On sets of first-order formulas axiomatizing representable relation algebras

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On sets of first-order formulas axiomatizing representable relation algebras

by

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This is to certify that the master's thesis of
Jeremy F. Alm
has met the thesis requirements of Iowa State University

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Preface

The purpose of computing is insight, not numbers.

Richard Hamming

The pure mathematician knows that pure mathematics has an end in itself which is more allied with philosophy.

Philip Jourdain, in the introduction to Georg Cantor’s Contributions to the Founding of the Theory of Transfinite Numbers

The idea of a representation of an algebra is of great mathematical interest, and indeed of philosophical interest, at least to mathematicians. (Whether philosophers would exhibit an interest in this more philosophical brand of mathematics I cannot say; whether the public at large would take such an interest we can all say with certainty.) Cayley’s theorem is perhaps the most famous of the representation theorems. It was, in a way, a great success: every group was found actually to be isomorphic to a set of permutations under the operation of functional composition. So this first-order formalization, the abstract algebraization of the notion of a set of permutations, admits the isomorphic closure of the class of all such sets of permutations. That is, the sets of permutations could be characterized formally, up to isomorphism. And this is the best that can be done with first-order languages, since they cannot distinguish among isomorphic “models.” It is only by our use of a stronger language our “ordinary talk about mathematics”, that we distinguish among isomorphic non-identical models. One wonders whether there could (or should) be any formal logical system, necessarily stronger than first-order predicate logic, that could capture this distinction. Going further, De Morgan hoped for an algebraic system that captured all the “forms of thought.” If this seems almost naïve, it is only because from our vantage point in the history of mathematics there is much more to be seen than there was in 1860. The whole development of mathematical
logic and the formalizations of set theory leave the author with a strong impression of the indispensability of the full arsenal of our mathematical language. It would seem that any "model of thought" designed for viewing the system from without, like first-order languages, will always and necessarily be in some way inadequate.

Lest we be thought to despair: first-order logic provides for some fascinating mathematics. Indeed, it is the purpose of this thesis to elucidate one small corner of that fascinating world. We will concern ourselves no longer with groups but with boolean algebras with additional operations, those that generalize the composition and conversion of binary relations. Alfred Tarski laid out algebraic axioms that hold in any field of binary relations, hoping to find a first-order characterization of fields of binary relations in the same way that sets of permutations were characterized by the group axioms. It turned out that Tarski’s axioms were insufficient, as Roger Lyndon found a relation algebra (an abstract algebra satisfying Tarski’s axioms) that was isomorphic to no field of binary relations. A natural question to ask at this point was, Did Tarski “forget” one? Or two? Could the definition of a relation algebra be strengthened by adding an axiom or two (or several) so that the models thereof would necessarily be isomorphic to fields of binary relations? Donald Monk answered this question negatively: no finite axiom set would suffice. (Tarski had shown previously that such an axiom set did indeed exist, but his proof did not speak to the size of such as set.) So the algebras of “real” binary relations could be characterized, up to isomorphism, by algebraic axioms, but only by infinitely many. This provides an interesting contrast: our linguistic description of a field of binary relations being fairly simple; a first-order characterization being necessarily infinite, and being only a characterization up to isomorphism, at that. Thus the discrepancy between what we can express using the full arsenal of our language and what we can formalize in first-order logic is before us again.

It was stated above that the Cayley representation theorem was a great success “in a way.” There are two reasons for this qualified statement. Upon one of them we have already expounded. The other is that the failure of a such a representation theorem to exist for relation algebras, although it could be considered a disappointment, admits so much interesting mathematics—mathematics that didn’t need to be developed for groups. This “failure” has
inspired fifty years of interesting mathematical research, some of which is contained in these pages. Had Tarski’s axioms “succeeded” in the way that the group axioms did, this thesis would not have been written. Of course, we must not go so far as to be glad for this “failure”; we must take the world as it is, and not let our romantic conceptions of what “would be more interesting or less interesting” skew our view of it. But we can relish what we get, and expect more and more interesting mathematics, for as long as people are inclined to its study.
1 Introduction

1.1 General algebra

Let $I$ be a non-empty set, and let $\rho: I \rightarrow \omega$ be a function. An algebra $A$ is a non-empty set $A$ together with functions $f_i: A^{\rho(i)} \rightarrow A$, $i \in I$. $\rho: I \rightarrow \omega$ is the type of $A$. Algebras are similar if they have the same type. Rings with identity and fields are similar, for example, but rings and groups are not (the operations do not "match up"). So for a group $G$ we write $G = \langle G, \ast, \ast^{-1}, e \rangle$; for a ring $R$, write $R = \langle R, +, -., \cdot, 0, 1 \rangle$. $A \subseteq B$ will denote ordinary set inclusion, while $A \subseteq B$ will denote the subalgebra relation. For an algebra $A$, and $X \subseteq A$, let $\text{Sg}_A(X)$ denote the subalgebra in $A$ generated by $X$. For a homomorphism between similar algebras we will always write $h: A \rightarrow B$ (as opposed to $h: A \rightarrow B$). While $h: A \rightarrow B$ could mean any function between the underlying sets of the algebras, $h: A \rightarrow B$ will always denote a homomorphism. $\circ$ will stand for composition of binary relations. (See Def. 2.1.1) For example, $A \cong | \subseteq B$ means that $A$ is isomorphic to a subalgebra of $B$, so in particular there is some $C \subseteq B$ with $C \cong A$.

We will often work with classes of similar algebras. $V$ will denote the class of all sets. $\text{Id}$ will denote the universal identity relation, $\{ (x, x) : x \in V \}$.

For a class $K$ of similar algebras, let

\[
\begin{align*}
\text{IK} & = \{ A : A \text{ is isomorphic to some member of } K \} \\
\text{HK} & = \{ A : A \text{ is a homomorphic image of some member of } K \} \\
\text{SK} & = \{ A : A \text{ is isomorphic to a subalgebra of some member of } K \} \\
\text{SK}' & = \{ A : A \text{ is a subalgebra of some member of } K \} \\
\text{PK} & = \{ A : A \text{ is isomorphic to a product of members of } K \}
\end{align*}
\]
For a class $K$, the following identities and inclusions hold:

\[
\begin{align*}
1 & : I_K = I_K \\
H & : HH_K = H K \\
S & : S S_K = S K \\
S'S' & : S'S'K = S'K \\
I & : IS'K = SK \\
P & : PPP_K = PK \\
S & : SH_K \subseteq HS_K \\
S' & : PS_K \subseteq SP_K \\
H & : PH_K \subseteq HP_K
\end{align*}
\]

The last three lines are not equalities since equality can fail to hold for various classes $K$. Applying the operators $H, S, P$ to a class $K$, $HSPK$ yields the largest class. For more on class operators see chapter 0 of [HenMonTar].

An equational class is a class $K = \{A : A \models \Sigma\}$, where $\Sigma$ is a set of equations and $\models$ denotes satisfaction of formulas. Birkhoff’s theorem says that $K$ is an equational class iff $K = HK = SK = PK$. See the appendix for more on Birkhoff’s theorem.

We have a general correspondence between homomorphisms $h : A \rightarrow B$ and congruences $C \subseteq A \times A$. Given $h : A \rightarrow B$, let $C = h | h^{-1} \subseteq A \times A$, where $|$ denotes composition of binary relations. (See definition 2.1.1) Then $C$ is a congruence relation. Conversely, given a congruence relation $C \subseteq A \times A$, let $h : A \rightarrow A/C$ given by $a \in A \rightarrow a/C$, where $A/C = \{a/C : a \in A\}$ and $a/C$ is the equivalence class of $a$.

For more on general algebra see [BurSan].
1.2 Boolean algebra

Boolean algebras are generalizations of algebras of sets under the operations of union (generalized by $+$), intersection (generalized by \cdot), complementation (generalized by $-$), and the constants $\emptyset$ and $U$, the universe (generalized by 0 and 1). BA is the class of all boolean algebras.

We write $B = (B, +, \cdot, -, 0, 1)$ for $B \in BA$. $B$ satisfies all of the following:

\[
\begin{align*}
  x + (y + z) &= (x + y) + z \\
  x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
  x + y &= y + x \\
  x \cdot y &= y \cdot x \\
  x \cdot (y + z) &= x \cdot y + x \cdot z \\
  x + (y \cdot z) &= (x + y) \cdot (x + z) \\
  x + (x \cdot y) &= x \\
  x \cdot (x + y) &= x \\
  x + -x &= 1 \\
  x \cdot -x &= 0
\end{align*}
\]

The order of operations is $-$, $\cdot$, $+$. Each of these axioms is easily seen to hold for algebras of sets, where + is interpreted as union, etc. Sometimes + is called join, and $\cdot$ meet.

The two-element boolean algebra is called trivial. The one-element boolean algebra is called degenerate. Every non-degenerate boolean algebra has a homomorphism onto the trivial algebra.

There is an alternate (but equivalent) definition, in which boolean algebras are given fewer fundamental operations. (See [Madd].) Let $B = (B, +, -)$, and $B$ satisfies

\[
\begin{align*}
  x + (y + z) &= (x + y) + z \\
  \quad \text{(assoc. of +)}
\end{align*}
\]
\[ x + y = y + x \quad \text{(comm. of \ +)} \]
\[ x = \overline{x + y} + \overline{x + y} \quad \text{(Huntington's axiom)} \]

Now we can define \( x \cdot y = \overline{x + y} \). The constants 0 and 1 can be defined by \( 1 = x + -x \) and \( 0 = x \cdot -x \) for any \( x \). (One can prove that \( x + -x = y + -y \) for any \( x, y \), so the definitions of 0 and 1 makes sense.) This first definition of BAs given is a little more natural, while the second is simpler. We adopt the second. Here we have written \( \overline{x} \) instead of \( -x \). We will do this when it is convenient.

We define a partial order \( \preceq \) where \( x \preceq y \) iff \( x + y = y \). (Think of \( X \preceq Y \) iff \( X \cup Y = Y \) for sets.) \( +, \cdot \) are monotone, i.e. \( x \preceq y \Rightarrow x + z \preceq y + z \).

We can also define the symmetric difference, \( x \Delta y = x \cdot \overline{y} + \overline{x} \cdot y \). One can prove \( x \Delta y = 0 \iff x = y \).

An atom is a minimal non-zero element in the partial ordering. An algebra is atomic if every non-zero element has an atom below it. An algebra is said to be complete if arbitrary meets and joins exist, i.e. given an arbitrary \( X \subseteq B \), inf \( X \) and sup \( X \) (with respect to the partial ordering) exist. We write \( \Sigma X = \sup X \) and \( \Pi X = \inf X \). In an atomic boolean algebra, we can write every element as the join of the atoms below it:

\[ x = \sum_{a \in \text{At } B} a \]

where \( \text{At } B \) is the set of all atoms of \( B \).

