A_3^* = \{6, ..., 10\}$, $\mathcal{M}_1 = A_5 \cap A_4^* = \{1, ..., 5\}$, *which is, of course, not the finest Morse decomposition of* Φ_G.

It remains to analyze the behavior of the semiflow Φ_G on a Morse set. Definition 63 and Remark 70 already point at a recurrence property that holds for $\omega-$limit sets: Note that by Definition 63 it holds: $\{i\} \in \mathcal{S}$ is recurrent iff $i \in \omega(\{i\})$ iff $i \in \lambda$ for some loop λ of G. Hence we obtain from Remark 70 for the finest Morse decomposition $\mathcal{M} = \{\mathcal{M}_\mu,\ \mu = 1, ..., k\}$ of Φ_G

$$\mathcal{R} = \cup_{\mu=1}^{k} \mathcal{M}_\mu. \tag{2.5}$$

The recurrent set is partitioned into the disjoint sets of the finest Morse decomposition under the following natural concept of connectedness (compare Definition 36 for the standard connectedness concepts for graphs):

Definition 78 *A set* $B \in \mathcal{S}$ *is called connected under* Φ_G *if for any* $i, j \in B$ *there exist* $n \in \mathbb{N}$ *and a map* $p : \{0, ..., n\} \to B$ *with the properties*

1. $p(0) = i$, $p(n) = j$
2. $p(m + 1) \in \Phi_G(1, \{p(m)\})$ *for* $m = 0, ..., n - 1$.

The flow Φ_G *is called strongly connected if the set of vertices* V *is connected under* Φ_G.

The following result then characterizes the behavior of the semiflow Φ_G on its Morse sets, compare Theorem 53 for continuous dynamical systems.

Theorem 79 *Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. The recurrent set $\mathcal{R}$ of Φ_G satisfies*

$$\mathcal{R} = \bigcap \{A \cup A^*, \ \ A \text{ is an attractor}\}$$

and the (finest) Morse sets of Φ_G coincide with the Φ_G-connected components of $\mathcal{R}$.

Proof. Assume that $x \in \mathcal{R}$, then $x \in \gamma$ for some loop γ of the graph G. Let A be an attractor for Φ_G, then if $x \in A$ we are done. Otherwise if $x \notin A$ then it holds that $\gamma \cap A = \varnothing$. But $\gamma \subset \omega(x)$ and therefore $\omega(x) \backslash A \neq \varnothing$, which means that $x \in A^*$. Conversely, if $x \in \cap\{A \cup A^*, \ A$ is an attractor$\}$, then x is in any attractor containing $\omega(x)$. Arguing as in the proof of Theorem 75, there exists a loop γ of the graph such that $x \in \gamma$, which shows that $x \in \mathcal{R}$.
The second statement of the theorem follows directly from Definition 78 and (2.5). ■

Example 80 *(Continuation of Example 3) We consider the same sequence of attractor-repeller pairs as in Example 77 and obtain*

$$\cap_{i=0}^{5} \{A_i \cup A_i^*\} = \{1, ..., 13, 15, ...19, 21\},$$

which is already the set of recurrent points of this graph.

As discussed in the paragraph on invariance above, forward invariance under Φ_G is a fairly strong requirement for a set $A \subset V$, and thus it appears that there are few sets to which one can restrict the semiflow Φ_G, namely (unions of positive) orbits. However, the idea of a subgraph from Section 2.1 invites the following definition:

Definition 81 *Let $G = (V, E)$ be an $L-$graph with associated semiflow Φ_G on $\mathcal{S} = \mathfrak{P}(V)$. Let $G' = (V', E_{V'})$ be the subgraph of G for a subset of vertices $V' \subset V$. The resulting semiflow $\Phi_{G'}$ on $\mathcal{S}' = \mathfrak{P}(V')$ is called the semiflow Φ_G restricted to V'.*

Note that if $\mathcal{M} \subset V$ is a Morse set of Φ_G, then $(\mathcal{M}, E_{\mathcal{M}})$ is an $L-$graph. This allows us to prove the following fact about Morse sets and recurrence:

Corollary 82 *Under the conditions of Theorem 79 the semiflow Φ_G restricted to any Morse set is recurrent.*

The proof follows directly from Theorem 68 and Definition 63.

As we have seen, most of the concepts used to characterize the global behavior of continuous dynamical systems can be adapted in a natural way to the positive semiflow of an $L-$graph, resulting in very similar characterizations. Indeed, the proofs for semiflows on a finite set are considerably simpler than the corresponding ones for continuous dynamical systems. What is missing in the context of semiflows is first of all the group property of a flow, and hence limit objects for $t \to -\infty$. This results in missing some of the invariance properties of crucial sets, such as limit sets, Morse sets, the (components of) the recurrent set, etc. And secondly, the use of the discrete topology implies that while all points in the (finest) Morse sets are limit points, not all $\omega-$limit points of the semiflow are contained in the (finest) Morse sets. But those exceptional limit points (and hence the set of all limit points) can be characterized, compare Remark 72.

2.4 Matrices Associated with Graphs and their Semiflows

In the previous sections we have analyzed the communication structure of graphs using two different mathematical languages, that of graph theory and that of dynamical systems. In this section we will briefly use yet another language, matrices and linear algebra. Connections between graphs and nonnegative matrices have been studies extensively in the literature, compare e.g. [5] and [6], or the survey [13] and the references therein. We will describe some connections and hint at algorithms that allow for the computation of the objects discussed in the previous sections.

Definition 83 *Given a graph $G = (V, E)$ with $\#V = d$. The adjacency matrix $A_G = (a_{ij})$ of G is the $d \times d$ matrix with elements*

$$a_{ij} = \begin{cases} 1 & \textit{if}(i,j) \in E \\ 0 & \textit{otherwise.} \end{cases}$$

Vice versa, denote the set of $d \times d$ matrices whose entries are in $\{0,1\}$ by $M(d,\{0,1\})$. Any matrix $A \in M(d,\{0,1\})$ is called an adjacency matrix and can be viewed as representing a graph. Continuing our notation from Section 2.1 we define:

Definition 84 *An adjacency matrix is called an $L-$matrix if each row has at least one entry equal to* 1.

Example 85 *(Continuation of Example 3) For this graph, the adjacency matrix is of the form*

$$A_G = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

which is seen to be an $L-$matrix.

