FLOQUET ANALYSIS OF LAMB WAVES PROPAGATING IN
PERIODICALLY-LAYERED COMPOSITES

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INTRODUCTION

In this paper, we present a general method for solving wave propagation problems in periodically layered composite plates. Such plates can be divided into periodic cells composed of two or more sublayers with different material parameters. Composite plates behave quite differently from homogeneous plates. Shull et al. [1] observed that guided wave dispersion curves for these plates, unlike their homogeneous counterparts, do not scale with frequency-thickness product. In a recent contribution, Auld et al. [2] presented an analysis for SH wave propagation in periodically layered plates and attributed this phenomenon to the pass-band and stop-band structure caused by the layering. The SH wave analysis is analogous to the electromagnetic case (e.g. periodic microwave filters). However, in ultrasonic practice, the more relevant case is the propagation of Lamb-type waves (polarized in the vertical plane) in periodic plates. This problem is characteristically different from the SH problem due to the coupling between quasi longitudinal and quasi shear wave components. In this paper, we will show how Floquet modes can be utilized to simplify significantly the solution to the plate wave problem with periodic structures. The Floquet modes, which are the characteristic modes for the infinite periodically layered medium, can be thought of as analogous to the longitudinal and shear modes for the infinite homogeneous isotropic medium.

Elastic waves in infinite periodic media has been studied extensively in the past [3-8]. There are two general approaches to solving this problem. One is the method of the characteristic matrix, which solves for wave components in all of the layers simultaneously. In this approach, the order of the matrix is a function of the number of layers. When the number of layers increases, the numerical burden and memory requirements can become excessively large. The other method is the transfer matrix approach in which a global transfer matrix is expressed in terms of multiplication of individual layer transfer matrices. The transfer matrix approach is advantageous since the order of the transfer matrix is independent of the number of layers. The maximum dimension necessary for a transfer matrix is six. On the other hand, the transfer matrix approach is known to have numerical instabilities in certain regimes. There have been some efforts over the past 30 years to
resolve the numerical instability problem [4, 9]. In this work, we will employ the transfer matrix approach.

In the following sections, we will present a brief review of the Floquet analysis of the infinite periodic medium. Then, a general method will be presented for solving wave propagation problems in periodic plates. Although all analysis will be presented for plates with isotropic layers, the same procedure can be applied to plates with anisotropic layer structures.

BACKGROUND THEORY

Wave propagation in isotropic elastic solids is governed by the following equations

\[ \tau_{ij,j} + \rho \omega^2 u_i = 0 \]  \hspace{1cm} (1)

\[ \tau_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \]  \hspace{1cm} (2)

where \( \tau \) is the stress tensor, \( u \) is displacement vector, \( \rho \) is the mass density, \( \lambda \) and \( \mu \) are Lamé constants, and \( \omega \) is circular frequency. In this work, we are concerned with structures that are composed of periodic layers as shown in Figure 1. The periodic nature of the structure implies that the coefficients of the governing equation of motion will be periodic. This problem can be solved using either method. However, one can simplify the solution by taking advantage of the periodic nature of the problem. The problem of wave propagation in periodic media arise in many disciplines including quantum mechanics, geophysics, and microwave engineering. In the area of ultrasonic wave propagation, extensive work has been carried out for both isotropic and anisotropic materials. Most of this work is focused on infinite periodic structures. In this paper, we focus on analyzing guided wave propagation in periodically layered plates. We present a general method for solving plate problems by extending Lamb waves from a homogeneous plate to plates with periodic structuring. We limit, however, the layer material to isotropic media and consider wave propagation with in-plane motion.

The relevant displacements and stresses are \( u_1, u_3, \tau_{33} \) and \( \tau_{13} \). Let us define a vector \( F \) containing these variables

\[ F = [ u_1 \hspace{1cm} u_3 \hspace{1cm} \tau_{33} \hspace{1cm} \tau_{13} ] \]. \hspace{1cm} (3)

To solve this problem, we choose the transfer matrix approach. The transfer matrix relating the variable vector \( F^+ \) at the top layer to \( F^- \) at the bottom layer of a homogeneous plate
may be expressed as

\[ F^+ = T F^- , \]  

where, for an isotropic plate with thickness \( d \), \( T \) is given by

\[ T = H^{-1} E H , \]  

where

\[
H = \begin{bmatrix}
ik_1 & ik_1 & -ik_3^T & ik_3^T \\
-ik_1^T & ik_1^T & -ik_3 & ik_3 \\
-\mu(k_1^2 - 2k_3^2) & \mu(k_1^2 - 2k_3^2) & -2\mu k_1 k_3^T & 2\mu k_1 k_3 \\
-2\mu k_1 k_3^T & 2\mu k_1 k_3 & \mu(k_1^2 - 2k_3^2) & -\mu(k_1^2 - 2k_3^2)
\end{bmatrix}
\]  

and

\[ E = \text{Diag} \left[ e^{ik_1^T d}, e^{-ik_1^T d}, e^{ik_3^T d}, e^{-ik_3^T d} \right] . \]  

