A technique for the well conditioning of the systems resulting from tree-cotree decomposition

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A technique for the well conditioning of the systems resulting from tree-cotree decomposition

by

Subramanian N. Lalgudi

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE

Major: Electrical Engineering

Program of Study Committee:
Shanker Balasubramaniam, Major Professor
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Iowa State University
Ames, Iowa
2003
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This is to certify that the master's thesis of
Subramanian N Lalgudi
has met the thesis requirements of Iowa State University

Signatures have been redacted for privacy
# TABLE OF CONTENTS

LIST OF TABLES ........................................ iv
LIST OF FIGURES ........................................ v
ABSTRACT ................................................ viii

CHAPTER 1. INTRODUCTION .............................. 1
  1.1 Numerical methods for 3D magnetostatic and eddy current problems ........................ 1
  1.2 Choice of basis functions for representing the unknown ........................................ 2
  1.3 Singular systems and tree-cotree decomposition ..................................................... 2
  1.4 Eddy current problem using IE method ................................................................. 4
  1.5 Organization ........................................ 5

CHAPTER 2. INTEGRAL EQUATION FORMULATION ....... 6
  2.1 Problem Statement .................................. 6
  2.2 Formulation of the problem ......................... 7
    2.2.1 SVD method ..................................... 10
    2.2.2 Tree-cotree method ............................. 11

CHAPTER 3. PERFORMANCE STUDY OF THE PROPOSED METHODS 14

CHAPTER 4. CONCLUSION .................................. 34

BIBLIOGRAPHY ............................................. 35

ACKNOWLEDGEMENTS ...................................... 40
### LIST OF TABLES

| Table 3.1 | Condition number of the interaction matrix from SVD and Tree-cotree methods for a conducting sphere of radius 1cm and conductivity of $3.1663 \times 10^6$ mho/m | 16 |
| Table 3.2 | Condition number of the interaction matrix from SVD and Tree-cotree methods for a conducting cube of 2cm and conductivity of $3.1663 \times 10^6$ mho/m | 16 |
| Table 3.3 | Divergence of solenoidal basis functions from tree-cotree method + Gram-Schmidt method and SVD method for a conducting sphere of radius 1cm | 17 |
| Table 3.4 | Divergence of solenoidal basis functions from tree-cotree method + Gram-Schmidt method and SVD method for a conducting cube of size 2cm | 17 |
| Table 3.5 | Condition number of the interaction matrix from SVD, Tree-cotree method, and Tree-cotree method with Gram-Schmidt Orthonormalization for a conducting sphere of radius 1cm and conductivity of $3.1663 \times 10^6$ mho/m | 18 |
| Table 3.6 | Condition number of the interaction matrix from SVD, Tree-cotree method, and Tree-cotree method with Gram-Schmidt Orthonormalization for a conducting cube of 2cm and conductivity of $3.1663 \times 10^6$ mho/m | 18 |
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>A homogeneous body immersed in an electric field.</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>Geometrical Parameters associated with the $f^{th}$ face.</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>Solenoidal basis function $f_e(r)$ as a linear combination of RWG basis functions.</td>
<td>12</td>
</tr>
<tr>
<td>3.1</td>
<td>Conducting sphere in a uniform magnetic field $B_z(r) = \hat{a}_z$.</td>
<td>20</td>
</tr>
<tr>
<td>3.2</td>
<td>Induced magnetic flux density measured outside the sphere (radius: 1 cm) at $r = 1.5$ cm in the $XZ$ plane from SVD method and tree-cotree method.</td>
<td>21</td>
</tr>
<tr>
<td>3.3</td>
<td>Total current density measured inside the sphere (radius: 1 cm) at $r = 0.75$ cm in the $XZ$ plane from SVD method and tree-cotree method.</td>
<td>22</td>
</tr>
<tr>
<td>3.4</td>
<td>Conducting spherical shell in a uniform magnetic field $B_z(r) = \hat{a}_z$.</td>
<td>22</td>
</tr>
<tr>
<td>3.5</td>
<td>Induced magnetic flux density measured outside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 4.0$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.</td>
<td>23</td>
</tr>
<tr>
<td>3.6</td>
<td>Total current density measured inside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 1.8725$ cm and $1.125$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.</td>
<td>24</td>
</tr>
<tr>
<td>3.7</td>
<td>Induced magnetic flux density measured outside the spherical shell at $r = 7.5$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.</td>
<td>25</td>
</tr>
<tr>
<td>3.8</td>
<td>Total current density measured inside the spherical shell at $r = 5.0625$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.</td>
<td>26</td>
</tr>
</tbody>
</table>
Figure 3.9 Condition number of the system matrix versus the choice of the tree from tree-cotree method, a) Sphere of radius 1 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 341 tetrahedrons, b) Cube of 2 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 361 tetrahedrons.

Figure 3.10 Condition number of the system matrix versus the choice of the tree from tree-cotree + Gram-Schmidt method. a) Sphere of radius 1 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 341 tetrahedrons, b) Cube of size 2 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 361 tetrahedrons.

Figure 3.11 Condition number of the system matrix versus the choice of the tree from a) tree-cotree method, b) tree-cotree + Gram-Schmidt method. Sphere of radius 1 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 1445 tetrahedrons.

Figure 3.12 Induced magnetic flux density measured outside the sphere (radius: 1 cm) at $r = 1.5$ cm in the XZ plane from tree-cotree + Gram-Schmidt method and tree-cotree method.

Figure 3.13 Total current density measured inside the sphere (radius: 1 cm) at $r = 0.75$ cm in the XZ plane from tree-cotree + Gram-Schmidt method and tree-cotree method.

Figure 3.14 Induced magnetic flux density measured outside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 4.0$ cm in the XZ plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.

