HYPERSINGULAR INTEGRAL EQUATIONS FOR CRACK PROBLEMS

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INTRODUCTION

The investigation of scattering of waves by cracks in an elastic medium and by thin scatterers in an acoustic medium, via analytical and experimental methods, seems to be of continuing importance to nondestructive evaluation. On the analytical side, formulation and numerical solution of crack scattering problems using boundary integral equations is popular and effective because of the very nature of a crack, but this approach still suffers some shortcomings of an analytical nature. That is, the governing equations in their primitive form involve a hypersingular kernel function, and the usual process of regularization to lower the kernel singularity usually introduces undesirable features in the analysis accompanied by computational difficulty.

In the present work the integral equation for a crack in a domain is formulated by taking the gradient of the representation integral for an interior point and moving it to the boundary through a careful limiting process. Two such limiting processes are discussed. In one method the boundary is augmented before the point is moved to the boundary and the augmented part is shrunk to zero. This results in finite part integrals which are then regularized to give a regular integral over the crack surface and finite part integrals which can be replaced by analytical expressions. In the other method, referred to as the new method, a Taylor series expansion and Stokes theorem are used before moving the point to the boundary, to arrive at the same integral and analytical expression as before. The smoothness requirements on the density function as demanded by the two methods are discussed. The validity of the two methods is demonstrated by solving 2-D and 3-D acoustic wave scattering problems and comparing with theory.

Acoustic Wave Scattering

The representation integral for the scattered field $\phi^S(\underline{\xi})$ at a point $\underline{\xi}$ from an arbitrarily shaped thin scatterer s^+, as in Fig. 1, is given by

$$\phi^S(\underline{\xi}) = \int_{s^+} \left[G^D(\underline{x},\underline{\xi}) \left(\frac{\partial\phi^+}{\partial n^+} + \frac{\partial\phi^-}{\partial n^-} \right) - (\phi^+ - \phi^-) \frac{\partial G^D(\underline{x},\underline{\xi})}{\partial n(\underline{x})} \right] ds(\underline{x}). \tag{1}$$

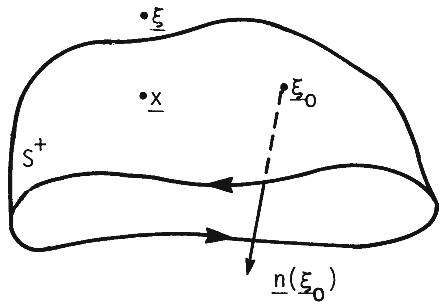

Fig. 1. Arbitrarily shaped thin scatterer.

The scattered field $\phi^S(\underline{\xi})$ is related to the total field $\phi(\underline{\xi})$ through $\phi^S(\underline{\xi}) = \phi(\underline{\xi}) - \phi^i(\underline{\xi})$ where $\phi^i(\underline{\xi})$ is the incident field at $\underline{\xi}$. The fundamental solution $G^D(\underline{x},\underline{\xi}) = e^{ikr}/4\pi r$ where $r = |\underline{x} - \underline{\xi}|$. The superscripts '+' and '-' correspond to the two sides of the scatterer. In the limit as $\underline{\xi} \to \underline{\xi}_o$, a point on s^+, the above equation results in the familiar boundary integral equation (BIE) which has improper integrals and the order of singularity is $1/r^2$. For scalar fields these improper integrals result in solid angles and for vector fields, Cauchy principal values (CPV). As the BIE for the thin scatterer problem contains too many unknowns, the gradient of the representation integral is necessary, which is given by

$$\frac{\partial\phi^S(\underline{\xi})}{\partial\xi_r} = \int_{s^+}\left[\frac{\partial G^D(\underline{x},\underline{\xi})}{\partial\xi_r}\left(\frac{\partial\phi^+}{\partial n^+} + \frac{\partial\phi^-}{\partial n^-}\right) - (\phi^+ - \phi^-)\frac{\partial^2 G^D(\underline{x},\underline{\xi})}{\partial\xi_r\,\partial n(\underline{x})}\right]ds(\underline{x}). \qquad (2)$$

In the limit as $\underline{\xi} \to \underline{\xi}_o$, the above integral is improper, and the order of singularity is $1/r^3$. This strongly singular integral will be referred to as hypersingular [2,3,6]. When the hypersingular integrals are interpreted in an ordinary sense or CPV sense, they are divergent and have no meaning. However, when interpreted in the finite part (FP) sense, they are finite and have meaning.

