Adjusting remanufacturing capacity using sales and return information

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by

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CHAPTER 1. INTRODUCTION

1.1 Overview

The advent of technological innovation has shortened the life-cycle of electronic products; therefore, these products are often replaced with newer versions before the end of their useful life. In addition, due to the high cost of labor in developed countries, a product that breaks is often replaced with a new product instead of being repaired. Landfill space is becoming scant, and disposal of electronic products in landfills is being prohibited due to the highly toxic chemicals that can be released, posing a threat to human health and environment; therefore, it is imperative to divert the waste stream by initiating operations to take back products from end users for reuse. Firms have faced a new challenge to manage the huge number of end of life products.

In the European Union (EU), a new regulation regarding environmental issues makes electronics manufacturers responsible for assisting in the recovery and recycling of end-of-life products and removing the hazardous substance content of products prior to disposal. Starting in August 2005, producers of electronics were required to comply with the WEEE (Waste Electrical and Electronic Equipment) Directive; moreover, as of July 2006, those producers needed to conform to RoHS (Restriction of Use of certain Hazardous Substances) Directive. Thus, remanufacturing is becoming more prevalent for the electronics industry. In March 2007, the Chinese government imposed the RoHS-like regulations used in EU. Unlike the EU RoHS, which applies to electronics manufacturers, the China RoHS applies to everyone in the supply chain such as manufacturers, distributors, importers, and retailers,
including the supply of second hand electronic equipment. Besides, the Chinese legislation requires that compliance can be verified only by testing done in accredited Chinese laboratories. In addition, many states in the United States have considered legislation aimed to support remanufacturing programs. For example, Washington State passed a bill that requires electronics manufacturers to take responsibility for the safe disposal of their products after they become obsolete.

Remanufacturing is the process of restoring or reusing parts from returned products as an input for manufacturing new products (Lund, 1984). Recently, several leading electronics companies have initiated remanufacturing programs. HP started its product take back program to allow either business or individual consumers around the world to return used toner-cartridges at no charge and Kodak offered a worldwide extensive program to reclaim single-use cameras from customers (Degher, 2002). IBM and Dell Corporations provided a channel for customers to turn in old PCs on an exchange program. Xerox initiated a remanufacturing program for used cartridges and end-of-lease copiers (Savaskan et al., 2004).

Remanufacturing and reselling returned products not only reduces the amount of new material consumption but also compensates for the costs of taking back products. It is estimated that the manufacturing cost of a new product can be reduced by 60 to 80% by remanufacturing (Guide et al., 2000). A study has shown that remanufacturing is a value-added business and its total annual sales are estimated at over 53 billion dollars (Guide et al., 2000). Ferrer (1999) addressed the viability and the potential market of remanufacturing computers. In that paper, a situation when the returns can be remanufactured to meet the demand for the new product is represented.
However, the management of the remanufacturing operations is more complex than traditional manufacturing operations (Fleischmann et al., 1997). The complexity in remanufacturing operations is due to the considerable uncertainty associated with the timing and amounts of future returns. Managing remanufacturing operations in the presence of uncertainty requires high investments because additional labor and machines are required (Guide and Van Wassenhove, 1997). There is a wide range of examples of this problem in numerous industries, e.g., reusable beverage containers, disposable cameras, batteries, toner cartridges, computers, and automobiles.

To be successful in the business of recovering value from used products, it is imperative for manufacturers to plan for future availability in an optimal way. Because it is difficult to estimate the amount of future returns, additional resources must be available to mitigate some of the risk inherent in an irregular stream. On the other hand, carrying too much capacity should be avoided because it creates unnecessary investment in unused capacity that potentially lowers profit. Nevertheless, if we fail to have enough capacity, we lose an opportunity to extract value from the returns due to limited capacity. To determine the optimal capacity level, we must be careful to balance the tradeoff between carrying excess capacity and risking shortage. In addition, we cannot afford to leave returned items waiting too long for reprocessing because of the high obsolescence rate in technology (Ferrer and Ketzenberg, 2004). Historical patterns of sales and returns may help to forecast future returns. Therefore, this research develops policies for the optimal capacity level for remanufacturing that take into account past sales and past return information.

Planning for future capacity availability involves decisions to expand and contract capacity. In a comprehensive review of major research in the area of remanufacturing,
Fleishmann et al. (1997) observed that there has been limited research considering capacity management in the remanufacturing context. In this framework, capacity refers to the processing abilities and limitations that stem from the scarcity of various processing resources. Also, capacity can be interpreted as some upper bound on processing quantities (Van Mieghem, 2003).

In an effort to maintain sufficient but not excessive capacity for processing returns before the end of their useful life, it is necessary for remanufacturers to adjust capacity over time according to expectations of the future returns. The purpose of capacity management is to facilitate decisions regarding the amounts by which capacity should be expanded or contracted and their associated times. To determine the optimal capacity policy for the remanufacturing operations, the sales history and the numbers of items outstanding in the market (i.e., sold but not yet returned) are used to give an estimate of future returns that constitute a demand for remanufacturing capacity. The criterion used to determine the best capacity policy is the maximization of the net profits from remanufacturing products while taking into account all relevant capacity adjustment costs.

1.2 Problem statement

This research takes the viewpoint of original equipment manufacturers (OEMs) who try to manage the returns of electronic products as well as take a benefit from material recovery of used products from the market. Those OEMs include major computer manufacturers, e.g., IBM, Dell, HP, etc. To assist operation planning activities, we explore how to use the information from sales history and returns history to adjust capacity in an optimal way.
Determining an optimal policy not only assists planning activities for future remanufacturing operations but also improves the viability of the remanufacturing business.

We assume that whenever the returns surpass the available capacity levels then it is required to expand capacity to meet the shortage on an emergency basis; for instance, by adding another shift. Also, there is a fixed cost associated with each capacity expansion that makes the establishment of the optimal capacity policy more complex. We consider a single facility with a finite planning horizon. We further assume that the manufacturer makes a decision to either expand or contract capacity or stay put with the current capacity level at discrete time periods. The adjustment of capacity can occur instantaneously. There is no physical degradation of capacity. That is, once the capacity is in place, its performance remains steady over time.

1.3 Research objective

The purpose of this research is to address the multi-period capacity management problem for remanufacturing a product with a limited lifecycle. The formulated model uses information about past sales and returns to estimate the future returns. The capacity is represented by the number of workers as well as workstations. Therefore, the capacity expansion consists of building workstations and hiring workers, whereas the capacity contraction refers to reducing workstations and re-tasking workers. Capacity can be increased or reduced at a given unit cost. The capacity decision must be made prior to observing actual returns for that period. We assume that when the returns surpass the available capacity level then it is required to expand capacity on an emergency basis to eliminate the shortage.
This research explores how to utilize the benefit of information from sales and returns history and apply this information to dynamically determine the optimal capacity adjustments. The objective of this research is to find the optimal policy for capacity decisions and study the effect on the capacity policy of different patterns of product lifecycle, time to return distribution, and capacity-related costs.

1.4 Thesis organization

The remainder of this research is organized into five chapters. In the next chapter the relevant research studies are presented. Chapter 3 is devoted to model formulation and Chapter 4 presents the simple capacity policy. Chapter 5 shows numerical examples that illustrate the implementation of the model and show the effect of different sales patterns, return rates and capacity-related costs on the optimal policy. Finally, Chapter 6 presents concluding remarks and describes the future directions for this research.
CHAPTER 2. LITERATURE REVIEW

2.1 Overview

Remanufacturing has been considered to be potentially an effective option to respond to new regulations and achieve environmental goals of electronics manufacturers. Fleishmann et al. (1997) and Guide et al. (2000) offered comprehensive reviews in the field of remanufacturing. The reviews indicate that although there has been significant research development in the remanufacturing, the study of capacity management for a remanufacturing environment is limited. Uncertainty in either timing or quantity of the returns impedes good management in capacity planning (Toktay et al., 2000). However, by using the information obtained from past sales and the current number of items outstanding in the market, we should have the ability to dynamically determine the optimal capacity planning policy.

2.2 Capacity planning

The capacity planning problem for remanufacturing operations has been addressed by a few papers found in the literature. Guide and Spencer (1997) developed a capacity planning model for workstations in a remanufacturing environment. The study addressed the issue of determining the required capacity by considering a stochastic material recovery rate and probabilistic routing. Guide et al. (1997) extended the previous work and presented a new capacity planning technique with consideration of uncertainties in routings, material recovery rates, and processing times. They evaluated the performance of the new capacity plan against
the traditional capacity techniques. The results indicated that the new techniques work better than the traditional ones when considering the variability in the remanufacturing environment.

Shih (2001) studied reverse logistics planning for electronic products in Taiwan. Using historical data, the author presented a model to determine the optimal capacity expansion plans of storage and disassembly facilities for different product take-back rates. A method for parameter estimation regarding the amount of future returns and relevant costs was discussed.

More recently, Franke et al. (2005) considered remanufacturing of mobile phones. They developed a model to determine the required capacities for remanufacturing operations. They used information about uncertainties in the amount and conditions of returns as well as combinatorial optimization to determine the capacities of work stations.

Our main contribution is to characterize the optimal capacity adjustment policy that can be applied under a remanufacturing environment using existing available information regarding past sales and past returns. Capacity expansion research has been conducted extensively in a variety of areas to give insight into investment decisions. Luss (1982) provided an excellent review of the capacity expansion problem literature. More recently, Van Mieghem (2003) presented a comprehensive literature review on capacity management and discussed the recent research developments in this field.

Rocklin et al. (1984) considered a stochastic capacity expansion and contraction model and found an optimal capacity policy when the capacity expansion on an emergency basis is allowed to satisfy the excess demand. Dynamic programming was used to obtain the optimal policy. They described some conditions under which the optimal capacity policy has a
(\(S', S''\)) type solution. That is, if the initial capacity is less than \(S'\), then it is optimal to bring the capacity up to \(S'\). If the current capacity is greater than \(S''\), then it is optimal to bring the capacity down to \(S''\). Otherwise, it is optimal to maintain the capacity at the same level.

Aneja and Chaouch (1993) extended the analysis of Rocklin et al. (1984) by incorporating the economies of scale in capacity expansion and a fixed penalty cost of an emergency expansion into the model and established the optimal policy. The properties of \(K-convex\) functions were used to establish a simple form of the optimal policy. In particular, they showed that if the demand distribution is concave, then an optimal capacity policy of the \((S', S'')\) type exists.

In this research, we consider the demand for the remanufacturing capacity to be dependent on the previous sales and the number of products sold but not returned from the market. Therefore, unlike in the model of Aneja and Chaouch, demands for capacity in successive periods are dependent. We show that the optimal capacity policy has a simple form and consists of threshold values and target levels that depend on the current capacity level and the numbers of items outstanding in the market (i.e., sold and not yet returned).

In addition, Angelus and Porteus (1996) extended the Rocklin et al. (1984) paper to consider capacity and production planning for short-life-cycle and produce-to-stock products. Their models characterize the capacity planning for production of perishable and durable products. That is, if leftover inventory cannot be re-sold in the following period, then the product is considered perishable, while the unsold units of the durable product can be carried over for possible sales in the subsequent periods. Their paper relaxes the requirements of expanding capacity to satisfy the demand deficit on an emergency basis and does not incorporate the economies of scale in capacity adjustments. However, the optimal policy for
either perishable or durable products is similar to the policy as described by Rocklin et al. (1984).

Angelus et al. (2000) developed capacity expansion strategies in the presence of correlated stochastic demand and positive expansion lead times. However, capacity contraction and emergency capacity expansion are not considered. Also, there is a constraint on an upper limit for installing additional capacity. They showed that there is a similarity between the optimal policy of the proposed model and the \((s, S)\) optimal inventory policy (Scarf, 1960). That is, if capacity is smaller than \(s\), then the optimal policy is to raise the capacity level to \(S\). Otherwise, maintain the same capacity level. In addition, the results indicate that both parameters of the optimal policy depend on the upper limit of total capacity allowance.

Angelus and Porteus (2006) considered the problem of assembling a single type of capacity over time from multiple components in which each component has a different lead time. For example, production capacity consists of equipment and workforce, each of which requires different lead times for installation (construction, acquisition, or hiring and training). In addition, the option of delaying the progress of previously ordered capacity expansions is considered. However, neither the capacity contraction option nor the economies of scale in the expansion are considered. They showed that although capacity is assembled from multiple individual components, under some conditions it is possible to treat it as a single state variable.

Chittamvanich and Ryan (2006) formulated a single period newsvendor-like model to set the remanufacturing capacity for a product nearing obsolescence. They explored the benefit of using information from early returns on the predictability of future returns. Utilizing
information on early returns, the capacity level that maximizes the expected profits generated from remanufacturing the returned products is determined. The model focused on a single-period expansion in which there is only one chance to make a decision to install new capacity assuming that there is no capacity available initially. The model showed that the uncertainty in the return parameter estimates causes a decision error but the capacity decisions would improve as additional information is acquired with the arrivals of more returns.

Recently, Vlachos et al. (2007) examined the capacity planning policies of remanufacturing facilities. The model and analysis were based on the system dynamics methodology, which is a modeling and simulation paradigm for studying the dynamic behavior of social and economic systems and searching for policies such that the system performance is improved (Forrester, 1961). They studied the impact on the optimal capacity planning policies of the number of times a product can be reused before its disposal and take-back obligations enforced by environmental regulatory legislation. The decision variables were the length of review periods and the capacity planning strategies, namely leading, trailing, and matching. Leading and trailing strategies mean that capacity expansion leads and lags behind the demand, respectively, while the matching strategy represents building capacity as closely as possible to its demand. However, the focus was on the maturity phase of the product lifecycle and limited to only the capacity expansion option.

The difference between this research and past works is that we develop policies for dynamically computing the optimal capacity level that can be applied under a remanufacturing environment by using past sales and return information from early returns. Using that information, our optimal policy consists of two state variables rather than the single state variable discussed in Aneja and Chaouch (1993). Moreover, we study the effects
of the different patterns of product sales and uncertainty in timing of the future returns on the optimal capacity policy.