Figure 3.15 Total current density measured inside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 1.8725$ cm and 1.125 cm in the XZ plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.
Figure 3.16  Induced magnetic flux density measured outside the spherical shell at 
$r = 7.5$ cm in the $XZ$ plane from the tree-cotree + Gram-Schmidt 
method and the tree-cotree method .......................... 32

Figure 3.17  Total current density measured inside the spherical shell at $r = 5.0625$ 
cm in the $XZ$ plane from the tree-cotree + Gram-Schmidt method and 
the tree-cotree method ............................................. 33
ABSTRACT

A popular integral equation formulation for eddy current problem is considered. In this formulation, an electric field integral equation is used and is casted in terms of the unknown current density. This current density is solenoidal in the quasistatic limit and is expressed in terms of solenoidal basis functions. Each of this basis function is obtained as the curl of the edge element shape function and is associated with each edge of the graph obtained by removing the edges of the tree. The resulting system is found to be ill-conditioned and the condition number of it is found to be dependent on the choice of the tree.

In this thesis, these two problems are studied and some approaches to solve them are proposed. In the proposed approaches, the solenoidal basis functions are obtained differently as a linear combination of non-solenoidal facet shape functions. The combination weights are computed in two different ways. In the first, they are computed numerically by finding the null space of the discretized divergence matrix through a singular value decomposition of this matrix. In the second, they are computed analytically using the tree-cotree decomposition of the mesh. It is shown that the basis functions from the second approach and from the popular formulation are the same. Then the condition number of the system from two approaches are compared. In the first, it is very low and the system is well-conditioned. While in the second, it is high and also depends on the choice of the tree. Next, the reason for this difference is analyzed. Based on the analysis, a technique is proposed to convert systems resulting from the second approach to systems which are well conditioned and whose condition number is independent of the choice of the tree. The accuracy of the different approaches is compared against some analytical results and a benchmark result. It is also shown that the new representation for the basis functions may help in developing a solution without any quasistatic assumptions.
1.1 Numerical methods for 3D magnetostatic and eddy current problems

The determination of currents or fields produced in magnetic or conducting bodies by a time-varying magnetic field is essential in the detection and characterization of cracks in nuclear reactors. This problem for a magnetic body at static frequencies is called a magnetostatic problem and for a conducting body at quasistatic frequencies it is called an eddy current problem. Numerical solution for the problem is essential in building robust systems and is based on the solution of stationary or quasi-stationary Maxwell's equations.

Any numerical solution to the problem involves modeling the problem domain (can be body itself or may also contain some region of freespace) by simple elements like cubes or tetrahedrons, assigning values for conductivity or permeability to these elements, modeling the unknown quantity (say eddy currents in conducting bodies) using suitable basis functions which satisfies the unknown's properties, choosing an equation describing the problem in terms of the unknown, testing this equation with suitable weighting functions, and solving the resultant system of equations to obtain the unknown quantity of interest.

The numerical solution can be obtained using two different methods. The first method is called the differential equation method e.g. finite element method (FEM). In this method, the problem domain consists of both the body and some portion of free space. The freespace is included to impose the farfield boundary condition so that fields decay as a function of distance from the body, resulting in zero field value at infinity. A differential equation describing the body is solved for the unknown fields in the whole problem domain. This method results in sparse matrices. The problems with this method are that: 1) the problem size is increased because of meshing even air regions; 2) the system may be inherently ill conditioned due
to huge variation in the material properties of the problem domain. Meshing the air region can be eliminated if radiation boundary condition for the fields is ensured explicitly through techniques like perfectly matched layers. The second method is the integral equation method (IE). IE methods treat the farfield boundary condition automatically through the Green’s function and hence only non-air regions need to be meshed. In this method, an integral equation describing the problem is solved for sources in the body. These sources are then used to find fields anywhere. This method results in dense matrices. The problem with this method are the cost involved in computing, storing the matrix elements and the cost involved in finding the solution to the resultant matrix equation.

1.2 Choice of basis functions for representing the unknown

Crucial in both the methods discussed thusfar is the choice of basis functions used to represent the unknown. These basis functions can be scalars or vectors. Scalar basis functions makes the unknown fully continuous across a material discontinuity. This makes them unsuitable for modeling vector unknowns which require only tangential continuity (ex: electric field) or unknowns which require only normal continuity (ex: electric and magnetic flux density). On the other hand, vector basis functions ensuring these properties can be chosen. Thus these basis functions can be used to model vector unknowns. Each such basis function is usually associated with an edge or a face of the tetrahedron.

1.3 Singular systems and tree-cotree decomposition

In magnetostatic and eddy current problems, the unknowns like magnetic flux density (B) or current density (J) are solenoidal (i.e., \( \nabla \cdot B = 0 \) for B and \( \nabla \cdot J = 0 \) for J). This is easily accomplished in the solution by obtaining these unknowns as the curl of a vector potential (i.e., \( B = \nabla \times A \), \( J = \nabla \times T \)). These vector potentials A and T are then modeled as a linear combination of suitable basis functions which satisfy their properties. The unknown degrees of freedom in the vector potential’s representation are then solved. The unknown B or J is then obtained using these degrees of freedom. For example in the magnetostatic problem [1] in terms
of unknown \( J = \nabla \times T \), \( T \) is represented using edge element shape functions \[2\]. These basis functions are associated with the edges and ensure tangential continuity of the unknown across a material discontinuity. And each degree of freedom is then the line integral of the unknown along the corresponding edge. Each basis of \( J \), obtained as curl of \( T \), is also associated with an edge and is obtained as the curl of the corresponding edge element shape function. This basis for \( J \) ensures its normal continuity across an interface. Though \( J \) is defined uniquely, the vector potential is not since the addition of the gradient of any scalar function to it does not modify its curl. The degrees of freedom do not form an independent set and make the resulting system singular. Then only specific type of iterative solvers like conjugate gradient method can be used to solve this system. Alternatively, the independent degrees of freedom for \( J \) can be obtained by assigning arbitrary values to the degrees of freedom corresponding to the edges of a tree spanned by the finite element mesh. The most straightforward choice is to assign zero value to the tree edges \[3\]. Unique \( T \) and non-singular system are obtained by solving for only the edges in the cotree. This procedure to find the independent degrees of freedom is called the tree-cotree decomposition. Though this decomposition results in non-singular systems, condition number of the system is dependent on the choice of the tree. Thus the resulting system will be ill-conditioned if a random tree is chosen as reported in \[1, 4, 5, 6, 7\].

The conditioning is improved in \[6, 7\] by selecting an optimal tree ensuring this. It is very difficult to optimize the choice of the tree to arrive at numerical stability.

