SCATTERING BY A PERIODIC ARRAY OF COLLINEAR CRACKS

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INTRODUCTION

This paper deals with the problem of scattering of an in-plane wave by a periodic array of collinear cracks in an elastic solid. The work is of interest for the detection of cracked zone in both quantitative evaluation of materials and seismology.

Scattering by elastic waves, both normally incident anti-plane longitudinal waves, by a Griffith crack was studied by Mal [1]. An integral transform method was used in [1]. Van der Hijden and Neerhoff [2] also investigated the problem of scattering of in-plane longitudinal and transverse waves by a Griffith crack, and derived a set of integral equations for the unknown crack-opening displacements. The Chebyshev polynomial expansions of the unknown crack-opening displacements were used to solve these integral equations.

In recent years, scattering of waves by a periodic structure has been intensively studied for the application in the quantitative nondestructive evaluation of materials. Scattering of scalar waves by a periodic array of screen was treated in [3]. An integral equation was derived for the unknown jump of the scalar potential across the screen, and this equation was again solved by an expansion in Chebyshev polynomials of the unknown jump. The boundary integral equation method has been used to solve the three-dimensional problem of reflection and transmission of a longitudinal wave by a doubly periodic array of spherical cavities [4]. The same method has been also used to solve the problem of scattering by a periodic array of inclined cracks [5], where the scattering by a periodic array of collinear cracks has been treated as a special case. The problem of scattering of waves by a periodic array of collinear cracks was first treated by Angel and Achenbach [6,7], where it was solved by using Fourier series expansion of the Green-Lame potentials. In this paper, we will solve the same problem by extending van der Hijden and Neerhoff's treatment [2] of in-plane wave scattering by a sigle crack. The merit of our treatment over Angel and Achenbach [6,7] will be discussed.

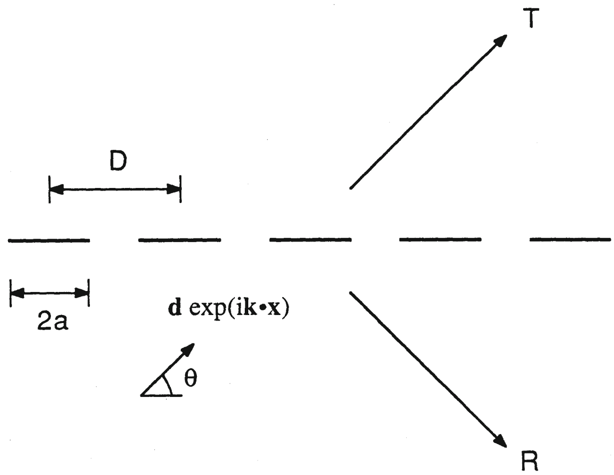

Fig. 1. In-plane wave incident on a periodic array of collinear cracks.

STATEMENT OF THE PROBLEM

The two-dimensional geometry is shown in Fig.1. Every crack is of width 2a, and located collinearly along the x-axis. The center points of the cracks are equally spaced with distance D apart along the x-axis. The propagation vector of the incident wave makes an angle θ with the x-axis.

Let us consider a time-harmonic displacement wave of the form u(x,y)exp(-iωt), where u(x,y) and ω are the amplitude and the angular frequency of the wave, respectively. The governing equation for u(x,y) is

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho\omega^2 u_i = 0 , \tag{1}$$

where μ and λ are Lame constants, and ρ is the mass density.

In the usual manner the displacement field is decomposed into two parts, the incident field u^{in}, and the scattered field u^{sc} :

$$u_i (x,y) = u_i^{in} (x,y) + u_i^{sc} (x,y) . \tag{2}$$

The incident field is taken as either a plane longitudinal wave or a plane transverse wave :

$$\mathbf{u}^{in} (\mathbf{x}) = \mathbf{u}^L (\mathbf{x}) \text{ or } \mathbf{u}^T (\mathbf{x}) , \tag{3}$$

where

$$\mathbf{u}^L (\mathbf{x}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \exp(i\mathbf{k}^L \bullet \mathbf{x}) ,$$
$$\mathbf{u}^T (\mathbf{x}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \exp(i\mathbf{k}^T \bullet \mathbf{x}) . \tag{4}$$