In the following chapter, we present the mathematical model. In chapter 4, we prove the optimality of a simple capacity policy in which the optimal thresholds and target levels in each period depend on the current capacity level and the current number of items outstanding in the market.
CHAPTER 3. CAPACITY ADJUSTMENT MODEL

3.1 Notation

We formulate the capacity adjustment problem as a discrete-time finite-horizon Markov decision process. Exploiting the sales and returns history, the problem is to determine an optimal capacity policy that maximizes the expected net profit generated from remanufacturing the returns less all the capacity-related costs.

The following describes notation used in this research:

- $T$: Total number of periods
- $t$: Decision epochs that correspond to the beginning of a period
  \[ t \in \{0,1,\ldots,T-1\} \]
- $a_1$: The unit cost of expanding capacity
- $b_1$: The unit cost of reducing capacity
- $K_a$: The fixed cost charged for capacity expanded
- $a_3$: The unit operating cost associated with maintaining production capacity
- $a_4$: The unit emergency expansion unit cost
- $K_s$: The fixed cost when an emergency expansion is required
- $\gamma$: Discount factor for one period \(0 < \gamma \leq 1\)
- $p_1$: Probability of an item being sold in a given time period
\( p_2 \) : Probability that an item outstanding is returned in the next period

\( S_t \) : Number of items sold during the \( t^{th} \) period

\( N_t \) : Number of items returned during the \( t^{th} \) period, in units of ten thousand.

\( V \) : The time from sale to return

\( x_{t-1} \) : Capacity available at the beginning of the \( t^{th} \) period in items/period.

\( y_{t-1} \) : Number of items sold but not returned at the beginning of the \( t^{th} \) period.

\( u_t \) : Amount of capacity added or removed at the beginning of the \( t^{th} \) period (decision variable)

\( z_t \) : New capacity at the beginning of the \( t^{th} \) period \( (z_t = x_{t-1} + u_t) \)

\( r_t \) : Net revenue from remanufacturing one item during the \( t^{th} \) period

\( R_t(y_{t-1}) \) : The expected revenue from processing the returns in period \( t \) given that the number of units outstanding in the market is \( y_{t-1} \) at the beginning of the period

\( L_t(z_t | y_t) \) : The expected operating and emergency expansion costs associated with capacity level \( z_t \) and the number of units outstanding in the market \( y_t \) at the beginning of the \( t^{th} \) period

\( V_t(x_{t-1}, y_{t-1}) \) : The maximum expected discounted profit for \( (T - t) \) remaining periods given that at the beginning of period \( t \) the state is \( (x_{t-1}, y_{t-1}) \)

\( G_t(x_{t-1}, y_{t-1}, z_t) \) : The expected profit from expansion with capacity level \( z_t \) given that at the beginning of period \( t \) the state is \( (x_{t-1}, y_{t-1}) \)
\( H_t(x_t, y_t, z_t) \): The expected profit from contraction with capacity level \( z_t \) given that at the beginning of period \( t \) the state is \((x_{t-1}, y_{t-1})\)

### 3.2 Product return model

In this section, we describe the model for sales over time and the distribution of the time lapse between sale and return of the units. The distribution of the sales over the life of electronic products can be described by the well-known product life cycle curve (Bayus, 1998; Bollen, 1999; Tibben-Lembke, 2002). The typical pattern of product sales can be characterized into five distinct phrases: introduction, growth, maturity, decline, and obsolescence. Solomon et al. (2000) studied the electronic product sales and described a similar pattern. Several researchers studied the distribution of the returns by utilizing real data. De Brito and Dekker (2003) suggested that a negative exponential distribution can be used to describe the time between the sale of an item and its return. Toktay et al. (2002) studied the return data of a single-use camera of Kodak and used a discrete time distributed lag model corresponding to either a geometric or a Pascal distribution. Chittamvanich (2003) modeled the time to sale and time from sale to return of each unit using the gamma distribution because of its shape flexibility that makes it possible to model a wide variety of distribution shapes.

In this research we model the sales that change over time according to the product life cycle stages. That is, the sales initially increase after the product is introduced to the market and then gradually decrease after reaching the peak in sales. The geometric distribution is chosen to represent the time lapse from purchase to return of the units or the number of
periods that the units spend with customers. Varying the distribution parameter of the geometric distribution we can model different time to return characteristics. A higher value of the distribution parameter corresponds to lower values of both mean and variance of the time to return. Given the number of items that have been sold but not yet returned, the memoryless property of the geometric distribution allows the characterization of the number of returns in a given time interval as a binomial random variable.

3.2.1 The amount of product sold

We assume that the distribution of product sales during the $t^{th}$ period $(S_t)$ is known at time $t$, and the realizations of sales are known for periods $1, 2, ..., t-1$. The sales over the life of a product are described by the classical product life cycle curve or the bell-shaped curve (Bayus, 1998; Bollen, 1999; Tibben-Lembke, 2002).

3.2.2 Time from sale to return

The time from sale to return of the units or the time that a unit spends with its purchaser are independent and identically distributed random variables having a geometric distribution with parameter $p_2$. That is, the probability mass function is

$$g(v) = p_2(1 - p_2)^{v-1}, \quad v = 1, 2, ..., \quad 0 < p_2 < 1.$$  

3.2.3 The return distribution

Let $N_t$ denote the number of items returned during the $t^{th}$ period. Let $y_t$ represent the number of items sold but not yet returned at the beginning of the $t^{th}$ period.
This quantity can be determined according to the following relationship:

\[ y_t = y_{t-1} + S_t - N_t. \]  

(2)

We further assume that \( y_0 = 0 \).

Using the memoryless property of the gamma distribution and the current number of items outstanding with the customers, we characterize the probability distribution of the number of returns as binomial. Therefore, the probability distribution of the number of items returned during the \( t^{th} \) period can be written as:

\[
\Pr[N_t = k | y_t] = \binom{y_t}{k} p_2^k (1 - p_2)^{y_t-k}, \text{ where } k = 0, 1, 2, ..., y_t. 
\]  

(3)

In this research, we assume that every item is eventually returned but not necessarily within the study horizon. Typically, it is unusual that all items will be returned. However, under extended producer responsibility, eventually an item will be returned to electronics manufacturers because they are responsible for all end-of-life products.

### 3.3 Markov decision model

We find a simple optimal policy for the capacity decisions in a Markov decision model where the current state of the system determines the set of actions available and their rewards, and the state and action together determine the probability distribution of the state of the system at the next decision epoch (Puterman, 1994).
3.3.1 Decision epochs

Let the decision epoch \( t \) denote the beginning of the \( (t+1)^{th} \) period. We use the convention that the decision epoch \( T \) represents the end of the study time horizon; therefore, there is no decision taken at this time.

3.3.2 States: \((x_{t-1}, y_{t-1})\)

The state variable \( x_{t-1} \) represents the available capacity at the beginning of the \( t^{th} \) period and \( y_{t-1} \) denotes the number of items sold but not yet returned, i.e., the number of items outstanding in the market at the beginning of the \( t^{th} \) period.

In our model, capacity is considered as an upper limit on the number of returns that can be remanufactured during the \( t^{th} \) period. Due to high obsolescence and rapid technological advances in electronic products, it is desirable to expedite the remanufacturing processes. Thus, we assume that if the returns surpass the available capacity, then additional capacity must be introduced to satisfy the deficit in demand. Therefore, the capacity is formulated as follows:

\[
x_i = \max\{x_{t-1} + u_i, N_t\}
\]  

(4)

where \( x_0 = 0 \).

3.3.3 Actions: \( u_i \)

At each decision epoch, the decision is to choose the amount of capacity expansion or contraction \( u_i \). We assume that capacity expansion and contraction occur instantaneously.
and the decision must be made before observing the actual returns during the $t^{th}$ period. Also, capacity cannot be shared across products. To ensure a nonnegative capacity level at any decision epoch, we require $u_t \geq -x_{t-1}$.

![Figure 1. The timeline of events in a period](image)

Figure 1 illustrates an overview picture of the decision model. At the beginning of a period $t$, we have capacity level $x_{t-1}$ available and observe an amount of items still outstanding in the market $y_{t-1}$. Using such information, the decision is to determine the capacity level $u_t$ to process the returns, $N_t$. At the end of the period, if the number of the returns exceeds the adjusted capacity level then the deficits will be satisfied by the emergency expansion.
3.3.4 Transition probabilities

From the distribution of the number of items sold $S_t$ and the binomial distribution of the random variable $N_t$, the transition probabilities between the values of $(x_{t-1}, y_{t-1})$ at consecutive decision epochs can be described by considering two cases. Let $x_{t-1} = i, y_{t-1} = j, u_t = k$ and $y_t = m$.

In the first case, enough capacity is guaranteed, i.e., $i + k \geq j$. Then we have

$$p_t(i + k, j + n|i, j: k) = \begin{cases} \sum_{m=-n}^{j} P(N_t = m)P(S_t = m + n) & ; -j \leq n \leq -1 \\ \sum_{m=0}^{j} P(N_t = m)P(S_t = m + n) & ; 0 \leq n \leq \max(S_t). \end{cases}$$

In the second case, enough capacity is not guaranteed, i.e., $i + k < j$. Then, we have

$$p_t(l, j + n|i, j: k) = P(N_t = l)P(S_t = l + n) ; i + k < l \leq j,$$

and

$$p_t(i + k, j + n|i, j: k) = \begin{cases} \sum_{m=-n}^{i+k} P(N_t = m)P(S_t = m + n) & ; -j \leq n \leq -1 \\ \sum_{m=0}^{i+k} P(N_t = m)P(S_t = m + n) & ; 0 \leq n \leq \max(S_t). \end{cases}$$

The proof that these expressions define a valid conditional probability distribution is presented in Appendix A.

3.3.5 Rewards

a) Let $r_t$ be the unit net revenue from selling a remanufactured item during the $t^{th}$ period. Because we assume that we must add capacity to satisfy all returns, the sales of remanufactured items equal the number of returns. Also, we assume that all remanufactured items will be sold.
\[ R_t(y_t) = r_t N_t. \] (5)

We assume that all items returned can be remanufactured profitably. In reality, not all items can be remanufactured and sold but some can be dismantled into parts for new products, and others may be recycled to recover materials. The revenue from processing the returns is considered as an average value across all return products. For example, some products may be returned in good-as-new condition and some may not. The net revenue is taken as an average value across all returned products. Certainly, considering different amounts of net revenue based on the different conditions of the returns is an interesting area for research but it is beyond the scope of this dissertation.

b) Capital cost:

\[ c_c[u_t] = \begin{cases} K_a + a_i u_t, & u_t > 0 \\ b_i u_t, & u_t \leq 0 \end{cases} \] (6)

where \( a_i \geq 0, b_i \leq 0, K_a > 0 \).

c) Operating cost

\[ c_m[x_{t-1}, u_t] = a_3 (x_{t-1} + u_t) \] (7)

where \( a_3 > 0 \).

d) Emergency expansion cost

\[ c_e[x_{t-1}, u_t, N_t] = K_a \delta(N_t - x_{t-1} - u_t) + a_4(N_t - x_{t-1} - u_t)^+ \] (8)

where

\[ \delta(w) = \begin{cases} 0 & \text{if } w \leq 0 \\ 1 & \text{if } w > 0 \end{cases} \] (9)

\((w)^+ \equiv \max(0, w), a_4 \geq 0, \text{ and } K_a > 0\).
In addition to the earlier requirements on the cost parameters, we further assume the following conditions:

\[ a_i > |b_i| \]  \hspace{1cm} (10)
\[ a_4 > a_i + a_3 \]  \hspace{1cm} (11)
\[ b_i + a_3 > 0 \]  \hspace{1cm} (12)
and \[ a_4 > |b_i| + a_3. \]  \hspace{1cm} (13)

Considering the cost requirement, (10) states that it is more expensive to expand capacity than to contract it. However according to (11), it is more economical to increase capacity and maintain a high level of capacity than expand on an emergency basis. Condition (12) is similar to one in Rocklin et al. (1984), where the capital reclamation of capacity is less than the operating denote the maximum expected discounted profit for the \((T-t)\) remaining periods given that at the beginning of period \(t\) the state is \((x_{t-1}, y_{t-1})\).

\[
V_t(x_{t-1}, y_{t-1}) = \max_{u_t} \{ r_t E_{N_t}[N_r] - c_a[u_t] - c_m[x_{t-1}, u_t] + \gamma \{ -E_{N_t}[c_x[x_{t-1}, u_t, N_r]] + E_{N_t,S}[V_{t+1}(\max\{x_{t-1} + u_t, N_r\}, y_{t-1} + S_t - N_r)] \} \}
\]  \hspace{1cm} (14)

for \(t = T-1, T-2, \ldots, 1, 0\) with \(V_T(x_T, y_T) = 0\).

The problem is to find the optimal decision sequence that maximizes the expected profit \(V_0(x_0, y_0)\). The solution can be founded by solving backwards starting from the last period to identify the optimal action for each state. However, when amount of sales is large the computation is difficult due to the large number of states and actions in each period to be evaluated. To reduce the computational burden of finding the optimal solution, we seek a
simple form for optimal policy that specifies the best actions to implement for each possible state.

In the next chapter, we show that an optimal policy has a simple form consisting of threshold values and target levels.
CHAPTER 4. OPTIMAL CAPACITY POLICY

4.1 Introduction

In previous research, Aneja and Chaouch (1993) considered a capacity adjustment model for a traditional facility rather than remanufacturing and showed that there is a simple optimal policy consisting of threshold and target levels. The demands for capacity in distinct periods were assumed to be independent, so that a Markov decision process could be formulated in terms of a single state variable for current capacity. In this research, the number of returns in period $t$, which constitutes the demand for capacity, depends on past sales and returns in previous periods. We extend the previous results to our model with two state variables so that the optimal decision depends on the past sales and returns information. In addition, the threshold and target levels depend on the number of items sold but not yet returned.

Generally, proving optimality is complex when there is a nonzero fixed cost associated with capacity expansion because it is not necessarily true that the optimal value function $V_t(.)$ is concave (Denardo, 1982, Rocklin et al., 1984, and Bertsekas, 1987). However, although $V_t(.)$ may not be concave, it does have the $K$-concave property corresponding to the notion of $K$-convexity introduced by Scarf (1960). Therefore, we can show the existence of the simple optimal policy using the properties of $K$-concavity.

