NONLINEAR WAVES IN ROCKS

Katherine R. McCall, Paul A. Johnson, and G. Douglas Meegan
Earth and Environmental Sciences Division
Los Alamos National Laboratory
Los Alamos, NM 87545

INTRODUCTION

We are interested in the nonlinear interaction of frequency components in large amplitude acoustic waves in rocks. As compared to other, more ordered solids, rocks are elastically highly nonlinear. The ratio of third order elastic constants to second order elastic constants in a typical rock is orders of magnitude greater than in solids such as iron [1]. This high degree of nonlinearity means that frequency components mix and elastic energy is transferred from the fundamentals to sum and difference frequencies. There are at least three reasons for our interest in these effects in rocks. 1) Accurate models of explosion and earthquake sources may depend on understanding nonlinear elastic effects. 2) Efficient frequency mixing in a highly nonlinear elastic material could lead to a low frequency seismic source generated from two high frequency input waves. 3) Accurate measurement of nonlinear coefficients in rock would provide a sensitive probe of physical characteristics such as consolidation and saturation.

Our goal in this work is to examine the extent to which source frequency content is modified during nonlinear elastic wave propagation in rock. We have taken both a theoretical and experimental approach. In the second section, we develop a theoretical framework for investigating nonlinear elastic wave propagation. We develop the solution to the equation of motion for the displacement field (to second order in the displacement, including attenuation) in terms of the exact solution of the linear problem, employing a Green function technique. The Green function method allows us to explore the effects of an arbitrary external source. The results have served as a guide to the experimental work, and the experimental results have allowed us to determine reasonable parameter values for the theory.

In the third section, we describe our ultrasonic experiments in Berea sandstone, designed to study the spectral changes that take place along the wave propagation path. Our initial experiments focused on confirming the theoretical result for a plane wave propagating in an elastic material with cubic anharmonicity, that is, the amplitude of the $2\omega$ harmonic is proportional to $x$ [2], where $x$ is the propagation distance. In the last section, we show, discuss, and compare our theoretical and experimental results for a pulsed, single frequency plane wave source. The experimental results are consistent with the theoretical Green function calculations.
The derivation of the equation of motion for a homogeneous solid, including first order nonlinear elastic terms and linear attenuation has been described many times. For simplicity, we restrict ourselves to the problem of a compressional plane wave propagating in the \(x\) - direction. Keeping terms to second order in the displacement field, the equation of motion for this field is [3],

\[
\rho \ddot{u}(x,t) = (\lambda+2\mu) \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{2} \left[ 3(\lambda+2\mu) + 2(\ell + 2m) \right] \frac{\partial}{\partial x} \left( \frac{\partial u(x,t)}{\partial x} \right)^2 + S(x,t),
\]

(1)

where \(\rho\) is the density, \(u(x,t)\) is the displacement, \(\lambda\) and \(\mu\) are second order elastic constants, \(\ell\) and \(m\) are third order elastic constants, and \(S(x,t)\) is the external source that initiates the response of the system. Equation (1) describes a system having linear and nonlinear elasticity (cubic anharmonicity). We introduce attenuation by allowing the spatial derivative of the displacement to have a component which is retarded in time [4]. In frequency space, this is equivalent to making the substitution

\[
\frac{\partial u(x,\omega)}{\partial x} \rightarrow \chi(\omega) \frac{\partial u(x,\omega)}{\partial x},
\]

(2)

where

\[
\chi(\omega) = 1 - \frac{\Delta}{1 - i\omega \tau},
\]

(3)

\(\Delta\) can be thought of as the fractional amount of the displacement derivative which is retarded in time, and \(\tau\) is the characteristic damping time. In the linear elastic case, the factor \(\chi(\omega)\) is equivalent to the complex factor multiplying real moduli in the standard solid model of a series Voigt unit and spring [5].

In the frequency domain, an analytic Green function can be found for the problem involving linear elasticity and linear attenuation. We write the equation of motion in the frequency domain

\[
g^{-1}(x,\omega) \ u(x,\omega) = -S(x,\omega) - \beta \frac{\partial}{\partial x} \left[ \frac{\partial \omega'}{2\pi} \chi(\omega') \frac{\partial u(x,\omega')}{\partial x} \right] \frac{\partial u(x,\omega-\omega')}{\partial x},
\]

(4)

where,

\[
g^{-1}(x,\omega) = \chi(\omega) \frac{\partial^2}{\partial x^2} + k^2,
\]

(5)

\[
\beta = \frac{3(\lambda+2\mu) + 2(\ell + 2m)}{2(\lambda + 2\mu)},
\]

(6)

\(k=\omega/c\) is the wavenumber, and \(c^2=(\lambda+2\mu)/\rho\) is the square of the compressional wave velocity. All of the linear terms in Eq. (4) are on the left hand side, in the operator \(g^{-1}(x,\omega)\). The right hand side of Eq. (4) has two kinds of sources, the external source \(S(x,\omega)\) and the internal sources due to the nonlinearity.