Here k^L and k^T are the propagation vectors, which are given by

$$\mathbf{k}^L = (k_L \cos\theta,\ k_L \sin\theta),$$
$$\mathbf{k}^T = (k_T \cos\theta,\ k_T \sin\theta), \tag{5}$$

where

$$k_L = \frac{\omega}{c_L}, \qquad c_L = [(\lambda+\mu)/\rho]^{\frac{1}{2}},$$
$$k_T = \frac{\omega}{c_T}, \qquad c_T = [\mu/\rho]^{\frac{1}{2}}. \tag{6}$$

The incident field u^{in} satisfies equation (1). Therefore, the scattered field u^{sc} must also satisfy that equation. Since the traction vanishes on the faces of the crack, the boundary conditions for the problem are given by

$$\sigma_{kl}^{sc}(x)\, n_l^- = -\,\sigma_{kl}^{in}(x)\, n_l^-, \qquad x \in \Gamma_m^-, \qquad m = 0, \pm 1, \pm 2, \cdots, \tag{7}$$

where Γ^-_m is the lower face of the m-th crack, and σ^{sc}_{kl}, σ^{in}_{kl} and n^-_l are given by

$$\sigma_{kl}^{sc}(x) = C_{klmn} \frac{\partial}{\partial x_n} u_m^{sc}(x), \tag{8}$$
$$\sigma_{kl}^{in}(x) = C_{klmn} \frac{\partial}{\partial x_n} u_m^{in}(x), \tag{9}$$
$$n_l^- = (0,\ 1). \tag{10}$$

Here, C_{klmn} is the elastic tensor of the material, and is given by

$$C_{klmn} = \mu(\delta_{km}\,\delta_{ln} + \delta_{kn}\,\delta_{lm}) + \lambda \delta_{kl}\,\delta_{mn}. \tag{11}$$

In addition to the equation (7), the scattered field must also satisfy the Sommerfeld radiation condition.

MATHEMATICAL FORMULATION

By using the reciprocity relation of elastodynamics [8], along with the Sommerfeld radiation condition, we may write the following integral representation for the scattered field u^{sc}.

$$u_k^{sc}(x) = \sum_{m=-\infty}^{\infty} \int_{\Gamma_m^-} ds'_m\, \psi_i^m(x')\, \sigma_{ij;k}^G(x,x')\, n_j^-, \tag{12}$$

where ψ^m_i (i = 1, 2) are the unknown crack-opening displacements for the m-th crack;

$$\psi_i^m(x') = u_i^{sc}(x')|_{x' \in \Gamma_m^+} - u_i^{sc}(x')|_{x' \in \Gamma_m^-}, \tag{13}$$

while the Green's stress tensor $\sigma^G_{ij;k}$ is given by

$$\sigma_{ij;k}^G(x,x') = C_{ijln} \frac{\partial}{\partial x_n^-} u_{l;k}^G(x,x'). \tag{14}$$

Here, $u^G_{l;k}$ is the displacement in the l-direction due to a unit force in the k-direction applied at x. The crack coordinate s'_m and the global coordinates (x', y') are related by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = s'_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} mD \\ 0 \end{pmatrix}, \qquad -a \le s'_m \le a . \tag{15}$$

Substitution of equation (15) into (12) yields

$$u_k^{sc}(x) = \sum_{m=-\infty}^{\infty} \int_{-a}^{a} ds'_m \, \psi_i^m(s'_m) \, \sigma^G_{ij;k}(x, s'_m ; m) \, n_j^- . \tag{16}$$

Since s'_m is a dummy variable of the integration, we can rewrite (16) as

$$u_k^{sc}(x) = \sum_{m=-\infty}^{\infty} \int_{-a}^{a} ds' \, \psi_i^m(s') \, \sigma^G_{ij;k}(x, s' ; m) \, n_j^- . \tag{17}$$

