ELASTIC WAVE SCATTERING BY MACRO-MICRO CRACK CONFIGURATIONS

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INTRODUCTION

Elastic wave scattering by several cracks in an infinite isotropic elastic solid has been studied by several investigators[1]-[6]. In general, a system of singular integral equations for the crack opening displacements or the dislocation densities has to be solved. From the crack-damage interaction point of view, a special case of some practical importance is the macro-micro crack configuration, especially when the microcracks are in the vicinity of the macrocrack tips. For example, a macrocrack tip in ceramics usually is accompanied by a cloud of microcracks. In this paper, an iteration method is developed for this special configuration. Instead of solving a system of singular integral equations simultaneously, we need to solve only one equation, for each crack, at each iterative step. This iteration approach is based on the elastodynamic integral representation theorem, and can be applied to both 2-D and 3-D cracks. Furthermore, it can also be extended to cracks in anisotropic solids, since the constitutive equation used in the derivation is the general Hooke's law. In the last section, applications of this iterative technique is given to a pair of collinear cracks under normal incidence of a longitudinal wave. The accuracy of the leading order solution is tested by comparing it with the exact numerical solutions in the literature.

STATEMENT OF THE PROBLEM

The configuration considered here consists of a macrocrack A and several microcracks S_n around the tip of the macrocrack, as shown in Fig. 1. The cracks are embedded in an isotropic and linearly elastic solid with Lame's constants λ, μ, and mass density ρ. A Cartesian coordinate system is introduced such that the origin of the coordinate system is located at the geometrical center of the macrocrack.

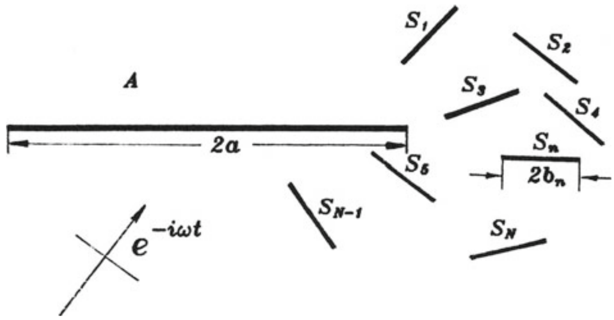

Fig. 1. Macro-micro crack configuration

For time-harmonic wave incidence ($\exp(-i\omega t)$), the reduced wave equation governing the motion of the solid is given by

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0 \quad , \tag{1}$$

where σ_{ij} defines the stress components, u_i denotes the displacement components, and ω is the circular frequency. The stresses are related to the displacements by Hooke's law

$$\sigma_{ij} = C_{ijmn} u_{m,n} \quad , \tag{2}$$

where, for the isotropic solid considered here, we have

$$C_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}) \quad .$$

The traction-free conditions on the faces of each crack give rise to the following boundary conditions for the total stress field

$$\sigma_{ij}\nu_j = 0 \quad , \qquad \underset{\sim}{x} \subset A \cup \sum_{n=1}^{N} S_n \quad , \tag{3}$$

where ν_j denote the components of the unit normal vectors of the crack faces and N is the number of microcracks.

To solve the scattering problem, equation (1) has to be solved together with the boundary condition (3). Because of the linearity of the governing equations and the boundary conditions, the solution to the boundary value problem (1)-(2) may be represented by a superposition of N+1 solutions, each of which corresponds to a problem involving only one crack loaded by unknown tractions on its faces. The unknown tractions are induced by the incident field and the field scattered by the other cracks. In other words, one may first solve the scattering problem for each crack as if there were no other cracks. Then, superposition of all the solutions yields the final result. The interaction among the cracks, or the influence of the other cracks on the particular one we are considering is taken into account by assuming that the crack is under the incidence of the original incident field plus some unknown fields, which are the very fields scattered by all the other cracks. Therefore the total stress field may be written as the following sum of N+2 terms

$$\sigma_{ij} = \sigma_{ij}^{in} + \sigma_{ij}^{(o)} + \sum_{n=1}^{N} \sigma_{ij}^{(n)} \tag{4}$$

where $\sigma_{ij}^{(o)}$ is the stress field scattered by the macrocrack A when its faces are loaded by

$$\{\sigma_{ij}^{in} + \sum_{n=1}^{N} \sigma_{ij}^{(n)}\}\nu_j^{(o)}$$

in the absence of all the microcracks. Here $\nu_j^{(o)}$ is the unit normal vector of the macrocrack A. Similarly, $\sigma_{ij}^{(n)}$ in (4) represents the stress field scattered by the n'th microcrack S_n when its faces are loaded by the tractions

$$\{\sigma_{ij}^{in} + \sigma_{ij}^{(o)} + \sum_{\substack{m=1 \\ m\neq n}}^{N} \sigma_{ij}^{(m)}\}\nu_j^{(n)}$$

and the scattered field is calculated in the absence of the macrocrack and the rest of the microcracks, where $\nu_j^{(n)}$ denotes the unit normal of the n'th microcrack S_n.