A homomorphism \( h : B \rightarrow B' \) is a map that preserves \( + \) and \( - \). A (boolean-algebraic) ideal in an algebra \( B \) is a set \( I \subseteq B \) such that \( x, y \in I \Rightarrow x + y \in I \) and \( y \in I \), \( x \leq y \Rightarrow x \in I \).\(^1\) For a homomorphism \( h \), let \( \text{ker } h \) denote the pre-image of 0 under \( h \). Then every kernel is an ideal, and every ideal is the kernel of some homomorphism. Given an ideal \( I \), \( C = \{ (x, y) : x \Delta y \in I \} \) is a congruence relation on \( B \). Similarly, given a congruence \( C \), the equivalence class of zero is an ideal.

\(^1\)A set-theoretic ideal is a set \( I \subseteq \mathcal{P}(U) \) such that \( X, Y \in I \Rightarrow X \cup Y \in I \) and \( Y \in I \), \( X \subseteq Y \Rightarrow X \in I \), where \( \mathcal{P}(U) \) is the power set of \( U \). Boolean-algebraic ideals are the abstract analogue of set-theoretic ideals. Despite the difference in cosmetics, they are really "the same."
1.3 Ultraproducts

A (set-theoretic) filter $\mathcal{F}$ on a set $U$ is $\mathcal{F} \subseteq \mathcal{P}(U)$ such that $X \in \mathcal{F}, X \subseteq Y \Rightarrow Y \in \mathcal{F}$ and $X, Y \in \mathcal{F} \Rightarrow X \cap Y \in \mathcal{F}$. An ultrafilter is a maximal proper filter. $\mathcal{F}$ is an ultrafilter iff for all $X \in \mathcal{P}(U), X \in \mathcal{F}$ or $U \setminus X \in \mathcal{F}$ but not both.

Let $\mathcal{F}$ be an ultrafilter on an indexing set $I$. Let $\prod_{i \in I} A_i$ be a product of sets, and $f, g \in \prod_{i \in I} A_i$. We define an equivalence relation $\sim_\mathcal{F}$ by $f/\mathcal{F} = g/\mathcal{F}$ iff $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$. We will write the equivalence class of $f$ as $f/\mathcal{F}$ rather than $f/\sim_\mathcal{F}$. Then

$$\prod_{i \in I} A_i/\mathcal{F} = \{f/\mathcal{F} : f \in \prod A_i\}$$

is called an ultraproduct. Considering an ultrafilter to be the collection of "big" subsets of $I$, then $f \sim_\mathcal{F} g$ iff $f$ and $g$ agree "almost everywhere," or on a "big set."

If the sets $A_i$ are algebras instead of just sets (and then we write $A_i$), then we define the operations in the ultraproduct in the usual way on representatives from the equivalence classes: $(f/\mathcal{F}) \ast (g/\mathcal{F}) := (f \ast g)/\mathcal{F}$ for a binary operation $\ast$ and where $(f \ast g)$ denotes the "pointwise" product. Here is an important result, known as Los' Lemma:

Let $\varphi$ be a first-order sentence. Then

$$\prod_{i \in I} A_i/\mathcal{F} \models \varphi \iff \{i \in I : A_i \models \varphi\} \in \mathcal{F}$$

In particular, ultraproducts preserve satisfaction of sentences.

For more on the ultraproduct construction, see [ChaKei].
2 Relation algebras

2.1 Definition of RA

In order to motivate the definition of a relation algebra, we discuss those structures of which they are the abstract analogue.

2.1.1 Algebras of relations

Let $\text{Sb}(X)$ denote the power set of $X$, and let $\text{Re}(X)$ denote the power set of $X \times X$.

**Definition 2.1.1.** An algebra of binary relations (or proper relation algebra) is an algebra $(A, \cup, -|, -^{-1}, \text{Id}_{E})$, where $A \subseteq \text{Sb}(E)$ for a nonempty equivalence relation $E$, and $\text{Id}_{E} = \text{Id} \cap E$.

The operation $-|$, called relative multiplication or composition, is given by $R -| S = \{(x, z) : \exists y (x, y) \in R \land (y, z) \in S\}$, and the operation $^{-1}$ is given by $R^{-1} = \{(x, y) : (y, x) \in R\}$.

The set of all such algebras is denoted by $\text{PRA}$ for proper relation algebra, and $\text{PRA} = \{\text{Sb}(E) : E = E|E^{-1}\}$. Here and everywhere, $\text{Sb}(E)$ denotes the algebra over the set $\text{Sb}(E)$.

**Theorem 2.1.2.** A set $E \subseteq U \times U$ (for some $U$) is an equivalence relation iff $E = E|E^{-1}$.

**Proof.** By definition, $E$ is an equivalence relation precisely when $E$ satisfies $E^{-1} = E$ (symmetry) and $E|E \subseteq E$ (transitivity). So to prove the "only if" direction, let $E$ be an equivalence relation. Then $E = E|E^{-1} \subseteq E$. Also, if $xEy$ then $xExEy$, since $E$ is reflexive over its field, and so we have $E \subseteq E|E = E|E^{-1}$. Thus $E = E|E^{-1}$ as desired. For the "if" direction, let $E = E|E^{-1}$. Then $E^{-1} = (E|E^{-1})^{-1} = E|E^{-1} = E$, so $E$ is symmetric. Also, $E|E = E|E^{-1} = E$, so $E$ is transitive. □
Algebras of the form $\text{Re}(U) = (\text{Re}(U), \cup, \setminus, 1, \text{Id}_{U \times U})$ and their subalgebras are called *square* PRAs, since the boolean unit has the form $U \times U$.

### 2.1.2 Examples

The following Hasse diagram shows the boolean structure of a particular proper subalgebra of $\text{Re}(\{x, y\})$. This algebra is *square* (the boolean unit is of the form $U \times U$) but not *full* (it is not a power set algebra).

Another example is $\text{Sb}(E)$, where $E = \{(x, x), (y, y), (z, z)\}$. In this instance, the boolean unit is also the relational identity. This algebra is *full* but not *square*. See the following Hasse diagram:
Any $\text{Re}(U)$ can be written $\text{Sb}(E)$, where $E = U \times U$. It is interesting that any subalgebra of an $\text{Sb}(E)$ is a subalgebra of a direct product of algebras of the form $\text{Re}(U)$.

Thus we have the following decomposition theorem.

**Theorem 2.1.3.** $\text{S}\{\text{Sb}(E) : E = E E^{-1}\} = \text{SP}\{\text{Re}(U) : U \in \mathcal{V}\}$.

**Proof.** We simply show $\text{I}\{\text{Sb}(E) : E = E E^{-1}\} = \text{P}\{\text{Re}(U) : U \in \mathcal{V}\}$. Let $E = \bigcup_{\alpha \in J}(U_\alpha \times U_\alpha)$ where the $U_\alpha$s are the disjoint equivalence classes of $E$. Then the maps

$$
R \in \text{Sb}(E) \mapsto (R \cap (U_\alpha \times U_\alpha) : \alpha \in J)
$$

$$(R_\alpha : \alpha \in J) \in \prod_{\alpha \in J} \text{Re}(U_\alpha) \mapsto \bigcup_{\alpha \in J} R_\alpha, \quad \text{where the } R_\alpha \text{s are disjoint}$$
establish the desired correspondence.

Thus every proper relation algebra has a decomposition into a subalgebra of a product of square algebras.

2.1.3 Abstract relation algebras

Definition 2.1.4. A relation algebra $\mathbf{A}$ is an algebra $\mathbf{A} = \langle A, +, -, ;, \sim, l' \rangle$ that satisfies

\[
\begin{align*}
(x + y) + z &= x + (y + z) \quad (2.1) \\
x + y &= y + x \quad (2.2) \\
x &= \overline{x} + y + \overline{y} + y \quad (2.3) \\
x; (y; z) &= (x; y); z \quad (2.4) \\
(x + y); z &= x; z + y; z \quad (2.5) \\
x; l' &= x \quad (2.6) \\
\tilde{x} &= x \quad (2.7) \\
(x + y)' &= \tilde{x} + \tilde{y} \quad (2.8) \\
(x; y)' &= \tilde{y}; \tilde{x} \quad (2.9) \\
\tilde{y} + \tilde{x}; \overline{x}; \overline{y} &= \overline{y} \quad (2.10)
\end{align*}
\]

The first three axioms say that $\mathbf{A}$ is a boolean algebra with additional operations. The remaining axioms are relational identities that hold in every PRA, translated into the abstract algebraic language. The class of all relation algebras is denoted by $\text{RA}$.

We can define a partial order $\leq$ on an RA by $x \leq y$ iff $x + y = y$ (iff $x \cdot y = x$). Also, $x < y$ will mean $x \leq y$ and $x \neq y$. An element $a$ is called an atom if $a > 0$ and $(\forall x)(x < a \Rightarrow x = 0)$. An RA is called atomic if every nonzero element has an atom below it. An RA is called symmetric if conversion is the identity function ($x = \tilde{x}$). An RA is called integral if $x = 0$ or $y = 0$ whenever $x; y = 0$. This last condition is equivalent to the condition that $l'$ be an atom. (See Th. 2.2.9)
2.1.4 An example

Consider the following RA with boolean structure as follows:

All elements of this algebra are self-converse \((\bar{x} = x)\). Relative multiplication is given by \(x; y = x \cdot y\). If such an equational definition of \(;\) is not available, composition for finite RAs can be specified by a multiplication table on the atoms. For this example we have

\[
\begin{array}{ccc}
; & a & b & c \\
\hline
a & a & 0 & 0 \\
b & 0 & b & 0 \\
c & 0 & 0 & c \\
\end{array}
\]

It is sufficient to specify \(;\) on the atoms because in a finite relation algebra every non-zero element is the join of all of the (finitely many) atoms below it, and \(;\) distributes over \(+\). (Of course, each entry in the table should be the join of some atoms.) For example, if \(x = a + b\) and
\[ y = b + c, \text{ then to compute } x; y \text{ we do the following: } x; y = (a + b); (b + c) = a; b + a; c + b; b + b; c. \]

Now each of these terms in the sum can be read off the multiplication table for atoms.

It is easy to see that this abstract relation algebra is isomorphic to the second algebra given in 2.1.2, which is a PRA.