The periodicity in the geometry implies that

$$\psi_i^m(s') = \exp(imDk_{L,T}\cos\theta) \, \psi_i(s') , \tag{18}$$

where $\psi_i(s')$ is the unknown crack-opening displacement for the zeroth crack, and

$$k_{L,T} = \begin{cases} k_L & \text{for an incident longitudinal wave} \\ k_T & \text{for an incident transverse wave} . \end{cases} \tag{19}$$

Substituting equation (18) into (17), we obtain

$$u_k^{sc}(x) = \int_{-a}^{a} ds' \, \psi_i(s') \sum_{m=-\infty}^{\infty} \exp(imDk_{L,T}\cos\theta) \, \sigma^G_{ij;k}(x, s'; m) \, n_j^- . \tag{20}$$

It also follows from the periodicity of the problem that we only have to consider one out of the infinite set of boundary conditions (7), since the conditions are equivalent. The stresses corresponding to (20) follow from (8), and (7) subsequently yields

$$f_k(x) = \int_{-a}^{a} ds' \, \psi_i(s') \, G_{ik}(x, s') , \qquad -a \le x \le a, \; k = 1, 2 , \tag{21}$$

where

$$f_k(x) = -\sigma^{in}_{kl}(x) \, n_l^- \big|_{x \in \Gamma_0^-} , \tag{22}$$

$$G_{ik}(x, s') = \sum_{m=-\infty}^{\infty} \exp(imDk_{L,T}\cos\theta) \, F_{ik}(x, s'; m) , \tag{23}$$

$$F_{ik}(x, s'; m) = [\, \mu(\sigma^G_{ij;k,l} + \sigma^G_{ij;l,k}) + \lambda\sigma^G_{ij;\alpha,\alpha} \, \delta_{kl} \,] \, n_j^- \, n_l^- \big|_{x \in \Gamma_0^-} . \tag{24}$$

Equation (21) is a pair of singular integral equations for the unknown crack-opening displacements ψ_i of the zeroth crack.

The integral equation (21) for ψ_i can be solved by using the Chebyshev polynomial expansions of the unknown crack-opening displacements [2]. The result is given as follows.

$$\psi_i(s') = \sum_{n=1}^{\infty} C_{in}\,\phi_n(s'), \qquad i = 1, 2, \tag{25}$$

where ϕ_n are the Chebyshev polynomials [8] and the expansion coefficients C_{in} are obtained by solving an infinite system of linear algebraic equations. The resulting infinite system is obtained for an incident P-wave as

$$d^L_{1j} = \sum_{n=1}^{\infty} K^L_{11jn} C_{1n}, \qquad d^L_{2j} = \sum_{n=1}^{\infty} K^L_{22jn} C_{2n}, \qquad j \in Z^+, \tag{26}$$

and for an incident SV-wave as

$$d^T_{1j} = \sum_{n=1}^{\infty} K^T_{11jn} C_{1n}, \qquad d^T_{2j} = \sum_{n=1}^{\infty} K^T_{22jn} C_{2n}, \qquad j \in Z^+, \tag{27}$$

where $Z^+ = \{1,2,3, \cdots\}$, and

$$d^L_{1j} = -2j \sin\theta\, J_j(ak_L \cos\theta),$$

$$d^L_{2j} = -\frac{j}{\cos\theta}\left[\left(\frac{k_T}{k_L}\right)^2 - 2\cos^2\theta\right] J_j(ak_L \cos\theta), \tag{28}$$

$$d^T_{1j} = -j\,\frac{\cos 2\theta}{\cos\theta}\, J_j(ak_T \cos\theta),$$

$$d^T_{2j} = -2j \sin\theta\, J_j(ak_T \cos\theta), \tag{29}$$

$$K^{L,T}_{11jn} = \frac{\pi}{2aDk_T^2}\, jn \sum_{m=-\infty}^{\infty} \frac{1}{(a\varsigma_m^{L,T})^2} J_j(a\varsigma_m^{L,T}) J_n(a\varsigma_m^{L,T}) M_1(a\varsigma_m^{L,T}),$$