To focus on the hypersingular integrals, the problem is simplified by assuming a rigid scatterer, i.e. $\partial\phi^+/\partial n^+ = \partial\phi^-/\partial n^- = 0$. Then Eq. (2) reduces to

$$\frac{\partial\phi^S(\underline{\xi})}{\partial\xi_r} = -\int_{s^+}\frac{\partial^2 G^D(\underline{x},\underline{\xi})}{\partial\xi_r\,\partial n(\underline{x})}\,u(\underline{x})ds \qquad (3)$$

where $u(\underline{x}) = \phi^+(\underline{x}) - \phi^-(\underline{x})$. In the limit as $\underline{\xi} \to \underline{\xi}_o$, the above integral is hypersingular. Now on subtracting and adding the fundamental solution for the static problem, i.e. $G(\underline{x},\underline{\xi}) = 1/(4\pi r)$ we have

$$\frac{\partial\phi^S(\underline{\xi})}{\partial\xi_r} = -\int_{s^+}\left[\frac{\partial^2 G^D(\underline{x},\underline{\xi})}{\partial\xi_r\,\partial n(\underline{x})} - \frac{\partial^2 G(\underline{x},\underline{\xi})}{\partial\xi_r\,\partial n(\underline{x})}\right]u(\underline{x})ds - \int_{s^+}\frac{\partial^2 G(\underline{x},\underline{\xi})}{\partial\xi_r\,\partial n(\underline{x})}\,u(\underline{x})ds. \qquad (4)$$

Now as $\underline{\xi} \to \underline{\xi}_o$ the first integral is regular in the limit while the second is hypersingular. The advantage of dealing with Eq. (4) as compared with Eq. (3) is that we are dealing with a hypersingular integral whose integrand is real and less complicated. This advantage is more pronounced in vector problems.

In the discussion to follow, the proposed new method is used to handle the hypersingular integral followed by the finite part approach. A comparison of the results from both these methods is analyzed.

New Method

Consider the hypersingular term of Eq. (4).

$$I,_r(\underline{\xi}) = -\int_{s^+} \frac{\partial^2 G(\underline{x},\underline{\xi})}{\partial \xi_r \, \partial n(\underline{x})} u(\underline{x}) ds(\underline{x}) \tag{5}$$

On subtracting and adding the first two terms of the Taylor series expansion of the density function u, about the point $\underline{\xi}_o$ and by making use of a special form of Stokes' integral theorem, Eq. (5) can be converted to line integrals, a surface integral and a solid angle as follows.

$$\begin{aligned} I,_r(\underline{\xi}) = &-\int_{s^+} \frac{\partial^2 G(\underline{x},\underline{\xi})}{\partial \xi_r \, \partial n(\underline{x})} \left[u(\underline{x}) - u(\underline{\xi}_o) - u,_p(\underline{\xi}_o)(x_p - \xi_{op})\right] ds \\ &- u,_k(\underline{\xi}_o) \int_{s^+} \frac{\partial G(\underline{x},\underline{\xi})}{\partial \xi_r} n_k(\underline{x}) ds + \frac{\partial u(\underline{\xi}_o)}{\partial \xi_r} \left(\frac{\Omega(\underline{\xi})}{4\pi}\right) \\ &+ u(\underline{\xi}_o)\, \varepsilon_{qkr} \oint_c \frac{\partial G(\underline{x},\underline{\xi})}{\partial x_k} dx_q + u,_p(\underline{\xi}_o)\, \varepsilon_{qrp} \oint_c G(\underline{x},\underline{\xi}) dx_q \\ &+ u,_p(\underline{\xi}_o)\, \varepsilon_{qkr} \oint_c \frac{\partial G(\underline{x},\underline{\xi})}{\partial x_k} (x_p - \xi_{op}) dx_q \end{aligned} \tag{6}$$