### 2.2 Arithmetic in RA

We derive some useful results that hold in RA. First we show that left-distributivity of \( ; \) over + follows from the axioms. Note that only right-distributivity is assumed explicitly.

**Theorem 2.2.1.** \( x; (y + z) = x; y + x; z. \)

We use \( (a; b)^\sim = \bar{a}; \bar{b} \) to “turn around” the composition, and use right-distributivity.

**Proof.**

\[
\begin{align*}
x; (y + z) &= ([x; (y + z)]^\sim)^\sim \\
&= ((y + z); \bar{x})^\sim \\
&= ((\bar{y} + \bar{z}); \bar{x})^\sim \\
&= (\bar{y}; \bar{x} + \bar{z}; \bar{x})^\sim \\
&= ((x; y)^\sim + (x; z)^\sim)^\sim \\
&= ([x; y + x; z]^\sim)^\sim \\
&= x; y + x; z \quad \text{by (1.7)}
\end{align*}
\]

\[ \square \]

**Theorem 2.2.2.** The operation \( ^\sim \) is an automorphism of the boolean reduct of any \( A \in RA. \)

In particular, \( \bar{x} = \bar{x} \) (or \( -\bar{x} = (\bar{x})^\sim \)).

**Proof.** The operation \( ^\sim \) is bijective, since \( \bar{x} = x \). \( (x + y)^\sim = \bar{x} + \bar{y} \) is axiom (1.8).

\[
x \leq y \iff x + y = y \iff \bar{x} + \bar{y} = (x + y)^\sim = \bar{y} \iff \bar{x} \leq \bar{y}
\]
so conversion is order-preserving (or "monotone").

Now since \( \preceq \) preserves order, we have \( \Gamma' \leq 1 \) and \( 1 = (\Gamma')' \leq \Gamma' \), so \( \Gamma' = 1 \). Also, we have \( 0'' \geq 0 \) and \( 0 = (0'')' \geq 0', \) so \( 0'' = 0' \). Hence \( \preceq \) preserves \( 0 \) and \( 1 \).

To prove \( \tilde{x} = \tilde{\tilde{x}} \), first note that \( x + \tilde{x} = 1 \), so \( \tilde{x} + \tilde{\tilde{x}} = (x + \tilde{x})' = \Gamma = 1 \). Therefore \( \tilde{x} \leq \tilde{\tilde{x}} \).

Similarly, \( \tilde{x} + \tilde{\tilde{x}} = 1 \), so by a parallel argument \( \tilde{\tilde{x}} \leq (\tilde{x})' \). By monotonicity of conversion, we have \( \tilde{x} \leq (\tilde{x})' = \tilde{\tilde{x}} \). Therefore \( \tilde{x} = \tilde{\tilde{x}} \).

Hence conversion is an automorphism of the boolean reduct. \( \square \)

**Theorem 2.2.3.** If \( x \leq y \) then \( x; z \leq y; z \) and \( z; x \leq z; y \) (monotonicity of \( ; \) )

**Proof.** Assume \( x \leq y \). Then \( x; z \leq x; y; z = (x + y); z = y; z. \) Also, \( z; x \leq z; x + z; y = z; (x + y) = z; y. \) \( \square \)

The following theorem is known as the Peircean Law, after C. S. Peirce (pronounced "purse").

**Theorem 2.2.4.** \( x; y \cdot \tilde{x} = 0 \iff y; z \cdot \tilde{x} = 0 \)

**Proof.** By (1.10), we have \( \overline{a; \overline{a; b}} \leq \overline{b} \) for any \( a \) and \( b \).

Suppose that \( x; y \cdot \tilde{x} = 0 \). This is equivalent to \( \overline{x; y} \geq \tilde{x} \). Then

\[
(\overline{x; y})' \geq z \quad \text{by Th 2.2.2}, \quad -(x; y) \geq z \quad \text{by Th 2.2.2 again}, \quad -y; \tilde{x} \geq z \quad \text{by (1.9)}. \]

Finally we have \( y; \overline{y; x} \geq y; z \) by monotonicity of \( ; \).

(1.10) tells us that \( y; \overline{y; x} \leq -\tilde{x} \). Combining the last two inequalities, we get \( y; z \leq y; \overline{y; x} \leq -\tilde{x} \), and hence \( y; z \cdot \tilde{x} = 0. \)

The converse follows by an alternate assignment of the roles of \( x, y, z. \) \( \square \)

**Theorem 2.2.5.** \( 1'' = 1', \quad 1'; x = x, \quad 0; x = 0 = x; 0 \)

Note that neither of the first two were assumed explicitly. (1' was assumed to be a right identity.)

**Proof.** \( 1'' = 1'; 1' = 1''; (1'')' = (1'')' = 1'' = 1' \). Also, \( 1'; x = (x; 1')'' = (x; 1')'' = \tilde{x} = x. \)

To prove \( 0; x = 0, \) note that by the previous theorem we have \( y; x \cdot \tilde{x} = 0 \iff x; z \cdot \overline{y} = 0. \) Now let \( x \) be arbitrary, let \( y = 0 = \tilde{0} \), and let \( z = 1 = \tilde{1}. \) Then we get \( 0; x \cdot 1 = 0 \iff \)
\(x; 1 \cdot 0 = 0\). Now the right side of this biconditional is always true; therefore the left is also. Thus \(0; x = 0; x \cdot 1 = 0\), as desired. The derivation of \(x; 0 = 0\) is similar. 

\[\square\]

**Theorem 2.2.6.** A symmetric algebra is commutative.

*Proof.* Suppose \(\hat{x} = x\) for all \(x\). Then \(x; y = (x; y) = (x; y) = \bar{y}; \hat{x} = y; x\). 

\[\square\]

**Theorem 2.2.7.** \(x \leq x; \hat{x}; x\)

*Proof.*

\[
x \cdot y; z = x \cdot y; (z \cdot 1)
\]

\[
= x \cdot y; (z \cdot (\bar{y}; x + \bar{x}; x))
\]

\[
= x \cdot y; (z \cdot \bar{y}; x) + x \cdot y; (\bar{y}; x \cdot z))
\]

\[
= x \cdot y; (z \cdot \bar{y}; x) + 0
\]

To justify the last step, \(x \cdot y; (\bar{y}; x \cdot z)) = 0\), note that by (1.10) we have \(y; \bar{y}; x \leq \hat{x}\) and hence \(x \cdot y; (\bar{y}; x \cdot z)) \leq x \cdot \hat{x} = 0\).

Now we use the above, reassigning the roles of \(y\) and \(z\): let \(y = x, z = 1'\). Then

\[
x = x \cdot x; 1'
\]

\[
= x \cdot x; (1' \cdot \hat{x}; x)
\]

by the above

\[
= x \cdot x; \hat{x}; x
\]

\[
\leq x; \hat{x}; x
\]

\[\square\]

**Theorem 2.2.8.** If \(x, y \leq 1', \) then \(\hat{x} = x\) and \(x; y = x \cdot y\).

*Proof.* By the previous theorem and monotonicity, \(x \leq x; \hat{x}; x \leq 1'; \hat{x}; 1' = \hat{x}\). So \(x \leq \hat{x}\). Then \(\hat{x} \leq \hat{x} = x\), and \(x = \hat{x}\).
For the second result, \( x; y \leq 1; y = y \) and \( x; y \leq x; l' = x \) by monotonicity, so \( x; y \leq x \cdot y \). Also, by monotonicity and the previous theorem \( x \cdot y \leq (x \cdot y); (x \cdot y) \leq x; (x \cdot y); y \leq x; (1; l') \; y = x; y \). Hence \( x; y = x \cdot y \).

\[ \square \]

**Theorem 2.2.9.** Let \( A \in RA \) be non-degenerate. Then \( A \) is integral iff \( 1' \in At A \).

**Proof.** First, show \( 1' \notin At A \implies A \) not integral.

By hypothesis, \( \exists x \; 0 < x < 1' \). Let \( y = \bar{x} \cdot 1' \). Note that \( y \neq 0 \). Then \( x; y = x \cdot y = x \cdot \bar{x} \cdot 1' = 0 \). So \( A \) is not integral.

Conversely, suppose \( 1' \in At A \). We want \( x; y \neq 0 \) for \( x \neq 0 \neq y \).

First we have \( 0 \neq \bar{x} = 1'; \bar{x} \cdot 1' \). By (2.2.4), \( \bar{x} \cdot 1' \neq 0 \). Since \( 1' \) is an atom, we have \( \bar{x}; x \geq 1' \). Therefore by monotonicity we have \( 1'; y \leq (\bar{x}; x); y \leq (\bar{x}; 1); 1 = \bar{x}; 1 \). So \( y = 1'; y \leq \bar{x}; 1 \), and \( \bar{x}; 1 \cdot y \neq 0 \). Then by (2.2.4), \( x; y \cdot 1 \neq 0 \), and \( x; y \neq 0 \).

\[ \square \]

**Theorem 2.2.10.** \((1; x; 1)\ast = 1; x; 1\)

**Proof.** \( x \leq x; \bar{x}; x \leq 1; \bar{x}; 1 \) by 2.2.7 and monotonicity of composition. So \( x \leq 1; \bar{x}; 1 \). Again by monotonicity, \( 1; x; 1 \leq 1; 1; \bar{x}; 1 = 1; \bar{x}; 1 \). Hence \( 1; x; 1 \leq 1; \bar{x}; 1 \) (\( \spadesuit \)). By letting \( \bar{x} \) take the role of \( x \) in (\( \spadesuit \)), we get \( 1; \bar{x}; 1 \leq 1; \bar{x}; 1 = 1; x; 1 \). Hence \( 1; \bar{x}; 1 = 1; x; 1 \), and \((1; x; 1)\ast = 1; \bar{x}; 1 = 1; x; 1 \) as desired.

\[ \square \]

It is useful to refer to \( 1; x; 1 \) as the closure of \( x \), especially in a proper relation algebra, where \( E[R]E = \bigcup \{ U_\alpha \times U_\alpha: U_\alpha \) is an equivalence class of \( E \), \((U_\alpha \times U_\alpha) \cap R \neq \emptyset \} \).

**Definition 2.2.11.** Let \( A \in RA \), and \( I \subseteq A \). \( I \) is said to be a **relational ideal** if

i. \( y \in I \), \( x \leq y \implies x \in I \)

ii. \( x, y \in I \implies x + y \in I \)

iii. \( x \in I \implies 1; x, x; 1, \bar{x} \in I \)

Note that an ideal \( I \) is proper iff \( 1 \notin I \), since \( I \) is “closed going down” (see i. above). It is a straightforward exercise to show that iii. above is equivalent to \( x \in I \implies 1; x; 1 \in I \).
Definition 2.2.12. An algebra \( A \) is simple if the only homomorphisms from \( A \) onto similar algebras are either isomorphisms or else are mappings from \( A \) to the degenerate (1-element) algebra.