The following definition and lemma are adapted from those concerning $K$-convexity in Bertsekas (1987).
For $K \geq 0$, a function $g(y)$ is said to be $K$-concave in $y$ if:

$$g(y + c) - K \leq g(y) + c \left[ \frac{g(y) - g(y - e)}{e} \right], \forall c \geq 0, e > 0.$$  \hspace{1cm} (15)

Figure 2 illustrates $K$-concavity; it shows the straight line passing through points $[a, g(a)]$ and $[a - e, g(a - e)]$. From (15) it requires the height of the line above $a + c$ to be no less than $g(a + c) - K$.

**Lemma 1.** Properties of a $K - concave$ function

(a) A concave function $g$ is also $K$-concave for all $K \geq 0$

(b) If $g_1(y)$ and $g_2(y)$ are $K$-concave and $L$-concave ($K \geq 0, L \geq 0$) respectively, then

$$\alpha g_1(y) + \beta g_2(y)$$

is $(\alpha K + \beta L)$-concave for all positive $\alpha$ and $\beta$. 
(c) Let $g$ be a continuous $K$-concave function and $g(y) \to +\infty$ as $y \to +\infty$. Define $S$ as

$$g(S) = \max_{y \geq 0} g(y),$$

and let $s$ be the smallest $y \geq 0$ such that $g(y) \geq g(S) - K$ with $s \leq S$

Then

i) If $s > 0$, $g(S) - K > g(y)$, for all $0 \leq y < s$

ii) If $s > 0$, $g(y)$ is an increasing function on $[0, s)$

iii) $g(y) \geq g(z) - K$ for all $y, z$ with $s \leq y \leq z$.

Figure 3 illustrates part (c) of Lemma 1.

---

4.2 Single period problem

Assume that at the beginning of period $T$ the capacity available is $x_{T-1}$ and the number of items outstanding in the market is $y_{T-1}$. The goal is to find a decision rule $z_T^* = x_{T-1} + u_T^*(x_{T-1}, y_{T-1})$ that maximizes the expected profit function
\[
J_T(x_{T-1}, y_{T-1}, z_T) = \begin{cases} 
R_T(y_{T-1}) - L_T(z_T | y_{T-1}), & z_T = x_{T-1} \\
R_T(y_{T-1}) - a_1(z_T - x_{T-1}) - L_T(z_T | y_{T-1}), & z_T > x_{T-1} \\
R_T(y_{T-1}) - b_1(z_T - x_{T-1}) - L_T(z_T | y_{T-1}), & z_T < x_{T-1}
\end{cases}
\]

where
\[
R_T(y_{T-1}) = r_T E[N_T] = r_T y_{T-1} p_2
\]

and
\[
L_T(z_T | y_{T-1}) = a_3 z_T + a_4 \sum_{n=z_T}^{y_{T-1}} (n - z_T) d_T(n | y_{T-1}) + K_z \sum_{n=z_T+1}^{y_{T-1}} d_T(n | y_{T-1})
\]

or equivalently
\[
L_T(z_T | y_{T-1}) = a_3 z_T + a_4 \sum_{n=z_T}^{y_{T-1}} n d_T(n | y_{T-1}) + (K_z - a_3 z_T) \sum_{n=z_T+1}^{y_{T-1}} d_T(n | y_{T-1}).
\]

Note that, \( R_T(y_{T-1}) \) represents the expected revenue that depends on \( y_{T-1} \)

\( L_T(z_T | y_{T-1}) \) represents the expected operating and emergency expansion cost that depend on \( y_{T-1} \)

\( d_T(n | y_{T-1}) \) represents the return distribution in period \( T \)

where \( d_T(n | y_{T-1}) \equiv P(N_T = n | y_{T-1}) \).

Let
\[
G_T(x_{T-1}, y_{T-1}, z_T) = R_T(y_{T-1}) - a_1(z_T - x_{T-1}) - L_T(z_T | y_{T-1})
\]

and
\[
H_T(x_{T-1}, y_{T-1}, z_T) = R_T(y_{T-1}) - b_1(z_T - x_{T-1}) - L_T(z_T | y_{T-1})
\]

where \( G_T(x_{T-1}, y_{T-1}, z_T) \) and \( H_T(x_{T-1}, y_{T-1}, z_T) \) respectively represent the expected profit from expansion excluding fixed cost and from contraction.
Then we can write $J_T(x_{T-1}, y_{T-1}, z_T)$ as

$$J_T(x_{T-1}, y_{T-1}, z_T) = \begin{cases} G_T(x_{T-1}, y_{T-1}, z_T) - K_a, & z_T > x_{T-1} \\ H_T(x_{T-1}, y_{T-1}, z_T), & z_T \leq x_{T-1}. \end{cases} \quad (19)$$

For notational convenience, we shall exclude the subscript for the moment and we may write the functions as follows:

$$G(x, y, z) = ryp_2 - [a(z - x) + L(z|y)] \quad (20)$$

$$H(x, y, z) = ryp_2 - [b(z - x) + L(z|y)]. \quad (21)$$

From (20) and (21), we notice that revenue does not affect the optimal level of new capacity $(z)$, but it does affect the amount earned at that level. Let us consider a condition that guarantees the existence of an optimal solution. We observe that $G(x, y, z)$ and $H(x, y, z)$ are $K_a$-concave functions in $z$ if the function $-L(z|y)$ is $K_a$-concave.

To determine the concavity of function $-L(z|y)$ in $z$, for a given $y$, from (15) we must show that

$$-L(z + c|y) - K_a \leq -L(z|y) + c \left[ \frac{-L(z|y) - L(z - e|y)}{e} \right] \text{ for } K_a \geq 0, \ c \geq 0 \text{ and } e > 0.$$

This inequality is equivalent to:

$$-K_a - a_3c - a_4 \sum_{n=z+1}^{x+y} nd(n|y) - K_s \sum_{n=z+1}^{x+y} d(n|y) + a_3 z \sum_{n=z+1}^{x+y} d(n|y) + a_4 c \sum_{n=z+c+1}^{x+y} d(n|y)$$

$$\leq c \left[ -a_3 c + a_4 \sum_{n=z-c}^{x} nd(n|y) + K_s \sum_{n=z-c+1}^{x} d(n|y) - a_3 z \sum_{n=z-c+1}^{x} d(n|y) + a_4 e \sum_{n=z-c+1}^{y} d(n|y) \right]$$
or,

\[-K_a - a_s \sum_{n=z+1}^{z+c} nd(n|y) - K_s \sum_{n=z+1}^{z+c} d(n|y) + a_{y} z \sum_{n=z+1}^{z+c} d(n|y) + a_{c} \sum_{n=z+c+1}^{z} d(n|y) - a_{c} \sum_{n=z-c+1}^{z} d(n|y) \]

\[\leq \frac{c}{e} \left[ a_s \sum_{n=z-e+1}^{z-c} nd(n|y) + K_s \sum_{n=z-e+1}^{z} d(n|y) - a_{y} z \sum_{n=z-e+1}^{z} d(n|y) \right],\]

which is true because \( a_{y} c \sum_{n=z+c+1}^{z} d(n|y) - a_{c} \sum_{n=z-c+1}^{z} d(n|y) \leq 0 \),

\[-a_s \sum_{n=z+1}^{z+c} nd(n|y) + a_{y} z \sum_{n=z+1}^{z+c} d(n|y) \leq 0, \text{ and from (18) we have that} \]

\[a_s \sum_{n=z-e+1}^{z-c} nd(n|y) + K_s \sum_{n=z-e+1}^{z} d(n|y) - a_{y} z \sum_{n=z-e+1}^{z} d(n|y) \geq 0.\]

We now describe the form of an optimal policy for the single period problem.

**Theorem 1.**

(i) An optimal policy for the single period problem is given by

\[ u^*_T(x_{T-1}, y_{T-1}) = \begin{cases} Q_T(y_{T-1}) - x_{T-1}, & x_{T-1} < q_T(y_{T-1}) \\ 0, & q_T(y_{T-1}) \leq x_{T-1} \leq T_T(y_{T-1}) \\ T_T(y_{T-1}) - x_{T-1}, & T_T(y_{T-1}) < x_{T-1} \end{cases} \quad (22) \]

where \( Q_T(y_{T-1}) \) and \( T_T(y_{T-1}) \) are the capacity levels that maximize the net profit from expansion and contraction, respectively, and \( q_T(y_{T-1}) \) is the smallest \( z \geq 0 \) such that

\[ G_T(x_{T-1}, y_{T-1}, q_T(y_{T-1})) \geq G_T(x_{T-1}, y_{T-1}, Q_T(y_{T-1})) - K_a. \]
(ii) The optimal profit-to-go function from state \((x_{T-1}, y_{T-1})\) given by

\[
V_T(x_{T-1}, y_{T-1}) = \begin{cases} 
R_T(y_{T-1}) - K_y - a_i(Q_T(y_{T-1}) - x_{T-1}) & x_{T-1} < q_T(y_{T-1}) \\
L_T(Q_T(y_{T-1})|y_{T-1}), & q_T(y_{T-1}) \leq x_{T-1} \leq T_T(y_{T-1}) \\
R_T(y_{T-1}) - L_T(x_{T-1}|y_{T-1}), & x_{T-1} > T_T(y_{T-1})
\end{cases}
\]

is \(K_a\)-concave in \(x_{T-1}\).

**Proof**

We shall first show that the optimal policy is of the form given in (22) and second that

\(V_T(x_{T-1}, y_{T-1})\) is \(K_a\)-concave in \(x_{T-1}\).

Define \(g(y, z) = R(y) - [a_i z + L(z | y)]\) and we have that \(g(y, z)\) is a \(K_a\)-concave function in \(z\). Let \(g(y, Q(y)) = \max_{z \geq 0} g(y, z)\) and \(q(y)\) be the smallest \(z \geq 0\) such that \(g(y, z) \geq g(y, Q(y)) - K_a\). We now describe the optimal policy for capacity expansion and contraction separately.

(i) An optimal policy for the single period problem

1. **Capacity expansion**

Considering the connection of the current state of capacity on the optimal policy for expansion; there are three possible cases, which are:

1) \(x \geq Q(y)\)
2) \( q(y) < x < Q(y) \)

3) \( 0 \leq x < q(y) \)

**Case 1.1.** \( x \geq Q(y) \).

We show that it is optimal not to expand capacity.

If capacity is increased to \( m \), the profit is \( R(y) - a_{1}(m-x) - L(m|y) - K_{a} \) where \( m = x + \sigma \) and \( \sigma > 0 \). If we do not expand capacity, the profit is \( R(y) - L(x|y) \).

Since \( x \geq Q(y) \), \( m > Q(y) \), and \( m > x \), we have \( g(y,x) > g(y,m) \).

Thus,
\[
R(y) - a_{1}x - L(x|y) > R(y) - a_{1}m - L(m|y)
\]
\[
R(y) - L(x|y) > R(y) - a_{1}(m-x) - L(m|y)
\]

or
\[
R(y) - L(x|y) > R(y) - a_{1}(m-x) - L(m|y) - K_{a}.
\]

**Case 1.2.** \( q(y) < x < Q(y) \).

We show that it is optimal not to expand capacity.

If capacity is increased to \( m \), the profit is \( R(y) - a_{1}(m-x) - L(m|y) - K_{a} \) where \( m = x + \sigma \) and \( \sigma > 0 \). If we do not expand capacity, the profit is \( R(y) - L(x|y) \).

By definition of \( g(y,z) \), \( g(y,Q(y)) \) is \( K_{a} \)-concave and the function of \( z \) has \( Q(y) \) as the maximum point.

Using part \((c)-(iii)\) of Lemma 1,

we have
\[
g(y,x) \geq g(y,Q(y)) - K_{a}
\]
or \[ R(y) - a_i x - L(x|y) \geq R(y, Q(y)) - a_i Q(y) - L(Q(y)|y) - K_a \]

so that \[ R(y) - L(x|y) \geq R(y) - a_i (Q(y) - x) - L(Q(y)|y) - K_a \]

and by definition of \( g(y, Q(y)) \), we also have

\[ R(y) - a_i Q(y) - L(Q(y)|y) \geq R(y) - a_i m - L(m|y) \]

thus \[ R(y) - a_i (Q(y) - x) - L(Q(y)|y) - K_a \geq R(y) - a_i (m - x) - L(m|y) - K_a \] .

**Case 1.3.** \( 0 \leq x < q(y) \).

We show that it is optimal to expand capacity up to \( Q(y) \).

If capacity is increased to \( m \), the profit is \( R(y) - a_i (m - x) - L(m|y) - K_a \) where \( m = x + \sigma \)

and \( \sigma > 0 \). If we do not expand capacity, the profit is \( R(y) - L(x|y) \).

Using part (c) - (i) of Lemma 1,

we have

\[ g(y, x) < g(y, Q(y)) - K_a \]

or \[ R(y) - a_i x - L(x|y) < R(y) - a_i Q(y) - L(Q(y)|y) - K_a \]

so that \[ R(y) - a_i (Q(y) - x) - L(Q(y)|y) - K_a > R(y) - L(x|y) \]

and by definition of \( g(y, Q(y)) \), we have

\[ R(y) - a_i Q(y) - L(Q(y)|y) \geq R(y) - a_i m - L(m|y) \]

thus \[ R(y) - a_i (Q(y) - x) - L(Q(y)|y) - K_a \geq R(y) - a_i (m - x) - L(m|y) - K_a \] .
2. Capacity contraction

We now consider the connection of the current state of capacity to the optimal policy for contraction; there are two possible cases, which are:

1) \( x > T(y) \)

2) \( x \leq T(y) \)

Define \( h(y, z) = R(y) - b_1 z - L(z|y) \) and \( h(y, T(y)) = \max_{z \geq 0} h(y, z) \). We have that \( h(y, z) \) is a \( K_{a_i} \)-concave function in \( z \) for a given \( y \). We first show that \( Q(y) \leq T(y) \). By definition, \( g(y, Q(y)) = \max_{z \geq 0} g(y, z) \) and \( h(y, T(y)) = \max_{z \geq 0} h(y, z) \).