where $\Omega(\underline{\xi})$ is the solid angle made by the surface s^+ from the point $\underline{\xi}$. If $\underline{\xi}_o$, the point on s^+, does not fall on the edge of the scatterer, and the density function at that point is at least $c^{1,\alpha}$, i.e. $|u(x) - u(\underline{\xi}_o) - u,_k(\underline{\xi}_o)(x_k - \xi_{ok})| \le B|\underline{x} - \underline{\xi}_o|^\beta$, $1 < \beta \le 2$, $|B| < \infty$, then in the limit as $\underline{\xi} \to \underline{\xi}_o$ all of the integrals in the last equation are regular or, at most, weakly singular.

Finite Part Approach

Consider the integral as in Eq. (5). If s_ε is a flat surface on s^+ which includes $\underline{\xi}_o$ as shown in Fig. 2, then the limit as $\underline{\xi} \to \underline{\xi}_o$ of the integral in Eq. (5) can be thought of as follows:

$$I,_r(\underline{\xi}_o) = -\lim_{s_\varepsilon \to 0} \left\{ \lim_{\underline{\xi} \to \underline{\xi}_o} \int_{s^+ - s_\varepsilon} \frac{\partial^2 G(\underline{x}, \underline{\xi})}{\partial \xi_r \, \partial n(\underline{x})} u(\underline{x}) ds \right.$$

$$\left. + \lim_{\underline{\xi} \to \underline{\xi}_o} \int_{s_\varepsilon} \frac{\partial^2 G(\underline{x}, \underline{\xi})}{\partial \xi_r \, \partial n(\underline{x})} \left[u(\underline{\xi}_o) + u,_p(\underline{\xi}_o)(x_p - \xi_{op}) + \ldots. \right] ds \right\} \qquad (7)$$

Performing the integration over s_ε and taking the limit as $\underline{\xi} \to \underline{\xi}_o$ results in two integrals which are unbounded in the limit as $s_\varepsilon \to 0$ and the rest are of order s_ε and vanish as $s_\varepsilon \to 0$. On retaining the terms which are unbounded and for $u(\underline{x})$ at least $c^{1,\alpha}$ at $\underline{\xi}_o$ the limit as $\underline{\xi} \to \underline{\xi}_o$ gives

$$I,_r(\underline{\xi}_o) = \lim_{s_\varepsilon \to 0} \left\{ - \int_{s^+ - s_\varepsilon} \frac{\partial^2 G(\underline{x}, \underline{\xi})}{\partial \xi_r \, \partial n(\underline{x})} u(\underline{x}) ds + u(\underline{\xi}_o) \frac{n_r(\underline{\xi}_o)}{4\pi} \int_0^{2\pi} \frac{d\theta}{\rho(\theta)} \right.$$

$$\left. - u,_p(\underline{\xi}_o) \frac{n_r(\underline{\xi}_o)}{4\pi} \int_0^{2\pi} \rho,_p(\theta) \ln|2\rho| d\theta \right\} \qquad (8)$$

where $\rho(\theta)$ is as shown in Fig. 2. If the density function $u(x)$ does not satisfy the smoothness requirement of being at least $c^{1,\alpha}$ at $\underline{\xi}_o$, then the limit $\underline{\xi} \to \underline{\xi}_o$ does not exist since the term multiplying $\ln[\underline{\xi} - \underline{\xi}_o]$ is not zero anymore, which means the limit process results in an unbounded quantity. On shrinking $s_\varepsilon \to 0$, the integral over $s^+ - s_\varepsilon$ is also unbounded. It can be shown that the infinities associated with the last two integrals cancel with the infinities associated with the first integral, thereby resulting in a finite quantity as $s_\varepsilon \to 0$. This is referred to as the finite part (FP) of the original integral and is expressed as

$$I,_r(\underline{\xi}_o) = - \,\mathrm{FP}\!\!\int_{s^+} \frac{\partial^2 G(\underline{x}, \underline{\xi}_o)}{\partial \xi_r \, \partial n(\underline{x})} u(\underline{x}) ds(\underline{x}). \qquad (9)$$