This next theorem provides a useful characterization of the simple relation algebras.

Theorem 2.2.13. Let \( A \in RA \). Then \( A \) is simple iff for all \( x \neq 0, 1; x; 1 = 1 \).

Note that if \( A \) is simple and \( g \) is a homomorphism with domain \( A \), then \( \ker g = \{ x \in A : g(x) = g(0) \} \) is either \( \{ 0 \} \) or \( A \). Thus the only relational ideals on \( A \) are \( \{ 0 \} \) and \( A \).

Proof. We prove both directions by contrapositive. Suppose \( A \) is not simple. Then there is a relational ideal \( I \) on \( A \) such that \( \{ 0 \} \subsetneq I \subsetneq A \). Thus there is some \( x \in I, x \neq 0 \) and \( 1; x; 1 \in I \).

But \( I \) is proper, so \( 1 \notin I \), and \( 1; x; 1 < 1 \).

Conversely, suppose that there is some \( x \neq 0 \) so that \( 1; x; 1 < 1 \). Then \( I = \{ z : z \leq 1; x; 1 \} \) is a relational ideal, and \( \{ 0 \} \subsetneq I \subsetneq A \). \( \square \)

2.3 Representable relation algebras

RAs are algebraic generalizations of PRAs. It is natural to ask whether every RA is isomorphic to some PRA.

Definition 2.3.1. A relation algebra is said to be representable if it is isomorphic to some proper relation algebra. The class of all representable relation algebras is denoted by \( RRA \).

So we have \( RRA = IPRA = IS'\{ Sb(E) : E = E\mid E^{-1} \} = S\{ Sb(E) : E = E\mid E^{-1} \} \).

Theorem 2.3.2 (Lyndon, 1950). \( RRA \neq RA \).

Proof. We exhibit a non-representable relation algebra. Lyndon found a large non-representable relation algebra. The following algebra, which is the smallest, is due to MacKenzie.

Let \( A \) be an algebra with four atoms \( 1', a, \bar{a}, b (b = \bar{b}) \). The multiplication table for diversity atoms is as follows:
We will show that this cannot be the multiplication table for a proper relation algebra. Suppose that $1', a, \bar{a}, b$ are real relations, and that $1'$ is an identity relation. All these atoms are non-zero, so they all contain a pair. Let $(x, y) \in b$.

\[
\begin{array}{c|ccc}
; & a & \bar{a} & b \\
\hline
a & a & 1 & a + b \\
\bar{a} & 1 & \bar{a} & \bar{a} + b \\
b & a + b & \bar{a} + b & \bar{b}
\end{array}
\]

$b \leq a; \bar{a} = \bar{a}; a$, so there exist $w, v$ so that

Then $(w, v) \in a; a = a$, so we can add the edge

Now $a \leq b; b$, so there exists $z$ which is distinct from $v, w, x$ such that
Note that $z \neq v, w, x$ since $\langle z, x \rangle \in b; a \leq 0'$ and $\langle z, v \rangle, \langle z, w \rangle \in b \leq 0'$. Now since $\langle z, x \rangle \in b; a$ and $A$ is a finite algebra, there is an atom that contains $\langle z, x \rangle$;\(^1\) likewise for $\langle z, y \rangle$. Hence we have the following edges that need labels:

Now $\langle x, z \rangle, \langle y, z \rangle \in \bar{a}; a; b = b$. Therefore the edges marked $\alpha, \beta$ can be labeled $b$. But then $x, y, z$ form a "monochromatic triangle":

\(^1\text{If } A \text{ were not finite this might not be the case!}
But then $b \cdot b; b \neq 0$, which is incompatible with the multiplication table, which says that $b; b = b$. Hence $A$ is not representable.
3 RRA is an equational class

In this chapter we will show that RRA is closed under H, S, and P. Thus by Birkhoff’s theorem the set of equations true in RRA axiomatizes the class.

This theorem has an interesting history. In [Lyn50] Roger Lyndon published a proof that RRA was not axiomatizable by quantifier-free formulas—namely equations, quasi-equations\(^1\), and the like. Five years later Tarski published a result that showed that RRA was axiomatizable by equations in [Tar55]! It turned out that Lyndon had made a mistake and that one of the algebras that he used in his proof which was thought not to be representable was in fact representable.

3.1 Closure under subalgebras and products

Theorem 3.1.1. RRA = SRRA.

Proof. \( \text{SRRA} = \text{SS}\{ \text{Sh}(E) : E = E|E^{-1} \} = \text{S}\{ \text{Sh}(E) : E = E|E^{-1} \} = \text{RRA.} \)

Theorem 3.1.2. RRA = PRRA.

Proof. Recall Th 2.1.2, which says that \( \text{S}\{ \text{Sh}(E) : E = E|E^{-1} \} = \text{SP}\{ \text{Re}(U) \} \).

Then \( \text{PRRA} = \text{PSP}\{ \text{Re}(U) \} \subseteq \text{SPP}\{ \text{Re}(U) \} = \text{SP}\{ \text{Re}(U) \} = \text{RRA,} \) where (*) holds since \( \text{PS} \subseteq \text{SP} \) in general.

3.2 Closure under homomorphic images

Closure under H turns out to be as challenging as closure under S and P was easy. The proof that we give here is due to Roger Maddux, and was presented by him in a universal

\(^1\)A quasi-equation is a quantifier-free formula of the form \( \varepsilon_0 \land \varepsilon_1 \land \ldots \land \varepsilon_{n-1} \implies \varepsilon_n \), where \( \varepsilon_0, \ldots, \varepsilon_n \) are all equations.
algebra course at Iowa State. This proof also appears in [Madd]. We have a series of lemmas, theorems, and definitions. Our first goal is to demonstrate that $\text{HS}\{\text{Sb}(E) : E = E | E^{-1}\} = \text{SH}\{\text{Sb}(E) : E = E | E^{-1}\}$.

**Definition 3.2.1.** We say that an algebra $A'$ is *congruence extensile*, or that it has the *congruence extension property*, if for all $A \subseteq A'$ and all congruences $C \subseteq A \times A$, $C$ extends to a congruence $C' \subseteq A' \times A'$ so that $C = C' \cap (A \times A)$.

**Lemma 3.2.2.** Suppose $A'$ has the congruence extension property. Then $\text{HS}\{A'\} = \text{SH}\{A'\}$.

*Proof.* We know in general that $\text{SH} \subseteq \text{HS}$. So suppose that $B \in \text{HS}\{A'\}$. So there is some $A \subseteq A'$, and $h : A \rightarrow B$. Let $C = h | h^{-1}$. Extend $C$ to $C' \subseteq A' \times A'$. $C'$ induces a homomorphism $h' : A' \rightarrow A'/C'$. Then $h' \cap (A \times V) : A \rightarrow A/C$ is a homomorphism, since $C = C' \cap (A \times A)$. So $A/C \in \text{SH}\{A'\}$. But $B \equiv A/C$, so $B \in \text{SH}\{A'\}$. \hfill \Box

**Lemma 3.2.3.** Let $A' \in \text{RA}$. Then $A'$ has the congruence extension property.

*Proof.* Let $A \subseteq A'$. Let $C \subseteq A \times A$ be a congruence. Let $I$ be the equivalence class of the boolean zero. $I$ is a relational ideal, i.e. satisfies i.–iii. below:

i. $y \in I$, $x \leq y \Rightarrow x \in I$

ii. $x, y \in I \Rightarrow x + y \in I$

iii. $x \in I \Rightarrow 1; x, 1; x \in I$

To prove i., let $y \in I$ (i.e. $y \in I$) and $x \leq y$. Then $x \cdot y = x$. Since $C$ is a congruence and $y \in C$, we have $x \cdot y \in C \cdot 0$. Hence $x \in C$.

To prove ii., let $x \in C$ and $y \in C$. Then $(x + y) \in C(0 + 0)$. To prove iii., let $x \in C$. Note that $1C1$ since $C$ is reflexive. Then $(x; 1)C(0; 1)$ and $0; 1 = 0$, so $(x; 1)C0$, and similarly for $1; x$. Also, since $xC0$, we have $xC0$. But $0 = 0$, so $xC0$. So $I$ is a relational ideal.

Now let $J = \{x \in A' : x \leq y \in I\} \subseteq A'$. Now $J$ is a relational ideal on $A'$. $J$ induces a congruence on $A'$, $C' = \{(x, y) : x \triangle y \in J\}$. Now $C = C' \cap (A \times A)$; the inclusion $\subseteq$ is clear.

\footnote{Recall that $h : A \rightarrow B$ (as opposed to $h : A \rightarrow B$) will always denote a homomorphism.}

\footnote{Recall that $\Delta$ denotes symmetric difference: $x \Delta y := x \cdot y + x \cdot y$}
If \((x, y) \in C' \cap (A \times A)\), then \(x, y \in J\) and \(x \Delta y \in J\). But \(J \cap A = I\), so \(x \Delta y \in I\), and \((x, y) \in C\). This concludes the proof.

**Theorem 3.2.4.** \(\text{HS}\{\text{Sh}(E) : E = E|E^{-1}\} = \text{SH}\{\text{Sh}(E) : E = E|E^{-1}\}\).

*Proof.* \(\text{Sh}(E) \in \text{RA}\). Apply previous two lemmas.

Next we wish to show \(\text{H}\{\text{Sh}(E) : E = E|E^{-1}\} \supset \text{RA}\).

**Definition 3.2.5.** Let \(E \neq \emptyset\) be an equivalence relation. We define the *points of* \(E\),

\[
\text{Pt}_E := \{p \subseteq E : E|p|E = E, \ p|E|p \subseteq \text{Id}_E\}
\]

where \(\text{Id}_E = \text{Id} \cap E\).

In the following lemma we have properties of the points of \(E\) that we will need. Note that

i. provides a characterization of the points.