In this thesis, we derive a technique to convert systems resulting from tree-cotree decomposition to systems which are well conditioned and whose condition number is independent of the choice of the tree. Pursuant to this objective, we consider an existing integral formulation for the eddy current problem in terms of unknown current density \( J \) using the tree-cotree decomposition. We then obtain the solenoidal basis functions for the unknown \( J \) numerically by finding the null space of the divergence operator via a singular value decomposition of this operator. We then solve the problem using these new basis functions. We show that the conditioning of the system differs greatly between the two approaches. We then identify the possible reason for this difference and propose a new technique to convert systems resulting
from tree-cotree decomposition to systems which are well conditioned and whose conditioning is independent of the choice of the tree.

1.4 Eddy current problem using IE method

IE methods to 3D eddy current problems in conducting media are discussed in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In [15], the electric field integral equation (EFIE) characterizing the body is cast in terms of the unknown conduction current density. The current density is approximated as a linear combination of solenoidal basis functions. Each basis function is obtained as the curl of the edge element shape function corresponding to an edge in the cotree joining all the nodes.

In our present work, the solenoidal basis functions are obtained as a linear combination of non-solenoidal shape functions associated with the face of the tetrahedron [18]. The reason for this new representation is that we think the subsequent formulation will later facilitate us in developing a solution to the 3D eddy current problem without any quasistatic approximations like $\nabla \cdot \mathbf{J} = 0$. This new solution is already proposed for homogeneous conducting bodies through a surface formulation in [19]. The problem will then be solved through a loop-star decomposition of the 3D current density similar to its surface counterpart discussed in detail in [20, 21, 22, 23, 24, 25, 26, 27]. Two different approaches to find the solenoidal basis functions are discussed. In both the approaches, the weights of the combination resulting in a solenoidal basis are determined. This is done by computing the null space of the divergence operator.

In the first approach, this null space is found numerically through a singular value decomposition of the discretized divergence matrix. In the second, it is found analytically from the face to cotree edge incidence matrix [28, 29] through a tree-cotree decomposition identical to [15]. The resulting basis function is same as that in [15]. It is to be noted that the linearly independent columns of the null space are also mutually orthogonal in the first approach while they are not in the second.

We find that the condition number of the system matrix constructed from these approaches differ significantly. In the first approach, the condition number is very low resulting in well
conditioned systems. In the second, it is very high resulting in ill conditioned ones.

We also find that transforming the linearly independent columns of the null space from the second approach to a mutually orthogonal set using Gram-Schmidt orthonormalization makes the system well conditioned. The resulting condition number is same as that obtained in the first approach. We use this new procedure (tree-cotree decomposition + Gram-Schmidt orthonormalization) as the third approach to obtain the solenoidal basis. The major shortcoming in the new approach is the cost of performing the Gram-Schmidt orthonormalization. Since the face to cotree edge incidence matrix is very sparse and can be structured, the cost of performing Gram-Schmidt orthonormalization on it can be reduced at least by an order of magnitude. Computing the basis in this way eliminates the cost of finding SVD in the first approach and the ill conditioning encountered in the second.

In the rest of the text, we denote solving the eddy current problem using the first approach as the SVD method, using the second as the tree-cotree method, using the third as the tree-cotree + Gram-Schmidt method.

1.5 Organization

The rest of the thesis is organized as follows: Chapter 2 outlines the volume integral equation formulation for the eddy current problem using the SVD method and the tree-cotree method. Chapter 3 presents the results validating the accuracy of our different methods, the results comparing the condition numbers from different methods. Finally, chapter 4 summarizes the contribution of the thesis.
CHAPTER 2. INTEGRAL EQUATION FORMULATION

In this chapter, the integral equation formulation for the eddy current problem using the SVD and the tree-cotree methods is explained.

The electric field integral equation (EFIE) describing the non-magnetic conducting body is cast in terms of the unknown current density. This current is due to both conduction and polarization in the body. For bodies with high conductivities in quasistatic limit, the magnitude of polarization current is much smaller compared to the conduction current and hence is neglected. Also, in this limit, the conduction current density is assumed to be solenoidal. This assumption does not affect its accuracy much. The current density is then modeled as a linear combination of solenoidal basis functions. These basis functions are in turn expressed as a combination of non-solenoidal facet shape functions. The combination weights resulting in a solenoidal basis are found both numerically and analytically. The EFIE is discretized using Method of Moments (MoM) with Galerkin testing. The resulting linear system of equations is solved using an iterative solver.

The rest of the chapter is organized as follows. In the first part, the eddy current problem using the IE method is stated. In the next part, the formulation of the problem is described. Also, the procedures to find the solenoidal basis numerically and analytically are explained.

2.1 Problem Statement

The eddy current problem in a non-magnetic conducting region $\Omega$ of volume $V$ (see Figure 2.1) can be described by the EFIE

$$\frac{J(r)}{\sigma(r)} + j\omega A(r) + \nabla \phi(r) = E_i(r)$$

(2.1)
Figure 2.1 A homogeneous body immersed in an electric field.

\[ \mathbf{A}(\mathbf{r}) = \frac{j\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{v}' \]

\[ \phi(\mathbf{r}) = -\frac{1}{j\omega 4\pi \varepsilon_0} \int_{V'} \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{v}' \]  

Also

\[ \mathbf{J}(\mathbf{r}) \in \gamma \]  

where \( \mathbf{r} \) and \( \mathbf{r}' \) are the position vectors of the observation and source points respectively, \( V' \) is the source volume in \( \Omega \), \( \omega \) is the angular frequency, \( \mathbf{J}(\mathbf{r}) \) is the conduction current density, \( \sigma(\mathbf{r}) \) is the conductivity, \( \mathbf{A}(\mathbf{r}) \) is the magnetic vector potential, \( \phi(\mathbf{r}) \) is the electric scalar potential, \( \mathbf{E}_i(\mathbf{r}) \) is the externally applied electric field, and \( \gamma \) is the set of the vector fields \( \nu \) defined in \( \Omega \) which satisfy the conditions:

\[ \nabla \cdot \nu = 0 \quad \text{in } \Omega \]

\[ \mathbf{n} \cdot \nu = 0 \quad \text{on } \partial \Omega \]

The objective is to solve for \( \mathbf{J}(\mathbf{r}) \) in (2.1) satisfying (2.4).