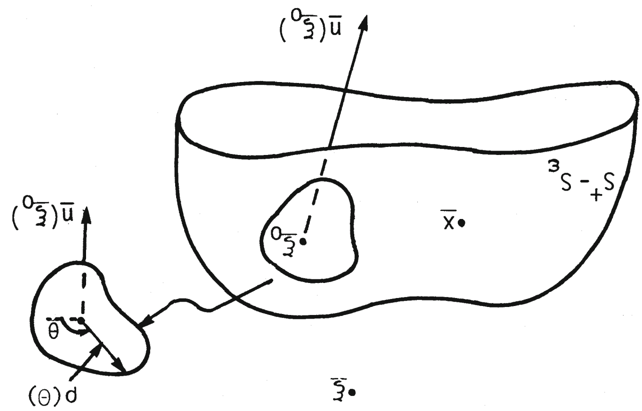

Fig. 2. Scatterer surface divided into two parts.

In the above definition the density function is part of the integrand. There are other forms of expressing $I,_r(\underline{\xi}_o)$ and the form most suited for numerical computation is under investigation. One form of expressing $I,_r(\underline{\xi}_o)$ in terms of FP and CPV which does not include the density function as part of the integrand is as follows.

$$I,_r(\underline{\xi}_o) = - \int_{S^+} \frac{\partial^2 G(\underline{x},\underline{\xi}_o)}{\partial \xi_r \partial n(\underline{x})} \left[u(\underline{x}) - u(\underline{\xi}_o) - u,_p(\underline{\xi}_o)(x_p - \xi_{op})\right] ds$$

$$- u(\underline{\xi}_o) \int_{S^+} \frac{\partial^2 G(\underline{x},\underline{\xi}_o)}{\partial \xi_r \partial n(\underline{x})} ds - u,_p(\underline{\xi}_o) \int_{S^+} \frac{\partial^2 G(\underline{x},\underline{\xi}_o)}{\partial \xi_r \partial n(\underline{x})} (x_p - \xi_{op}) ds. \qquad (10)$$

There are two methods one could use to compute FP and CPV integrals. One method is to use special numerical techniques, e.g. [4], and the other is to convert the FP and CPV integrals to regular integrals.

If $I,_n(\underline{\xi}_o) = I,_r(\underline{\xi}_o)\, n_r(\underline{\xi}_o)$, then the expression for $I,_n(\underline{\xi}_o)$ starting from Eq. (6) or Eq. (10), i.e. by using the new method of the finite part approach, is the same and is given by

$$I,_n(\underline{\xi}_o) = - \int_{S^+} \frac{\partial^2 G(\underline{x},\underline{\xi}_o)}{\partial n(\underline{\xi}_o) \partial n(\underline{x})} \left[u(\underline{x}) - u(\underline{\xi}_o) - u,_p(\underline{\xi}_o)(x_p - \xi_{op})\right] ds$$

$$- u,_k(\underline{\xi}_o) \int_{S^+} \frac{\partial G(\underline{x},\underline{\xi}_o)}{\partial \xi_r} n_k(\underline{x}) ds + u(\underline{\xi}_o) n_r(\underline{\xi}_o) \varepsilon_{qkr} \oint_C \frac{\partial G(\underline{x},\underline{\xi}_o)}{\partial x_k} dx_q$$

$$+ u,_p(\underline{\xi}_o) n_r(\underline{\xi}_o) \varepsilon_{qkr} \oint_C \frac{\partial G(\underline{x},\underline{\xi}_o)}{\partial x_k} (x_p - \xi_{op}) dx_q$$

$$+ u,_p(\underline{\xi}_o) n_r(\underline{\xi}_o) \varepsilon_{qrp} \oint_C G(\underline{x},\underline{\xi}_o) dx_q. \qquad (11)$$

Numerical Example

From our previous analysis the expression for the normal gradient of the scattered field from a thin scatterer is given by

$$\frac{\partial \phi^S(\underline{\xi}_o)}{\partial n(\underline{\xi}_o)} = - \frac{\partial \phi^i(\underline{\xi}_o)}{\partial n(\underline{\xi}_o)} = - \int_{S^+} \left[\frac{\partial^2 G^D(\underline{x},\underline{\xi}_o)}{\partial n(\underline{\xi}_o) \partial n(\underline{x})} - \frac{\partial^2 G(\underline{x},\underline{\xi}_o)}{\partial n(\underline{\xi}_o) \partial n(\underline{x})}\right] u(\underline{x}) ds$$

$$+ I,_n(\underline{\xi}_o) \qquad (12)$$

Results for the 2-D counterpart of this problem are discussed in [7].