**Lemma 3.2.6.** i. \(p \in \text{Pt}_E\) iff for all equivalence classes \(U\) of \(E\), \(\exists u \in U \ p \cap U^2 = \{(u, u)\}\).

ii. \(p \in \text{Pt}_E \Rightarrow p = p^{-1} \subseteq \text{Id}_E\).

iii. \(R, S \subseteq E, p, q \in \text{Pt}_E;\) then
   a. \(E|p|R|q|E \cap E|p|S|q|E = E|p|(R \cap S)|q|E\)
   b. \(E|R|p|E \cap E|p|S|E = E|R|p|S|E\)
   c. \(E|p|R|q|E = E|q|R^{-1}|p|E\)

iv. \(\forall R \subseteq E \ \exists p, q \in \text{Pt}_E \ E|R|E = E|p|R|q|E\)

v. \(\forall R, S \subseteq E \ \exists p \in \text{Pt}_E \ E|R|S|E = E|R|p|S|E\)

*Proof.* For this proof we will abbreviate \(\langle u, v \rangle \in p\) by \(upv\). So the string \(upvEupv\) indicates that the following holds:

![Graph Diagram]

This will reduce somewhat the number of graphs that need to be drawn.
i. \((\Rightarrow)\): Assume \(p \in \text{Pt}_E\). Let \(U\) be a (nonempty) equivalence class of \(E\). \(E[p]E = E\) implies that \(p \cap U^2\) is nonempty. So choose \(\langle u, v \rangle \in p \cap U^2\). Then \(u p v E u p v\) since \(\langle u, v \rangle \in p\) and \(u E v\). But \(p|E|p \subseteq \text{Id}_E\) by hypothesis, so \(\langle u, v \rangle \in \text{Id}\) and \(u = v\). Therefore \(p \cap U^2 \subseteq \text{Id}_E\). Now suppose that \(\langle u, u \rangle, \langle v, v \rangle \in p \cap U^2\). Then \(u p v E u p v\), so \(u(p|E|p)v\), but \(p|E|p \subseteq \text{Id}\), so \(u = v\).

\((\Leftarrow)\): Suppose for all equivalence classes \(U\), \(\exists u p \cap U^2 = \{\langle u, u \rangle\}\). Show \(E[p]E = E\): The inclusion \(\subseteq\) always holds. To show \(\supseteq\), let \(\langle x, y \rangle \in E\). Then there is some equivalence class \(U\) so that \(x, y \in U\). We also have \(p \cap U^2 = \{\langle u, u \rangle\}\). Then \(x E u p v E y\), and so \(\langle x, y \rangle \in E[p]E\).

Show \(p|E|p \subseteq \text{Id}_E\): Let \(\langle x, y \rangle \in p|E|p\). Then \(\langle x, x \rangle, \langle y, y \rangle \in p\) and \(x p x E y p y\). But since \(x E y\), \(x\) and \(y\) are in the same equivalence class \(U\). So then \(\langle x, y \rangle \in p \cap U^2\), which implies \(x = y\). Therefore \(p|E|p \subseteq \text{Id}_E\).

ii. follows from i.

iii. a. We want \(E[p|R|q|E \cap E[p|S|q|E = E[p|(R \cap S)|q|E\). So let \(\langle u, v \rangle \in E[p|R|q|E \cap E[p|S|q|E\). Then there exists points (1)-(8) such that

\[
\begin{array}{c}
(1) & p & (2) \quad R \\
(3) & p & (4) \quad S \\
& (5) & q & (6) \\
& (7) & q & (8) \\
(9) & u & (10) \quad v \\
\end{array}
\]

By i., (1),(2),(3),(4) are all the same point. Likewise, (5),(6),(7),(8) are all the same point. So then \(\langle u, v \rangle \in E[p|(R \cap S)|q|E\).

\(^4\)Edges labeled \(p\) are now drawn undirected in light of ii.
The proof of $\supseteq$ is trivial by $\vdash$-monotonicity.

b. The proof of $E[R|p|E \cap E|p|S|E = E[R|p|S|E$ is similar to the previous.

c. $E[p|R|q|E = E[q|R^{-1}|p|E:

Let $(u, v) \in E[p|R|q|E:

\begin{tikzpicture}
    \node (u) at (0,0) [circle, fill] {u};
    \node (p) at (1,0) [circle, fill] {p};
    \node (r) at (2,0) [circle, fill] {R};
    \node (q) at (3,0) [circle, fill] {q};
    \node (e) at (4,0) [circle, fill] {E};
    \node (v) at (5,0) [circle, fill] {v};

    \draw (u) -- (p);
    \draw (p) -- (r);
    \draw (r) -- (q);
    \draw (q) -- (e);
    \draw (e) -- (v);
\end{tikzpicture}

This gives

\begin{tikzpicture}
    \node (u) at (0,0) [circle, fill] {u};
    \node (p) at (1,0) [circle, fill] {p};
    \node (x) at (2,0) [circle, fill] {x};
    \node (r) at (3,0) [circle, fill] {R};
    \node (y) at (4,0) [circle, fill] {y};
    \node (q) at (5,0) [circle, fill] {q};
    \node (e) at (6,0) [circle, fill] {E};
    \node (v) at (7,0) [circle, fill] {v};

    \draw (u) -- (p);
    \draw (p) -- (x);
    \draw (x) -- (r);
    \draw (r) -- (y);
    \draw (y) -- (q);
    \draw (q) -- (e);
    \draw (e) -- (v);
\end{tikzpicture}

Now $uE y$ and $vEx$, so we can write $uEy q y R^{-1} x p x E v$, and $(u, v) \in E[q|R^{-1}|P|E$.

The inclusion $\subseteq$ is similar.

So then $E[p|R|q|E = E[q|R^{-1}|p|E$.

iv. Let $\{U_\alpha\}_{\alpha \in I}$ be the equivalence classes of $E$. Let $R \subseteq E$. Let $R_\alpha = R \cap U_\alpha^2$. For all non-empty $R_\alpha$, pick $(u_\alpha, v_\alpha) \in R_\alpha$. For $R_\alpha$ empty, let $(u_\alpha, v_\alpha) \in U_\alpha$. Let $p := \{(u_\alpha, u_\alpha) : \alpha \in I\}$ $q := \{(v_\alpha, v_\alpha) : \alpha \in I\}$. Then $E[R|E = E[p|R|q|E$.

v. Let $R, S \subseteq E$. Let $\{U_\alpha\}_{\alpha \in I}$ be as above. When $(R|S) \cap U_\alpha^2 \neq \emptyset$, pick $(x, y) \in (R|S) \cap U_\alpha^2$.

For every such alpha, $\exists u_\alpha \in U_\alpha, xRu_\alpha Sy$. Let $p := \{(u_\alpha, u_\alpha) : \alpha \in I\}$. Then $E|R|S|E = E[R|p|S|E$.
Lemma 3.2.7. Let $\mathcal{B} = \langle B, +, -, \cdot, ;, 1 \rangle$ be a nondegenerate algebra of relational type. Let $h : \mathcal{Sh}(E) \rightarrow \mathcal{B}$ be a homomorphism onto $\mathcal{B}$ with maximal kernel. Define $\sigma : \mathcal{Sh}(E) \rightarrow \mathcal{R}(\text{Pt}_E)$ by

$$\sigma(R) = \{ (p, q) \in \text{Pt}_E \times \text{Pt}_E : h(E) = h(E[p|R|q|E]) \}$$

Then

i. $\sigma(\emptyset) = \emptyset$

ii. $\sigma(E) = \text{Pt}_E \times \text{Pt}_E$

iii. $R \subseteq S \Rightarrow \sigma(R) \subseteq \sigma(S)$

iv. $\sigma(R \cup S) = \sigma(R) \cup \sigma(S)$

v. $\sigma(E \setminus R) \cap \sigma(R) = \emptyset$

vi. $\sigma(E \setminus R) \cup \sigma(R) = \text{Pt}_E \times \text{Pt}_E$, and consequently $\sigma(E \setminus R) = \sigma(E) \setminus \sigma(R)$

vii. $\sigma(R|S) = \sigma(R)|\sigma(S)$

viii. $\sigma(R^{-1}) = \sigma(R)^{-1}$

ix. $\sigma(\text{Id}_E) \supseteq \text{Id} \cap (\text{Pt}_E \times \text{Pt}_E)$

i.-ix. say that $\sigma$ is almost a homomorphism; it would be if equality held in ix. We will call a function that satisfies i.-ix. a near-homomorphism.

Note: the maximality of the kernel of $h$ is needed only for iv. and vi.

Proof. i. $\sigma(\emptyset) = \{ (p, q) : h(E) = h(E[p|\emptyset|q|E]) = \emptyset$, since $h(E) \neq h(\emptyset)$.

ii. $E = E[p|E|q|E]$ for all $p, q$, so $h(E) = h(E[p|E|q|E])$, and ii. holds.
iii. $R \subseteq S \Rightarrow \sigma(R) \subseteq \sigma(S)$:

Let $R \subseteq S$, so $S \cap R = R$. Let $(p, q) \in \sigma(R)$. Then $h(E) = h(E[p|R|q|E])$, and so

$$h(E[p|S|q|E]) = h(E[p|S|q|E \cap E])$$

$$= h(E[p|S|q|E]) \cdot h(E)$$

$$= h(E[p|S|q|E]) \cdot h(E[p|R|q|E]) \quad (p, q \in \sigma(R))$$

$$= h(E[p|S|q|E \cap E[p|R|q|E])$$

$$= h(E[p|(R \cap S)|q|E]) \quad (h \text{ a hom.})$$

$$= h(E[p|R|q|E]) \quad (S \cap R = R)$$

$$= h(E) \quad p, q \in \sigma(R)$$

iv. $\sigma(R \cup S) = \sigma(R) \cup \sigma(S)$:

From iii. we get $\sigma(R), \sigma(S) \subseteq \sigma(R \cup S)$, so $\supseteq$ holds.

For $\subseteq$, let $(p, q) \in \sigma(R \cup S)$. Then

$$h(E) = h(E[p|(R \cup S)|q|E])$$

$$= h(E[p|R|q|E \cup E[p|S|q|E])$$

$$= h(E[p|R|q|E] + h(E[p|S|q|E])$$

(h a hom.)