Therefore, we have

\[
R(y) - a_1 Q(y) - L(Q(y)|y) \geq R(y) - a_1 T(y) - L(T(y)|y)
\]

(24)

and

\[
R(y) - b_1 T(y) - L(T(y)|y) \geq R(y) - b_1 Q(y) - L(Q(y)|y).
\]

(25)

From (25), it follows that

\[-L(T(y)|y) + L(Q(y)|y) \geq b_1 T(y) - b_1 Q(y) .\]

Since \( a_i > |b_1| \) and suppose \( T(y) < Q(y) \), we have

\[-L(T(y)|y) + L(Q(y)|y) \geq b_1 T(y) - Q(y) > a_i (T(y) - Q(y))\]

or equivalently \( R(y) - a_1 T(y) - L(T(y)|y) > R(y) - a_1 Q(y) - L(Q(y)|y) \), which contradicts (24).

Case 2.1. \( x > T(y) \).

We show that it is optimal to contract capacity to \( T(y) \).

If capacity is decreased to \( T(y) \), the profit is \( R(y) - b_1 (T(y) - x) - L(T(y)|y) \).
If we do not contract capacity, the profit is \( R(y, x) - L(y, x) \). By definition of \( h(y, T(y)) \), we have

\[
h(y, x) \leq h(y, T(y))
\]

or

\[
R(y) - b_1 x - L(x|y) \leq R(y) - b_1 T(y) - L(T(y)|y)
\]

so that

\[
R(y) - L(x|y) \leq R(y) - b_1 (T(y) - x) - L(T(y)|y).
\]

Also, if capacity is decreased to other level, the profit is \( R(y) - b_1 (m - x) - L(m|y) \)

where \( m = x + \sigma \) and \( \sigma < 0 \).

By definition of \( h(y, T(y)) \), we have

\[
R(y) - b_1 m - L(m|y) \leq R(y) - b_1 T(y) - L(T(y)|y)
\]

so that

\[
R(y) - b_1 (m - x) - L(m|y) \leq R(y) - b_1 (T(y) - x) - L(T(y)|y).
\]

**Case 2.2.** \( x \leq T(y) \).

We show that it is optimal not to contract capacity.

If capacity is decreased to \( m \), the profit is \( R(y) + b_1 (m - x) - L(m|y) \) where \( m = x + \sigma \)

and \( \sigma < 0 \). If we do not contract capacity, the profit is \( R(y) - L(x|y) \).

Since \( x \leq T(y), m < T(y), \) and \( m < x \), then we have \( h(y, x) > h(y, m) \).

Thus,

\[
R(y) - b_1 x - L(x|y) > R(y) - b_1 m - L(m|y)
\]

or equivalently

\[
R(y) - L(x|y) > R(y) - b_1 (m - x) - L(m|y) - K_a.
\]
The above results describe the optimal policy consisting threshold values and target levels for the single period problem. The next step is to show that $V_T(x_{T-1},y_{T-1})$ is $K_a$-concave in $x_{T-1}$.

(ii) $K_a$-concavity of the optimal profit-to-go function

1. Capacity expansion

Using the properties of $K_a$-concavity to verify the optimal profit-to-go functions (23) for the capacity expansion, let us write the function for the capacity expansion ($V_e$) as follows:

$$V_e(x,y) = \begin{cases} g(y,Q(y)) + a_i x - K_a, & x < q(y) \\ g(y,x) + a_i x, & x \geq q(y). \end{cases}$$

We wish to show that $V_e(x,y)$ is $K_a$-concave in $x$ based on the fact that $g(y,z)$ is $K_a$-concave in $z$. Thus, we must verify that

$$V_e(x+c,y) - K_a \leq V_e(x,y) + c \left[ \frac{V_e(x,y) - V_e(x-e,y)}{e} \right] \text{ for all } c \geq 0, e > 0, x.$$ (27)

We recognize three cases:

Case 1.1. $x \geq q(y)$.

If $x - e \geq q(y)$ then we must show

$$g(y,x+c) + a_i (x+c) - K_a \leq g(y,x) + a_i x + c \left[ \frac{g(y,x) + a_i x - g(y,x-e) - a_i (x-e)}{e} \right] \text{ or}$$

$$g(y,x+c) - K_a \leq g(y,x) + c \left[ \frac{g(y,x) - g(y,x-e)}{e} \right]$$

which is true since $g(y,z)$ is $K_a$-concave.
If \( x - e < q(y) \) then we must show

\[
g(y, x + c) + a_i(x + c) - K_a \leq g(y, x) + a_i x + c \left[ \frac{g(y, x) + a_i x + K_a - g(y, Q(y)) - a_i (x - e)}{e} \right]
\]
or

\[
g(y, x + c) - K_a \leq g(y, x) + c \left[ \frac{g(y, x) - g(y, Q(y)) + K_a}{e} \right]
\]

which we can write as

\[
g(y, x + c) - K_a \leq g(y, x) + c \left[ \frac{g(y, x) - g(y, q(y))}{e} \right].
\]

If \( g(y, x) > g(y, q(y)) \) then we must show

\[
g(y, x + c) - K_a \leq g(y, x) + c \left[ \frac{g(y, x) - g(y, q(y))}{e} \right].
\]

This inequality is true because of part \((c) - (iii)\) of Lemma 1,

\[
g(y, x + c) - K_a \leq g(y, x).
\]

If \( g(y, x) \leq g(y, q(y)) \) then we wish to show

\[
g(y, x + c) - K_a \leq g(y, x) + c \left[ \frac{g(y, x) - g(y, q(y))}{e} \right]
\]

which is true since \( g(y, z) \) is \( K_a \)-concave.

**Case 1.2.** \( x \leq x + c \leq q(y) \).

We wish to show

\[
g(y, Q(y)) + a_i(x + c) - K_a - K_a
\]

\[
\leq g(y, Q(y)) + a_i x - K_a + c \left[ \frac{g(y, Q(y)) + a_i x - K_a - g(y, Q(y)) - a_i (x - e) + K_a}{e} \right],
\]

which is equivalent to \( -K_a \leq 0 \).
Case 1.3. \( x < q(y) < x + c \).

We wish to show

\[
g(y, x + c) + a_1(x + c) - K_a \\
\leq g(y, Q(y)) + a_1x - K_a + c \left[ \frac{g(y, Q(y)) + a_1x - g(y, Q(y)) - a_1(x - e) + K_a}{e} \right]
\]

, which we can write as

\[
g(y, x + c) + a_1(x + c) - K_a \\
\leq g(y, q(y)) + a_1x + c \left[ \frac{g(y, q(y)) + a_1x - g(y, q(y)) - a_1(x - e)}{e} \right]
\]

or equivalently \( g(y, x + c) - K_a \leq g(y, q(y)) \).

This inequality is true because of part \((c) - (iii)\) of Lemma 1.

2. Capacity contraction

We recognize that \( h(y, z) \) is a \( K_a \)-concave function of \( z \) for a given \( y \). Hence from (26), we can write the optimal profit-to-go functions for the contraction \((V_c)\) as follows:

\[
V_c(x, y) = \begin{cases} 
  h(y, x) + b_i x, & x \leq T(y) \\
  h(y, T(y)) + b_i x, & x > T(y).
\end{cases}
\] (28)

We can verify that \( V_c \) is a \( K_a \)-concave function of \( x \) based on the fact that \( h(y, z) \) is a \( K_a \)-concave in \( z \).
From the previous analysis, we show that \( V_x(x, y) \) and \( V_y(x, y) \) are \( K_a \)-concave. If we combine (26) and (28) and reinstate the subscript, we obtain (23), which we can write as follows:

\[
V(x, y) = \begin{cases} 
G(x, y, Q(y)) - K_a, & x < q(y) \\
H(x, y, w(x, y)), & q(y) \leq x \leq T(y) \\
H(x, y, T(y)), & x > T(y)
\end{cases}
\]

The proof that \( V(x, y) \) is \( K_a \)-concave depends on the remaining cases:

1) \( x < q \leq T < x + c \)

2) \( x - e < q \leq x \leq T \leq x + c \)

3) \( x - e < q \leq T \leq x \)

**Case 1.** \( x < q \leq T < x + c \)

we wish to show:

\[
H(x + c, y, T) - K_a \leq G(x, y, Q) - K_a + \frac{G(x, y, Q) - G(x - e, y, Q) + K_a}{e}
\]

or, \( H(x + c, y, T) \leq G(x, y, Q) + a_1c \)

equivalently, \( H(x + c, y, T) - G(x, y, Q) - a_1c \leq 0 \)

or, \( H(x + c, y, T) - G(x, y, Q) - a_1c \leq H(x + c, y, T) - G(x, y, T) - a_1c \)

or, \( H(x + c, y, T) - G(x, y, T) - a_1c = (a_1 - b_1)(T - x - c) \leq 0 \)

because of \( T < x + c \) and \( a_1 > b_1 \).

**Case 2.** \( x - e < q \leq x \leq T \leq x + c \)
We wish to show:

\[ H(x + c, y, T) - K_a \leq H(x, y, w(x, y)) + c \left[ \frac{H(x, y, w(x, y)) - G(x - e, y, Q) + K_a}{e} \right] \]

or,

\[ R(y) - b_1(T - x - c) - L(T) - K_a \leq R(y) - b_1(x(x, y) - x) - L(x(x, y)) + c \left[ \frac{R(y) - b_1(x(x, y) - x) - L(x(x, y)) - R(y) + a_1(Q - x + e) + L(Q) + K_a}{e} \right] \]

equivalently,

\[ R(y) - b_1(T - x - c) - L(T) - K_a \leq R(y) - L(x) + c \left[ \frac{R(y) - L(x) - R(y) + a_1(Q - x + e) + L(Q) + K_a}{e} \right] \]

or,

\[ 0 \leq R(y) - L(x) - [R(y) - b_1(T - x - c) - L(T) - K_a] + c \left[ \frac{R(y) - L(x) - R(y) + a_1(Q - x + e) + L(Q) + K_a}{e} \right] \]

which is true because of the following

From capacity contraction case 2.1, we know that

\[ R(y) - b_1(T - c) - L(T) \leq R(y) - b_1x - L(x), \text{ where } x \geq T - c. \]

Thus, \( R(y) - L(x) - [R(y) - b_1(T - x - c) - L(T) - K_a] \geq 0. \)

From capacity expansion case 1.2, we know that

\[ R(y) - L(x) \geq R(y) - a_1(Q - x) - L(Q) - K_a \]

Thus, \( R(y) - L(x) - R(y) + a_1(Q - x) + L(Q) + K_a \geq 0. \)
which implies, $R(y) - L(x) - R(y) + a_1(Q - x + e) + L(Q) + K_a \geq 0$.

**Case 3.** $x - e < q \leq T \leq x$

We wish to show:

$$H(x+c, y, T) - K_a \leq H(x, y, T) + c\left[\frac{H(x, y, T) - G(x - e, y, Q) + K_a}{e}\right]$$

or,

$$R(y) - b_1(T - x - c) - L(T) - K_a \leq R(y) - b_1(T - x) - L(T)$$

$$+ c\left[\frac{R(y) - b_1(T - x) - L(T) - R(y) + a_1(Q - x + e) + L(Q) + K_a}{e}\right]$$

equivalently,

$$0 \leq R(y) - b_1(T - x) - L(T) - [R(y) - b_1(T - x - c) - L(T) - K_a]$$

$$+ c\left[\frac{R(y) - b_1(T - x) - L(T) - R(y) + a_1(Q - x + e) + L(Q) + K_a}{e}\right]$$

, which is true because of the following:

$$R(y) - b_1(T - x) - L(T) - [R(y) - b_1(T - x - c) - L(T) - K_a] = -b_1c + K_a > 0.$$  

We have $R(y) - a_1(Q - x) - L(Q) > R(y) - a_1(Q - x + e) - L(Q)$.  

From capacity contraction case 2.1, we know that 

$$R(y) - b_1(T - x) - L(T) \geq R(y) - L(x).$$

From capacity expansion case 1.1, we know that 

$$R(y) - L(x) \geq R(y) - a_1(Q - x) - L(Q).$$

Thus, 

$$R(y) - b_1(T - x) - L(T) \geq R(y) - a_1(Q - x) - L(Q) > R(y) - a_1(Q - x + e) - L(Q).$$
So that \( R(y) - b_1(T - x) - L(T) - [R(y) - a_1(Q - x + e) - L(Q)] > 0 \)
which implies, \( R(y) - b_1(T - x) - L(T) - [R(y) - a_1(Q - x + e) - L(Q)] + K_a > 0 \).

End of proof of Theorem 1.

Theorem 1 describes the structure of an optimal policy for the one period problem. The optimal policy for a single period has a simple form and the policy consists of threshold values \( T(y) \) and \( q(y) \) as well as target levels \( T(y) \) and \( Q(y) \). The next section, we extend the one period result to the multi-period case using the \( K_a \)-concavity of the optimal value function.

### 4.3 Multi-period problem

We shall show that there is an optimal capacity policy, at any stage \( t < T \), of a form similar to the single period case. The optimal policy of period \( t \) is given by:

\[
u^*_t(x_{t-1}, y_{t-1}) = \begin{cases} 
Q_t(y_{t-1}) - x_{t-1}, & x_{t-1} < q_t(y_{t-1}) \\
w_t(x_{t-1}, y_{t-1}) - x_{t-1}, & q_t(y_{t-1}) \leq x_{t-1} < T_t(y_{t-1}) \\
T_t(y_{t-1}) - x_{t-1}, & T_t(y_{t-1}) \leq x_{t-1}.
\end{cases}
\] 

(29)

where \( T_t(y_{t-1}), Q_t(y_{t-1}), q_t(y_{t-1}), \) and \( w_t(x_{t-1}, y_{t-1}) \) are defined by

\[
G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) = \max_{z_t \geq 0} G_t(x_{t-1}, y_{t-1}, z_t),
\] 

(30)

\[
H_t(x_{t-1}, y_{t-1}, T_t(y_{t-1})) = \max_{z_t \geq 0} H_t(x_{t-1}, y_{t-1}, z_t),
\] 

(31)

\( q_t(y_{t-1}) \) is the smallest \( z_t \geq 0 \) such that \( G_t(x_{t-1}, y_{t-1}, z_t) \geq G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K_a \).
And for \( q_t(y_{i-1}) \leq x_{i-1} \leq T_t(y_{i-1}) \), \( w_t(x_{i-1}, y_{i-1}) \) is \( z_t \) that maximizes \( H_t(x_{i-1}, y_{i-1}, z_t) \) s.t. \( 0 \leq z_t \leq x_{i-1} \). We will show that \( q_t(y_{i-1}) \leq w_t(x_{i-1}, y_{i-1}) \).