Preliminary work for a penny-shaped scatterer for a crude discretization with five nonconforming quadratic elements gave an error of 11% at nodes close to the edges and 1.9% error at nodes closest to the centre. Finer discretization to thirtythree elements lowers the edge error to 1.4%.

Elastic Wave Scattering

In the case of acoustic wave scattering, the new method and the finite part approach were used to handle the strongly singular term and the resulting expressions were shown to be the same. Hence, in the following discussion on elastic wave scattering, the new method only will be used and steps involved are the same as in the previous case, with the only difference being that this is a vector problem.

The representation integral for scattered elastic waves from an arbitrarily shaped thin crack s^+ as in Fig. 1 is given by

$$u_m^s(\underline{\xi}) = C_{ijkl} \int_{s^+} \left\{ G_{im}^D \left(\frac{\partial u_k^+}{\partial x_l} - \frac{\partial u_k^-}{\partial x_l} \right) n_j^+(\underline{x}) - \frac{\partial G_{km}^D}{\partial x_l} (u_i^+ - u_i^-) n_j^+(\underline{x}) \right\} ds \quad (13)$$

where u_m^s is the scattered field, u_m^i the incident field, $u_m = u_m^s + u_m^i$ the total field, G_{im}^D the components of the fundamental solution, and C_{ijkl} material constants. For a stress-free crack surface the above equation reduces to

$$u_m^s(\underline{\xi}) = - C_{ijkl} \int_{s^+} \frac{\partial G_{km}^D}{\partial x_l} \Delta u_i(\underline{x}) n_j(\underline{x}) ds \quad (14)$$

where $\Delta u_i(\underline{x}) = u_i^+(\underline{x}) - u_i^-(\underline{x})$ and the superscripts + and - denote the two sides of the crack. The above equation in the limit as $\underline{\xi} \to \underline{\xi}_o$ results in the familiar BIE. As this equation by itself is insufficient to solve the problem, it is necessary to take the gradient of the representation integral, which is

$$\frac{\partial u_m^s(\underline{\xi})}{\partial \xi_r} = - C_{ijkl} \int_{s^+} \frac{\partial^2 G_{km}^D(\underline{x},\underline{\xi})}{\partial \xi_r \partial x_l} \Delta u_i(\underline{x}) n_j(\underline{x}) ds. \quad (15)$$

In the limit as $\underline{\xi} \to \underline{\xi}_o$, the above integral is improper and the order of singularity is $1/r^3$, i.e. hypersingular. On subtracting and adding the fundamental solution corresponding to the equivalent static problem, as for acoustic wave scattering, gives

$$\frac{\partial u_m^s(\underline{\xi})}{\partial \xi_r} = - C_{ijkl} \int_{s^+} \left[\frac{\partial^2 G_{km}^D(\underline{x},\underline{\xi})}{\partial \xi_r \partial x_l} - \frac{\partial^2 G_{km}(\underline{x},\underline{\xi})}{\partial \xi_r \partial x_l} \right] n_j(\underline{x}) \Delta u_i(\underline{x}) ds$$

$$- C_{ijkl} \int_{s^+} \frac{\partial^2 G_{km}(\underline{x},\underline{\xi})}{\partial \xi_r \partial x_l} n_j(\underline{x}) \Delta u_i(\underline{x}) ds. \quad (16)$$