$$K^{L,T}_{22jn} = \frac{\pi}{2aDk_T^2}\, jn \sum_{m=-\infty}^{\infty} \frac{1}{(a\varsigma_m^{L,T})^2} J_j(a\varsigma_m^{L,T}) J_n(a\varsigma_m^{L,T}) M_2(a\varsigma_m^{L,T}), \tag{30}$$

$$M_1(t) = \frac{[(ak_T)^2 - 2t^2]^2}{\sqrt{(ak_T)^2 - t^2}} + 4t^2\sqrt{(ak_L)^2 - t^2},$$

$$M_2(t) = \frac{[(ak_T)^2 - 2t^2]^2}{\sqrt{(ak_L)^2 - t^2}} + 4t^2\sqrt{(ak_T)^2 - t^2}, \tag{31}$$

$$\varsigma_m^{L,T} = k_{L,T}\cos\theta + \frac{2\pi}{D} m. \tag{32}$$

RESULTS AND DISCUSSION

In Figs. 2-5, reflection coefficients and transmission coefficients for the incident longitudinal and transverse waves are shown for both normal incidence and oblique incidence. In Fig. 2, RPP stands for the absolute value of the reflection coefficient of an incident P-wave for reflected longitudinal wave. Similarly, in Fig. 4, TPS stands for the absolute value of the transmission coefficient of an incident P-wave for transmitted transverse wave. It has been found that the present results coincides with Angel and Achenbach [6,7]. It should be, however, emphasized here that, while Angel and Achenbach treated normal incidence [6], and oblique incidence [7] separately, our results have been obtained in a unified manner for both normal incidence and oblique incidence as has been shown in the previous section.

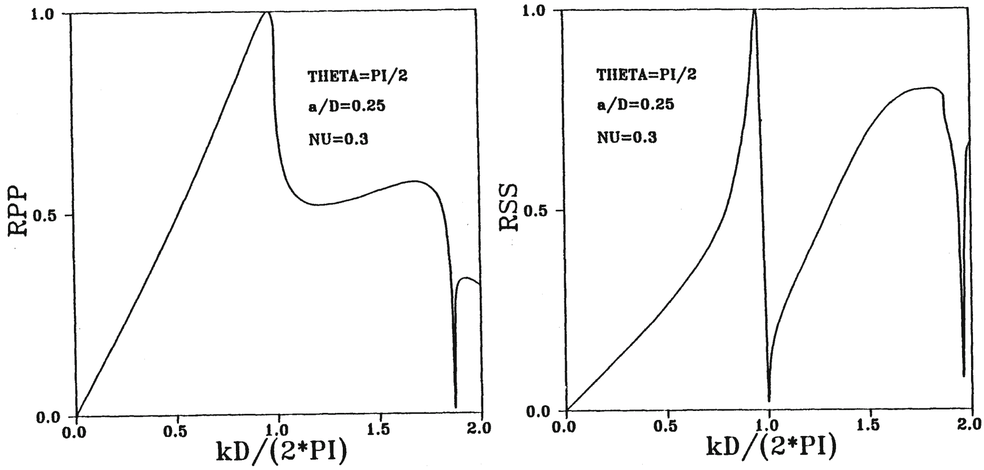

Fig. 2. Absolute values of reflection coefficients of normally incident longitudinal and transverse waves v.s. $k_T D/2\pi$ for $a/D = 0.25$, $\nu = 0.3$.