Since the kernel of $h$ is maximal, any image of $\text{Sb}(E)$ under $g$ (namely $B$) is simple by general algebraic considerations.\(^5\) For all $X \in \text{Sb}(E)$, either $h(X) = h(E)$ or $h(X) = h(E[p|S|q|E])$.\(^5\)

\(^5\)If $A$ is an algebra and $\theta$ is a congruence on $A$, then there is a 1-1 correspondence between congruences $\theta' \supseteq \theta$ and congruences on $A/\theta$. Therefore if $\theta$ is maximal then $A/\theta$ is simple.
Since \( h(E) \neq h(\emptyset) \), it is not the case that \( h(E|p|R|q|E) = h(E|p|S|q|E) = h(\emptyset) \). So at least one of \( h(E|p|R|q|E) \), \( h(E|p|S|q|E) \) is equal to \( h(E) \). Therefore \( \langle p, q \rangle \) is in either \( \sigma(R) \) or \( \sigma(S) \), hence in their union.

\[ h(E|p|R|q|E) = h(E|p|R|q|E) \cdot h(E) \]

\[ = h(E|p|R|q|E) \cdot h(E|p|(E \setminus R)|q|E) \quad \text{(hypothesis)} \]

\[ = h(E|p|R|q|E \cap E|p|(E \setminus R)|q|E) \quad \text{(h a hom.)} \]

\[ = h(E|p|(R \cap (E \setminus R))|q|E) \quad \text{(3.2.6)} \]

\[ = h(E|p|\emptyset|q|E) \]

\[ = h(\emptyset) \neq h(E) \]

So \( \langle p, q \rangle \notin \sigma(R) \). Thus \( \sigma(E \setminus R) \subseteq \text{Pt}_E^2 \setminus \sigma(R) \), and the intersection is empty.

\[ \sigma(E \setminus R) \cup \sigma(R) = \text{Pt}_E \times \text{Pt}_E, \text{ and consequently } \sigma(E \setminus R) = \sigma(E) \setminus \sigma(R): \]

\( \sigma(E \setminus R) \cup \sigma(R) = \text{Pt}_E^2 \) follows from iv. From \( \sigma(E \setminus R) \cap \sigma(R) = \emptyset \) and \( \sigma(E \setminus R) \cup \sigma(R) = \text{Pt}_E^2 \) it follows that \( \sigma(E \setminus R) \) is the boolean complement of \( \sigma(R) \), hence \( \sigma(E \setminus R) = \sigma(E) \setminus \sigma(R) \).

\[ \sigma(R|S) = \sigma(R)|\sigma(S): \]
Let \( (p, q) \in \sigma(R|S) \). So \( h(E) = h(E|p|R|S|q|E) \). By v. of lemma 3.2.6, \( \exists r \in \text{Pt}_E \) \( E|p|R|S|q|E = E|p|R|r|S|q|E \). Therefore \( E|p|R|S|q|E = E|p|R|r|S|q|E \subseteq E|p|R|r|E \subseteq E \) (since \( S|q|E \subseteq E \)), and \( E|p|R|S|q|E = E|p|R|r|S|q|E \subseteq E|r|S|q|E \subseteq E \) (since \( E|p|R \subseteq E \)). So then

\[
\begin{align*}
  h(E) & = h(E|p|R|S|q|E) \\
       & = h(E|p|R|S|q|E \cap E|p|R|r|E) \quad 3.2.6 \\
       & = h(E|p|R|S|q|E) \cdot h(E|p|R|r|E) \\
       & = h(E) \cap h(E|p|R|r|E) \quad \text{h a hom.} \\
       & = h(E|p|R|r|E) \quad \text{h a hom.}
\end{align*}
\]

And so \( (p, r) \in \sigma(R) \). Similarly, \( (r, q) \in \sigma(S) \). So \( (p, q) \in \sigma(R) \sigma(S) \).

Conversely, let \( (p, q) \in \sigma(R) \sigma(S) \). Then \( \exists r \in \text{Pt}_E \) \( (p, r) \in \sigma(R) \), \( (r, q) \in \sigma(S) \). So \( h(E|p|R|r|E) = h(E) = h(E|r|S|q|E) \). Then

\[
\begin{align*}
  h(E) & = h(E|E) \\
       & = h(E); h(E) \quad \text{h a hom.} \\
       & = h(E|p|R|r|E); h(E|r|S|q|E) \quad \text{hyp.} \\
       & = h(E|p|R|r|E|E|r|S|q|E) \quad \text{h a hom.} \\
       & = h(E|p|R|r|S|q|E) \quad (r|E|E|r = r|E|r = r, r \in \text{Pt}_E) \\
       & = h(E|p|R|S|q|E) \quad \text{hyp.}
\end{align*}
\]

Hence \( (p, q) \in \sigma(R|S) \).
viii. \( \sigma(R^{-1}) = \sigma(R)^{-1} \):

\[
\langle q, p \rangle \in \sigma(R^{-1}) \iff h(E) = h(E[q|R^{-1}|p|E]) = h(E[p|R|q|E])
\]

\[
\iff \langle p, q \rangle \in \sigma(R)
\]

\[
\iff \langle q, p \rangle \in \sigma(R)^{-1}
\]

ix. \( \sigma(\text{Id}_E) \supset \text{Id} \cap (\text{Pt}_E \times \text{Pt}_E) \):

If \( \langle p, p \rangle \in \text{Id} \cap \text{Pt}_E^2 \), then \( h(E) = h(E[p|\text{Id}_E]|p|E) \) since \( E = E[p|E] = E[p|\text{Id}_E]|E = E[p|\text{Id}_E]|(E[p|E]) = E[p|\text{Id}_E]|p|E \). So then \( \langle p, p \rangle \in \sigma(\text{Id}_E) \).

\[\square\]

**Lemma 3.2.8.** Let \( A = (A, +, -, ;, \cdot, \text{Id}) \) with \( A \models 1'' = 1' \wedge x; 1' = 1'; x = x \). Let \( g : A \to \text{Re}(U) \) be a near-homomorphism for some set \( U \). Then \( g(1') \) is an equivalence relation and \( h : A \to \text{Re}(U/g(1')) \) given by \( h(a) = \{ (r/g(1'), s/g(1')) : (r, s) \in g(a) \} \) is a homomorphism. Furthermore, \( h \) is injective if \( g \) is.

**Proof.** First we show that \( g(1') \) is an equivalence relation: \( g(1')|g(1')^{-1} = g(1')|g(1') = g(1')|g(1') = g(1'; 1') = g(1') \), hence \( g(1') \) is an equivalence relation over its field. From now on we will denote \( g(1') \) by \( E \).

Now we show that \( h \) is a homomorphism.

\( h(a + b) = h(a) \cup h(b) \):

\[
\langle r/E, s/E \rangle \in h(a + b) \iff \langle r, s \rangle \in g(a + b) = g(a) \cup g(b)
\]

\[
\iff \langle r, s \rangle \in g(a) \text{ OR } \langle r, s \rangle \in g(b)
\]

\[
\iff \langle r/E, s/E \rangle \in h(a) \text{ OR } \langle r/E, s/E \rangle \in h(b)
\]

\[
\iff \langle r/E, s/E \rangle \in h(a) \cup h(b)
\]

\( h(\bar{a}) = (U/E)^2 \setminus h(a) \):

\[
\langle r/E, s/E \rangle \in h(\bar{a}) \iff \langle r, s \rangle \in g(\bar{a}) = U^2 \setminus g(a)
\]
\[ h(a; b) = h(a) h(b): \]

\[ \langle r/E, s/E \rangle \in h(a; b) \iff \langle r, s \rangle \in g(a; b) = g(a) g(b) \]

\[ \iff \exists t \langle r, t \rangle \in g(a) \text{ AND } \langle t, s \rangle \in g(b) \]

\[ \iff \exists t \langle r/E, t/E \rangle \in h(a) \text{ AND } \langle t/E, s/E \rangle \in h(b) \]

\[ \iff \langle r/E, s/E \rangle \in h(a) h(b) \]

\[ h(\bar{a}) = h(a)^{-1}: \]

\[ \langle r/E, s/E \rangle \in h(\bar{a}) \iff \langle r, s \rangle \in g(\bar{a}) = g(a)^{-1} \]

\[ \iff \langle s, r \rangle \in g(a) \]

\[ \iff \langle s/E, r/E \rangle \in h(a) \]

\[ \iff \langle r/E, s/E \rangle \in h(a)^{-1} \]

\[ h(1') = \text{Id} \cap (U/E)^2: \]

\[ \langle r/E, s/E \rangle \in h(1') \iff \langle r, s \rangle \in g(1') = E, \text{ (an equivalence relation)} \]

\[ \iff rEs \]

\[ \iff r/E = s/E \]

\[ \iff \langle r/E, s/E \rangle \in \text{Id} \cap (U/E)^2 \]

Now we suppose that \( g \) is 1-1, and prove that \( h \) is also. Let \( a, b \in A, a \neq b \). Then \( g(a) \neq g(b) \). Suppose without loss of generality that \( g(a) \setminus g(b) \neq \emptyset \). We want to show that \( h(a) \neq h(b) \). It will suffice to show that if \( \langle x, y \rangle \in g(a) \setminus g(b) \), then \( \langle x/E, y/E \rangle \in h(a) \setminus h(b) \). So let \( \langle x, y \rangle \in g(a) \setminus g(b) \). We want \( \langle x/E, y/E \rangle \) to be distinct from all \( \langle r/E, s/E \rangle \),
where \( (r, s) \in g(b) \). Suppose by way of contradiction that there is some \( (r, s) \in g(b) \) so that 
\( \langle x/E, y/E \rangle = \langle r/E, s/E \rangle \). Then \( xEr \) and \( yEs \). So we have

\[
(x, y) \in E \mid (g(b) \mid E = g(1') \mid g(b) \mid g(1'))
\]

\[
= g(1' \mid b \mid 1')
\]

\[
= g(b)
\]

This stands in contradiction to the assumption that \( \langle x, y \rangle \in g(a) \setminus g(b) \). Therefore \( \langle x/E, y/E \rangle \in h(a) \setminus h(b) \), and \( h(a) \neq h(b) \), and so \( h \) is 1-1 also.

\[
\square
\]

**Lemma 3.2.9.** Let \( g : Sb(E) \rightarrow B \) be a homomorphism onto a non-degenerate algebra \( B \) such that \( g \) has a maximal kernel. Then \( B \in RRA \).

**Proof.** Let \( \sigma : Sb(E) \rightarrow \text{Re}(Pt_E) \), \( \sigma(R) = \{ (p, q) \in Pt_E : g(E) = g(E[p|R|q|E]) \} \). Consider \( g^{-1} | \sigma \subseteq B \times \text{Re}(Pt_E) \).

\[
\begin{array}{ccc}
Sb(E) & \xrightarrow{\sigma} & \text{Re}(Pt_E) \\
g & \downarrow & \\
B & \xrightarrow{g^{-1} | \sigma} \\
\end{array}
\]

Note that since \( g \) is surjective, the domain of \( g^{-1} | \sigma \) is all of \( B \). It is easy to check that \( g^{-1} | \sigma \) is functional and is a near-homomorphism (just use the definition of \( \sigma \)). Then by the
previous lemma, there is an \( f : B \rightarrow \text{Re}(\text{Pt}_E/g(\text{Id}_E)) \) that is a homomorphism, and \( f \) is 1-1 if \( g^{-1}\sigma \) is 1-1.