Remark 86 *Alternatively, the entries* $a_{ij} \in A_G$ *can be viewed as paths in* G *of length* 1*:* $a_{ij} = 1$ *iff there exists an edge* $\gamma \in \Gamma^1$ *with* $\pi_0(\gamma) = i$ *and* $\pi_1(\gamma) = j$*. Continuing this thought we have the following relationship between paths of length* $n \geq 1$ *and entries of* A_G^n*, the* $n-$*th power of* A_G*:* $a_{ij}^{(n)} \in A_G^n$ *is exactly the number of (different) paths* $\gamma \in \Gamma^n$ *from* i *to* j *in* G*. Hence the* $i-$*th row of* A_G^n *describes exactly the vertices that can be reached from* i *via a path of length* n*. In complete analogy, the* $i-$*th row of* $(A_G^T)^n$ *describes exactly the vertices from which* i *can be reached using a path of length* n*. Here* A^T *denotes the transpose of a matrix* A*.*

The communication concepts developed in Sections 2.1 and 2.2 only depend on the existence of paths connecting certain vertices and not on the number of such paths. We define logical addition $+^*$ and multiplication $\cdot^*$ to describe these ideas and results. These logical operations on the set $\{0, 1\}$ are given by

$$\begin{aligned} 0 \cdot^* 0 &= 0 & \qquad 0 +^* 0 &= 0 \\ 0 \cdot^* 1 &= 0 & \qquad 0 +^* 1 &= 1 \\ 1 \cdot^* 0 &= 0 & \qquad 1 +^* 0 &= 1 \\ 1 \cdot^* 1 &= 1 & \qquad 1 +^* 1 &= 1. \end{aligned}$$

We extend this notion to addition $+^*$ and multiplication $\cdot^*$ of matrices in $M(d, \{0, 1\})$ in the obvious way, denoting by A^{n*} the $n-$th logical product of $A \in M(d, \{0, 1\})$ with itself. Note that $M(d, \{0, 1\})$ is closed under logical addition and multiplication. Since computer calculations involving $+^*$ and $\cdot^*$ are very fast, we obtain efficient algorithms for the computation of orbits and communicating classes:

Let $G = (V, E)$ be a graph with $\#V = d$ and adjacency matrix A. For a vertex $i \in V$ its positive and negative orbits are given by

$$O^+(i) = \{j \in V,\ A_{ij}^{n*} = 1 \text{ for some } n = 1, ..., d\}$$
$$O^-(i) = \{j \in V,\ (A^T)_{ij}^{n*} = 1 \text{ for some } n = 1, ..., d\}.$$

The proof of these facts follows directly from Section 2.2 and Remark 86 above. The communicating classes of G can now be computed as $C = \mathcal{O}^+(i) \cap \mathcal{O}^-(i)$ for $i \in V$, compare Theorem

11. The order among communicating classes can be determined directly from the computation of $O^+(i)$ (or $O^-(i)$), compare Remark 25 and Definition 27. Communicating classes and their order are also sufficient to compute the quotient graphs G_q and G_Q of a given graph G. Hence all the objects analyzed in Section 2.2 can be computed effectively using the matrix ideas described above.

The concepts of irreducibility and aperiodicity play an important role in the analysis of non-negative matrices. We briefly introduce these concepts and discuss their use in the analysis of communication structures in graphs.

Definition 87 *A matrix $A \in M(d, \{0,1\})$ is said to be irreducible if it is not permutation similar to a matrix having block-partition form*

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

with A_{11} and A_{22} square.

Lemma 88 *Let G be a graph with adjacency matrix A. Then A is irreducible iff G consists of exactly one communicating class.*

A proof of this lemma can be found, e.g., in [6]. Note that the adjacency matrix of any $L-$graph is permutation equivalent to a matrix of the form

$$A_Q = \begin{pmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1l} \\ 0 & A_{22} & & & & A_{2l} \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & A_{ll} \end{pmatrix} \tag{2.6}$$

where the square blocks A_{ii}, $i = 1, ..., l$ correspond to the vertices within the communicating class C_i for $i = 1, ..., k$ and to the transitory vertices, and the blocks A_{ij} for $j > i$ determine the order structure among the communicating classes. Hence A_Q "is" the adjacency matrix of the extended quotient graph G_Q, compare Section 2.2.3. For additional characterizations of irreducible nonnegative matrices we refer to [13], Chapter 9.2, Fact 2.

Definition 89 *Let $A \in M(d, \{0,1\})$ be irreducible with associated graph G. The period of A is defined to be the greatest common divisor of the length of loops of G. If this period is 1, the matrix is said to be aperiodic. We say that a graph is periodic of period p (or aperiodic) if its adjacency matrix has this property.*

Lemma 90 *A matrix $A \in M(d, \{0,1\})$ is aperiodic iff $A^n > 0$ (has all elements > 0) for some $n \in \mathbb{N}$.*

A proof of this lemma can be found, e.g., in [6]. For additional characterizations of aperiodic matrices we refer to [13], Chapter 9.2, Fact 3. One can extend this definition to any communicating class of a graph G: Let $C \subset V$ be a communicating class of G and A_C its diagonal block in the representation (2.6) of the adjacency matrix A_G. Note that A_C is irreducible and we define the period of C to be the period of A_C.

Example 91 *(Continuation of Example 3) For this graph we have that the communicating classes C_2, C_3, C_4, C_7, and C_8 are aperiodic, while C_1 and C_6 have period 2 and C_5 has period 3.*

The rest of this section is devoted to studying some of the connections between the semiflow of a graph and the adjacency matrix. Since the semiflow is a sequence of maps $\Phi_G(n, \cdot) : \mathfrak{P}(V) \to \mathfrak{P}(V)$, $n \in \mathbb{N}$, we first need to define the analogue of $\mathfrak{P}(V)$. For a graph $G = (V, E)$ with $\#(G) = d$, we can proceed as follows:

For a subset $A \subset V$ let χ_A denote its characteristic function, i.e.

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

Let e_i be the $i - th$ canonical basis vector of $\mathbb{R}^d$. Define $\iota : \mathfrak{P}(V) \to \mathbb{R}^d$ by

$$\iota(A) = \sum\nolimits_{i=1}^{d} \chi_A(i) e_i.$$

We denote by $\mathbb{Q}^d$ the (vertex set of the) unit cube in $\mathbb{R}^d$. Note that $\iota : \mathfrak{P}(V) \to \mathbb{Q}^d$ is bijective and hence we can identify $\mathfrak{P}(V)$ with $\mathbb{Q}^d$ as sets. We will use the same notation for the two versions of the map ι.

With these notations we can express the semiflow Φ_G in terms of the adjacency matrix A: For paths of G of length 1, i.e. for edges we have $\mathcal{O}_1^+(i) := \{j \in V,\ (i,j) \in E\} = \iota^{-1}(\iota(\{i\})^T \cdot^* A)$. The set $\mathcal{O}_1^+(i)$ is also called the orbit of i at time 1. Similarly we have for $W \subset V$: $\mathcal{O}_1^+(W) := \{j \in V,\ (i,j) \in E \text{ for some } i \in W\} = \iota^{-1}(\iota(W)^T \cdot^* A)$. By Remark 86 we obtain for paths of length $n \geq 1$

$$\mathcal{O}_n^+(W) := \left\{ \begin{array}{c} j \in V, \quad \text{there are } i \in W \text{ and } \gamma \in \Gamma^n \text{ such} \\ \text{that } \pi_0(\gamma) = i \text{ and } \pi_n(\gamma) = j \end{array} \right\} = \iota^{-1}(\iota(W)^T \cdot^* A^{n*}).$$

The definition of the associated (positive) semiflow Φ_G in Equation (2.3) now yields the following alternative way of describing this semiflow

$$\Phi_G(n, W) = \iota^{-1}((\iota(W)^T \cdot^* A^{n*})^T). \tag{2.7}$$

This observation justifies the following definition:

Definition 92 *Consider a matrix $A \in M(d, \{0,1\})$ and let $\mathbb{Q}^d \subset \mathbb{R}^d$ be the $d-$dimensional unit cube. The map*

$$\Psi_A : \mathbb{N} \times \mathbb{Q}^d \to \mathbb{Q}^d,$$

$$\Psi_A(n, q) = (q^T \cdot^* A^{n*})^T$$

is called the positive semiflow of A. Similarly, the map

$$\Psi_A^- : \mathbb{N}^- \times \mathbb{Q}^d \to \mathbb{Q}^d,$$

$$\Psi_A^-(n, q) = (q^T \cdot^* (A^T)^{n*})^T$$

is called the negative semiflow of A.