2.2 Formulation of the problem

The EFIE is discretized using MoM [30] and resultant system of equations is solved to obtain \( \mathbf{J}(\mathbf{r}) \). For this, the unknown \( \mathbf{J}(\mathbf{r}) \) is expressed as

\[ \mathbf{J}(\mathbf{r}) = \sum_{e=1}^{N_{\text{basis}}} I_e \mathbf{f}_e(\mathbf{r}) \]
where \( N_{\text{basis}} \) is the total number of independent basis functions and \( I_e \) is the unknown coefficient corresponding to the \( e \)th basis function \( f_e(r) \). The condition (2.4) is satisfied if

\[
f_e(r) \in \gamma
\]  

(2.8)

Testing (2.1) with weighting (or testing) basis \( w_k(r) \), we obtain

\[
< w_k(r), \frac{J(r)}{\sigma(r)} > + j\omega < w_k(r), A(r) > + < w_k(r), \nabla \phi(r) > = < w_k(r), E_i(r) > \tag{2.9}
\]

where

\[
< f(r), g(r) > = \int_V f(r) \cdot g(r) \, dv
\]  

(2.10)

and \( V \) is the test volume in \( \Omega \). The inner product \( < w_k(r), \nabla \phi(r) > \) can be expanded as

\[
< w_k(r), \nabla \phi(r) > = \int_V \nabla \cdot (\phi(r)w_k(r)) \, dv - \int_V \phi(r) \nabla \cdot w_k(r) \, dv
\]  

(2.11)

where \( S \) is the surface bounding \( V \). The computation of \( < w_k(r), \nabla \phi(r) > \) is not required if \( w_k(r) \cdot \hat{n} \) is continuous across \( S \) and \( \nabla \cdot w_k(r) = 0 \) for \( r \in \Omega \). We will show later that these are the properties of the source basis function \( f_e(r) \). Hence, while testing (2.1), we use \( w_k = f_k \).

Substituting (2.7) in (2.1), and testing the resultant equation with weighting functions \( f_k \), a linear system of equations is obtained. This system can be expressed in matrix form as

\[
Z I = V
\]  

(2.12)

where \( Z \) is a \( (N_{\text{basis}} \times N_{\text{basis}}) \) system (or interaction) matrix, \( I \) is the vector \( (N_{\text{basis}} \times 1) \) of unknown coefficients \( I_e \) and \( V \) is the vector \( (N_{\text{basis}} \times 1) \) due to the incident field. Each element of \( Z \) and \( V \) is

\[
Z_{ke} = < f_k(r), \frac{f_e(r)}{\sigma(r)} > + j\omega < f_k(r), A_e(r) >
\]  

(2.13)

\[
V_k = < f_k(r), E_i(r) >
\]

where

\[
A_e(r) = \frac{\mu_0}{4\pi} \int_{V'} \frac{f_e(r')}{|r - r'|} \, dv'
\]  

(2.14)

The matrix equation (2.12) is then solved using an iterative solver like transpose free quasi minimal residual [31] to obtain \( I \). Then, all the quantities of interest like current density \( J(r) \) from (2.7) and magnetic flux density \( (\nabla \times A(r)) \) from (2.2) can be found.
The region $\Omega$ is assumed to be meshed with tetrahedrons. We define the following: $V$, $E$, $F$, and $T$ denote the total number of nodes, edges, faces, and tetrahedrons in the mesh, respectively; $V_b$, $E_b$, and $F_b$ denote the number of boundary nodes, edges and faces in the mesh, respectively; and $V_{nb}$, $E_{nb}$, and $F_{nb}$ denote the number of non-boundary nodes, edges and faces in the mesh, respectively.

Each basis $f_e(r)$ is expressed as

$$f_e(r) = \sum_{f=1}^{F} \alpha_{fe} \mathbf{R}_f(r)$$

where $\mathbf{R}_f(r)$ is the facet shape function associated with face $f$ (see Figure 2.2) and is defined [18] as

$$\mathbf{R}_f(r) = \begin{cases} \frac{a_f}{3V_f^+} \rho_f^+ & r \in T_f^+ \\ \frac{a_f}{3V_f^-} \rho_f^- & r \in T_f^- \\ 0 & \text{elsewhere} \end{cases}$$

$$\nabla \cdot \mathbf{R}_f(r) = \begin{cases} \frac{a_f}{V_f^+} & r \in T_f^+ \\ -\frac{a_f}{V_f^-} & r \in T_f^- \\ 0 & \text{elsewhere} \end{cases}$$

where $a_f$ is the area of face $f$, $V_f^+$ and $V_f^-$ are volumes of tetrahedrons $T_f^+$ and $T_f^-$ on either side of $f$, $\rho_f^+(r) = r - r_f^+$, is the vector from the node opposite to face $f$ in $T_f^+$ to the observation point $r$ where $r \in T_f^+$, and $\rho_f^-(r) = r_f^- - r$ for $r \in T_f^-$. The boundary condition in (2.8) can be accomplished by neglecting the boundary faces in the representation of $f_e(r)$.

$$f_e(r) = \sum_{f=1}^{F_{nb}} \alpha_{fe} \mathbf{R}_f(r)$$

The solenoidality condition in (2.8) can be accomplished by computing suitable weights $\alpha_{fe}$ which ensures zero divergence of $f_e(r)$. These weights can be found either numerically or analytically. These two ways to obtain $\alpha$ are described respectively in the sections below.
2.2.1 SVD method

The weights \( \alpha_{fe} \) are found numerically by imposing the zero divergence of basis in all tetrahedrons, computing the null space of the resulting discretized divergence matrix. From (2.18) and (2.17), we have for \( f_e(r) \) in tetrahedron \( T_m \)

\[
\nabla \cdot f_e(r) = \sum_{f=1}^{F_{nb}} \alpha_{fe} \nabla \cdot R_f(r) \\
= \sum_{f=1}^{F_{nb}} \alpha_{fe} \beta_{fm} \frac{a_f}{V_m}
\]

where

\[
\beta_{fm} = \begin{cases} 
1 & \text{if } T_m = T_f^+ \\
-1 & \text{if } T_m = T_f^- \\
0 & \text{otherwise}
\end{cases}
\]