A systematic Green function solution to Eq. (4) is possible if we use a parameter \(\eta\) to keep track of powers of the internal source, and expand the displacement and source functions in powers of \(\eta\): \(u(x,\omega) = u_0(x,\omega) + \eta \ u_1(x,\omega) + \eta^2 \ u_2(x,\omega) + \ldots\), and \(f(x,\omega) = f_0(x,\omega) + \eta \ f_1(x,\omega) + \eta^2 \ f_2(x,\omega) + \ldots\). For the first few terms in the expansion of the source function, we find

2106
\[ f_0(x, \omega) = S(x, \omega) \]  
\[ f_1(x, \omega) = \beta \frac{\partial}{\partial x} \left( \frac{d\omega'}{2\pi} \chi(\omega') \frac{\partial u_0(x, \omega')}{\partial x} \right) \]  
\[ f_2(x, \omega) = 2\beta \frac{\partial}{\partial x} \left( \frac{d\omega'}{2\pi} \chi(\omega') \frac{\partial u_1(x, \omega')}{\partial x} \right) \]  

Matching powers of \( \eta \), we obtain a hierarchy of equations which allow us to solve systematically for the displacement field to any desired order. The displacement field is found in the form of an integral over a product of a Green function and a source function,

\[ u_n(x, \omega) = \int dx' g(x, x', \omega) f_n(x', \omega) , \]  

where \( g(x, x, \omega) \) is the Green function which satisfies

\[ g^{-1}(x, \omega) g(x, x', \omega) = -\delta(x-x') . \]  

The solution to Eq. (11) for the Green function of an infinite medium is

\[ g(x, x', \omega) = i\chi^{-1}(\omega) e^{iA(\omega)lx-x'l}, \]  

where

\[ A(\omega) = \kappa(\omega) + i\alpha(\omega) , \]  

\[ \kappa(\omega) = k \sqrt{\frac{r+s}{2[(1-\Delta)^2 + (\omega\tau)^2]}}, \]  

\[ \alpha(\omega) = l k^l \sqrt{\frac{r-s}{2[(1-\Delta)^2 + (\omega\tau)^2]}}, \]  

\[ r^2 = s^2 + (\omega\tau)^2, \] and \( s = (1-\Delta)^2 + (\omega\tau)^2 \). Given the Green function in Eq. (12) and the source functions in Eqs. (7) - (9), the systematic solution for the displacement field begins with the terms involving the external source \( f_0(x, \omega) \). We write \( f_0(x, \omega) \) as a source at the origin, of the form

\[ f_0(x, \omega) = -2i \chi(\omega) A(\omega) U \delta(x) F(\omega) , \]  

where \( U \) is the longitudinal displacement field produced at the origin by the source, and \( F(\omega) \) is the frequency spectrum of the source with units of time. Substituting \( f_0(x, \omega) \) into Eq. (10), we find the linear component of the displacement is

\[ u_0(x, \omega) = U e^{iA(\omega)lx} F(\omega) . \]  

The source function \( f_1(x, \omega) \), Eq. (8), for the first order nonlinear component of the displacement is determined by the linear displacement term in Eq. (16). This first order source function yields a first order nonlinear displacement of the form

\[ u_1(x, \omega) = \frac{\beta U^2 k^2 x}{8} E(x, \omega) , \]  

where
\[
E(x,\omega) = \int \frac{d\omega'}{2\pi} C(\omega,\omega') \frac{\chi(\omega') \chi(\omega - \omega')}{\chi^2(\omega)} F(\omega') F(\omega - \omega') \times e^{i[A(\omega') + A(\omega - \omega')]|x|} - e^{iA(\omega)|x|} \\
\left[ A(\omega') + A(\omega - \omega') - A(\omega) \right] |x|}
\]