It follows from (2.7), Proposition 55 and from bijectivity of $\iota : \mathfrak{P}(V) \to \mathbb{Q}^d$ that Ψ_A and Ψ_A^- are, indeed, semiflows. This allows us to reinterpret all concepts and results from Sections

2.3.2 - 2.3.4 for semiflows of square $\{0,1\}-$matrices. Alternatively, we could have developed the theory for semiflows of the type Ψ_A and then translated the results to graphs. The key condition in Sections 2.3.2 - 2.3.4 is for a graph to be an $L-$graph, which translates into $L-$matrices, see Definition 84.

To complete this chapter, we mention a few connections between the semiflow Φ_G of an $L-$graph and concepts from matrix theory:

Let $G = (V, E)$ be an $L-$graph with associated semiflow $\Phi : \mathbb{N} \times \mathfrak{P}(V) \to \mathfrak{P}(V)$. Let A be the adjacency matrix of G with associated semiflow $\Psi_A : \mathbb{N} \times \mathbb{Q}^d \to \mathbb{Q}^d$. Then it holds:

1. G has exactly one communicating class iff Φ has only the trivial Morse decomposition $\{\varnothing, V\}$ iff A is irreducible.

2. A vertex $i \in V$ is in a communicating class $C \subset V$ iff $\{i\}$ is recurrent under Φ iff $(\sum_{n=1}^{d-1} A^{n*})_{ii} > 0$.

3. For two communicating classes $C_\mu \preceq C_\nu$ holds iff their corresponding Morse sets satisfy $\mathcal{M}_\mu \preceq \mathcal{M}_\nu$ iff the adjacency matrix $A_{\mu\mu}$ of the subgraph corresponding to C_μ has a smaller index in the representation (2.6) than $A_{\nu\nu}$.

In the next chapter we will study Markov chains and interpret many of the results we have obtained so far in that context.

CHAPTER 3. Markov Chains

Markov chains are discrete time stochastic processes for which the future is conditionally independent of the past, given the presence. Hence they are the stochastic analogue of (deterministic) difference equations (and of differential equations in the continuous time setting). They play an important role in many science, engineering and social sciences applications as the simplest (except for iid sequences) models of random systems evolving in time. Recently, Markov chains have gained new attention because of their connections to hidden Markov models, switching systems, hybrid systems, information-control systems and other models in science and engineering.

If the state space of a Markov chain is a finite set, its probabilistic behavior can be analyzed using specific graphs and/or matrices. This mathematical connection goes back to Frobenius (around 1910) and Kolmogorov (in the 1930s). The goal of this chapter is to utilize the concepts and theory developed in Chapter 2 for the analysis of finite state Markov chains. This allows us to restate some well-known properties of Markov chains using graphs and semiflows, and to show a few new connections.

The first section of this chapter contains the review of some concepts and facts for finite state Markov chains as they are usually presented in a first graduate course on stochastic processes. The second section sets up the framework in which we achieve some translation from graph and semiflow language into Markov chain language. The last section basically explains a "three-column dictionary" of equivalent objects and results in the three languages.

3.1 Review of Markov Chains

In this section we present, without proofs, a short review of the theory of Markov chains on a finite state space. We refer the reader to, e.g., [10], [14], or [15] for more details on finite chains, and to [3], Chapter 14.1 for a thorough discussion of chains on countable state spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and S a finite set with cardinality $\#(S) = d$. A discrete time stochastic process on Ω with values in the state space S is a sequence of random variables $X_n : \Omega \to S$, $n \in \mathbb{N}$. For each $\omega \in \Omega$ the sequences $(X_n(\omega), n \in \mathbb{N})$ are called trajectories of the process. In general, for a probability measure Q on a measurable space $(A, \mathcal{A})$ we denote by $Q\{\cdot \mid B\}$ the conditional probability of Q given the event $B \in \mathcal{A}$, and by $\mathbb{E}(\cdot \mid \mathcal{B})$ the conditional expectation of Q given the $\sigma-$algebra $\mathcal{B} \subset \mathcal{A}$.

Definition 93 *A Markov chain with values in S is a discrete time stochastic process satisfying the Markov property, i.e.*

$$\mathbb{P}\{X_{n+1} = j \mid X_0 = i_0, ..., X_{n-1} = i_{n-1}, X_n = i\} = \mathbb{P}\{X_{n+1} = j \mid X_n = i\}$$

for all times $n \in \mathbb{N}$ and all states $i_0, ..., i_{n-1}, i, j \in S$.

Using the notation $p^{n,n+1}(i,j) = \mathbb{P}\{X_{n+1} = i_{n+1} \mid X_n = i\}$ for the one-step transition probabilities, these satisfy for all $i, j \in S$ the properties $0 \leq p^{n,n+1}(i,j) \leq 1$ and $\sum_{j \in S} p^{n,n+1}(i,j) = 1$. We will only deal with Markov chains for which the one-step transition probabilities do not depend on n. These chains are called time homogeneous and we will use the notation

$$p(i,j) = \mathbb{P}\{X_{n+1} = j \mid X_n = i\} = \mathbb{P}\{X_1 = j \mid X_0 = i\}.$$

The probabilistic properties of (time homogeneous) Markov chains are completely determined by the transition mechanism, i.e., the one-step transition probabilities $p(i,j)$, and the initial value, i.e. the initial distribution $\pi_0 = \mathcal{D}(X_0)$ on S, where we denote by $\mathcal{D}(\cdot)$ the distribution of a random variable. From these we compute the joint distributions as follows

$$\begin{aligned}
&\mathbb{P}\{X_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}, X_n = i_n\} \\
&\qquad = \mathbb{P}\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}\} \times \mathbb{P}\{X_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}\} \\
&\qquad = \mathbb{P}\{X_n = i_n \mid X_{n-1} = i_{n-1}\} \times \mathbb{P}\{X_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}\} = ... = \\
&\qquad = \mathbb{P}\{X_0 = i_0\} p(i_0, i_1) \cdots p(i_{n-2}, i_{n-1}) p(i_{n-1}, i_n) \\
&\qquad = \pi_0(i_0) p(i_0, i_1) \cdots p(i_{n-2}, i_{n-1}) p(i_{n-1}, i_n).
\end{aligned}$$

This implies, in particular, the so-called Chapman-Kolmogorov equation for the n-step transition probabilities

$$p_n(i,j) = \mathbb{P}\{X_n = j \mid X_0 = i\} = \sum_{k \in S} p_r(i,k) p_{n-r}(k,j) \tag{3.1}$$

for $1 < r < n$.