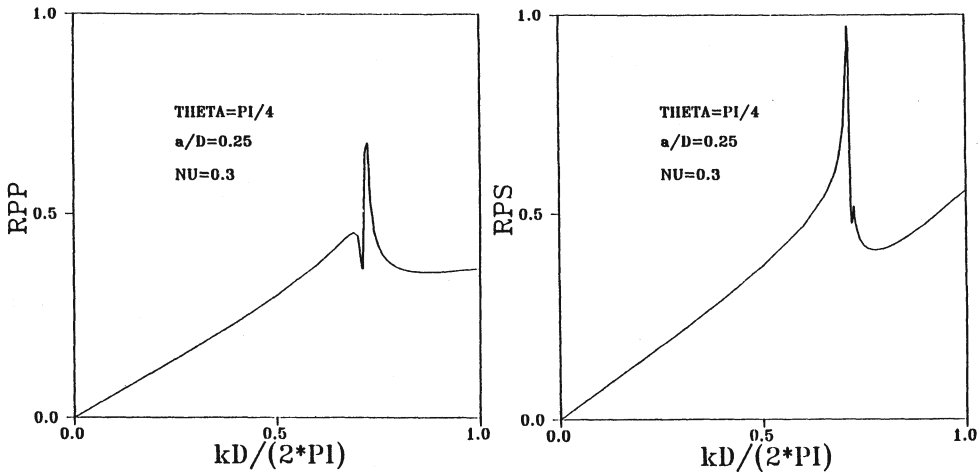

Fig. 3. Absolute values of reflection coefficients (RPP and RPS) of an obliquely incident longitudinal wave v.s. $k_T D/2\pi$ for $\theta = \pi/4$, $a/D = 0.25$, $\nu = 0.3$.

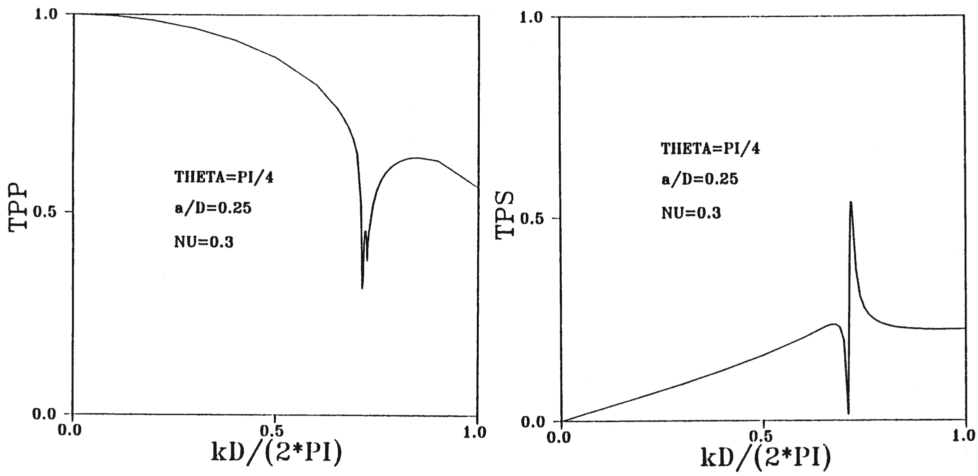

Fig. 4. Absolute values of transmission coefficients (TPP and TPS) of an obliquely incident longitudinal wave v.s. $k_T D/2\pi$ for $\theta = \pi/4$, $a/D = 0.25$, $\nu = 0.3$.

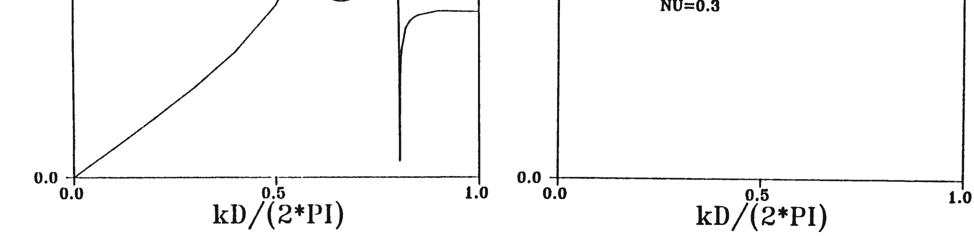

Fig. 5. Absolute values of reflection (RSS) and transmission (TSS) coefficients of an obliquely incident transverse wave v.s. $k_T D/2\pi$ for $\theta = \pi/4$, $a/D = 0.25$, $\nu = 0.3$.

CONCLUSION

By generalizing van der Hijden and Neerhoff's results [2], scattering of in-plane longitudinal and transverse waves by a periodic array of collinear cracks has been solved.

The merit of our treatment over Angel and Achenbach's results [6,7] is that (i) the formulation is relatively simple, and unified for both normal and oblique incidence, and more importantly that (ii) there is no need for numerical integrations, because all the integrations involved in the formulation have been analytically evaluated.

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