To verify the form of the optimal policy and the \( K_a \)-concavity of the optimal profit-to-go function from state \((x_{k-1}, y_{k-1})\) for \( k = t+1, \ldots, T-1 \), we employ an induction proof on period \( t \). We assume the following: 1) All future optimal policies \( u_k^*(x_{k-1}, y_{k-1}) \), \( k \geq t+1 \), have the form given by (29) with \( k \) substituting for \( t \).

2) We assume that the optimal profit-to-go function, which has the following form:

\[
V_k(x_{k-1}, y_{k-1}) = \begin{cases} 
G_k(x_{k-1}, y_{k-1}, q_k(y_{k-1})) - K_a, & x_{k-1} < q_k(y_{k-1}) \\
H_k(x_{k-1}, y_{k-1}, w_k(x_{k-1}, y_{k-1})), & q_k(y_{k-1}) \leq x_{k-1} \leq T_k(y_{k-1}) \\
H_k(x_{k-1}, y_{k-1}, T_k(y_{k-1})), & x_{k-1} > T_k(y_{k-1})
\end{cases}
\]  

(32)

is \( K_a \)-concave in \( x_{k-1} \) for \( k \geq t+1 \).

As part of the induction hypothesis, finally we assume that the functions \( G_k(x_{k-1}, y_{k-1}, z_k) \) and \( H_k(x_{k-1}, y_{k-1}, z_k) \) are \( K_a \)-concave in \( z_k \), \( k \geq t+1 \).

We shall show the following: First, \( G_t(x_{i-1}, y_{i-1}, z_t) \) and \( H_t(x_{i-1}, y_{i-1}, z_t) \) are \( K_a \)-concave functions of \( z_{i-1} \). Second, the optimal policy \( u_t^*(x_{i-1}, y_{i-1}) \) has the form shown in (30).

3) The optimal profit-to-go \( V_t(x_{i-1}, y_{i-1}) \) is \( K_a \)-concave in \( x_{i-1} \). To prove that \( G_t(x_{i-1}, y_{i-1}, z_t) \) and \( H_t(x_{i-1}, y_{i-1}, z_t) \) are \( K_a \)-concave functions of \( z_t \) we need the following Lemmas.
Lemma 2. adapted from Aneja and Chaouch (1993)

Let $g(y, z)$ be a $K$-convex function of $z$ for a given $y$ with $g(y, z) \to \infty$ as $z \to \infty$.

Assume $g(y, z)$ is also $K$-monotone non-decreasing in $z$, i.e., $g(y, z) \leq g(y, z + a) + K$ for all $z \geq 0$ and $a \geq 0$ for a given $y$.

Then $f(y, x) = \min_{0 \leq z \leq x} [-B(z - x) + g(y, z)]$ for $B > 0$

is $K$-convex and $K$-monotone non-decreasing in $x$.

Lemma 3.

Let $M_t(x, y) = a_t x + \overline{V}_{t+1}(x, y)$, where $a_t = \frac{a_1}{\gamma}$. Then $M_t(x, y)$ is $K_a$-convex and $K_a$-monotone non-decreasing in $x$.

Proof

By the induction hypothesis of the form of optimal policy $u^*_t(x_t, y_t)$ and $K_a$-concavity in $z_{t+1}$ of $G_{t+1}(x_t, y_t, z_{t+1})$ and $H_{t+1}(x_t, y_t, z_{t+1})$, we have that $V_{t+1}(x_t, y_t)$ is $K_a$-concave in $x_t$,

therefore $\overline{V}_{t+1}(x_t, y_t)$ is $K_a$-convex in $x_t$, where $\overline{V}_{t+1}(x_t, y_t) = -V_{t+1}(x_t, y_t)$. $M_t(x, y)$ is the sum of a linear function and $K_a$-convex function, thus it is $K_a$-convex. To show $K_a$-monotone non-decreasing, we wish to show:

$M_t(x, y) \leq M_t(x + a, y) + K_a$ for all $x \geq 0$, $y \geq 0$ and $a \geq 0$.

We have

$$\overline{V}_{t+1}(x_t, y_t) = \begin{cases} -G_{t+1}(x_t, y_t, Q_{t+1}(y_t)) + K_a, & x_t < q_{t+1}(y_t) \\ -H_{t+1}(x_t, y_t, w_{t+1}(x_t, y_t)), & x_t \geq q_{t+1}(y_t). \end{cases}$$

Thus,
We have three cases:

**Case 1.** \( x < x + a < q(y) \)

We wish to show: \( M_j(x, y) \leq M_j(x + a, y) + K_a \)

or, \( a_s x - G_{t+1}(x, y, Q_{t+1}(y)) + K_a \leq a_s (x + a) - G_{t+1}(x + a, y, Q_{t+1}(y)) + K_a + K_a \)

equivalently, \( G_{t+1}(x, y, Q_{t+1}(y)) - G_{t+1}(x + a, y, Q_{t+1}(y)) + a_s a + K_a \geq 0 \).

We can show that

\[
G_{t+1}(x, y, Q_{t+1}(y)) - G_{t+1}(x + a, y, Q_{t+1}(y)) + a_s a + K_a = (a_s - a_t) a + K_a \geq 0
\]

, which is true since \( a_s > a_t > a_t \).

**Case 2.** \( x < q(y) \leq x + a \)

We wish to show: \( M_j(x, y) \leq M_j(x + a, y) + K_a \)

or, \( a_s x - G_{t+1}(x, y, Q_{t+1}(y)) + K_a \leq a_s (x + a) - H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + K_a \)

equivalently, \( -G_{t+1}(x, y, Q_{t+1}(y)) \leq a_s a - H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + K_a \).

We can show that \( G_{t+1}(x, y, w_{t+1}(x, y)) + a_t a = G_{t+1}(x + a, y, w_{t+1}(x + a, y)) \)

and

\[
-H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + G_{t+1}(x + a, y, w_{t+1}(x + a, y)) + a_t a
\]

\[
= (b_t - a_t)(w_{t+1}(x + a, y) - x_{t+2} - a).
\]

Thus, \( -H_{t+1}(x + a, y, w_{t+1}(x + a, y)) = (b_t - a_t)(w_{t+1}(x + a, y) - x_t - a) \)

\[
-G_{t+1}(x, y, w_{t+1}(x + a, y)) - a_t a.
\]
From \[-G_{t+1}(x, y, Q_{t+1}(y)) \leq a_3 a - H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + K_a,
\]
equivalently, \[-G_{t+1}(x, y, Q_{t+1}(y)) \leq a_3 a + (b_i - a_i)(w_{t+1}(x + a, y) - x_t - a)\]
\[-G_{t+1}(x, y, w_{t+1}(x + a, y)) - a_i a; \]
\[-G_{t+1}(x, y, Q_{t+1}(y)) \leq (a_5 - a_i)a + (b_i - a_i)(w_{t+1}(x + a, y) - x_t - a)\]
\[-G_{t+1}(x, y, w_{t+1}(x + a, y))\]
or, \[0 \leq (a_5 - a_i)a + (b_i - a_i)(w_{t+1}(x + a, y) - x_t - a)\]
\[+G_{t+1}(x, y, Q_{t+1}(y)) - G_{t+1}(x, y, w_{t+1}(x + a, y))\]
, which is true since \(a_5 > a_4 > a_i > b_i, w_{t+1}(x + a, y) \leq x_t + a, \) and
\[G_{t+1}(x, y, Q_{t+1}(y)) \geq G_{t+1}(x, y, w_{t+1}(x + a, y)).\]

**Case 3.** \(q(y) \leq x < x + a\)

We wish to show: \(M_j(x, y) \leq M_j(x + a, y) + K_a\)
or, \[a_3 x - H_{t+1}(x, y, w_{t+1}(x, y)) \leq a_3 (x + a) - H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + K_a\]
We have \(H_{t+1}(x, y, w_{t+1}(x, y)) = \max_{0 \leq z \leq x} H_{t+1}(x, y, z),\)
thus \(-H_{t+1}(x, y, w_{t+1}(x, y)) = \min_{0 \leq z \leq x} -H_{t+1}(x, y, z).\)

From \(-H_{t+1}(x, y, w_{t+1}(x, y)) \leq -H_{t+1}(x + a, y, w_{t+1}(x + a, y)) + K_a,\)
we can write as \(b_i(w_{t+1}(x, y) - x_t) + g(y, w_{t+1}(x, y)) \leq b_i(w_{t+1}(x + a, y) - x_t - a)\)
\[+g(y, w_{t+1}(x + a, y)) + K_a\]
, where \(g(y, w_{t+1}(x, y)) = (a_3 - a_d)w_{t+1}(x, y) + K_s \sum_{n=t+1}^{y} p_{t+1}(n|[y, y])\)
\[ +\gamma E_{N, t_1}^* \{ M_{t+2} \left( \max \{ w_{t+1}(x, y), N_{t+1} \}, y + S_{t+1} - N_{t+1} \} \} \]

and

\[ g(y_t, w_{t+1}(x_t + a, y_t)) = (a_3 - a_4)w_{t+1}(x_t + a, y_t) + K \sum_{n=\zeta_{t+1}}^y p_{t+1}(n|y_t) \]

\[ +\gamma E_{N, t_1}^* \{ M_{t+2} \left( \max \{ w_{t+1}(x_t + a, y_t), N_{t+1} \}, y_t + S_{t+1} - N_{t+1} \} \} \].

By the induction hypothesis, since \( \max \{ w_{t+1}(x, y), N_t \} \) is convex and non-decreasing in \( w_{t+1} \), and \( M_{t+2}(x, y) \) is \( K_a \)-convex in \( x \), then the composite function

\[ M_{t+2} \left( \max \{ w_{t+1}(x, y), N_t \}, y_t + S_{t+1} - N_{t+1} \right) \] is \( K_a \)-convex and non-decreasing in \( w_{t+1} \)

(Rockafellar, 1970). The properties of \( K_a \)-convexity are preserved under summation and expectation (Denardo, 1982, p.151). Therefore, we have that

\[ E_{N, t_1}[M_{t+2} \left( \max \{ w_{t+1}(x, y), N_{t+1} \}, y_t + S_{t+1} - N_{t+1} \} \] is \( K_a \)-convex and non-decreasing in \( w_{t+1} \).

By the \( K_a \)-convexity of \( L_t(z_t|y_t) \), we have that \((a_3 - a_4)w_{t+1}(x_t, y_t) + K \sum_{n=\zeta_{t+1}}^y p_{t+1}(n|y_t) \) is a \( K_a \)-convex function in \( w_{t+1} \). By part (b) of the Lamma 1, we have that \( K \)-convexity is preserved under addition. Thus, \( g(y_t, w_{t+1}(x, y)) \) is also \( K_a \)-convex and non-decreasing in \( w_{t+1} \). By Lemma 2, we have that

\[ -H_{t+1}(x_t, y_t, w_{t+1}(x_t, y_t)) \leq -H_{t+1}(x_t + a, y_t, w_{t+1}(x_t + a, y_t)) + K_a. \]

Thus, \( a_5x - H_{t+1}(x_t, y_t, w_{t+1}(x_t, y_t)) \leq a_5(x + a) - H_{t+1}(x_t + a, y_t, w_{t+1}(x_t + a, y_t)) + K_a. \)

End of proof of Lemma 3.
At this point, we are now ready to prove that $G_t(x_{t-1}, y_{t-1}, z_t)$ and $H_t(x_{t-1}, y_{t-1}, z_t)$ are $K_a$-concave functions of $z_t$.

**Theorem 2**

The functions $G_t(x_{t-1}, y_{t-1}, z_t)$ and $H_t(x_{t-1}, y_{t-1}, z_t)$ are $K_a$-concave functions of $z_t$.

**Proof**

We can express as $L_t(z_t | y_{t-1}) = (a_3 - a_4)z_t + K_s \sum_{n=z_t+1}^{y_{t-1}} p_t(n | y_{t-1}) + a_4E_{N_t} \max \{z_t, N_t\}$

Thus, we can write $G(.)$ and $H(.)$ as follows:

$$G_t(x_{t-1}, y_{t-1}, z_t) = R_t(y_{t-1}) - \left[ a_t(z_t - x_{t-1}) + (a_3 - a_4)z_t + K_s \sum_{n=z_t+1}^{y_{t-1}} p_t(n | y_{t-1}) + \gamma E_{N_t,S_t} [M_t(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)] \right]$$

$$H_t(x_{t-1}, y_{t-1}, z_t) = R_t(y_{t-1}) - \left[ b_t(z_t - x_{t-1}) + (a_3 - a_4)z_t + K_s \sum_{n=z_t+1}^{y_{t-1}} p_t(n | y_{t-1}) + \gamma E_{N_t,S_t} [M_t(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)] \right].$$

Since $\max \{z_t, N_t\}$ is convex and non-decreasing in $z_t$, and $M_t(x, y)$ is $K_a$-convex and $K_a$-monotone non-decreasing in $x$ (Lemma 3), then we have that

$M_t(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)$ is $K_a$-convex and $K_a$-monotone non-decreasing in $z_t$. Also, $E_{N_t,S_t} [M_t(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)]$ is $K_a$-convex in $z_t$ (Denardo, 1982, p.151). By the $K_a$-convexity of $L_t(z_t | y_{t-1})$, we have that $(a_3 - a_4)z_t + K_s \sum_{n=z_t+1}^{y_{t-1}} p_t(n | y_{t-1})$ is $K_a$-convex function in $z$. By part (b) of Lemma 1, we have that $K_a$-convexity is preserved under
addition. Thus, 

\[(a_3 - a_4)z_t + K_a \sum_{n=\zeta+1}^{\zeta+1} p_t(n|y_{t-1}) + \gamma E_{N_t,S_t}[\max\{z_t, N_t\}, y_{t-1} + S_t - N_t]\]

is also \(K_a\)-convex and \(K_a\)-monotone non-decreasing in \(z_t\). Thus, \(G_t(x_{t-1}, y_{t-1}, z_t)\) and

\(H_t(x_{t-1}, y_{t-1}, z_t)\) are \(K_a\)-concave in \(z_t\).

End of proof of Theorem 2.

In the next step we show that the optimal policy \(u_t^*(x_{t-1}, y_{t-1})\) has a simple form consisting of thresholds and target levels.