The formulas above can be written in compact matrix form: We identify the state space S with the set $\{1, ..., d\}$ and define the transition probability matrix P of $(X_n)_{n \in \mathbb{N}}$ by setting $p_{ij} = p(i,j)$ for $i, j = 1, ..., d$. The initial distribution π_0 can then be identified with a probability vector in $\mathbb{R}^d$, i.e. a vector $q \in \mathbb{R}^d$ that satisfies $0 \leq q_i \leq 1$ and $\sum q_i = 1$. The Chapman-Kolmogorov equation now takes the form

$$(p_n(i,j))_{i,j=1...d} = P^n, \text{ for } n \geq 1,$$

and letting π_n denote the distribution of X_n we have

$$\pi_n^T = \pi_{n-1}^T P = \pi_0^T P^n.$$

We also define for future use $P^0 := I$, the $d \times d$ identity matrix.

We now introduce some concepts from the theory of Markov chains that lead to the classification of states and to the characterization of invariant distributions for chains. The key ideas are those of communication and of transience/recurrence.

Definition 94 *A state $i \in S$ has access to a state $j \in S$ if $p_n(i,j) > 0$ for some $n \geq 0$. A state $i \in S$ communicates with a state $j \in S$ if $p_n(i,j) > 0$ and $p_m(j,i) > 0$ for some $n, m \geq 0$.*

Note that according to this definition, which is standard in the theory of Markov chains, the communication relation is an equivalence relation, and hence S partitions into equivalence classes, called communicating classes.

Definition 95 *A set of states $A \subset S$ is said to be stochastically closed if $p(i,j) = 0$ for all $i \in A$ and all $j \notin A$, or equivalently*

$$\sum_{j \in A} p(i,j) = 1 \text{ for all } i \in A.$$

A closed set is called irreducible if it is a communicating class. A Markov chain is said to be irreducible if the state space S is irreducible.

Remark 96 *The adjacency matrix A_P of a transition matrix P is defined by $(A_P)_{ij} = 1$ iff $P_{ij} > 0$. Note that the adjacency matrix A_P of a Markov chain $(X_n)_{n\in\mathbb{N}}$ is irreducible (in the sense of Definition 87) iff the chain is irreducible (in the sense of Definition 95).*

Definition 97 *For a state $i \in S$ we define its period by*

$$\delta(i) = \gcd\{n \geq 1,\, p_n(i,i) > 0\},$$

where gcd *denotes the greatest common divisor. Then i is called periodic if $\delta(i) > 1$, and aperiodic if $\delta(i) = 1$. A Markov chain is said to be aperiodic if all points $i \in S$ are aperiodic.*

Remark 98 *Note that the adjacency matrix A_P of a Markov chain $(X_n)_{n\in\mathbb{N}}$ is aperiodic (in the sense of Definition 89) iff the chain is aperiodic (in the sense of Definition 97).*

A crucial idea in the analysis of the qualitative behavior of stochastic processes is that of reachability, i.e., trajectories starting at one point reach (a neighborhood of) another point. For Markov chains this idea takes the form of hitting times: For $A \subset S$ we define the first hitting time of A as the random variable

$$\tau_A(\omega) := \inf\{n \geq 1,\, X_n(\omega) \in A\},$$

with the understanding that $\tau_A(\omega) = \infty$ if the inf does not exist. We often drop the variable ω and denote the (conditional) distributions of the first hitting times by

$$f_n(i,A) := \mathbb{P}\{\tau_A = n \mid X_0 = i\}.$$

Using this idea we can generalize the Chapman-Kolmogorov equation (3.1) to random intermediate time points as follows (recall that we use the notation $P^0 = I$)

$$p_n(i,j) = \sum_{r=1}^{n} f_r(i,j) p_{n-r}(j,j) \quad \text{for } n \geq 1. \tag{3.2}$$

In particular, the stochastic process with transition probability matrix P and initial variable τ_A is again a Markov chain for any $A \subset S$.

The idea of a first hitting time leads to the definition of a sequence of random variables of subsequent visits to a state or set of states: Define $\tau_A^{(m+1)}(\omega) := \inf\{n > \tau_A^{(m)}(\omega),\, X_n(\omega) \in A\}$ for $m \geq 1$, with $\tau_A^{(1)}(\omega) = \tau_A(\omega)$. With this notation we can write the number of visits of a set $A \subset S$ up to time $m \geq 1$ as

$$N_A(m,\omega) := \sum_{n=1}^{m} \chi_A(X_n(\omega)),$$

and the total number of visits as

$$N_A(\omega) := \sum_{n \geq 1} \chi_A(X_n(\omega)) = \#\{m \geq 1,\, \tau_A^{(m)} < \infty\},$$

where χ_A denotes again the characteristic function of a set A.

Two other concepts derived from first hitting times play a role in the analysis of Markov chains: The first one is the (conditional) probability of reaching a set of states A from a state i, i.e.

$$\mathbb{P}\{\omega \in \Omega, X_n(\omega) \in A \text{ for some } n \geq 1 \mid X_0 = i\} = \sum_{n \geq 1} f_n(i,A) =: F(i,A).$$

The other useful probabilistic concept is that of moments, where we will use only the first moment, i.e. the mean first hitting time

$$\mu(i,A) := \sum_{n \geq 1} n f_n(i,A).$$

With these preparations we can define the ideas of recurrence and transience. When talking about points $j \in S$ we often use the notation $f_n(i,j)$, $F(i,j)$, $\mu(i,j)$ instead of $f_n(i,\{j\})$ etc.

Definition 99 *A state $i \in S$ is called recurrent, if $\mathbb{P}\{\tau_i < \infty \mid X_0 = i\} = F(i,i) = 1$. States that are not recurrent are called transient.*

A recurrent state $i \in S$ satisfying $\mathbb{E}(\tau_i \mid X_0 = i) = \mu(i,i) < \infty$ is called positive recurrent, the other recurrent states are called null recurrent. Here we denote by $\mathbb{E}(Y \mid X_0 = i)$ the conditional expectation of a random variable Y under the measure $\mathbb{P}(\cdot \mid X_0 = i)$.

We list some standard results regarding the classification of states in Markov chains, compare e.g., [10] or [14] for the proofs.

Theorem 100 *Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space S.*

1. *A state $j \in S$ is recurrent iff*
$$\sum_{n\in\mathbb{N}} p_n(j,j) = \infty.$$
2. *If the state $j \in S$ is recurrent then*
$$\mathbb{P}\{N_j = \infty \mid X_0 = j\} = 1.$$
3. *A state $j \in S$ is transient iff*
$$\sum_{n\in\mathbb{N}} p_n(j,j) < \infty.$$
4. *If the state $j \in S$ is transient then it holds for any $i \in S$*
$$\mathbb{P}\{N_j < \infty \mid X_0 = i\} = 1 \quad \textit{and}$$
$$\mathbb{E}(N_j \mid X_0 = i) < \infty.$$
5. *If the state $j \in S$ is recurrent and periodic of period δ then*
$$\lim_{n\to\infty} p_{n\delta}(j,j) = \frac{\delta}{\mu(j,j)},$$
in particular, for aperiodic states we have
$$\lim_{n\to\infty} p_n(j,j) = \frac{1}{\mu(j,j)}.$$
6. *If the state $j \in S$ is positive recurrent and aperiodic, then for $i \in S$ arbitrary we have*
$$\lim_{n\to\infty} p_n(i,j) = \frac{F(i,j)}{\mu(j,j)}.$$

For irreducible chains the qualitative behavior is uniform for all points, leading to the following results:

Theorem 101 *Let $(X_n)_{n\in\mathbb{N}}$ be an irreducible Markov chain on the finite state space S. We fix $j \in S$, then for all $i \in S$ it holds that*

1. *If j has period δ, then so has i.*

2. *If j is transient (recurrent, positive recurrent), then so is i. In fact, all states are either transient or positive recurrent.*

3. $\mathbb{P}\{\lim\limits_{m\to\infty} \frac{1}{m} N_j(m) = \frac{1}{\mu(j,j)} \mid X_0 = i\} = 1.$

4. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} p_k(i,j) = \frac{1}{\mu(j,j)}.$

5. *If j is periodic of period δ then $\lim\limits_{n\to\infty} p_{n\delta}(i,j) = \frac{\delta}{\mu(j,j)}$ and $\lim\limits_{n\to\infty} \frac{1}{\delta} \sum\limits_{k=n}^{n+\delta-1} p_k(i,j) = \frac{1}{\mu(j,j)}$, in particular for aperiodic states j we have $\lim\limits_{n\to\infty} p_n(i,j) = \frac{1}{\mu(j,j)}$. In all cases convergence is geometric with rate $r < 1$, where $r = \max\{|\lambda|, \lambda$ is an eigenvalue of P with $|\lambda| < 1\}$, i.e. the ergodicity coefficient of the transition matrix P.*

For irreducible Markov chains the long term behavior for $n \to \infty$ is described by ergodicity and stationarity. Both types of behavior can be formulated using invariant measures of the chain.

Definition 102 *Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the state space S. A probability distribution π^* on S is called invariant for the chain if $\pi^* = \mathcal{D}(X_n)$ for all $n \in \mathbb{N}$. Here $\mathcal{D}(\cdot)$ denotes again the distribution of a random variable.*

Remark 103 *If $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with transition probability matrix P on the finite state space $S = \{1, ..., d\}$, then the distribution π^* on S is invariant iff $\pi^* \simeq (\pi_k^*, k = 1, ..., d) \in \mathbb{R}^d$ satisfies $\pi^{*T} = \pi^{*T} P$, i.e. if π^* is a left eigenvector of P corresponding to the (real) eigenvalue 1. For irreducible chains this eigenvalue is a simple root of the characteristic polynomial of P, and it is the only one with absolute value equal to 1.*

Remark 104 *Note that a Markov chain $(X_n)_{n\in\mathbb{N}}$ is (strictly) stationary (i.e., all its finite-dimensional distributions are invariant under time shift) iff its initial variable X_0 has distribution $\mathcal{D}(X_0) = \pi^*$ for some invariant distribution π^*.*

Theorem 105 *Let $(X_n)_{n\in\mathbb{N}}$ be an irreducible Markov chain on the finite state space S with transition probability matrix P.*

1. *The Markov chain has a unique invariant distribution π^* on S.*
2. *The invariant distribution $\pi^* \simeq (\pi_k^*,\ k = 1, ..., d) \in \mathbb{R}^d$ satisfies $\pi_k^* = \frac{1}{\mu(k,k)}$, where $\mu(k,k)$ denotes again the mean first return (hitting) time from k to k.*
3. *In particular, all states are positive recurrent.*

3.2 Markov Chains, Graphs, and Semiflows

Usually, presentations about finite state Markov chains develop the theory for irreducible chains, i.e. the chain consists of one communicating class. According to Theorems 101 and 105, the states of an irreducible chain behave uniformly in their limit behavior and thus no coexistence of transient and (positive) recurrent states is possible. We are interested in studying the qualitative behavior of general finite state Markov chains. Chapter 9.8 of [13] presents some results from a matrix point-of-view. We will do so by using the results from Chapter 2. In this section, we develop the mechanisms that allow us to translate many of those results to the context of Markov chains.

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space S with transition probability matrix P. We associate with P a weighted graph $G = (V, E, w)$, where $V = S$, $(i, j) \in E$ iff $p(i, j) > 0$, and $w : E \to [0, 1]$ defined by $w((i, j)) = p(i, j)$. The adjacency matrix A_G is defined as in Definition 83 for the graph $G = (V, E)$. Note that the graph $G = (V, E)$ is automatically an $L-$graph. Vice versa, let $G = (V, E)$ be a graph with a weight function $w : E \to \mathbb{R}$ that satisfies the properties (i) $w : E \to [0, 1]$ and (ii) $\sum\limits_{j\in V} w((i, j)) = 1$ for all $i \in V$, then $G = (V, E, w)$ can be identified with the probability transition matrix P of a Markov chain on the state space V via $p(i, j) = w((i, j))$.

With this construction, we can try to interpret all concepts and results from Section 2.2 in the context of Markov chains. Basically, the key ingredient in all proofs is the following simple observation: Let $\gamma = \langle i_0, ..., i_n \rangle$ be a path in $G = (V, E)$, then the probability $\mathbb{P}(\gamma)$ that this path occurs as a (finite length) trajectory of the chain $(X_n)_{n\in\mathbb{N}}$ with transition probability matrix P and initial distribution π_0 is given by the joint probability

$$\begin{aligned}\mathbb{P}(\gamma) &= \mathbb{P}\{X_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}, X_n = i_n\} \\ &= \pi_0(i_0)p(i_0, i_1) \cdots p(i_{n-2}, i_{n-1})p(i_{n-1}, i_n).\end{aligned} \tag{3.3}$$

In particular, we obtain for any finite sequence $i_0, ..., i_n$ of vertices in G: $\langle i_0, ..., i_n \rangle$ is a path in G iff $p(i_\alpha, i_{\alpha+1}) > 0$ for $\alpha = 0, ..., n-1$.

This observation implies a probabilistic argument that we will need for our results in the next section. It is closely related to the no-cycle property of points for semiflows, compare Definition 64.

Lemma 106 *Let P be the transition probability matrix of a Markov chain on the state space S, and let $G = (V, E)$ be its associated graph. Consider a set $B \subset V$ and a point $i \in B$ with the properties*

1. *There exists a path $\gamma \in \Gamma^n$ with $\pi_0(\gamma) = i$ and $\pi_n(\gamma) =: k \notin B$ for some $n \geq 1$,*

2. *$\mathcal{O}^+(k) \cap B = \varnothing$.*

Then $\lim_{n\to\infty} p_n(i, j) = 0$ for all $j \in B$ uniformly at a geometric rate.