(18)

and

\[
C(\omega,\omega') = \frac{8\chi(\omega) A(\omega') A(\omega - \omega') [A(\omega') + A(\omega - \omega')]}{k^2 A(\omega') + A(\omega - \omega') + A(\omega)}.
\]

(19)

This result is arranged in the form of an amplitude times an envelope function. The amplitude is the result for \(\omega\) for a monochromatic source, \(F(\omega) = 2\pi \delta(\omega - \omega_0)\), with no attenuation [then \(E(x,\omega) = 2\pi e^{ik|x|} \delta(\omega - 2\omega_0)\)]. The amplitude is proportional to \((\omega k)^2\) as it is due to two waves launched by the source, and proportional to \(x\) as the internal source works at all points between the point of observation and the external source point, \(x=0\). The envelope function \(E(x,\omega)\) describes the decay of the amplitude due to attenuation.

In the limit of zero attenuation, a continuous sine wave source

\[
F(\omega) = 2\pi \frac{1}{2i} \delta(\omega - \omega_0) - \delta(\omega + \omega_0)
\]

(20)

produces a displacement field in the time domain,

\[
u(x,t) = u_0(x,t) + u_1(x,t) = U \sin(k_0 x - \omega_0 t) - \frac{\beta U^2 k_0^2 x}{4} \left[ \cos(2k_0 x - 2\omega_0 t) + 1 \right],
\]

(21)

where \(k_0 = \omega_0/c\). Thus, the primary signature of nonlinear elastic wave propagation is the generation of frequency components at frequencies \(\omega = \omega_0 + \omega_0\) and \(\omega = \omega_0 - \omega_0\). These frequency components have amplitudes which grow linearly in the propagation distance and as the square of the source frequency and source displacement amplitude.

**EXPERIMENT**

The experimental apparatus is shown in Fig. 1. The sample is a 2-m long by 6-cm diameter rod of Berea sandstone. One end of the sample was tapered in order to minimize...
reflections. To accommodate the detectors (pin transducers), 11 small holes were drilled into the Berea sandstone rod at 5 cm intervals, up to a distance of 58 centimeters from the source. The source is a drive transducer composed of a piezoelectric crystal and a backload [7]. A small diameter hole was drilled through the center of the backload and transducer and a fiber optic probe was positioned in the hole to directly measure the rock displacement at the source. This probe is sensitive to $10^{-9}$ meters over a frequency range of 0 - 200 kHz.

The source transducer was driven by a Hewlett Packard 3314A function generator amplified by a Crown PSA-2 power amplifier. A single frequency, amplitude modulated, N cycle wave train was input to the source transducer; N ranged from 8 to 24. Frequencies of 8 to 24 kHz were used and care was taken to assure that the measured signals were not contaminated by reflections from the opposite end of the sample. Detected signals were output to a 16 bit Analogic 6100B/652 waveform analyzer. Source displacements ranged from $10^{-9}$ - $10^{-6}$ m. Strain levels at the source were measured to be $6 - 60 \times 10^{-6}$, and, at the limit of the measurement range (58 cm), strain levels were $0.6 - 9 \times 10^{-6}$.

RESULTS

The most compelling experimental evidence for nonlinear elastic behavior in the sample is shown in Fig. 2. In Fig. 2a, we show the frequency spectrum measured at the source for a drive frequency $\omega/2\pi$ of 13.75 kHz. The three curves correspond to three amplitudes of the source transducer varying over a factor of approximately 50. The source

![Fig. 2](image)

Fig. 2 Displacement spectra at increasing applied voltages, indicated by the different line types. Drive is at 13.75 kHz. The Nyquist critical frequency is 500 kHz for 1024 data points. (a) Displacement at the source. (b) Displacement 58 cm from the source.
displacement spectrum is relatively monochromatic containing only a small fraction of \(2\omega\) at large drive levels due to electronic distortion. In Fig. 2b, we show the displacement frequency spectrum at 58 cm at the same drive levels. The displacement frequency spectrum at 58 cm is rich in harmonics not present at the source. Further, these higher harmonic displacement fields have amplitudes that are a sensitive function of the drive amplitude. Similar results were obtained for 30 drive frequencies in the range 8 - 24 kHz.