If $\underline{\xi}_o$ is a point on s^+ which does not fall on the crack edge, and the density function is at least $c^{1,\alpha}$ at that point, then the special form of Stokes theorem can be used to express the above equation in the limit as $\underline{\xi} \to \underline{\xi}_o$ as

$$\frac{\partial u_m^S(\underline{\xi}_o)}{\partial \xi_r} = - C_{ijkl} \int_{s^+} \left(\frac{\partial^2 G_{km}^D}{\partial \xi_r \partial x_l} - \frac{\partial^2 G_{km}}{\partial \xi_r \partial x_l} \right) n_j(\underline{x}) \Delta u_i(\underline{x}) ds$$

$$- C_{ijkl} \int_{s^+} \frac{\partial^2 G_{km}}{\partial \xi_r \partial x_l} n_j(\underline{x}) \left[\Delta u_i(\underline{x}) - \Delta u_i(\underline{\xi}_o) - \Delta u_{i,p}(\underline{\xi}_o)(x_p - \xi_{op}) \right] ds$$

$$+ C_{ijkl} \int_{s^+} \frac{\partial G_{im}}{\partial x_r} \frac{\partial \Delta u_k(\underline{\xi}_o)}{\partial \xi_r} n_j(\underline{x}) ds + \frac{\partial \Delta u_m(\underline{\xi}_o)}{\partial \xi_r} \left(\frac{\Omega(\underline{\xi}_o)}{4\pi} \right)$$

$$+ \Delta u_i(\underline{\xi}_o) \, \varepsilon_{jrq} \, C_{ijkl} \oint_c \frac{\partial G_{km}}{\partial x_l} dx_q$$

$$+ \frac{\partial \Delta u_i(\underline{\xi}_o)}{\partial \xi_j} \varepsilon_{rlq} \, C_{ijkl} \oint_c G_{km} dx_q$$

$$+ \frac{\partial \Delta u_i(\underline{\xi}_o)}{\partial \xi_p} \varepsilon_{jrq} \, C_{ijkl} \oint_c \frac{\partial G_{km}}{\partial x_l} (x_p - \xi_{op}) dx_q$$

$$- \frac{\partial \Delta u_j(\underline{\xi}_o)}{\partial \xi_r} \frac{1}{8\pi} \oint_c \varepsilon_{mji} \, R_{,pp} \, dx_i$$

$$- \frac{\partial \Delta u_i(\underline{\xi}_o)}{\partial \xi_r} \frac{1}{8\pi(1-\nu)} \oint_c \varepsilon_{jpi} \, R_{,pm} \, dx_i \qquad (17)$$

where $R = |\underline{x} - \underline{\xi}_o|$. All the integrals in the last equation are regular or, at most, weakly singular.

The numerical formulation and computational strategy for solving Eq. (17) are underway.

Applications

The variety of physical problems where one has to deal with hypersingular integrals can be classified into two groups. In one group are problems where BIE is not sufficient to solve for all unknowns and so the gradient expression, which is hypersingular, is required. Such problems occur in elastic wave scattering from cracks and very thin inclusions, acoustic wave scattering from thin bodies, diffraction by a thin screen in electromagnetism, antenna design, etc. In the other group of problems are those where the BIE alone can solve the problem but using the gradient expression, which has hypersingular integrals, is physically more appealing and computationally more efficient. Such problems are known to occur in ground water flow, thermal problems and, no doubt, in other areas.

CONCLUSION

So far, in the literature, wave scattering from an arbitrarily shaped (not flat) crack surface has not been examined due to the difficulties encountered in getting an analytical expression for finite parts due to the curvature of the surface. In this paper we have not only overcome the above difficulties, but also have proposed a new method which does not require any FP or CPV interpretation and eliminates any numerical computation of FP or CPV integrals. In the proposed new method, Stokes' theorem is used not to reduce the order of singularity by transferring a derivative on the fundamental solution to the density function, but to convert surface integrals to line integrals. The new method is valid regardless of how the surface is discretized and whether the surface is open or closed. The smoothness demanded of the density function by this method is required by the nature of the hypersingularity and hence needs to be satisfied no matter what method is used. Indeed, without this smoothness requirement, we believe that no meaningful value can be given to the hypersingular integrals involved in the gradient BIE (Eq. (2)) by any method. Preliminary computations using the new method show excellent agreement with analytical solution.

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