Use of the superposition (4) in conjunction with the boundary condition (3) yields

$$\sigma_{ij}^{(o)}\nu_j^{(o)} = -\{\sigma_{ij}^{in} + \sum_{n=1}^{N} \sigma_{ij}^{(n)}\}\nu_j^{(o)} \quad , \quad \underset{\sim}{x} \subset A \tag{5}$$

$$\sigma_{ij}^{(n)}\nu_j^{(n)} = -\{\sigma_{ij}^{in} + \sigma_{ij}^{(o)} + \sum_{\substack{m=1 \\ m\neq n}}^{N} \sigma_{ij}^{(m)}\}\nu_j^{(n)} \quad , \quad \underset{\sim}{x} \subset S_n \quad , \quad n=1,2,\ldots N \tag{6}$$

Equation (1) with boundary conditions (2) or (5)-(6) has no analytical solution. In order to obtain numerical or asymptotic solutions, the boundary value problem will be transformed into a system of integral equations in the next section.

THE INTEGRAL EQUATIONS FOR THE COD

For crack problems, the scattered displacement field u_i can be expressed by the following integral representation [7]

$$u_i(\underset{\sim}{x}) = \int_S \Delta u_s(\underset{\sim}{x}')\tau^i_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu_t(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{7}$$

where S is the insonified crack face, τ^i_{st} is the Green's tensor, Δu_i is the crack opening displacement (COD) and ν_t is the unit normal vector of S.

The corresponding stress components can be computed by substituting (7) into the constitutive equation (2)

$$\sigma_{ij}(\underset{\sim}{x}) = T_{ijk} \int_S \Delta u_s(\underset{\sim}{x}')\tau^k_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu_t(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{8}$$

where T_{ijk} is a differential operator defined by

$$T_{ijk} = C_{ijkl} \frac{\partial}{\partial x_l}$$

It is noted that the governing equation (1) is satisfied automatically by (7) and (8). The remaining equations to determine the COD are the boundary conditions (5)-(6). It follows from (8) that

$$\sigma^{(o)}_{ij}(\underset{\sim}{x}) = T_{ijk} \int_A \Delta u^{(o)}_s(\underset{\sim}{x}')\tau^k_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu^{(o)}_t(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{9}$$

$$\sigma^{(n)}_{ij}(\underset{\sim}{x}) = T_{ijk} \int_{S_n} \Delta u^{(n)}_s(\underset{\sim}{x}')\tau^k_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu^{(n)}_t(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \tag{10}$$

where $\Delta u^{(o)}_s$ and $\Delta u^{(n)}_s$ are the crack opening displacements of the macrocrack A and the n'th microcrack S_n, respectively. Substitution of (9)-(10) into (5)-(6) yields the following system of N+1 singular integral equations for the N+1 unknown COD's $\Delta u^{(n)}_s$, n=0,1,2,...N,

$$\nu^{(o)}_j(\underset{\sim}{x})T_{ijk} \int_A \Delta u^{(o)}_s(\underset{\sim}{x}')\tau^k_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu^{(o)}_t(\underset{\sim}{x}')d\underset{\sim}{x}'$$

$$= -\nu^{(o)}_j\{\sigma^{in}_{ij}(\underset{\sim}{x})+T_{ijk}\sum_{n=1}^{N} \int_{S_n} \Delta u^{(n)}_s(\underset{\sim}{x}')\tau^k_{st}(\underset{\sim}{x},\underset{\sim}{x}')\nu^{(n)}_t(\underset{\sim}{x}')d\underset{\sim}{x}'\} \tag{11}$$

for $\underset{\sim}{x} \subset A$ and

$$\nu_j^{(n)}(\underset{\sim}{x})T_{ijk}\int_{S_n} \Delta u_s^{(n)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(n)}(\underset{\sim}{x}')d\underset{\sim}{x}' = -\sigma_{ij}^{in}(\underset{\sim}{x})\nu_j^{(n)}(\underset{\sim}{x})$$

$$- \nu_j^{(n)}(\underset{\sim}{x})T_{ijk}\sum_{\substack{m=1\\m\neq n}}^{N} \int_{S_m} \Delta u_s^{(m)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(m)}(\underset{\sim}{x}')d\underset{\sim}{x}'$$

$$-\nu_j^{(n)}(\underset{\sim}{x})\ T_{ijk} \int_{A} \Delta u_s^{(o)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(o)}(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \qquad (12)$$

for $\underset{\sim}{x} \subset S_n$, $n=1,2,...N$.