Imposing (2.19) for all \( f_e(r) \) and in all tetrahedrons, we get a matrix equation

\[
\mathcal{M} \alpha = 0
\]

(2.21)

where \( \mathcal{M} \) is a matrix of dimension \((T \times F_{nb})\), \( \alpha \) is a matrix \((F_{nb} \times F_{nb})\) of coefficients \( \alpha_{fe} \) in (2.18) and 0 is the zero matrix \((T \times F_{nb})\). An element of \( \mathcal{M} \) corresponding to tetrahedron \( T_m \) and face \( f \) is

\[
M_{mf} = \beta_{fm} \frac{a_f}{V_m}
\]

(2.22)

The matrix \( \alpha \) is the null space of \( \mathcal{M} \). And \( N_{basis} \) is the rank of \( \alpha \) (denoted by \text{Ran}(\alpha))}. This null space is found by first computing the singular value decomposition of \( \mathcal{M} \) and extracting
the $N_{\text{basis}}$ columns of the right singular matrix corresponding to the $N_{\text{basis}}$ smallest singular values. Basis $f_e(r)$ computed using $\alpha$ from (2.21) will satisfy (2.8). The rank of $\alpha$ is given by the difference between the maximum of row or column dimension of $\alpha$ and the rank of $\mathcal{M}$. This makes $N_{\text{basis}} = F_{nb} - T + 1$.

### 2.2.2 Tree-cotree method

In the previous method, the matrix $\alpha$ is obtained by numerically computing the null space of $\mathcal{M}$ through a SVD of $\mathcal{M}$. In what follows, we describe a way to avoid this numerical computation. We determine $\alpha$ analytically from the face to cotree edge incidence matrix through a tree-cotree decomposition of the mesh. This requires only finding a cotree connecting the nodes in the mesh.

The rest of the section is organized as follows. At first, the relation between the divergence and curl operators is expressed discretely using the incidence matrices between tetrahedrons and faces, faces and edges, respectively in a form similar to (2.21). Next, the basis function $f_e(r)$ is redefined so that $\alpha$ can be computed directly from the face to edge incidence matrix. Finally, $N_{\text{basis}}$ is shown to be equal to the number of edges in the cotree joining all the nodes.

1. If $D_{\text{div}}$ is the incidence matrix ($T \times F$) of tetrahedron to faces, $R_{\text{rot}}$ is the incidence matrix ($F \times E$) of faces to edges, then $\nabla \cdot \nabla \times () = 0$ can be discretely represented as

$$D_{\text{div}} R_{\text{rot}} = 0$$  \hspace{1cm} (2.23)

where $m$, $f$, and $e$ are the indices corresponding to $m$th tetrahedron, $f$th face and $e$th edge respectively. If $f = \{a, b, c\}$ where $a$, $b$ and $c$ are the vertices of face $f$, in that order, then each element of $D_{\text{div}}$ and $R_{\text{rot}}$ is

$$D_{\text{div}}(m, f) = \beta_{fm}$$  \hspace{1cm} (2.24)

$$R_{\text{rot}}(f, e) = \begin{cases} 1 & f = \{a, b, c\} \text{ and } e = \{a, b\} \\ -1 & f = \{a, b, c\} \text{ and } e = \{b, a\} \\ 0 & e \text{ does not belong to } f \end{cases}$$  \hspace{1cm} (2.25)
The representation (2.23) is same as that in (2.21). Also, the matrices $D_{\text{div}}$ and $M$ are almost the same except for the extra factor in the latter. These suggest that $\alpha$ may be obtained from $\mathcal{R}_{\text{rot}}$.

2. $\alpha$ can be computed from $\mathcal{R}_{\text{rot}}$ if $f_e(r)$ (see Figure 2.3) is redefined as

$$f_e(r) = \sum_{f=1}^{F_{\text{nb}}} R_{\text{rot}}(f,e) \frac{l_e}{a_f} R_f(r) = \nabla \times N_e(r)$$

(2.26)

where $e$ is the edge, $l_e$ its length and $N_e(r)$ its edge element shape function [2, 15]. The new $f_e(r)$ can also be computed by taking curl of the edge element shape functions and is exactly same as the basis functions used in [15]. Hence the redefined basis $f_e(r)$ satisfies (2.8).

3. If $\Omega$ is a simply connected region, then using the relation $F_{\text{nb}} = E_{\text{nb}} - V_{\text{nb}} + T - 1$,

$$N_{\text{basis}} = E_{\text{nb}} - V_{\text{nb}} = N_{\text{fincotreeedges}}$$

where $N_{\text{fincotreeedges}}$ is the number of non-boundary cotree edges in the meshed body. Now $\alpha$ is the submatrix of $\mathcal{R}_{\text{rot}}$ considering only columns that correspond to the cotree edges.
Once $\alpha$ is known, the inner products in (2.13) can be written as

$$< f_k(r), \frac{f_e(r)}{\sigma(r)} > = \sum_{j=1}^{F_{nb}} \alpha_{fk} \sum_{g=1}^{F_{nb}} \alpha_{ge} < R_f(r), \frac{R_g(r)}{\sigma(r)} >$$

$$< f_k(r), A_e(r) > = \sum_{j=1}^{F_{nb}} \alpha_{fk} \sum_{g=1}^{F_{nb}} \alpha_{ge} < R_f(r), A_g(r) > \quad (2.27)$$

$$< f_k(r), E_i(r) > = \sum_{j=1}^{F_{nb}} \alpha_{fk} < R_f(r), E_i(r) >$$

Define $Z^{face}$ as the interaction matrix ($F_{nb} \times F_{nb}$) and $V^{face}$ as the vector ($F_{nb} \times 1$) similar to $Z$ and $V$, respectively; their elements

$$Z^{face}_{fg} = < R_f(r), \frac{R_g(r)}{\sigma(r)} > + j\omega < R_f(r), A_g(r) >$$

$$A_g(r) = \frac{\mu_0}{4\pi} \int_{\Omega'} \frac{R_g(r')}{|r - r'|} \, dv' \quad (2.28)$$

$$V^{face}_f = < R_f(r), E_i(r) >$$

Then the system in (2.12) can be alternatively obtained as follows:

$$Z = \alpha^T Z^{face} \alpha$$

$$V = \alpha^T V^{face}$$

$$\alpha^T Z^{face} \alpha I = \alpha^T V^{face} \quad (2.29)$$
CHAPTER 3. PERFORMANCE STUDY OF THE PROPOSED METHODS