Proof. Denote $\mathbb{P}(\gamma) = \rho$, then by (3.3) we have $p_n(i, k) \geq \rho$ and hence

$$\sum_{j\in B} p_n(i, j) \leq 1 - \rho.$$

Using the Chapman-Kolmogorov equation (3.1) we compute for all $j \in B$

$$\begin{aligned}
p_{2n}(i,j) &= \sum_{l \in S} p_n(i,l) p_n(l,j) \\
&= \sum_{l \neq k} p_n(i,l) p_n(l,j) + p_n(i,k) p_n(k,j) \\
&\leq \sum_{l \neq k} p_n(i,l) \sum_{l \neq k} p_n(l,j) + 0 \\
&\leq (1-\rho) \sum_{l \neq k} p_n(l,j) \\
&\leq (1-\rho)^2.
\end{aligned}$$

Repeating this argument we obtain for all $m \geq 1$

$$p_{mn}(i,j) \leq (1-\rho)^m.$$

By assumption 2 of the lemma, we see that $p_{mn+\alpha}(i,j) \leq (1-\rho)^m$ for $0 \leq \alpha \leq n-1$, which proves the assertion. Note that the argument above even shows $\lim_{n\to\infty} \sum_{j \in B} p_n(i,j) = 0$ at the geometric rate $(1-\rho)$. ■

Next we comment briefly on the connection between Markov chains and semiflows defined by products of matrices, i.e. linear iterated function systems. The standard connection is constructed as follows: Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space $S = \{1, ...d\}$ with transition probability matrix P. We identify the probability measures on S with the set of probability vectors in $\mathbb{R}^d$, defined as $V = \{v \in \mathbb{R}^d,\ v_i \geq 0 \text{ for all } i = 1, ..., d \text{ and } \sum v_i = 1\}$. Then $\Upsilon : \mathbb{N} \times V \to V$, defined by $\Upsilon(n, v) = (v^T P^n)^T$ is a semiflow. If $v_0 = \mathcal{D}(X_0)$, this semiflow describes the evolution of the $1-$dimensional (and the $n-$dimensional) distributions of the Markov chain, thanks to the Chapman-Kolmogorov equation (3.1). Vice versa, if Υ is a linear, iterated function system on $\mathbb{R}^d$ that leaves V invariant, then Υ can be interpreted as a Markov chain on the state space $S = \{1, ..., d\}$. In standard texts, many results about finite state Markov chains are proved using this connection.

Our discussion in Section 2.4 suggests another matrix semiflow associated to a Markov chain, namely the semiflow $\Psi : \mathbb{N} \times \mathbb{Q}^d \to \mathbb{Q}^d$, $\Psi(n, q) = (q^T \cdot^* A^{n*})^T$ on the unit cube $\mathbb{Q}^d \subset \mathbb{R}^d$, compare Definition 92. Here $A \in M(d, \{0, 1\})$ is defined by $a_{ij} = 1$ if $p_{ij} > 0$, and $a_{ij} = 0$

otherwise. This semiflow does not propagate the distributions of the Markov chain, just its reachability or $\{0,1\}-$structure. It follows from Section 2.4 that all concepts and results for $L-$graphs can be interpreted in terms of this semiflow, e.g. Equation (3.3) and Lemma 106 have obvious translations to the context of Ψ.

Finally we consider semiflows on power sets: Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space $S = \{1, ...d\}$ with transition probability matrix P. Define $\mathcal{S} := \mathfrak{P}(S)$ and a semiflow $\Phi : \mathbb{N} \times \mathcal{S} \to \mathcal{S}$, through $\Phi(n, A) = \{j \in S$, there exists $i \in A$ such that $p_n(i, j) > 0\}$. The flow Φ is of the type (2.3) and hence it satisfies all the properties studied in Sections 2.3.2 - 2.3.4. In particular, Equation (3.3) and Lemma 106 have obvious translations to the context of Φ.

Note that, in contrast to the semiflow Υ (and the weighted graph (V, E, w)), the semiflows Ψ and Φ (and the graph $G = (V, E)$) do not, by definition, carry all the statistical (or distributional) information of the Markov chain $(X_n)_{n\in\mathbb{N}}$, just the information about certain basic events (such as paths) occurring with probability zero or with positive probability. Hence when using this graph or these semiflows to analyze the Markov chain, we can only hope for some statements of the kind “an event defined by the Markov chain has positive probability or probability 0”. We will see in the next section that the graph and the semiflows do characterize a surprisingly wide array of properties of the Markov chain.

3.3 Characterization of Markov Chains via Graphs and Semiflows

In this section let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space $S = \{1, ...d\}$ with transition probability matrix P. When we talk about probability measures on S we always think of S as endowed with the discrete $\sigma-$algebra $\mathcal{S} := \mathfrak{P}(S)$. Associated with $(X_n)_{n\in\mathbb{N}}$ are the graph $G = (S, E)$ and the semiflows $\Phi : \mathbb{N} \times \mathcal{S} \to \mathcal{S}$ and $\Psi : \mathbb{N} \times \mathbb{Q}^d \to \mathbb{Q}^d$, as defined in the previous section. The goal of this section is to develop a ”three-column dictionary” of equivalent objects and results in the three languages of Markov chains, graphs, and semiflows. To avoid confusion between the semiflows $\Upsilon : \mathbb{N} \times V \to V$, defined on the set of probability vectors in $\mathbb{R}^d$ and containing all the probabilistic information of the chain, and $\Psi : \mathbb{N}\times\mathbb{Q}^d \to \mathbb{Q}^d$, defined on the unit cube in $\mathbb{R}^d$ and containing only the reachability information of the chain, we will formulate our observations in terms of the semiflow Φ. The results immediately carry over to Ψ using the correspondence from Section 2.4.

3.3.1 Paths, Orbits, Supports of Transition Probabilities, and First Hitting Times

For a probability measure μ on $(S, \mathcal{S})$ the support suppμ is defined as the smallest subset $S' \subset S$ such that $\mu(S') = 1$. Note that the $n-$step transition probabilities $p_n(\cdot, \cdot)$ define probability measures $P(n, i, \cdot)$ on S via $P(n, i, A) := \sum_{j \in A} p_n(i, j)$. I.e., each row of the matrix P^n "is" a probability measure for all $n \in \mathbb{N}$.

Fact 1: A finite sequence of points $(i_0, ..., i_n)$ in S is a path of G iff $p_n(i_0, i_n) > 0$ iff $i_n \in \Phi(n, \{i_0\})$. Each of these statements is equivalent to $i_n \in \text{supp}P(n, i_0, \cdot)$.
The proof of this fact follows directly from (3.3). An immediate consequence of this fact is

Fact 2: Fix a point $i \in S$. Then for $j \in S$ we have: $j \in \mathcal{O}^+(i)$ iff $j \in \text{supp}P(n, i, \cdot)$ for some $n \geq 1$ iff $j \in \bigcup_{n \geq 1} \Phi(n, \{i\})$.

Fact 3: A subset $A \subset S$ is forward invariant for G iff A is stochastically closed for the Markov chain iff $\Phi(n, A) \subset A$ for all $n \geq 1$.
The proof of this fact follows directly from Fact 2. This fact also allows us to characterize certain properties of first hitting times for Markov chains.

Fact 4: Let $i \in S$ and $A \subset S$. The first hitting time distribution $(f_n(i, A),\ n \geq 1)$ satisfies for any $n \in \mathbb{N}$: $f_n(i, A) > 0$ iff there exists a path $\gamma \in \Gamma^n$ with $\pi_0(\gamma) = i$, $\pi_n(\gamma) \in A$, and $\pi_m(\gamma) \notin A$ for all $m = 1, ..., n-1$. Furthermore, $F(i, A) > 0$ iff $\mathcal{O}^+(i) \cap A \neq \varnothing$.
This result follows immediately from (3.3) and Fact 2. The definition of the semiflow Φ in Equation (2.3) allows directly for an equivalent statement in terms of Φ. Note that we did not define "first hitting times" for graphs or semiflows because this concept is hardly, if ever, used in graph theory. However, if we define for the graph $G = (S, E)$ the notion $\sigma_A(i) := \inf\{n \geq 1,$ there exists $\gamma \in \Gamma^n$ with $\pi_0(\gamma) = i$ and $\pi_n(\gamma) \in A\}$ as the first hitting time of a set $A \subset S$ for

paths starting in $i \in S$, then we obviously have from Fact 4 that $\sigma_A(i) = n$ implies $f_n(i, A) > 0$, but the converse is, in general, not true.