This system of integral equations has been solved numerically by Zhang and Achenbach [4] for N=1, by using the boundary element method. When N increases, the numerical solution to (11)-(12) becomes very difficult. Especially when N is very large and the microcracks are randomly distributed. In the next section, some simplifications will be introduced to reduce (11)-(12) into a manageable form.

In closing this section, we should mention that (11)-(12) can also be obtained directly from the integral representation (8) and the boundary condition (3). But, in doing so, the physical meanings get lost in the mathematical manipulation.

APPROXIMATE METHOD OF SOLUTION

Several length parameters are involved in the scattering process: the characteristic dimension (radius for a penny-shaped crack or half-length for a Griffith crack) of the macrocrack, a, the average characteristic dimension of the microcracks, b, the average distance between the macrocrack and the microcracks, d, and the incident wave length Λ. For reasons of practical interests, we assume that $b \ll \Lambda \ll a$ and $d \ll a$. Rather than going through lengthy algebra, in what follows we will utilize physical intuition to simplify (11)-(12) under the foregoing assumptions for the length parameters.

The difficulty of solving (11)-(12) lies in the coupling among the equations. This coupling reflects the interactions between the cracks. The interactions can be classified as (a) the interactions between the macrocrack and the microcracks; (b) the interaction among the microcracks. For the former, it is conceivable that the presence of one microcrack will have little impact on the macrocrack because of $b \ll a$. Observation of (11) reveals that the summation on the right hand side represents the influence of the microcracks on the main crack. Thus, based on the above argument, (11) can, in the first instance, be approximated by

$$\nu_j^{(o)}(\underset{\sim}{x}) T_{ijk} \int_A f_s^{(o)}(\underset{\sim}{x}') \tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}') \nu_t^{(o)}(\underset{\sim}{x}') d\underset{\sim}{x}' = -\sigma_{ij}^{in} \nu_j^{(o)}, x \subset A \qquad (13)$$

where $f_s^{(o)}$ is the O'th order approximation to $\Delta u_s^{(o)}$. It is obvious that $f_s^{(o)}$ is nothing but the crack opening displacement of the main crack, A, when all the microcracks are absent. A numerical technique for solving singular integral equations similar to (13) has been presented by Zhang and Achenbach [4]. If we can also ignore the interactions among the microcracks in our first order approximation, the summation on the right hand side of (12) becomes negligible. Therefore, we have

$$\nu_j^{(n)}(\underset{\sim}{x}) T_{ijk} \int_{S_n} g_{(n)s}^{(1)}(\underset{\sim}{x}') \tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}') \nu_t^{(n)}(\underset{\sim}{x}') d\underset{\sim}{x}' = -\sigma_{ij}^{in}(\underset{\sim}{x}) \nu_j^{(n)}(\underset{\sim}{x})$$

$$-\nu_j^{(n)}(\underset{\sim}{x}) \; T_{ijk} \int_A f_s^{(o)}(\underset{\sim}{x}') \tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}') \nu_t^{(o)}(\underset{\sim}{x}') d\underset{\sim}{x}' \qquad (14)$$

where $g_{(n)s}^{(o)}$ is the O'th order approximation of $\Delta u_s^{(n)}$. It represents the crack opening displacement of the n'th microcrack when it stays alone in the infinite medium and is insonified by two incident fields: (1) the original incident wave, which correspond to the first term on the right hand side of (14); (2) the wave field scattered by the main crack, A, which is represented by the second term. It should be noted that the right hand side of (14) is regular as long as there is no overlap between the macrocrack and the microcracks. Thus the numerical procedure to solve (14) is identical to that of solving (13).