In this chapter, the accuracy of the solution from the SVD method, the tree-cotree method and the tree-cotree + Gram-Schmidt method is validated against analytical results of sphere and a benchmark result. The conditioning of the system in all the three methods is studied in detail. The rest of the chapter is organized as follows. At first, the accuracy of the SVD method and the tree-cotree method is validated. Next, the condition number of the system matrix from the above methods is compared. Next, the reason for the difference in the conditioning in these methods is found and is used to obtain well conditioned systems even with the tree-cotree method. Finally, the feasibility of this new tree-cotree method is compared with SVD and tree-cotree methods.

The numerical results obtained from the SVD method and the tree-cotree method are compared against the analytical results for a conducting sphere [32], a conducting spherical shell [33] and against the results of Benchmark TEAM problem 6 [34]. In the first example, a conducting sphere of radius $a = 1$ cm and conductivity $\sigma = 3.1663 \times 10^6$ mho/m is immersed in a uniform sinusoidal magnetic field of frequency $f = 50$ Hz and magnetic flux density $B_i(r) = B_0 \mathbf{\hat{a}}_z \ T$ (see Figure 3.1). The sphere was discretized using 1445 tetrahedrons. In Figure 3.2 (a-d), the induced magnetic flux density computed numerically outside the sphere from the different methods is plotted against that computed analytically. In Figure 3.3 (a-b), the total current density inside the sphere is plotted.

In the second example, a conducting spherical shell of outer, inner radii $a_1 = 2$ cm and $a_2 = 1$ cm, respectively and conductivity $\sigma = 1.2667 \times 10^7$ mho/m is immersed in a uniform magnetic field of frequency $f = 50$ Hz and magnetic flux density $B_i(r) = \mathbf{\hat{a}}_z \ T$ (see Figure 3.4).
The shell was discretized using 2176 tetrahedrons. In Figure 3.5 (a-d), the numerically and analytically computed induced magnetic flux density outside the body are plotted. In Figure 3.6 (a-b), the total current density inside the shell is plotted.

In the third example, TEAM problem 6 is solved. A conducting spherical shell of outer, inner radii $a_1 = 5.5$ cm and $a_2 = 5.0$ cm, respectively and conductivity $\sigma = 5 \times 10^8$ mho/m is immersed in a uniform magnetic field of frequency $f = 50$ Hz and magnetic flux density $B_z(r) = \hat{a}_z T$ (see Figure 3.4). The shell was discretized using 3204 tetrahedrons. In Figure 3.7, the numerically and analytically computed induced magnetic flux density outside the body are plotted. In Figure 3.8 (a-b), the total current density inside the shell is plotted. The numerical results show a good agreement with the analytical results and serve to demonstrate the validity of the SVD method and the tree-cotree method.

Though both the SVD method and the tree-cotree method produce accurate solutions, we find a significant difference in the time required to solve (2.29) between these methods. For a given accuracy, the number of iterations in the iterative solution from the SVD method is far less than the tree-cotree method. This suggests that there is a difference in the conditioning of the system between the two methods. Hence, we compared the condition number of the system matrix $Z$ in (2.29) from the two methods. For this, we found the condition number for several simple geometries. In the first example, a conducting sphere of radius 1 cm and conductivity of $3.1663 \times 10^6$ mho/m was considered. The sphere was meshed with 341 tetrahedrons. The condition number of $Z$ (denoted as $\text{cond}(Z)$) from the SVD method and tree-cotree method are 12.552 and 4308.1845, respectively. In the second example, a cube of size 2 cm and conductivity of $3.1663 \times 10^6$ mho/m was considered. It was meshed with 361 tetrahedrons. The condition numbers are 9.3347434 and 4761.969, respectively. These values show that system obtained from the SVD method is well conditioned while it is not in the tree-cotree method.

Since the $\text{cond}(Z)$ from the tree-cotree method also depends on the choice of the tree, we found them for the above examples for different choices of trees. This was done to find the swing in values $\text{cond}(Z)$ can take due to different tree choices. The different trees in the graph were enumerated sequentially following the algorithm described in [35]. In Figure 3.9 (a-b),
condition numbers for the sphere and cube, respectively are shown as a function of the choice of the tree. It can be seen that though there is a swing in the condition number values, this swing is within a small margin. The condition numbers from the tree-cotree method are still very high compared to the those from the SVD method.

We also found the condition number for the sphere and the cube for different mesh densities. Table 3.1 and Table 3.2 shows the condition numbers for sphere and cube, respectively for varying number of unknowns \(N_{\text{basis}}\). The results clearly show that the system from the SVD method is extremely well conditioned compared to the tree-cotree method.

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>SVD</th>
<th>Tree-Cotree</th>
</tr>
</thead>
<tbody>
<tr>
<td>270</td>
<td>12.552433</td>
<td>4308.1845</td>
</tr>
<tr>
<td>524</td>
<td>8.8521</td>
<td>10622.419</td>
</tr>
<tr>
<td>1273</td>
<td>11.78690</td>
<td>30854.328</td>
</tr>
</tbody>
</table>

Table 3.1 Condition number of the interaction matrix from SVD and Tree-cotree methods for a conducting sphere of radius 1cm and conductivity of \(3.1663 \times 10^6\) mho/m

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>SVD</th>
<th>Tree-Cotree</th>
</tr>
</thead>
<tbody>
<tr>
<td>286</td>
<td>9.3347434</td>
<td>4761.969</td>
</tr>
<tr>
<td>671</td>
<td>16.089794</td>
<td>17769.32</td>
</tr>
<tr>
<td>1298</td>
<td>10.210632</td>
<td>57537.67</td>
</tr>
</tbody>
</table>