Above, \( k_1 \) is the component of the wavenumber \( k_\alpha \) in the \( x_1 \) direction, \( k_\alpha^2 = k_\alpha^2 - k_\alpha^2 \) for \( \alpha = \omega/c_\alpha \), \( c_\alpha \) is the velocity and \( \alpha = l, t \) stands for longitudinal and shear components, respectively. The equivalent transfer function for \( m \) layers is given by

\[ T = \prod_{i=1}^{m} T_i , \]  

where \( T_i \)’s are the transfer functions for the individual homogeneous layers. Now, if a medium is composed of an infinite number of these cells, then according to the Floquet theorem, the solution \( F \) can be expressed as

\[ F(x_3) = e^{ik_3 p} S(x_3) , \]  

where \( S(x_3 + p) = S(x_3) \). For the case where the cell is composed of \( m \) layers each with thickness \( d_i \) and \( d = \sum_{i=1}^{m} d_i \) then

\[
\begin{align*}
F(d) &= e^{ikd} F(0) \\
TF(0) &= TF(0)
\end{align*}
\]  

From Eqs (10) and (11) it is clear that \( e^{ikd} \) is an eigenvalue of the cell transfer matrix \( T \). Generally these eigenvalues are calculated numerically. Consequently, eigenvectors of the transfer matrix \( T \) may also be found. Furthermore, since \( T \) is \( k \)-unitary, it can be expressed as

\[ T = H^{-1} E H \]  

where \( H \) is the orthonormal matrix containing eigenvectors of \( T \), \( E \) is a diagonal matrix containing eigenvalues of \( T \), and \( H^{-1} = H^\dagger K \), where \( \dagger \) indicates a Hermitian conjugate operation, and matrix \( K \) is given by

\[ K = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} . \]  

Matrix \( K \), which is also referred to as a metric, has a physical significance. It is defined such that the norm of the vector is a meaningful physical quantity. The norm of the vector \( F \) is expressed as

\[ < F, F > = F^\dagger K F = (\tau_{31}^* u_1 + \tau_{33}^* u_3 + \tau_{31} u_1^* + \tau_{33} u_3^*) . \]  

Hence for this case, power is the quantity which is associated with the norm of the vector and is conserved. More detail on the topic of metrics can be found in \([10]\). Eigenvalues of \( T \)
can be found by solving the quadratic secular equation required for existence of roots of the homogeneous system

\[
\det[T - xI] = 0. \tag{15}
\]

Matrix \( T \) has symmetric invariants \( I_k \), i.e. \( I_0 = I_4 = 1 \) and \( I_3 = I_1 \). This property yields a characteristic polynomial of the form

\[
(x^2 + x^{-2}) - I_1(x + x^{-1}) + I_2 = 0. \tag{16}
\]

As a consequence, if \( x = e^{ikl} \) is an eigenvalue then \( 1/x = e^{-ikl} \) is also an eigenvalue resulting in upward and downward propagating wave components with wavenumbers \( \pm k \). These wave components can be calculated by solving the quadratic equation found by substituting \( y = 2 \cos kl = x + 1/x \).

\[
y^2 - I_1 y + I_2 - 2 = 0. \tag{17}
\]

The two wave components may be expressed as

\[
y_1/2 = \cos k_1 l = \frac{I_1 + \sqrt{I_1^2 - 4I_2 + 8}}{4}, \tag{18}
\]

\[
y_2/2 = \cos k_2 l = \frac{I_1 - \sqrt{I_1^2 - 4I_2 + 8}}{4}, \tag{19}
\]

where \( k_1 \) and \( k_2 \) are the wavenumbers for the Floquet wave that can freely propagate in an infinite periodic medium composed of cells all having transfer function \( T \). These components are similar to \( k_{13}^p \) and \( k_{13}^q \) wavenumbers for infinite homogeneous medium. One can view the homogeneous medium as a special case of the periodic structure, where all layers in the cell have the same material properties. The edges of the Brillouin zone correspond to the \((\omega, k_1)\) points that yield \( \cos k_1 l = \pm 1, \) \( k_1 = m\pi/p \) where \( m = 0, 1, 2, \ldots \).

For the SH case, these edges of the Brillouin zone satisfy the rigid and traction-free boundary conditions at the middle of the sublayers [2]. However, due to the mode coupling in the Lamb case, no such simple relationship exists.