As a final observation for this section, we note that Definition 97 of the period of a state $i \in S$ only uses the property of $p_n(i,i) > 0$. Hence according to Fact 1, this is really a pathwise property of the graph G: Consider the graph $G = (S, E)$ and a vertex $i \in S$, we define the period of i as $\eta(i) = \gcd\{n \geq 1$, there exists a path $\gamma \in \Gamma^n$ with $\pi_0(\gamma) = i$ and $\pi_n(\gamma) = i\}$. Then, by Fact 1, we have $\eta(i) = \delta(i)$, the period of i as a state of the Markov chain. It seems, however, that periods of vertices are rarely, if ever, used in graph theory.

3.3.2 Communication and Communicating Classes

We have defined communicating classes for graphs in Definition 9 and for Markov chains after Definition 94. These definitions differ slightly: For graphs we required paths of length $n \geq 1$, while we followed standard practice for Markov chains and allowed $n = 0$. The Markov chain practice renders the communication relation $\sim$ an equivalence relation, but it also leads to trivial communicating classes of the form $\{i\}$ with $p_n(i,i) = 0$ for all $n \geq 1$. In the context of graphs we called such vertices transitory, compare Definition 13. Hence we obtain the following results.

Fact 5: Consider two points $i, j \in S$. i and j communicate via the graph G iff i and j communicate via the Markov chain and $p_n(i,j) > 0$ and $p_m(j,i) > 0$ for some $n, m \geq 1$.

Fact 6: A point $i \in S$ is a transitory vertex of the the graph G iff $p_n(i,i) = 0$ for all $n \geq 1$.

Fact 7: A set $A \subset S$ is weakly invariant for the semiflow Φ iff for all $n \in \mathbb{N}$ there exist $i_n, j_n \in A$ with $p_n(i_n, j_n) > 0$.

Fact 8: A subset $C \subset S$ is a communicating class of the graph G iff C is a nontrivial communicating class of the Markov chain iff C is a finest Morse set of the semiflow Φ.

Fact 9: A subset $C \subset S$ is a maximal communicating class of the graph G iff C is a stochastically closed communicating class of the Markov chain iff C is a minimal (with respect to set inclusion) attractor of Φ.

Fact 10: A point $i \in S$ satisfies $i \in C$ for some communicating class C of the graph G iff there exists a sequence $n_k \to \infty$ with $p_{n_k}(i,i) > 0$ iff $i \in \omega(A)$, the $\omega-$limit set of some $A \subset S$ under the semiflow Φ iff $\{i\}$ is a recurrent point of Φ.
As we will see below, recurrence for semiflows (as defined in Definition 63) and for Markov chains (as defined in Definition 99) are two different concepts, as are the concepts of transitory points of semiflows and transient points of Markov chains.

3.3.3 Recurrence, Transience, and Invariant Measures for Markov Chains

Let $M \subset S$ be a stochastically closed set for Markov chain $(X_n)_{n\in\mathbb{N}}$ with transition probability matrix P. The restriction of P to M defines a new Markov chain $(X_n^M)_{n\in\mathbb{N}}$, whose transition probability matrix we denote by P^M. Since M is stochastically closed, we have for all $i, j \in M$ that $p_n(i,j) = p_n^M(i,j)$ holds for all $n \in \mathbb{N}$. Therefore we conclude from Theorem 100: $i \in M$ is a transient (recurrent, positive recurrent) point for X iff it is transient (recurrent, positive recurrent) for the restricted chain X^M. And Theorems 101 and 105 imply that μ is an invariant probability measure of X with $\mathrm{supp}\mu \cap M \neq \varnothing$ iff μ is an invariant probability measure for X^M. With these preparations we can characterize the long term behavior of states in a Markov chain:

Fact 11: A point $i \in S$ is a transient point of the Markov chain $(X_n)_{n\in\mathbb{N}}$ iff $i \in S \backslash \cup C$, C a maximal communicating class of the graph $G = (S, E)$.
Proof. Maximal communicating classes C of G are forward invariant by Remark 34. Hence the Markov chain X^C is irreducible and all states $i \in C$ are (positive) recurrent by Theorem 105.3. From the observation above we then have that i is (positive) recurrent for X, which proves the $\Rightarrow$ direction. Vice versa, if $i \in S \backslash \cup C$, then Lemma 106 with $B = \cup C$ implies $\lim\limits_{n\to\infty} p_n(i,i) = 0$

at a geometric rate. Hence $\sum_{n\in\mathbb{N}} p_n(i,i) < \infty$ and therefore the state i is transient by Theorem 100.3. ■

Note that the transitory points for the graph $G = (S, E)$ are a subset of the transient states of the Markov chain: Only those points are transitory that are not element of any communicating class.

Fact 12: A point $i \in S$ is a recurrent (and positive recurrent) point of the Markov chain $(X_n)_{n\in\mathbb{N}}$ iff $i \in \cup C$, C a maximal communicating class of the graph $G = (S, E)$.

The proof of this fact is immediate from Fact 11 and Theorems 100 and 105.3. Note that the semiflow concept of recurrence is different from that for Markov chains: Under the semiflow Φ the points in any communicating class are exactly the $\Phi-$recurrent points.

We now turn to invariant distributions of the Markov chain $(X_n)_{n\in\mathbb{N}}$. If μ_1 and μ_2 are two invariant probability measures of a Markov chain, then any convex combination $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for $\alpha \in [0, 1]$ is obviously again an invariant probability measure. An invariant probability measure μ is called extreme if it cannot be written as a convex combination $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ of two different invariant probability measures $\mu_1 \neq \mu_2$ for $\alpha \in (0, 1)$.

Fact 13: Each maximal communicating class C_ν of the graph $G = (S, E)$ is the support of exactly one invariant distribution μ_ν of the Markov chain $(X_n)_{n\in\mathbb{N}}$. The chain has exactly l extreme invariant distributions $\mu_1, ...\mu_l$, where l is the number of maximal communicating classes of the graph G. Any invariant distribution μ of the chain is a convex combination $\mu = \sum_{\nu=1}^{l} \alpha_\nu \mu_\nu$ with $\sum \alpha_\nu = 1$.

Proof. Maximal communicating classes C_ν of G are stochastically closed, hence the Markov chain restricted to C_ν is irreducible and therefore C_ν is the support of a unique invariant distribution μ_ν by Theorem 105.1. Now the other claims follow directly from Fact 9. ■

Fact 14: The Markov chain $(X_n)_{n\in\mathbb{N}}$ has a unique (in the distributional sense) stationary solution $(X_n^\nu)_{n\in\mathbb{N}}$ on each maximal communicating class C_ν of the graph $G = (S, E)$, obtained by taking initial random variables X_0 with distribution $\mathcal{D}(X_0) = \mu_\nu$. All other stationary solutions are convex combinations of the X^ν, $\nu = 1, ..., l$.