\( \sigma^{-1} \) is 1-1: Let \( b, c \in B, b \neq c. \) Let \( R \rightarrow b, S \rightarrow c. \) \( R \neq S, \) so either \( R \cap (E \setminus S) \neq \emptyset \) or \( S \cap (E \setminus R) \neq \emptyset. \) Suppose that \( R \cap (E \setminus S) \neq \emptyset. \) Choose \( p, q \in \text{Pt}_E, E = E[p[(R \cap (E \setminus S))]]|q|E. \) Then

\[
g(E) = g(E)p[R|q|E \cap E[p((E \setminus S)|q|E)]
\]

\[
= g(E)p[R|q|E \cdot g(E)p((E \setminus S)|q|E)
\]

Hence \( g(E)p[R|q|E) = g(E) = g(E)p[(E \setminus S)|q|E], \) and so \( (p, q) \in \sigma(R) \setminus \sigma(S) = g^{-1}\sigma(b) \setminus g^{-1}\sigma(c). \) Therefore \( g^{-1}\sigma(b) \setminus g^{-1}\sigma(c) \neq \emptyset, \) and hence \( g^{-1}\sigma(b) \neq g^{-1}\sigma(c). \)

We then conclude that \( f \) is 1-1 also; so \( f \) embeds \( B \) into a square relation algebra: \( B \cong | \subseteq \text{Re}(\text{Pt}_E/g(\text{Id}_E)). \)

\[ \Box \]

**Theorem 3.2.10.** \( \text{H}\{Sb(E) : E = E|E^{-1}\} \subseteq \text{RRA}. \)

**Proof.** Let \( B \in \text{H}\{Sb(E) : E = E|E^{-1}\}. \) Then there is some homomorphism \( g : Sb(E) \rightarrow B \) for some \( E. \) Let \( I = \ker g = g^{-1}[0]. \) \( I \) is a relational ideal. Let \( b, c \in B, b \neq c. \) Then \( \exists R, S \subseteq E, R \rightarrow b, S \rightarrow c. \) Let \( T := E[(R \triangle S)|E]. \)

If \( T \in I, \) then extend \( I \) to a maximal relational ideal \( J \) (use Zorn's Lemma). If \( T \notin I, \) then define \( I' = \{X \subseteq E : \exists X_1 \in I, X \subseteq X_1 \cup E[T|E]\}. \) It is straightforward to check that \( I' \) is a relational ideal containing \( I. \) To see that \( I' \) is proper, suppose the contrary, so that \( E \in I'. \)

Then there is some \( X_1 \in I \) so that \( E = X_1 \cup E[T|E]. \) Then \( X_1 \supseteq E[T|E], \) and so

\[
X_1 \supseteq E[T|E]
\]

\[
= E[(R \triangle S)|E]
\]

\[
def. \text{of } T
\]

\[
= E[X|E]
\]

But \( X_1 \in I, \) and since \( E[(R \triangle S)|E \subseteq X_1, \) we have that \( E[(R \triangle S)|E \in I \) also, and
consequently that $R \Delta S \in I$. But that means that $g(R) = g(S)$, contrary to assumption. Therefore $I'$ is proper. Since $I'$ is proper, it is included in a maximal relational ideal $J$ (use Zorn's Lemma). Thus whether or not $T \in I$, we get a maximal relational ideal $J \supseteq I \cup \{T\}$.

Then $\exists h_J : \text{Sb}(E) \to \text{Sb}(E)/J$ with (maximal) kernel $J$. By the previous lemma, $\text{Sb}(E)/J$ is isomorphic to a square proper relation algebra $Q_{b,c} \subseteq \text{Re}(U)$. Then $g^{-1}|h_j : B \to Q_{b,c}$ is a homomorphism that separates $b,c$:

Suppose for $R, S \in \text{Sb}(E)$, $g(R) = g(S)$. Then $g(R \Delta S) = 0$. So $R \Delta S \in I \subseteq J$. Now $R \Delta S \in J$, so $h_J(R \Delta S) = 0$, and $h_J(R) = h_J(S)$. So $g^{-1}|h_J$ is functional. It is straightforward to check that $g^{-1}|h_J$ is a homomorphism. To show that $g^{-1}|h_J$ separates $b,c$, recall that $g(R) = b, g(S) = c$, and $J \supseteq I \cup \{T\}$. Thus

$$T = E[(R \Delta S)|E \in J \implies E[(R \Delta S)|E \xrightarrow{h_J} 0$$

$$\implies E[(R \Delta S)|E \xrightarrow{h_J} 1$$

$$\implies \text{it is not the case that } (R \Delta S \xrightarrow{h_J} 0)$$

$$\implies R \Delta S \notin J$$

$$\implies h_J(R) \neq h_J(S)$$

So for each $b \neq c$ we get a separating homomorphism $g^{-1}|h_J$ to a square proper relation algebra. Thus we have a homomorphism

$$h : B \to \prod_{b,c \in B \atop b \neq c} Q_{b,c}$$

given by $h(x) = (g^{-1}|h_J(x) : b,c \in B, b \neq c)$. $h$ is an embedding into a product of proper relations algebras. Hence $B \in \text{SPRRA} = \text{RRA}$. $\square$

**Theorem 3.2.11.** HRRA = RRA.

*Proof.* $B \in \text{HRRA}$. $\exists A \in \text{RRA}, B \in \text{H}\{A\}$. Now $A \cong \emptyset \subseteq \text{Sb}(E)$ for some $E$, and so $B \in \text{HS}\{\text{Sb}(E) : E = E|E^{-1}\}$. So then $B \in \text{HS}\{\text{Sb}(E) : E = E|E^{-1}\} = \text{SH}\{\text{Sb}(E) : E = E|E^{-1}\} = \text{HRRA} = RRA$. $\square$
$E|E^{-1}| \subseteq \text{SRRA} = \text{RRA}.$

We now conclude that $\text{RRA} = \text{HRRA} = \text{SRRA} = \text{PRRA}$, and consequently that RRA is definable by equations by Birkhoff’s theorem (see [Bir35], the appendix).
4 Other axiomatizations of RRA

In the previous chapter we saw that RRA has an equational axiomatization. Now we ask whether RRA is axiomatizable by finitely many equations, or finitely many first order sentences, or by infinitely many equations but using only finitely many variables. The answer to each will be "no." First we establish the existence of a countable set of finite algebras with special properties.

4.1 Relation algebras and projective geometries

Both Lyndon and Jónsson developed connections between projective geometry and relation algebra. See [Jón59] and [Lyn61]. The following definition is from [Lyn61].

Definition 4.1.1. A (projective) geometry is a set of points $P$ and a set of lines $L$ such that $\ell \in L \Rightarrow \ell \subseteq P$ and satisfying

i. $L \neq \emptyset$, and $\forall \ell \in L, \ |\ell| \geq 4$.

ii. $\forall p, q \in P, p \neq q, \exists \ell \in L \ p, q \in \ell$. We write $\ell = pq$.

iii. if $p, q, r \in P, p \neq q \neq r \neq p$, and $\exists \ell \in L \ell \cap \overline{pq} \neq \emptyset, \ell \cap \overline{pr} \neq \emptyset$, but $(\ell \cap \overline{pq}) \cap (\ell \cap \overline{pr}) = \emptyset$, then $\ell \cap \overline{qr} \neq \emptyset$.

A projective line is a geometry such that $|L| = 1$.

A projective plane is a geometry such that every line contains $n + 1$ points, and every point lies on $n + 1$ lines. $n$ is said to be the order of the plane.

We are interested in building relation algebras from projective lines. Given a finite projective line $G(P, L)$, Let $i \notin P$, and let $A(G) = \langle Sb(P \cup \{i\}), \cup, -, ;, \{i\} \rangle$, where conversion is
identification \((\tilde{x} = x)\), and \(\cdot\) is given on the atoms \(i\) and all \(p \in P\) by \(\{p\}; \{p\} = \{p\} \cup \{i\}\) and for \(p \neq q\), \(\{p\}; \{q\} = \{r : p \neq r \neq q\}\).

Now we wish to give a more abstract (and more general) definition. Let \(\gamma \subseteq \{1, 2, 3\}\). Then \(E^\gamma_\alpha\) is a complete atomic symmetric integral relation algebra on \(\alpha\) atoms. For finite algebras, we write \(E^\gamma_{n+1}\), \(\alpha = n + 1\). In this case there are \(n + 1\) atoms and \(n\) diversity atoms. When \(\alpha\) is infinite we leave off the "+1". Relative multiplication is given on the atoms by

\[
a; a = \sum \{c \in \text{At}E^\gamma_\alpha : |\{a, c\}| \in \gamma\} \cup \{1'\}
\]

and for \(a \neq b\),

\[
a; b = \sum \{c \in \text{At}E^\gamma_\alpha : |\{a, b, c\}| \in \gamma\}
\]

If \(\gamma = \{1, 3\}\), then \(E^\gamma_{n+1}\) is identical to the algebra defined above, the Lyndon algebra of a projective line of order \(n - 1\).

**Theorem 4.1.2 (Lyndon, '61).** \(E^{\{1,3\}}_{n+1}\) ∈ RRA iff there exists a projective plane of order \(n - 1\). (See [Lyn61].)

**Theorem 4.1.3 (Bruck-Ryser '49).** There exist infinitely many integers such that there is no projective plane of that order. (See [BruRys49].)

By these two theorems we establish the existence of a countable set of arbitrarily large finite non-representable relation algebras. This collection will be central to the proofs of the theorems of both Monk and Jónsson.

### 4.2 Non-finite axiomatizability

That RRA is not finitely axiomatizable is a theorem due to Monk. The key to the proof is the standard model-theoretic ultraproduct construction.

**Lemma 4.2.1.** Let \(\{A_i\}_{i \in \omega}\) be a collection of finite sets such that \(|A_n| \geq 2^n\). Let \(\mathcal{F}\) be a non-principal ultrafilter on \(\omega\). Then

\[
\prod_{i \in \omega} A_i / \mathcal{F} = 2^{\aleph_0}
\]
Proof. We embed \( 2^\omega \) in the ultraproduct. This is sufficient since \( |\prod A_i| = 2^{\aleph_0} \), so we only need to establish that the ultraproduct is at least that big.

Let \( \alpha \in 2^\omega \). We will construct \( f_\alpha \in \prod A_i \) such that for \( \alpha \neq \beta \), we will have \( f_\alpha \not\equiv f_\beta \). Thus \( f_\alpha \) and \( f_\beta \) will be in distinct equivalence classes in the ultraproduct.