### PLATES WITH PERIODIC STRUCTURE

In this section, we will present a general method for solving the boundary condition equations for a plate composed of periodic layers. The major task is to obtain a transfer function for the plate. As shown in the previous section the plate transfer function containing \( n \) cells can be written as

\[
T = H^{-1}E^n H. \tag{20}
\]

Eq. 20 shows the main advantage in using the Floquet modes to solve the layered plate problem. The transfer function for the \( n \)-cell plate is gotten, by inspection, from the single-cell transfer function. Once the \( n \)-cell transfer function is known the problem can be solved simply. Our approach is similar to the one used to derive the solution given for Lamb waves in a homogeneous plate. The boundary conditions for the internal layers are satisfied automatically by the transfer matrix \( T \). The additional boundary conditions on the top and the bottom layers are prescribed separately. Generally, the boundary conditions are specified in terms of a linear combination of displacements and stresses. However, most common boundary conditions are either rigid or traction-free boundary conditions. For example, for the traction-free case, the transfer matrix \( T \) can be divided into submatrices \( T_{11}, T_{12}, T_{21}, \) and \( T_{22}, \) and vector of variables \( F \) can be written in terms of subvectors \( U \) and \( \Gamma \).

\[
\begin{bmatrix}
U^+ \\
\Gamma^+
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} \begin{bmatrix}
U^- \\
\Gamma^-
\end{bmatrix} \tag{21}
\]
Figure 2: Dispersion curves for a plate containing 1 layer of aluminium and 1 layer of Kevlar-epoxy (1/1 plate).

The traction-free boundary condition on the top and bottom surface implies that \( \Gamma^+ = \Gamma^- = 0 \) resulting in

\[
U^+ = T_{11} U^- \quad \text{(22)}
\]

\[
0 = T_{21} U^- \quad \text{(23)}
\]

Eq. (23) defines the dispersion relation for propagating waves in the plate, i.e.

\[
\det[T_{21}] = 0 \quad \text{(24)}
\]

Figure 2 shows the dispersion curves for a plate composed of a single cell. This case involves a unit cell with 2 layers. The layers are made of aluminum and another isotropic material with parameters close to that of Kevlar-epoxy with the plate wavevector along the fiber direction. The material properties are collected in [11]. The dispersion curves obtained from the Floquet method are compared with the results from a characteristics matrix method. As expected, both methods produce the same dispersion curve. These curves specify the frequency-angle coordinates of propagating guided modes. The main difference between these dispersion curves and that of the homogeneous layer is the bend in the curve at phase velocity close to the shear wave velocity. This feature is present for all modes. Each mode seems to approach an asymptote at the mixture shear velocity but as frequency increases, it drops to the shear velocity of the slower medium. Figures 3 shows dispersion curves for the two-cell plate. Figure 4 shows dispersion curves for both 1/1 and 2/2 plates. It can be seen that some of the modes in the 2/2 case are the same modes that propagate in the 1/1 case. For the 2/2 case, as shown in Figure 3, it can be seen that the dispersion curves fill only parts of the velocity-frequency space and other parts are completely empty. These zones are well known and are referred to as the pass-band and stop-band regions. These regions can be seen more clearly in Figure 5 where dispersion curves from cases 1/1, 2/2, and 3/3 are superimposed. As the number of cells increases the density of the dispersion curves in the pass bands increases to the point where the pass-band may be considered as a continuum.

MEDIA CONTAINING PERIODIC STRUCTURES

In most applications, the structure of interest is not itself periodic but contains periodic structures. Figure 6 shows a general case where a periodic medium is embedded in a
Figure 3: Dispersion curves for a plate containing 2 layers of aluminium and 2 layers of Kevlar-epoxy (2/2 plate).

Figure 4: Superposition of dispersion curves for a 1/1 and 2/2 plates indicating similar behavior for some modes.
nonperiodic structure. In this case, the overall transfer function $T$ can be written as

$$T = T_I T_{II} T_{III} = H_I^{-1} E_I H_I H_I^{-1} E_N H H_{II}^{-1} E_{III} H_{III}.$$ \hspace{1cm} (25)

An example of such structures are composite plates in which both top and bottom layers are of the same material, i.e. the plate is $(N + 1)/N$. Although such plates are not periodic, the periodic solution can be utilized effectively to simplify the transfer function. For this case, the overall transfer function may be written as

$$T = T_I T_{II} = H_I^{-1} E_I H_I H_I^{-1} E_N H,$$ \hspace{1cm} (26)

where $T_I$ is the transfer functions for the top layer and $T_{II}$ is the transfer function for the remaining layers which is periodic.

SUMMARY AND CONCLUSION

In this paper, we have presented a general scheme to solve the wave propagation problem for a periodically layered plate. To do this, we have utilized Floquet modes which are free propagating waves in the infinite periodic structure. It is shown that by utilizing the Floquet modes, the $n$-cell transfer function is obtained from the single-cell transfer function with no additional effort. We have compared the solutions obtained with the Floquet method with those gotten from the characteristic equation. We have shown that these two methods yield identical results. The main difference is that the Floquet approach takes advantage of the periodicity of the structure, resulting in a simplified form for the transfer function. The simpler form also provides an opportunity to develop schemes that are numerically stable.
REFERENCES

11. Layer 1: $c_l = 6.31 \text{ km/s}$, $c_t = 3.14 \text{ km/s}$ and $\rho = 2.72 \text{ gr/cm}^3$; layer 2: $c_l = 3.4 \text{ km/s}$, $c_t = 1.2 \text{ km/s}$ and $\rho = 1.6 \text{ gr/cm}^3$. 

Figure 6: Non-periodic plates containing periodic structures (for example periodic structures immersed in fluid).