Once (14) has been solved for $g_{(n)s}^{(o)}$, one can then substitute $g_{(n)s}^{(o)}$ into (11) to compute $f_s^{(1)}$, the next order approximation of $\Delta u_s^{(o)}$. By using the newly computed $f_s^{(1)}$ in (12), the next order approximation to $\Delta u_s^{(n)}$ can be obtained in the same manner. In general, for J=1,2,..., we have

$$\nu_j^{(o)}(\underset{\sim}{x}) T_{ijk} \int_A f_s^{(J-1)}(\underset{\sim}{x}') \tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}') \nu_t^{(o)}(\underset{\sim}{x}') d\underset{\sim}{x}'$$

$$= -\nu_j^{(o)} \{\sigma_{ij}^{in}(\underset{\sim}{x}) + T_{ijk} \sum_{n=1}^{N} \int_{S_n} g_{(n)s}^{(J-1)}(\underset{\sim}{x}') \tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}') \nu_t^{(n)}(\underset{\sim}{x}') d\underset{\sim}{x}'\} \qquad (15)$$

for $\underset{\sim}{x} \subset A$ and

$$\nu_j^{(n)}(\underset{\sim}{x})T_{ijk}\int_{S_n} g_{(n)s}^{(J)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(n)}(\underset{\sim}{x}')d\underset{\sim}{x}' = -\sigma_{ij}^{in}(\underset{\sim}{x})\nu_j^{(n)}(\underset{\sim}{x})$$

$$- \nu_j^{(n)}T_{ijk}\sum_{\substack{m=1\\m\neq n}}^{N}\int_{S_m} g_{(m)s}^{(J-1)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(m)}(\underset{\sim}{x}')d\underset{\sim}{x}'$$

$$-\nu_j^{(n)}\ T_{ijk}\int_A f_s^{(J-1)}(\underset{\sim}{x}')\tau_{st}^k(\underset{\sim}{x},\underset{\sim}{x}')\nu_t^{(o)}(\underset{\sim}{x}')d\underset{\sim}{x}' \quad , \qquad (16)$$

for $\underset{\sim}{x} \subset S_n$, n=1,2,...N. It is understood that

$$g_{(n)s}^{(o)}(\underset{\sim}{x}) = 0 \quad .$$

Equations (15)-(16) describe an iteration procedure to solve for the crack opening displacements. Each higher order of iteration introduces one more interaction among the cracks. For each iteration, one needs to solve only one integral equation for each crack, for the desired crack opening displacement.

EXAMPLE

To test the validity of the iteration formalism developed in the previous section, a simple example is considered here. The configuration is two dimensional and consists of only two collinear cracks. The half lengths of the macrocrack and the microcrack are a and b, respectively. The distance between the tip of the two cracks is denoted by d, see Fig. 2. A longitudinal wave is normally incident from infinity. The exact numerical solution to this problem can be found in [4]. Comparison of the approximate solution and the exact solution of the backscattered displacement amplitude is presented in Fig.3, for d/a = 0.1; $k_T a = 1$. The approximate solution shown in Fig.3 is the first order iteration only.

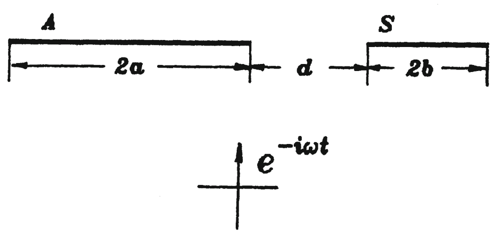

Fig. 2. Two-dimensional collinear cracks

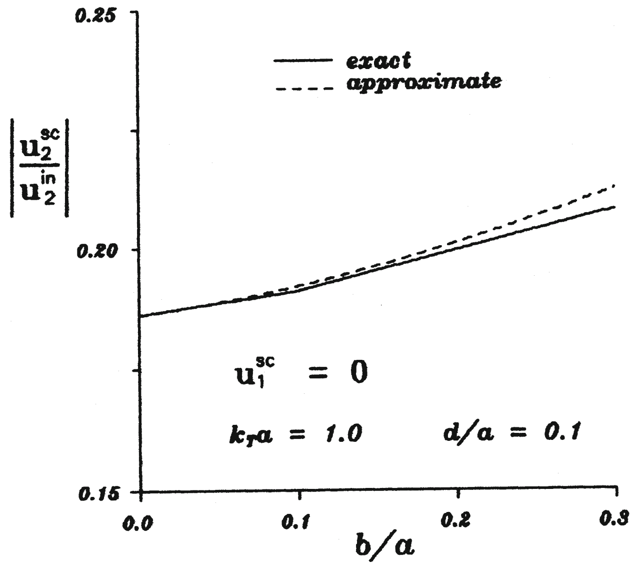

Fig. 3. Backscattered displacements for the configuration of Fig. 2 for normal incidence of a plane L wave

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