**Theorem 3.**

The optimal policy at the beginning of period \(t\) is given by

\[
\begin{align*}
    u_t^*(x_{t-1}, y_{t-1}) &= \begin{cases} 
    Q_t(y_{t-1}) - x_{t-1}, & x_{t-1} < q_t(y_{t-1}) \\
    w_t(x_{t-1}, y_{t-1}) - x_{t-1}, & q_t(y_{t-1}) \leq x_{t-1} \leq T_t(y_{t-1}) \\
    T_t(y_{t-1}) - x_{t-1}, & T_t(y_{t-1}) < x_{t-1}
    \end{cases}
\end{align*}
\]

(33)

**Proof**

We consider the optimal policy for expansion and contraction separately.

**1. Capacity expansion**

Considering the connection of the current state of capacity to the optimal policy for expansion; there are four possible cases, which are:

1) \(0 \leq x_{t-1} < q_t(y_{t-1})\)

2) \(q_t(y_{t-1}) \leq x_{t-1} < Q_t(y_{t-1})\)

3) \(Q_t(y_{t-1}) \leq x_{t-1} \leq T_t(y_{t-1})\)
4) \( x_t > T_t(y_{t-1}) \)

**Case 1.** \( 0 \leq x_t < q_t(y_{t-1}) \).

We show that it is optimal to expand capacity up to \( Q_t(y_{t-1}) \).

By definition of \( G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) \) we have

\[
R_t(y_{t-1}) - a_t(z_t - x_{t-1}) - L_t(z_t \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)] \\
\leq R_t(y_{t-1}) - a_t(Q_t(y_{t-1}) - x_{t-1}) - L_t(Q_t(y_{t-1}) \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{Q_t(y_{t-1}), N_t\}, y_{t-1} + S_t - N_t)].
\]

Thus, \( R_t(y_{t-1}) - a_t(z_t - x_{t-1}) - L_t(z_t \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)] - K_a \)

\[
\leq R_t(y_{t-1}) - a_t(Q_t(y_{t-1}) - x_{t-1}) - L_t(Q_t(y_{t-1}) \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{Q_t(y_{t-1}), N_t\}, y_{t-1} + S_t - N_t)] - K_a.
\]

**Case 2.** \( q_t(y_t) \leq x_t < Q_t(y_t) \).

We show that it is optimal not to expand capacity. For \( z_t > x_{t-1} \), we have

\[
R_t(y_{t-1}) - a_t(z_t - x_{t-1}) - L_t(z_t \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)] - K_a \\
\leq R_t(y_{t-1}) - b_t(z_t - x_{t-1}) - L_t(z_t \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)]
\]

and by definition of \( H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \) we have

\[
R_t(y_{t-1}) - b_t(z_t - x_{t-1}) - L_t(z_t \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)].
\]

\[
\leq R_t(y_{t-1}) - b_t(w_t(x_{t-1}, y_{t-1}) - x_{t-1}) - L_t(w_t(x_{t-1}, y_{t-1}) \mid y_{t-1}) + \gamma E_{N_t,S_t}[V_{t+1}(\max\{w_t(x_{t-1}, y_{t-1}), N_t\}, y_{t-1} + S_t - N_t)].
\]
Case 3. \( Q_t(y_{t-1}) \leq x_{t-1} \leq T_t(y_{t-1}) \). It is optimal not to expand capacity, similar to the proof in case 2.

Case 4. \( x_{t-1} > T_t(y_{t-1}) \). It is optimal not to expand capacity, similar to the proof in case 2.

2. Capacity contraction

Considering the connection of the current state of capacity to the optimal policy for contraction; there are four possible cases, which are:

1) \( 0 \leq x_{t-1} < q_t(y_{t-1}) \)
2) \( q_t(y_{t-1}) \leq x_{t-1} < Q_t(y_{t-1}) \)
3) \( Q_t(y_{t-1}) \leq x_{t-1} \leq T_t(y_{t-1}) \)
4) \( x_{t-1} > T_t(y_{t-1}) \)

Case 1. \( 0 \leq x_{t-1} < q_t(y_{t-1}) \).

We show that it is optimal not to contract capacity. For \( z_t \leq x_{t-1} \), we have

\[
R_t(y_{t-1}) - b_t(z_t - x_{t-1}) - L_t(z_t | y_{t-1}) + \gamma E_{N_t,S_t} [V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)]
\]

\[
\leq R_t(y_{t-1}) - a_t(z_t - x_{t-1}) - L_t(z_t | y_{t-1}) + \gamma E_{N_t,S_t} [V_{t+1}(\max\{z_t, N_t\}, y_{t-1} + S_t - N_t)], a_t \geq 0, b_t \leq 0
\]

\[
\leq R_t(y_{t-1}) - a_t(q_t(y_{t-1}) - x_{t-1}) - L_t(q_t(y_{t-1}) | y_{t-1}) + \gamma E_{N_t,S_t} [V_{t+1}(\max\{q_t(y_{t-1}), N_t\}, y_{t-1} + S_t - N_t)] + K_a
\]

\[
= R_t(y_t) - a_t(Q_t(y_t) - x_t) - L_t(Q_t(y_t) | y_t) + \gamma E_{N_t,S_t} [V_{t+1}(\max\{Q_t(y_t), N_t\}, y_t + S_t - N_t)]
\]

, by definition of \( G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) \).
**Case 2.** \( q_t(y_{i-1}) \leq x_{i-1} < Q_t(y_{i-1}) \).

We show that it is optimal to contract capacity to \( w_t(x_{i-1}, y_{i-1}) \). For \( z_t \leq x_{i-1} \), by definition of 
\( H_t(x_{i-1}, y_{i-1}, w_t(x_{i-1}, y_{i-1})) \) we have

\[
R_t(y_{i-1}) - b_t(z_t - x_{i-1}) - L_t(z_t | y_{i-1}) + \gamma E_{N_t,S_t} [V_{t+1} (\max \{ z_t, N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
\leq R_t(y_{i-1}) - b_t(w_t(x_{i-1}, y_{i-1}) - x_{i-1}) - L_t(w_t(x_{i-1}, y_{i-1}) | y_{i-1})
\]

\[
+ \gamma E_{N_t,S_t} [V_{t+1} (\max \{ w_t(x_{i-1}, y_{i-1}), N_t \}, y_{i-1} + S_t - N_t)].
\]

We can show that \( q_t(y_{i-1}) \leq w_t(x_{i-1}, y_{i-1}) \). Because if \( w_t(x_{i-1}, y_{i-1}) < q_t(y_{i-1}) \), using part (c)-(ii) of Lemma 1 we have the following:

\[
R_t(y_{i-1}) - a_t(q_t(y_{i-1}) - x_{i-1}) - L_t(q_t(y_{i-1}) | y_{i-1}) + \gamma E_{N_t,S_t} [V_{t+1} (\max \{ q_t(y_{i-1}), N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
\geq R_t(y_{i-1}) - a_t(w_t(x_{i-1}, y_{i-1}) - x_{i-1}) - L_t(w_t(x_{i-1}, y_{i-1}) | y_{i-1})
\]

\[
+ \gamma E_{N_t,S_t} [V_{t+1} (\max \{ w_t(x_{i-1}, y_{i-1}), N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
> R_t(y_{i-1}) - b_t(w_t(x_{i-1}, y_{i-1}) - x_{i-1}) - L_t(w_t(x_{i-1}, y_{i-1}) | y_{i-1})
\]

\[
+ \gamma E_{N_t,S_t} [V_{t+1} (\max \{ w_t(x_{i-1}, y_{i-1}), N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
\geq R_t(y_{i-1}) - b_t(z - x_{i-1}) - L_t(z | y_{i-1}) + \gamma E_{N_t,S_t} [V_{t+1} (\max \{ z, N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
\geq R_t(y_{i-1}) - a_t(z - x_{i-1}) - L_t(z | y_{i-1}) + \gamma E_{N_t,S_t} [V_{t+1} (\max \{ z, N_t \}, y_{i-1} + S_t - N_t)]
\]

\[
\geq R_t(y_{i-1}) - a_t(q_t(y_{i-1}) - x_{i-1}) - L_t(q_t(y_{i-1}) | y_{i-1}) + \gamma E_{N_t,S_t} [V_{t+1} (\max \{ q_t(y_{i-1}), N_t \}, y_{i-1} + S_t - N_t)]
\]

since \( w_t(x_{i-1}, y_{i-1}) < q_t(y_{i-1}) \leq x_{i-1} \), which is a contradiction.

**Case 3.** \( Q_t(y_{i-1}) \leq x_{i-1} \leq T_t(y_{i-1}) \).
We show that it is optimal to contract capacity to $w_i(x_{t-1}, y_{t-1})$. For $z_t \leq x_{t-1}$, by definition of

$$H_i(x_{t-1}, y_{t-1}, w_i(x_{t-1}, y_{t-1}))$$

we have

$$R_i(y_{t-1}) - b_i(z_t - x_{t-1}) - L_i(z_t \mid y_{t-1}) + \gamma E_{N_S} [V_{t+1}(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)]$$

$$\leq R_i(y_{t-1}) - b_i(w_i(x_{t-1}, y_{t-1}) - x_{t-1}) - L_i(w_i(x_{t-1}, y_{t-1}) \mid y_{t-1})$$

$$+ \gamma E_{N_S} [V_{t+1}(\max \{w_i(x_{t}, y_{t}), N_t\}, y_{t-1} + S_t - N_t)].$$

**Case 4.** $x_t > T_i(y_{t-1})$.

We show that it is optimal to contract capacity to $T_i(y_{t-1})$. For $z_t \leq x_{t-1}$, by definition of

$$H_i(x_{t-1}, y_{t-1}, w_i(x_{t-1}, y_{t-1}))$$

we have

$$R_i(y_{t-1}) - b_i(z_t - x_{t-1}) - L_i(z_t \mid y_{t-1}) + \gamma E_{N_S} [V_{t+1}(\max \{z_t, N_t\}, y_{t-1} + S_t - N_t)]$$

$$\leq R_i(y_{t-1}) - b_i(T_i(y_{t-1}) - x_{t-1}) - L_i(T_i(y_{t-1}) \mid y_{t-1}) + \gamma E_{N_S} [V_{t+1}(\max \{T_i(y_{t-1}), N_t\}, y_{t-1} + S_t - N_t)].$$

End of proof of Theorem 3.

We will verify that the optimal profit-to-go $V_i(x_{t-1}, y_{t-1})$ is $K_a$-concave in $x_{t-1}$. The proof relies on the following Lemma.

**Lemma 4.**

Let $g(y, z)$ be a $K$-convex function in $z$ for a given $y$ with $g(y, z) \to \infty$ as $y \to \infty$ and let $\bar{g}(y, z) \equiv -g(y, z)$. Assume $g(y, z)$ is also non-decreasing in $z$. Define a

function $\bar{f}(y, x) = \max_{0 \leq z \leq x} \bar{g}(y, z)$. Then $\bar{f}(y, x)$ is $K$-concave in $x$. 
Proof

From Aneja and Chaouch (1993), we have that \( f(y, x) = \min_{0 \leq z \leq x} g(y, z) \) is \( K \)-convex in \( x \).

Then \( \bar{f}(y, x) = -f(y, x) \) is \( K \)-concave in \( x \).

End of proof of Lemma 4.

Now we are ready to prove the remaining result that the optimal profit-to-go function, \( V_t(x_{t-1}, y_{t-1}) \), is \( K_a \)-concave in \( x_{t-1} \).

Theorem 4.

The optimal profit-to-go from state \( (x_{t-1}, y_{t-1}) \) for period \( t \)

\[
V_t(x_{t-1}, y_{t-1}) = \begin{cases} 
G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K_a, & x_{t-1} < q_t(y_{t-1}) \\
H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})), & q_t(y_{t-1}) \leq x_{t-1} \leq T_t(y_{t-1}) \\
H_t(x_{t-1}, y_{t-1}, T_t(y_{t-1})), & x_{t-1} > T_t(y_{t-1})
\end{cases}
\]  \( (34) \)

is \( K_a \)-concave in \( x_{t-1} \).

Proof

We observe that \( H_t(x_{t-1}, y_{t-1}, T_t(y_{t-1})) = H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \) for all \( x_{t-1} \geq q_t(y_{t-1}) \).

Thus, we can write \( V_t(x_{t-1}, y_{t-1}) \) as follows:

\[
V_t(x_{t-1}, y_{t-1}) = \begin{cases} 
G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K_a, & x_{t-1} < q_t(y_{t-1}) \\
H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})), & x_{t-1} \geq q_t(y_{t-1})
\end{cases}
\]

We wish to show:
\[ V_t(x_{t-1} + c, y_{t-1}) - K \leq V_t(x_{t-1}, y_{t-1}) + c \left[ \frac{V_t(x_{t-1}, y_{t-1}) - V_t(x_{t-1} - e, y_{t-1})}{e} \right] \text{ for } c > 0. \]

There are three cases:

**Case 1.** \( x_{t-1} < x_{t-1} + c < q_t(y_{t-1}) \)

We wish to show: \( G_t(x_{t-1} + c, y_{t-1}, Q_t(y_{t-1})) - K \leq G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K \)

\[ + c \left[ \frac{G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K - G_t(x_{t-1} - e, y_{t-1}, Q_t(y_{t-1})) - K}{e} \right] \]

equivalently, \( -K \leq 0. \)

**Case 2.** \( x_{t-1} < q_t(y_{t-1}) \leq x_{t-1} + c \)

We wish to show: \( H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - K \leq G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K \)

\[ + c \left[ \frac{G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - K - G_t(x_{t-1} - e, y_{t-1}, Q_t(y_{t-1})) - K}{e} \right] \]

or, \( H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) \leq G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) + a_t c \)

equivalently, \( H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - a_t c \leq 0 \)

or, \( H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - G_t(x_{t-1}, y_{t-1}, Q_t(y_{t-1})) - a_t c \)

\[ \leq H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - G_t(x_{t-1}, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - a_t c \]

and we can show that

\( H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - G_t(x_{t-1}, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - a_t c \)

\[ = (a_t - b_t)(w_t(x_{t-1} + c, y_{t-1}) - x_{t-1} - c) \leq 0. \]

**Case 3.** \( q_t(y_{t-1}) \leq x_{t-1} < x_{t-1} + c \)
We wish to show: \[ H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - K \leq H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \]

\[ + c \left[ \frac{H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) - H_t(x_{t-1} - e, y_{t-1}, w_t(x_{t-1} - e, y_{t-1}))}{e} \right] \]

Because \( H(x, y, w(x, y)) \) is an increasing function of \( x \), we have that

\[ H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - K - H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \]

\[ - c \left[ \frac{H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) - H_t(x_{t-1} - e, y_{t-1}, w_t(x_{t-1} - e, y_{t-1}))}{e} \right] \]

\[ \leq H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - K - H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) . \]

From Lemma 4, we have that the function \( H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \) is \( K \)-concave in \( x \).