We have calculated the linear and first order nonlinear displacement response to a source similar to that of the experiment, using Eqs. (16) and (17). Our model source is a sine wave modulated by a Gaussian envelope, thus

\[
F(\omega) = \frac{T}{\pi} \sqrt{\pi/2} \left( e^{i(\omega + \omega_0)t_0} e^{-(\omega + \omega_0)^2T^2/2} - e^{i(\omega - \omega_0)t_0} e^{-(\omega - \omega_0)^2T^2/2} \right)
\]

where \(\omega_0\) is the center frequency, \(t_0\) is the center of the time series, and \(T\) is the width of the time series. In Fig. 3, we show the displacement frequency spectra for values of the source displacement corresponding to Fig. 2, and for \(\omega_0/2\pi=13.8\) kHz, \(t_0=20\pi/\omega_0\), \(T=4\pi/\omega_0\), \(c=2.75\) km/s, \(x=58\) cm, \(\tau=1.5\) \(\mu\)s, \(\Delta=0.1\), and \(\beta=-1000\).

From Eq. (21), we expect the amplitude of the displacement field at \(2\omega\) to increase linearly with source - receiver distance \(x\) for low attenuation. A representative experimental measurement of the displacement field at \(2\omega\) as a function of distance from the source, is shown in Fig. 4. We have plotted the relative amplitude \(R\), where \(R\) is the ratio of the amplitude \(u_1\) at \(2\omega\) (source frequency of \(\omega\)) to the amplitude \(u_0\) at \(2\omega\) (source frequency of \(2\omega\)). This ratio was taken in order to correct for detector site effects and intrinsic attenuation. According to Eq. (21), this ratio is proportional to the propagation distance:

\[
R = \frac{\left| u_1(x,2\omega) \right|}{\left| u_0(x,2\omega) \right|} \propto x
\]

The results in Fig. 4 confirm this prediction. Measurements throughout the frequency range 8 - 24 kHz showed similar results. (We have reason to believe the fluctuations about a line in Fig. 5 are due to positional and frequency dependent elastic scattering from the array of detectors. These scatterers cause rapid spatial fluctuations in wave amplitude along the length of the rod and produce an effective increase in absorption in the rod [6]). The line through the data is the theoretical prediction for \(R\) from Eqs. (16) and (17), using the same values of parameters as for Fig. 3.

![Fig. 3 Calculated displacement spectra for increasing source displacement. From Eqs. (16) and (17), \(u(x,\omega) = u_0(x,\omega) + u_1(x,\omega)\). \(F(\omega)\) is given by Eq. (22). Note that \(\omega\) combines with itself in two ways: \(\omega + \omega = 2\omega\) and \(\omega - \omega = 0\). Higher order terms (higher harmonics) may be included using the method described in the theory section.](image-url)
Fig. 4 Harmonic amplitude versus propagation distance. R is given by Eq. (23). The circles are experimental measurements (drive frequency of 14.6 kHz); the line is the theoretical prediction from Eqs. (16) and (17). For low attenuation, we expect R ∝ x.

The coefficient of the 2ω harmonic in Eq. (21) is proportional to the square of the source displacement U and the square of the drive frequency ω, as well as the propagation distance x. We have measured the 2ω harmonic amplitude u₁ relative to the amplitude of the fundamental u₀ over a range of source displacements U. The trend of u₁ as a function of u₀ is consistent with theoretical expectation; u₁ grows as u₀ to a power between 3/2 and 2 over our measurement range. We have also measured u₁ as a function of drive frequency ω. To within the variability introduced by the frequency dependent response of the pinducers, u₁ grows as ω².

The behavior of the 2ω harmonic amplitude u₁ in our experiments agrees well with theoretical predictions given in Eqs. (17) and (21). Therefore, we are confident that a significant portion of the observed response is due to cubic anharmonicity in the elastic response of the rock. We have observed harmonics higher than 2ω (see Fig. 2) and intermodulation terms such as 2ω₁ - ω₂ when the source is excited at two distinct frequencies. The strong growth of odd harmonics as a function of source amplitude suggests that higher order terms (i.e. quartic anharmonicity) in the stress strain relationship may be important to a complete description of nonlinear elastic behavior in rock. The theoretical method presented here may be expanded in a straightforward manner to explore and predict the effects of quartic anharmonicity with and without attenuation.

ACKNOWLEDGEMENTS

This work is supported by the Offices of Basic Energy Research and Advanced Concepts, U.S. Department of Energy with the University of California. Discussions with R. A. Guyer, R. J. O'Connell, J. B. Minster, B. P. Bonner, T. J. Shankland, and S. R. Taylor are gratefully acknowledged. Special thanks go to Chris Angel for help during the preparation of the manuscript.

REFERENCES