Table 3.2 Condition number of the interaction matrix from SVD and Tree-cotree methods for a conducting cube of 2cm and conductivity of \(3.1663 \times 10^6\) mho/m

It is to be noted that the only difference between the SVD and tree-cotree methods is the way \(\alpha\) is obtained. In the former, it is obtained by computing the null space of \(\mathcal{M}\) in (2.21) through a SVD of \(\mathcal{M}\). The linearly independent columns of \(\alpha\) are also mutually orthonormal. In the latter, it is obtained from the face to cotree edge incidence matrix as described in (2.26). Here, the linearly independent columns of \(\alpha\) are not mutually orthogonal. This prompted us to convert the columns of \(\alpha\) from the tree-cotree method to a set of mutually orthonormal columns and to solve the problem using this new \(\alpha\). We employed Gram-Schmidt orthonormalization procedure to make columns of \(\alpha\) mutually orthonormal. The resulting \(\alpha\) is
different for different choices of tree and is not same as the $\alpha$ from the SVD method. Though orthonormalization creates the same number of independent columns in $\alpha$, we were not sure whether the solenoidality of the basis function is still maintained. Hence, we compared the divergence of the basis function from tree-cotree + Gram-Schmidt method and the SVD method for some simple geometries. In the first example, the sphere of radius 1cm was considered. The divergence of the basis functions were found for different mesh densities of the sphere. Table 3.3 and Table 3.4 compare the divergence values for the sphere and cube, respectively. Each divergence value is the average of absolute value of divergence of all basis functions computed at the center of a tetrahedron chosen at random. From the results, it is clear that the solenoidality of the basis functions is not affected by the orthonormalization.

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>Tree-cotree + Gram-Schmidt</th>
<th>SVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>270</td>
<td>9.84E-15</td>
<td>1.13E-13</td>
</tr>
<tr>
<td>524</td>
<td>7.02E-15</td>
<td>1.29E-13</td>
</tr>
<tr>
<td>1273</td>
<td>2.47E-14</td>
<td>1.72E-13</td>
</tr>
</tbody>
</table>

Table 3.4  Divergence of solenoidal basis functions from tree-cotree method + Gram-Schmidt method and SVD method for a conducting cube of size 2cm

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>Tree-cotree + Gram-Schmidt</th>
<th>SVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>286</td>
<td>6.93E-15</td>
<td>8.96E-14</td>
</tr>
<tr>
<td>671</td>
<td>1.49E-15</td>
<td>1.15E-13</td>
</tr>
<tr>
<td>1298</td>
<td>7.58E-15</td>
<td>1.56E-13</td>
</tr>
</tbody>
</table>

We then solved the problem using this new $\alpha$ and computed the induced fields, current density shown in Figure 3.2 through Figure 3.8. Figure 3.12 through Figure 3.17 shows these results from the tree-cotree + Gram-Schmidt method. It can be seen that the solution matches well with the previous results.

Since the accuracy of the tree-cotree + Gram-Schmidt method is validated, we found the $\text{cond}(Z)$ for the sphere and cube examples using the tree-cotree + Gram-Schmidt method.
The resulting condition number for the sphere with 341 tetrahedrons is 12.5552407 while it is 9.334737 for the cube meshed with 361 tetrahedrons. These values are almost the same as those in the SVD method and show that the system from the tree-cotree + Gram-Schmidt method is also well conditioned. Table 3.5 and Table 3.6 shows the condition number from all the three methods for the sphere and cube with varying mesh densities.

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>SVD</th>
<th>Tree-Cotree</th>
<th>Tree-Cotree + Gram-Schmidt</th>
</tr>
</thead>
<tbody>
<tr>
<td>270</td>
<td>12.552433</td>
<td>4308.1845</td>
<td>12.5552407</td>
</tr>
<tr>
<td>524</td>
<td>8.8521</td>
<td>10622.419</td>
<td>8.822117</td>
</tr>
<tr>
<td>1273</td>
<td>11.78690</td>
<td>30854.328</td>
<td>11.786976</td>
</tr>
</tbody>
</table>

Table 3.5 Condition number of the interaction matrix from SVD, Tree-cotree method, and Tree-cotree method with Gram-Schmidt Orthonormalization for a conducting sphere of radius 1cm and conductivity of $3.1663 \times 10^6$ mho/m

<table>
<thead>
<tr>
<th>No of unknowns</th>
<th>SVD</th>
<th>Tree-Cotree</th>
<th>Tree-Cotree + Gram-Schmidt</th>
</tr>
</thead>
<tbody>
<tr>
<td>286</td>
<td>9.3347434</td>
<td>4761.969</td>
<td>9.334737</td>
</tr>
<tr>
<td>671</td>
<td>16.089794</td>
<td>17769.32</td>
<td>16.089817</td>
</tr>
<tr>
<td>1298</td>
<td>10.210632</td>
<td>57537.67</td>
<td>10.210651</td>
</tr>
</tbody>
</table>

Table 3.6 Condition number of the interaction matrix from SVD, Tree-cotree method, and Tree-cotree method with Gram-Schmidt Orthonormalization for a conducting cube of 2cm and conductivity of $3.1663 \times 10^6$ mho/m

Since $\alpha$ after orthonormalization is different for different trees, it is possible that the condition number of $(Z)$ might vary for different tree choices. So we also found the cond $(Z)$ for the sphere and cube examples for different choices of tree. Figure 3.10 (a-b) show these results for the sphere and cube, respectively. From these figures, it can be seen that orthonormalization still results in the same condition numbers irrespective of the choice of the tree. Figure 3.11 (a-b) compares the condition number computed from both the tree cotree and tree-cotree + Gram-Schmidt method as a function of choice of the tree for the sphere of 1cm radius and discretized with 1445 tetrahedrons. The figure shows that the condition number from tree-cotree + Gram-Schmidt method is far less than that of tree-cotree method and also independent of the choice of the tree. Thus, from the results, it can be concluded that difference in the
conditioning from the SVD method and the tree-cotree method is due to the fact that the columns of $\alpha$ are also mutual orthogonal in the former while they are not in the latter. And the tree-cotree + Gram-Schmidt method can be used to get well conditioned systems. The resulting condition number is also independent of the choice of the tree.