This fact is a combination of Remark 104 and Fact 13.

3.3.4 Global Behavior and Multistability

This section is devoted to the convergence behavior of a Markov chain $(X_n)_{n\in\mathbb{N}}$ as $n \to \infty$. We have already characterized the transient and (positive) recurrent points. It remains to clarify how the chain behaves starting in any of these points. When one deals with difference (or differential) equations, one first looks for fixed points (and other simple limit sets) and then tries to find the initial values from which the system converges towards these fixed points (or more generally, limit sets). The study of the global behavior of dynamical systems clarifies these issues, compare, e.g., [4] for the case of flows, and Section 2.3 for an adaptation to specific semiflows related to $L-$graphs and to Markov chains.

Markov chains, in general, do not have fixed points. According to Fact 10, their long term behavior is determined by the communicating classes. And Fact 12 suggests that the maximal communicating classes determine the behavior of a Markov chain $(X_n)_{n\in\mathbb{N}}$ as $n \to \infty$. The following facts show that this is, indeed, true. The special case of a fixed point is recovered when a maximal communicating class consists of exactly one state; such states are called absorbing.

Fact 15: Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ with transition probability matrix P, initial random variable X_0 and initial distribution $\mathcal{D}(X_0) = \pi_0$. Let C be a maximal communicating class of the graph $G = (S, E)$ with invariant probability μ^*. Assume that π_0 is concentrated on C, i.e. $\pi_0(i) > 0$ iff $i \in C$. Then $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} \mathcal{D}(X_k) = \mu^*$ in the distributional sense, i.e. as vectors in $\mathbb{R}^d$. If, furthermore, C has period δ then $\lim\limits_{n\to\infty} \frac{1}{\delta}\sum\limits_{k=n}^{n+\delta-1} \mathcal{D}(X_k) = \mu^*$, in particular for C aperiodic we have $\lim\limits_{n\to\infty} \mathcal{D}(X_n) = \mu^*$. All convergences are at a geometric rate.

This fact follows immediately from Theorems 101.5 and 105.2, and the remarks at the beginning of Section 3.3.3.

Fact 16: Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$. Let $C = \cup C_\nu$, C_ν a maximal communicating class of the graph $G = (S, E)$, and $D := S\backslash C$. Pick $i \in D$. Then the first hitting time τ_C satisfies $\mathbb{P}\{\tau_C < \infty \mid X_0 = i\} = 1$ and $\mathbb{E}(\tau_C) < \infty$.

Proof. Assume, under the given assumptions, that $\mathbb{P}\{\tau_C < \infty \mid X_0 = i\} < 1$, i.e. $\mathbb{P}\{\tau_C = \infty \mid X_0 = i\} > 0$. Then there exists a state $j \in S\backslash C$ with $P\{N_j = \infty \mid X_0 = j\} > 0$. By the characterization from Theorem 100.4 the state j cannot be transient, and hence by the dichotomy in Theorem 101.2 j is recurrent, which contradicts Fact 12. Hence $\mathbb{P}\{\tau_C < \infty \mid X_0 = i\} = 1$ and then $\mathbb{E}(\tau_C) < \infty$ follows from geometric convergence in Lemma 106. ■

We define the probability that the chain, starting in $D = S\backslash C$, hits the maximal communicating class C_ν by $p_\nu := \mathbb{P}\{X_{\tau_C} \in C_\nu \mid X_0 \in D\}$. Note that $\sum p_\nu = 1$. Then the long-term behavior of $(X_n)_{n\in\mathbb{N}}$ with $\mathcal{D}(X_0)$ concentrated in D is as follows: X_n will leave the set D of transient states in finite time (even with finite expectation) and enter into the set C of (positive) recurrent points, where it may enter one or more of the C_ν's depending on whether p_ν is positive or not. By the random version of the Chapman-Kolmogorov equation (3.2), the process continues as a Markov chain, and hence in each C_ν follows the behavior described in Fact 15. In particular, if all C_ν are aperiodic, we obtain:

Fact 17: Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ with maximal communicating classes $C_1, ..., C_l$ and extreme invariant measures $\mu_1, ..., \mu_l$. Assume that all C_ν, $\nu = 1, ..., l$ are aperiodic. Then we have as limit behavior of the chain $\lim_{n\to\infty} \mathcal{D}(X_n) = \sum_{\nu=1}^{l} p_\nu \mu_\nu$, where the p_ν are as defined above.

This fact leads to the definition of multistable states: A state $i \in S$ is called multistable for the Markov chain $(X_n)_{n\in\mathbb{N}}$ if there exist maximal communicating classes C_1 and C_2 with $C_1 \neq C_2$ such that $p_\nu > 0$ for $\nu = 1, 2$. We now obtain the following fact as a criterion for the existence of multistable states. Recall the definition of a connected graph from Definition 36.

Fact 18: Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with connected graph G. Then the chain has multistable states iff G has at least two maximal communicating classes.

Example 107 *(Continuation of Example 3) The graph of this example has two connected components, the vertex sets* $Z_1 = \{1, ..., 20\}$ *and* $Z_2 = \{21\}$. *The component* Z_1 *has two maximal communicating classes,* $C_6 = \{15, 16\}$ *and* $C_7 = \{17, 18, 19\}$, *and hence it has bistable points, namely the set* $\{1, ..., 13\}$.

Multistable, and specifically bistable states play an important role in many applications of stochastic processes in the natural sciences and in engineering, compare, e.g. [7].

Bibliography

[1] Albertini, F., E.D. Sontag, Some connections between chaotic dynamical systems and control systems, in Proc. European Control Conference, Grenoble, 1991, 158-163.

[2] Arnold, L., Random Dynamical Systems, Springer, 1998.

[3] Athreya, K.B., S.N. Lahiri, Measure Theory and Probability Theory, Springer, 2006.

[4] Ayala-Hoffmann, J., P. Corbin, K. McConville, F. Colonius, W. Kliemann, J. Peters, Morse decompositions, attractors and chain recurrence, Proyecciones Journal of Mathematics 25 (2006), 79-109.

[5] Berman, A., M. Neumann and R.J. Stern, Nonnegative Matrices in Dynamic Systems, John Wiley & Sons, 1989.

[6] Berman, A., R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, 2nd edition SIAM 1994 (originally Academic Press, 1979).

[7] Colonius, F., F.J. de la Rubia, and W. Kliemann, Stochastic models with multistability and extinction levels, SIAM J. Applied Mathematics 56 (1996), 919-945.

[8] Colonius, F. and W. Kliemann, The Dynamics of Control, Birkhäuser, 2000.

[9] Katok, A., B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.

[10] Medhi, J., Stochastic Processes, 2nd edition J. Wiley, 1994.

[11] Potter, M., Sets, An Introduction, Clarendon Press, 1990.

[12] Roseman, D., Elementary Topology, Prentice Hall, 1999.

[13] Rothblum, U.G., Nonnegative Matrices and Stochastic Matrices, Chapter 9 of Handbook of Linear Algebra (L. Hogben, ed.), CRC Press, 2006.

[14] Schinazi, R., Classical and Spatial Stochastic Processes, Birkhäuser, 1999.

[15] Taylor, H.M., S. Karlin, An Introduction to Stochastic Modeling, Academic Press, revised edition 1994.