Using part (c)-(iii) of Lemma 1, then we have

\[ H_t(x_{t-1} + c, y_{t-1}, w_t(x_{t-1} + c, y_{t-1})) - K - H_t(x_{t-1}, y_{t-1}, w_t(x_{t-1}, y_{t-1})) \leq 0 . \]

End of proof of Theorem 4.

In the next chapter, the implementation of the model discussed in this chapter is presented and the study of the effect on the capacity policy of different patterns of product life cycle, time to return distribution, and capacity-related costs is discussed.
CHAPTER 5. NUMERICAL EXAMPLES

5.1 Introduction

In this chapter, the optimal capacity policy presented in the previous chapter is demonstrated. We consider two types of product sales patterns depending on the projected sales volume over time. For each type of sales patterns, there are two types of sales forecasts depending on the information about the future sales. For returns, we consider three different cases that describe the time to return distribution of different source of returns; e.g., corporation, education institution or household use. The objective is to study the effect of: (1) the different patterns of product lifecycle, (2) time to return distribution, and (3) capacity-related costs on the optimal capacity policy. We construct an example based on data collected for the remanufacturing of personal computers (Grenchus et al., 2002).

5.2 Factors to be explored

5.2.1 Sales distribution

Two different patterns of product lifecycle are constructed: short and long life cycle. Short life cycle describes a product that is slow to gain popularity at the introduction, and then sells fast, but its acceptance diminishes shortly after the peak in sales. Long life cycle describes a product that is successful in sales volume soon after it is introduced to the market and its sales stay at a steady high for long time before decreasing. For the rest of this section, we use the terms “peaked” and “level” sales to represent the short and long life cycle, respectively.
5.2.2 Sales forecast

For each pattern of product life cycle, we consider two types of sales forecast: deterministic and stochastic. In the deterministic case, the forecast is based on sales history of similar product \((d_t)\). In this case, we assume that product sales in period \(t\) are known or, equivalently, \(\Pr\{S_t = d_t\} = 1\). That is, we capture a situation of planning for capacity investment based on a single sales projection. This assumption correspondingly provides some computational tractability. We also assume that the product life cycle is two years and set the sales over the life of a product to approximate the shape of a product life cycle curve (Tibben-Lembke, 2002). In the stochastic case, we relax this assumption to allow for the situation in which the estimate of sales is stochastic. We use a binomial distribution with parameters \(\left(\left\lfloor \frac{d_t}{p_1} \right\rfloor, p_1 \right)\) and set \(p_1\) equal to 0.5 for maximum variability in the sales forecast.

We choose the \(\frac{d_t}{p_1}\) parameter to match mean of the binomial distribution to \(d_t\) i.e., set \(\frac{d_t}{p_1}\) to the integer closest to \(2d_t\).

The following are the scaled revenue and discount factor for one period used in the numerical study: \(T = 10\) periods, each of which represents 6 months, \(\eta = 20\) and \(\gamma = 0.97\). We further assume that revenue decreases by 10% each period. The decision epochs \((t)\) are 0, 1, ..., 9. Table 1 summarizes the parameters of the sales forecasts where each unit represents ten thousand PCs sold.
5.2.3 Time lapse from sales to return

For each sales forecast, we model differences in the time to return distribution by varying the parameter $p_2$ of the geometric distribution. In this research, we focus on reselling of used items; therefore the parameters were set up in favor of reselling option. Grenchus et al. (2002) reported that it is preferred to resell a used PC if it has been used for less than 3 years.

Different time to return distributions are described by varying $p_2$ to 0.2, 0.3, and 0.4. As we mentioned earlier, these cases are attributed to different source of returns; e.g., household use, education institution or corporation, respectively. For each value of $p_2$, we can calculate the expected proportion of all items sold in period 1-4 that will be returned by period 10 ($w$) as follows $w = \sum_{i=1}^{4} d_i \Pr\{v \leq 10-t\}/\sum_{i=1}^{4} d_i$.

For example, for the case of deterministic forecast with peaked sales pattern, using $p_2 = 0.3$ we find $w$ equals 0.95. A higher value of $p_2$ corresponds to lower values of both mean and variance in the time to return distribution. The expected proportions of items sold that will have been returned within the study horizon are shown in Figure 4 along with the decision times.

<table>
<thead>
<tr>
<th>Table 1. The parameters of the sales forecast</th>
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<tbody>
<tr>
<td>Pattern</td>
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<tr>
<td>Pattern</td>
</tr>
<tr>
<td>Peaked</td>
</tr>
<tr>
<td>Level</td>
</tr>
</tbody>
</table>
Figure 4. Expected proportion of items sold that will be returned using $p_2 = 0.2$, 0.3, and 0.4

5.2.4 Investment costs

We study the sensitivity to capacity-related costs; specifically, capital, operating, and emergency expansion, on the optimal policy and the expected profit.

The values of capacity costs and revenue parameters are derived accordingly to those used in Angelus and Porteus (1996). In that paper, they considered the problem of capacity planning for manufacturing a new electronics product that requires high investment in equipment. We focus on remanufacturing of used electronics product that involves high-labor intensive operations; therefore, the cost and revenue parameters are obtained by considering the ones using in Angelus and Porteus (2000) as highest limit. Given that limit, we scale down the parameters proportionally and adjust the values according to the cost constraints, (10)-(13). The following are the scaled cost parameters used in the study $a_1 = 2$, $a_3 = 3$, $a_4 = 6$, $b_1 = -1$, $K_o = 15$ and $K_i = 25$. We vary each of the categories independently by ±15%.
The calculations of the optimal policies and the optimal profit-to-go function over the study horizon are conducted in *Mathematica* (Wolfram, 2003). For the base case, we use the parameter value of $p_2 = 0.3$. For the sake of clarity, although we perform the calculations over 10 periods, we show the results for only time 3 because the deterministic sales case has only a small number of states compared to stochastic sales at the first two decision epochs and there is a similar patterns of the results from decision time 3 and subsequent decision times.

### 5.3 Optimal capacity

#### 5.3.1 Different patterns of product lifecycle

Figures 5 and 6 illustrate the effect at time 3 of deterministic and stochastic forecasts on the optimal capacity level with $p_2 = 0.3$. We assign the symbols $+$, $O$, and $Y$ to represent expansion up to $Q_t(y_t)$, contraction to $w_t(x_t, y_t)$ and contraction to $T_t(y_t)$, respectively. The values of $x_{t+1}$ corresponding to $Q_t(y_t)$, $w_t(x_t, y_t)$, and $T_t(y_t)$ are given in the Figures. The dashed line shown in peaked sales indicates maximum number of items outstanding for the level sales pattern. The polygons shown in Figure 6 delineate the sets of feasible states for the deterministic forecast (Figure 5).

The results illustrate how the optimal policies consist of expansion, staying put, and contraction. That is, when the current capacity is relatively small compared to the number of items outstanding, expansion is preferred. For a medium-range capacity level compared to the number of items outstanding, then staying put with the current capacity level is advantageous. Finally, for a large capacity level compared with the number of items still left
in the market then we should contract capacity. Although for a medium-range capacity level
the proof of Theorem 4 shows that it is optimal to slightly contract or maintain capacity, in
our case we only observe the case of maintaining capacity, that is \( w_t(x_t, y_t) \) equal to \( x_t \).

The results in Figures 5 and 6 show the influence of the sales pattern and the product sales
forecast on the optimal capacity policies. We observe that the level sales pattern requires
optimal capacity no lower than the peaked sales case. For the deterministic case, at state (6,
12), it is optimal to maintain capacity at 6 in the level sales, while it is optimal to contract
capacity to 5 in the peaked sales case. In addition, at states (7, 12), and (8, 12), it is optimal to
contract capacity to 6 for the level sales, while it is optimal to contract capacity by 1 unit for
the peaked sales. We observe the same pattern in stochastic sales. That is, at states (1, 5), (2,
11), and (3, 16), it is optimal for the level sales to expand capacity; while maintaining the
same capacity is optimal for the peaked sales in those states. In addition, at states (3, 3), (4,
6), and (6, 12), it is optimal for level sales to maintain the same capacity, whereas contraction
is optimal for peaked sales.

Considering the comparison between deterministic and stochastic sales, we observe that
the optimal policies in the peaked sales case are identical for states that are common.
However, there is a slight difference in the policy for level sales. That is, it is optimal for
deterministic case to expand capacity to 4 at state (2, 10), whereas it is optimal to maintain
capacity at that state in stochastic sales.

Table 2 compares the optimal expected profit for different types of sales forecast and sales
pattern at time 0 for state (0, 0) in the base case. The results reveal that, the optimal expected
profit in the peaked sales pattern is greater than that in the level pattern. Considering the
comparison between stochastic and deterministic, the optimal expected profit in the former is
greater than in the latter case. One explanation is that when we compare the results between the peaked and the level patterns, the total sales volume is fixed so that higher capacity in the level sales requires higher capacity investment resulting in lower profit than the peaked sales. But on the other hand, when we compare stochastic and deterministic for each sales pattern the profit increases as the number of items outstanding increases because of our emergency expansion assumption that we always meet the demands. For example in deterministic case, at state (1, 13), (1, 14), and (1, 15), the optimal expected profit for each state equals 50.86, 56.97, and 62.78 respectively. We observe the same pattern in stochastic case sales. In addition, the range of items outstanding in stochastic is larger than deterministic. For the deterministic case, the optimal expected profit ranges from a low of 54.96 at state (0, 14) to a high of 69.76 at state (3, 15). In the stochastic case, the optimal expected profit ranges from a low of -11.72 at state (0, 0) to a high of 193.79 at state (10, 30). Therefore, in the stochastic case there are more chances to obtain higher expected profits that outweigh for the potentially lower optimal expected profits than in the deterministic case. That is, we can potentially resell more items in stochastic than deterministic forecast and obtain higher profit to compensate for a large capacity costs. Tables 3-5 illustrate the optimal expected profit for different types of sales forecast and sales pattern with respect to different capacity-related costs. We observe that there is similar pattern of the results and the base case (Table 2).
Table 2. The optimal expected profit for different sales forecast and sales pattern (base case)

<table>
<thead>
<tr>
<th>Sales Pattern</th>
<th>Deterministic</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peaked</td>
<td>58.17</td>
<td>58.65</td>
</tr>
<tr>
<td>Level</td>
<td>57.69</td>
<td>58.10</td>
</tr>
</tbody>
</table>
Table 3. The optimal expected profit for different sales forecast and sales pattern with respect to different capital costs

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Capital</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-15% Level</td>
<td>61.14</td>
<td>61.66</td>
</tr>
<tr>
<td>Peaked</td>
<td>62.03</td>
<td>62.41</td>
</tr>
<tr>
<td>Level</td>
<td>61.42</td>
<td>61.66</td>
</tr>
<tr>
<td>+15% Level</td>
<td>54.25</td>
<td>54.61</td>
</tr>
<tr>
<td>Peaked</td>
<td>54.55</td>
<td>54.98</td>
</tr>
<tr>
<td>Level</td>
<td>54.25</td>
<td>54.61</td>
</tr>
</tbody>
</table>

Table 4. The optimal expected profit for different sales forecast and sales pattern with respect to different operating costs

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Operating</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-15% Level</td>
<td>67.03</td>
<td>67.84</td>
</tr>
<tr>
<td>Peaked</td>
<td>68.01</td>
<td>68.24</td>
</tr>
<tr>
<td>Level</td>
<td>67.03</td>
<td>67.84</td>
</tr>
<tr>
<td>+15% Level</td>
<td>48.59</td>
<td>48.69</td>
</tr>
<tr>
<td>Peaked</td>
<td>49.31</td>
<td>49.56</td>
</tr>
<tr>
<td>Level</td>
<td>48.59</td>
<td>48.69</td>
</tr>
</tbody>
</table>

Table 5. The optimal expected profit for different sales forecast and sales pattern with respect to different emergency expansion costs

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Emergency</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-15% Level</td>
<td>62.39</td>
<td>62.48</td>
</tr>
<tr>
<td>Peaked</td>
<td>62.77</td>
<td>63.04</td>
</tr>
<tr>
<td>Level</td>
<td>62.39</td>
<td>62.48</td>
</tr>
<tr>
<td>+15% Level</td>
<td>53.23</td>
<td>54.25</td>
</tr>
<tr>
<td>Peaked</td>
<td>54.77</td>
<td>55.00</td>
</tr>
<tr>
<td>Level</td>
<td>53.23</td>
<td>54.25</td>
</tr>
</tbody>
</table>
5.3.2 Time to return distribution

The results show the effect of the time to return distribution parameters on the optimal capacity policy. As expected, in all cases a higher value of $p_2$ encourages capacity expansion to higher levels and maintenance of higher capacity levels than a lower value of $p_2$. The complete results for all cases are shown in Appendix B. For illustration, Figure 7 shows an example of the influence of $p_2$ values on the optimal capacity level of peaked sales.

Overall, we observe more cases of expansion but fewer of contraction with a higher value of $p_2$. For instance, at state (3, 20) for $p_2$ equal to 0.2 and 0.4, it is optimal to expand capacity to 5 and 9 respectively, while it is optimal to expand to 7 in the base case. At state (6, 12) for $p_2$ equal to 0.2 and 0.4, it is optimal to contract capacity to 4 and maintain capacity at 6 respectively, while it is optimal to contract capacity to 5 in the base case. In addition, with a current capacity of 3 units it is optimal to expand capacity when the number of items outstanding reaches 18 for the base case, while it is optimal to expand with 26 and 14 items outstanding when $p_2$ equal to 0.2 and 0.4, respectively.