We think that the tree-cotree + Gram-Schmidt method can be used to solve the problem compared to the SVD method and tree-cotree methods because of the following. In the SVD method, the condition number is low. But the limiting factor here is the cost of finding the SVD. This cost scales as $O(N^3)$, where $N$ is the number of degrees of freedom for the problem. In the tree-cotree method, the condition number is very high leading to ill conditioned systems. The limiting factor is these ill conditioned systems. In the tree-cotree + Gram-Schmidt method, the condition number is same as that in the SVD method. The limiting factor here is cost of performing the Gram-Schmidt orthonormalization. This cost also scales as $O(N^3)$. But this cost can be reduced considering the sparse nature of the incidence matrix (maximum of three columns in a row are non-zero) which can also be structured. Gram-Schmidt orthonormalization for a sparse structured matrix can be found in $O(N^2)$ cost. The tree-cotree + Gram-Schmidt method thus eliminates the high cost for computing the SVD in the SVD method, ill conditioning of the tree-cotree method. Thus tree-cotree method can be used to obtain well conditioned system, obtain solution to the problem in $O(N^2)$ time and memory complexity.
Figure 3.1 Conducting sphere in a uniform magnetic field $B_1(r) = \hat{a}_z$. 
Figure 3.2 Induced magnetic flux density measured outside the sphere (radius: 1 cm) at $r = 1.5$ cm in the $XZ$ plane from SVD method and tree-cotree method.
Figure 3.3  Total current density measured inside the sphere (radius: 1 cm) at \( r = 0.75 \) cm in the \( XZ \) plane from SVD method and tree-cotree method.

Figure 3.4  Conducting spherical shell in a uniform magnetic field \( B_1(r) = \hat{a}_z \).
Figure 3.5 Induced magnetic flux density measured outside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 4.0$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.
Figure 3.6  Total current density measured inside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 1.8725$ cm and 1.125 cm in the $XZ$ plane from the SVD method and the tree-cotree method.
Figure 3.7  Induced magnetic flux density measured outside the spherical shell at $r = 7.5$ cm in the $XZ$ plane from the SVD method and the tree-cotree method.
Figure 3.8 Total current density measured inside the spherical shell at \( r = 5.0625 \) cm in the \( XZ \) plane from the SVD method and the tree-cotree method.

Figure 3.9 Condition number of the system matrix versus the choice of the tree from tree-cotree method, a) Sphere of radius 1 cm, conductivity of \( 3.1663 \times 10^6 \) mho/m, meshed with 341 tetrahedrons, b) Cube of 2 cm, conductivity of \( 3.1663 \times 10^6 \) mho/m, meshed with 361 tetrahedrons.
Figure 3.10  Condition number of the system matrix versus the choice of the tree from tree-cotree + Gram-Schmidt method. a) Sphere of radius 1 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 341 tetrahedrons, b) Cube of size 2 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 361 tetrahedrons.

Figure 3.11  Condition number of the system matrix versus the choice of the tree from a) tree-cotree method, b) tree-cotree + Gram-Schmidt method. Sphere of radius 1 cm, conductivity of $3.1663 \times 10^6$ mho/m, meshed with 1445 tetrahedrons.
Figure 3.12  Induced magnetic flux density measured outside the sphere (radius: 1 cm) at \( r = 1.5 \) cm in the \( XZ \) plane from tree-cotree + Gram-Schmidt method and tree-cotree method
Figure 3.13  Total current density measured inside the sphere (radius: 1 cm) at $r = 0.75$ cm in the $XZ$ plane from tree-cotree + Gram-Schmidt method and tree-cotree method
Figure 3.14 Induced magnetic flux density measured outside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 4.0$ cm in the $XZ$ plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.
Total current density measured inside the spherical shell (outer radius: 2 cm, inner radius: 1 cm) at $r = 1.8725$ cm and $1.125$ cm in the $XZ$ plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.
Figure 3.16 Induced magnetic flux density measured outside the spherical shell at \( r = 7.5 \text{ cm} \) in the \( XZ \) plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.
Figure 3.17  Total current density measured inside the spherical shell at $r = 5.0625$ cm in the $XZ$ plane from the tree-cotree + Gram-Schmidt method and the tree-cotree method.
CHAPTER 4. CONCLUSION

In this thesis, we considered an existing integral equation formulation for the eddy current problem using the tree-cotree decomposition. In this formulation, the electric field integral equation is cast in terms of the unknown current density. This current density is solenoidal in the quasistatic limit and is expressed in terms of the solenoidal basis functions. These basis functions are obtained as curl of the edge element shape functions. We obtained these basis functions as a linear combination of non-solenoidal facet shape functions. This new representation results in a formulation that will facilitate in the future development of solution to this problem without any quasistatic approximations. We then computed the combination weights resulting in a solenoidal basis in two different ways. In the first, the weights are computed numerically by finding the null space of the discretized divergence matrix through a singular value decomposition of this matrix. In the second, the weights are computed analytically using the tree-cotree decomposition of the mesh. We showed that this second approach is same as the existing formulation for the problem. We then compared the condition number of the system matrix resulting from the two approaches. The condition number from the first approach is very low resulting in well conditioned systems while that from the second approach is very high, depends on the choice of the tree resulting in ill conditioned systems. Next, we studied the reason for this difference. Based on the study, we proposed a technique to convert systems resulting from tree-cotree decomposition to systems which are well conditioned and whose condition number is independent of the choice of the tree. We validated the accuracy of the different approaches against some analytical results and a benchmark result.
BIBLIOGRAPHY


ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First and foremost, Dr. Shanker Balasubramanian for his guidance and support throughout this research and the writing of this thesis. Also, Dr. Antonello Tamburrino for his invaluable suggestions in my research. I would also like to thank my committee members for their efforts and contributions to this work: Dr. Aleksandar Dogandzic and Dr. Paul Sacks. I owe much to the present and past members of the Computational Electromagnetics Group: Sridharan Balasubramaniam, Pratima Kunapareddy, Gao Jun and Dr. Gregory Kobidze for the helpful discussions I had with them.