Figure 8 illustrates the optimal expected profit for different values of $p_2$. As expected, the expected profit increases as a function of $p_2$. Thus, it is beneficial to provide incentives to encourage faster returns from the market as long as the cost of providing these incentives does not outweigh the gain obtained in the expected revenue.
Figure 7. Optimal capacity levels of peaked sales with stochastic forecast at time 3 (\( p_2 = 0.2, 0.3, 0.4 \))

Figure 8. Comparison of the optimal expected profit with different values of \( p_2 \)
5.3.3 Sensitivity to capacity-related costs

The results show the effect of capacity-related costs on the optimal capacity policy and the optimal expected profit. In all cases as the capital and operating costs are increased, capacity contraction and maintenance of a lower capacity level become more favorable; while as the capital and operating costs are decreased, expansion and maintenance of a higher capacity level become more favorable. In contrast, as the emergency expansion becomes expensive, capacity expansion is preferred to buffer against the irregular stream of returns while if the emergency expansion is inexpensive, it is more favorable to keep capacity low and rely more on the emergency expansion.

For illustration, we focus on the peaked sales with stochastic forecast. The complete results can be found in Appendix B. Figures 9-11 illustrate an example of the sensitivity in each category of capacity-related costs.

Overall, in Figure 9 we observe more cases of expansion but less of contraction for a lower capital cost. The similar pattern is observed in the case of operating costs (Figure 10). For example, at state (3, 18) for the base case it is optimal to expand capacity to 6. Also, it is optimal to maintain capacity at 3 and expand capacity to 7 when the capital or operating costs increase and decrease by 15%, respectively. In addition, with a current capacity of 3 units it is optimal to expand capacity when the number of items outstanding reaches 18 for the base case, while it is optimal to expand with 15 items outstanding when the capital cost is decreased by 15%. However, capacity expansion is not recommended when the capital cost is increased by 15%.

For the emergency expansion cost, the results in Figure 11 show opposite effects. That is, the increase in emergency expansion costs encourages capacity expansion and maintenance
of a higher capacity level. Also, as we decrease the emergency expansion cost, it becomes favorable to keep capacity low and rely more on the emergency expansion.

We observe that there are more cases of expansion but less of contraction when emergency expansion is expensive. On the other hand, it is clear that there are more cases of maintaining the same capacity with no expansion when emergency expansion cost is inexpensive. For example, at state (3, 21), it is optimal to expand capacity to 7 in the base case. It is optimal to expand capacity to 8 with an increase in the emergency expansion cost by 15% and maintain capacity at 3 with a decrease of 15% in the emergency expansion cost. Also, with a current capacity of 3 units it is optimal to expand capacity when number of items outstanding reaches 18 for the base case, while it is optimal to expand with 15 items outstanding when the emergency expansion cost is increased by 15%. However, it is not recommended to expand capacity when the emergency expansion cost is decreased by 15%.

Figure 12 illustrates the sensitivity of the optimal expected profit to the capacity-related costs. It shows that the optimal profit is most sensitive to changes in the operating cost. Considering capital and emergency expansion costs, we observe that as the costs are increased by 15% the capital has at least as much influence as emergency expansion cost. However, as the costs are decreased by 15%, emergency expansion has a slightly larger influence on the optimal expected profit than capital cost. Thus, to obtain a higher profit we need to direct most attention on reducing operating costs. Also, it is more beneficial to focus on reducing emergency expansion costs rather than capital costs.
Figure 9. Optimal capacity levels of peaked sales with stochastic forecast at time 3
(capital cost = ±15%)

Figure 10. Optimal capacity levels of peaked sales with stochastic forecast at time 3
(operating cost = ±15%)
Figure 11. Optimal capacity levels of peaked sales with stochastic forecast at time 3

(emergency expansion cost = ±15%)

Figure 12. Sensitivity analysis of the optimal expected profit with respect to different capacity-related costs
CHAPTER 6. SUMMARY AND CONCLUSION

6.1 Summary

This paper contributes to the management of remanufacturing operations through developing an optimal policy for capacity decisions. We evaluated factors associated with different patterns of product lifecycle, time to return distribution, and capacity-related costs, to see how they impact the optimal capacity policy. We formulated this problem as a discrete-time finite-horizon Markov decision model. One important contribution of this research is to show that there exists a simple form for an optimal policy depending on the past sales and past return and it consists of threshold values and target levels.

A numerical study was carried out to evaluate the effects of the following factors on the optimal policy and the expected profit: (1) different patterns of product sales, (2) different time to return distribution and (3) capital-related costs, specifically, capital, operating, and emergency expansion. Also, we considered two types of sales forecasts depending on the information about the future sales: deterministic and stochastic.

6.2 Conclusions

In the numerical study that we carried out on the optimal capacity policy of the different sales patterns, the results suggest that the level sales pattern requires optimal capacity no lower than the peaked sales pattern. In addition, the higher capacity in the level sales requires higher capacity investment resulting in lower profit than the peaked sales. Therefore, it is beneficial to introduce incentives to encourage higher sales after product introduction until
the peak in sales. For the comparison between stochastic and deterministic sales with respect to different capacity-related costs, the optimal expected profit in the former case is greater than in the latter case. One explanation is that the optimal expected profit increases with the number of items outstanding because of our emergency expansion assumption. When we compare the results between stochastic and deterministic sales in each pattern, the range of items outstanding in stochastic is larger than deterministic. Therefore, in the stochastic case there are more chances to obtain higher expected profits that outweigh for the potentially lower optimal expected profits than in the deterministic case. That is, we can potentially resell more items in stochastic than deterministic forecast and obtain higher profit to compensate for a large capacity costs.

For the effect of the different time to return distributions, the results indicate that a higher value of the time to return distribution parameter, corresponding to lower values of both mean and variance, encourages capacity expansion to higher levels and maintenance of higher capacity levels. Our results also show that the expected profit increases with the value of the time to return distribution parameter. Thus, it is worthwhile to invest in the development of a process to offer incentives to accelerate returns from the market as long as the cost of providing these incentives does not outweigh the gain obtained in the expected revenue.

The effects of the capacity-related costs on the optimal policy are intuitive. We observe more instances of expansion but fewer of contraction when capital cost and operating cost decrease. On the other hand, the emergency expansion cost shows the opposite effects. For sensitivity analysis of the optimal profit to the capacity-related costs, the results show that the
operating cost has the most impact. Furthermore, more direct attention needs to be paid to reducing the emergency expansion rather than capital costs to obtain a higher profit.

6.3 Future research

An important extension is to consider different amounts of net revenue based on the conditions of the returns. In this research, we consider the net revenue as an average value across all returned products. In fact, newer returned products could convey higher value than those that have been used for a long period of time. At the first step, this model only focuses on managing a single return. However, future research could extend this study by incorporating the sales of returned products that have been returned several times. The main challenge for this analysis is to incorporate the sales of returned products that have been returned several times into the distribution of product sales. This analysis can be performed by incorporating the number of previous returned products into the product sales distribution.

Developing an analysis of how to allocate capacity an optimal way to different product types is also a future research area to consider. This extension is important because most third party logistics firms, working as subcontractor for OEMs, have processing lines that are capable of processing a wide variety of product types. This analysis can be performed by developing strategies to allocate resources with consideration of lifecycle stages and revenue from processing the returns for each product type.

One interesting extension is to determine the robustness of the optimal policy over fluctuations in returns. That is, we could find a policy that minimizes cost by maintaining barely sufficient capacity to handle the expected returns, but if the actual returns are much
higher, the loss from emergency expansion cost is considerable. This analysis allows identification of the sensitivity of the optimal policy to changes in returns. The challenge for this analysis is to develop a tractable approach to finding the optimal capacity policy in the presence of return uncertainty. In addition, another interesting extension is to find the optimal policy that incorporates the uncertainty character of the returns without making any assumption on its distribution. Bertsimas and Thiele (2006) studied the optimal robust policy for an inventory model and proposed a general methodology that takes into account the uncertainty in the optimal policy.

Another possible extension is to relax the assumption of emergency expansion, to represent the case where delaying expansion and carrying returns over to the next period are possible. To verify that the optimal capacity policy has a simple form, one needs to check condition under which the $K$-concavity of the optimal profit-to-go function is preserved and account for number of carry over returns in the state variables.

Finally, it would be interesting to apply a Bass diffusion model. That is, the sales of the product depend on advertising or other marketing efforts, and also by interacting with existing customers or owners of the product. In addition, the Bass diffusion model can incorporate a wide variety factors observed in practice such as competition among products and multiple purchases. Future research should study how different patterns of product diffusion can affect capacity planning decisions. However, this extension is rather complex to incorporate in this model because future sales would depend on current and past sales.
APPENDIX A. PROOF OF TRANSITION PROBABILITIES
Proof of Transition probabilities:

1. Enough capacity is guaranteed

Case 1.1 $N_i \leq S_i$

If $i + k \geq j$ then for $0 \leq n \leq \max(S_i)$

$$p_i(i + k, j + n) = \sum_{m=0}^{j} P(N_i = m)P(S_i = m + n)$$

Case 1.2 $N_i > S_i$

If $i + k \geq j$ then for $-j \leq n \leq -1$

$$p_i(i + k, j + n) = \sum_{m=-n}^{j} P(N_i = m)P(S_i = m + n)$$

We wish to show: $\sum_{n=-j}^{1} P(N_i = m)P(S_i = m + n) + \sum_{m=0}^{\max(S_i)} P(N_i = m)P(S_i = m + n) = 1$

or, $\sum_{m=0}^{\max(S_i)} P(N_i = m)P(S_i = m + n) = 1$

Let $s = m + n$

we have $\sum_{m=0}^{j} P(N_i = m) \sum_{x=0}^{\max(S_i)+m} P(S_i = s)$

equivalently, $\sum_{m=0}^{j} P(N_i = m) \sum_{x=0}^{\max(S_i)} P(S_i = s)$

or, $\left(\sum_{m=0}^{j} P(N_i = m)\right) \left(\sum_{x=0}^{\max(S_i)} P(S_i = s)\right) = 1$. 
2. Enough capacity is not guaranteed

Case 2.1 \( N_i \leq S_i \)

If \( i + k < j \) then for \( 0 \leq n \leq i + k \)

\[
p_t(i + k, j + n) = \sum_{m=0}^{i+k} P(N_t = m)P(S_t = m + n)
\]

and for \( i + k + 1 \leq l \leq j \)

\[
p_t(l, j + n) = P(N_t = l)P(S_t = l + n)
\]

Case 2.2 \( N_i > S_i \)

If \( i + k < j \) then for \( -j \leq n \leq i + k \)

\[
p_t(i + k, j + n) = \sum_{m=-n}^{i+k} P(N_t = m)P(S_t = m + n)
\]

and for \( i + k + 1 \leq l \leq j \)

\[
p_t(l, j + n) = P(N_t = l)P(S_t = l + n)
\]

We wish to show:

\[
\sum_{n=-j}^{i+k} \sum_{m=-n}^{i+k} P(N_t = m)P(S_t = m + n) + \sum_{n=0}^{\text{max}(S_t)} \sum_{m=0}^{i+k} P(N_t = m)P(S_t = m + n) + \sum_{n=-j}^{\text{max}(S_t)} \sum_{l=i+k+1}^{j} P(N_t = l)P(S_t = l + n) = 1
\]

or,
\[
\sum_{n=-j}^{-1} \sum_{m=-n}^{i+k} P(N_i = m)P(S_i = m + n) + \sum_{n=0}^{\text{max}(S_i)} \sum_{m=0}^{i+k} P(N_i = m)P(S_i = m + n)
\]

\[+
\sum_{n=0}^{\text{max}(S_i)} \sum_{m=0}^{l} P(N_i = l)P(S_i = l + n) + \sum_{n=0}^{\text{max}(S_i)} \sum_{m=0}^{l} P(N_i = l)P(S_i = l + n) = 1
\]
equivalently,
\[
\sum_{n=-j}^{-1} \sum_{m=-n}^{i} P(N_i = m)P(S_i = m + n) + \sum_{n=0}^{\text{max}(S_i)} \sum_{m=0}^{i} P(N_i = m)P(S_i = m + n)
\]
or,
\[
\sum_{m=0}^{\text{max}(S_i)} \sum_{n=-m}^{i} P(N_i = m)P(S_i = m + n)
\]
Let \( s = m + n \)

we have
\[
\sum_{m=0}^{i} P(N_i = m) \sum_{s=0}^{\text{max}(S_i)+m} P(S_i = s)
\]
or,
\[
\sum_{m=0}^{i} P(N_i = m) \sum_{s=0}^{\text{max}(S_i)} P(S_i = s)
\]
equivalently,
\[
\left( \sum_{m=0}^{i} P(N_i = m) \right) \left( \sum_{s=0}^{\text{max}(S_i)} P(S_i = s) \right) = 1.
\]
APPENDIX B. SCENARIO RESULTS
We show the complete results for the effect of the time to return distribution parameters and capacity-related costs on the optimal capacity policy that we have discussed in our analysis.

Figure 13. Optimal capacity levels of peaked sales with deterministic forecast at time 3

\( p_2 = 0.2, 0.3, 0.4 \)

Figure 14. Optimal capacity levels of level sales with deterministic forecast at time 3

\( p_2 = 0.2, 0.3, 0.4 \)
Figure 15. Optimal capacity levels of level sales with stochastic forecast at time 3 ($p_2 = 0.2, 0.3, 0.4$)

Figure 16. Optimal capacity levels of peaked sales with deterministic forecast at time 3
(capital cost = ±15%)
Figure 17. Optimal capacity levels of level sales with deterministic forecast at time 3 
(capital cost = ±15%)

Figure 18. Optimal capacity levels of level sales with stochastic forecast at time 3 
(capital cost = ±15%)
Figure 19. Optimal capacity levels of peaked sales with deterministic forecast at time 3

(operating cost = ±15%)

Figure 20. Optimal capacity levels of level sales with deterministic forecast at time 3

(operating cost = ±15%)
Figure 21. Optimal capacity levels of level sales with stochastic forecast at time 3

(operating cost = ±15%)

Figure 22. Optimal capacity levels of peaked sales with deterministic forecast at time 3

(emergency expansion cost = ±15%)
Figure 23. Optimal capacity levels of level sales with deterministic forecast at time 3
(emergency expansion cost = ±15%)

Figure 24. Optimal capacity levels of level sales with stochastic forecast at time 3
(emergency expansion cost = ±15%)
REFERENCES