So we want \( f_\alpha : \omega \to \bigcup A_i \), so that \( f_\alpha(n) \in A_n \). We denote the elements of \( A_i \) as follows: \( \{a_i^0, a_i^1, a_i^2, \ldots, a_i^{2^i-1}\} \subseteq A_i \), since \( A_i \) has at least \( 2^i \) elements. Then for any \( \alpha \), let \( f_\alpha(0) = a_i^0 \), the one element guaranteed to be in \( A_0 \). Let

\[
\begin{align*}
f_\alpha(1) &= \begin{cases} 
a_1^1, & \alpha(1) = 1 
    
a_1^0, & \alpha(1) = 0
\end{cases}
\end{align*}
\]

So \( f_\alpha(1) \in A_1 \). Let

\[
\begin{align*}
f_\alpha(2) &= \begin{cases} 
a_2^3, & \alpha(1) = 1 \land \alpha(2) = 1 
    
a_2^2, & \alpha(1) = 0 \land \alpha(2) = 1 
    
a_2^1, & \alpha(1) = 1 \land \alpha(2) = 0 
    
a_2^0, & \alpha(1) = 0 \land \alpha(2) = 0
\end{cases}
\end{align*}
\]

So \( f_\alpha(2) \in A_2 \). In general, let \( f_\alpha(n) = a_i^k \), where \( 0 \leq k \leq 2^n - 1 \), and where \( k \) is the integer given by the binary digits \( \alpha(1)____\alpha(n) \) read from left to right. For example, if \( \alpha = 10110 \) (after dropping the first digit \( \alpha(0) \)), then

\[
\begin{align*}
f_\alpha(1) &= a_1^1 \in A_1 
    
f_\alpha(2) &= a_2^1 \in A_2 
    
f_\alpha(3) &= a_3^5 \in A_3 
    
f_\alpha(4) &= a_4^{13} \in A_4 
    
f_\alpha(5) &= a_5^{13} \in A_5 
    
\vdots
\end{align*}
\]

The following binary tree depicts the "path" of \( f_{10110} \) through the algebras \( A_0, A_1, \ldots \).
(If $\alpha(i) = 1$, "go up." If $\alpha(i) = 0$, "go down.")

Now if $\alpha \neq \beta$, then there is some $n$ for which $\alpha(n) \neq \beta(n)$. Notice that $f_\alpha(k) \neq f_\beta(k) \forall k \geq n$ by construction. Since $\mathcal{F}$ is nonprincipal, $X \in \mathcal{F} \Rightarrow |X| = \omega$. Since $f_\alpha$ and $f_\beta$ agree on at most finitely many integers, the set on which they agree is not in the ultrafilter, hence $f_\alpha \not\in \mathcal{F} f_\beta$. Thus the map $\alpha \mapsto f_\alpha/\mathcal{F}$ is injective.

**Corollary 4.2.2.** Let $\{A_i\}_{i \in \omega}$ be a set of finite sets such that the sizes of the $A_i$'s is unbounded.
Then a non-principal ultraproduct of the $A_i$'s has cardinality $2^{|\omega|}$.

**Lemma 4.2.3.** Consider $\{E_{n+1}^{(1,3)}\}_{n \in I}$, $I \subseteq \omega$, $I$ infinite. Let $\mathcal{F}$ be a non-principal ultrafilter on $I$. Then $\prod_{n \in I} E_{n+1}^{(1,3)} / \mathcal{F}$ embeds in $E_{2^\omega}^{(1,3)}$, where the set of atoms of $E_{2^\omega}^{(1,3)}$ is $\prod_{n \in I} \operatorname{At} E_{n+1}^{(1,3)} / \mathcal{F}$.

**Proof.** Each $E_{n+1}^{(1,3)}$ is atomic. The property of being atomic is expressible by the sentence $\varphi = (\forall x) ( (y = 0) \Rightarrow (\exists y) ( (x = 0) \land x + y = x ) \land (\forall z) (z + y = y \Rightarrow z = 0) )$. Since each $E_{n+1}^{(1,3)} \models \varphi$, $\prod_{n \in I} E_{n+1}^{(1,3)} / \mathcal{F} \models \varphi$, since satisfaction of sentences is preserved under the ultraproduct construction (by Los' lemma—see [ChaKei]). Hence the ultraproduct is atomic.

Let $f \in \prod E_{n+1}^{(1,3)}$; let $[f]$ be the equivalence class of $f$ in the ultraproduct. If $[f]$ is an atom of the ultraproduct, then $\{n \in I : f(n)\} = \operatorname{At} E_{n+1}^{(1,3)} \in \mathcal{F}$. Thus $\exists g \in \prod \operatorname{At} E_{n+1}^{(1,3)} \subseteq \prod E_{n+1}^{(1,3)}$, where $[f] = [g]$. So there is a 1-1 correspondence between atoms of the ultraproduct and the atoms of $E_{2^\omega}^{(1,3)}$. If $[I]$ is the identity in the ultraproduct, then $I(n) = I_{A_n}$ "almost everywhere." So $[I]$ corresponds to some $g \in \prod \operatorname{At} E_{n+1}^{(1,3)}$, namely the "constant" function $I \mapsto I_{A_n}$, and $[I] = [I \mapsto I_{A_n}]$. So the correspondence preserves the identity.

Since conversion is identification, it is preserved.

To see that relative multiplication is preserved, notice that it is defined in terms of the boolean operations. Now $E_{2^\omega}^{(1,3)}$ is complete, so the correspondence on atoms extends to an injective homomorphism from the ultraproduct to $E_{2^\omega}^{(1,3)}$ that preserves the boolean operations, hence $;$. Therefore $E_{2^\omega}^{(1,3)}$ contains an isomorphic copy of the ultraproduct $\prod_{n \in I} E_{n+1}^{(1,3)} / \mathcal{F}$. $\square$

**Lemma 4.2.4.** $E_{2^\omega}^{(1,3)}$ is representable over $\mathbb{R}^2$.

**Proof.** The algebra $E_{2^\omega}^{(1,3)}$ has $2^{\aleph_0}$ atoms. Associate each diversity atom $a$ with a number in $\mathbb{R} \cup \{\infty\}$. Send $1' \in E_{2^\omega}^{(1,3)}$ to $\{(x, y), (x, y) : (x, y) \in \mathbb{R}^2\}$. For an atom $a \leq 0'$, send $a$ to $\{(x_0, y_0), (x_1, y_1) : (x_0, y_0) \neq (x_1, y_1), \frac{y_1 - y_0}{x_1 - x_0} = a \in \mathbb{R} \cup \{\infty\}\}$. (We take $\frac{y_1 - y_0}{x_1 - x_0} = \infty$ when $y_1 \neq y_0 \land x_1 = x_0$.) So two points in the plane are related via $a$ iff they are connected by a line of slope $a$. Clearly, the (representations of) the atoms are disjoint and their union is all of $\mathbb{R}^2 \times \mathbb{R}^2$. It is easy to see that conversion is identification. To see that $;$ works properly,

---

1The author may be forgiven, he hopes, for referring to $\infty$ as a "number," a clear abuse of the word. At least he has not claimed that rational functions have discontinuities, as in done in so many calculus texts.
suppose we have \( \alpha, \beta \in \mathbb{R}^2 \), and \( \langle \alpha, \beta \rangle \in a;a \). Then either \( \alpha = \beta \) or \( \alpha \) and \( \beta \) are connected by a line of slope \( a \). So \( \langle \alpha, \beta \rangle \in a \), and \( a,a = 1'+a \). If \( a,b \) are diversity atoms, \( a \neq b \), and \( \langle \alpha, \beta \rangle \in a;b \), then there is some point \( \gamma \) in the plane so that \( \alpha \) is connected to \( \gamma \) by a line of slope \( a \) and \( \gamma \) is connected to \( \beta \) by a line of slope \( b \). We cannot have \( \alpha = \beta \); also it is clear that \( \alpha \) and \( \beta \) could be connected by some \( \gamma \) given any \( c;d \). So \( a;b = 0' \cdot a + b \) as desired.

Thus we have represented the atoms. By the completeness (in the boolean sense) of \( \text{Re}(\mathbb{R}^2) \), we represent any \( x \in E^{(1,3)}_{2^{\omega}} \) as the join of the (representations of) the atoms below it. \( \square \)

**Theorem 4.2.5.** RRA is not finitely axiomatizable in first-order logic.

**Proof.** Let \( I = \{ n \in \omega : n \geq 5 \) and there is no projective plane of order \( n-1 \} \). By [BruRys49], \( I \) is infinite. By 4.1.2, \( \{ E^{(1,3)}_{n+1} \}_{n \in I} \) is a collection of finite non-representable relation algebras of unbounded sizes. By lemma 4.2.4, a non-principal ultraproduct \( \prod_{n \in I} E^{(1,3)}_{n+1} / \mathcal{F} \) has an isomorphic copy inside \( E^{(1,3)}_{2^{\omega}} \). \( E^{(1,3)}_{2^{\omega}} \) is representable, and then so is \( \prod_{n \in I} E^{(1,3)}_{n+1} / \mathcal{F} \). Let RRA\(^c\) denote the complement of RRA relative to the class of all algebras of relational type. If RRA were finitely axiomatizable, then it would be axiomatizable by some sentence \( \varphi \). (Just let \( \varphi \) be the conjunction of the finitely many axioms.) In this case, RRA\(^c\) would be axiomatized by \( \neg \varphi \), hence closed under ultraproducts, since that construction preserves satisfaction of sentences. Since RRA\(^c\) is not closed under ultraproducts, there is no such sentence \( \varphi \) and RRA is not finitely axiomatizable. \( \square \)

### 4.3 Equational bases

In this section we will show that any equational axiomatization of RRA must be unbounded in the number of variables in the equations. Consider \( E^{(1,3)}_{n+1} \) as defined previously.

**Lemma 4.3.1.** \( A \subseteq E^{(1,3)}_{n+1} \implies A \in \text{RRA} \).

**Proof.** Since \( A \) is finite, \( A \) is atomic. Let \( a_1, a_2, \ldots, a_n \) be the diversity atoms of \( E^{(1,3)}_{n+1} \). At least one atom of \( A \) must be the join of at least two atoms of \( E^{(1,3)}_{n+1} \). (If not, then \( A = E^{(1,3)}_{n+1} \).) Suppose this atom is \( (a_1 + a_2) \in \text{At} A \). This atom is “big.” Give \( a_1 + a_2 \) the name \( a \). Then \( a \)
satisfies

\[ a; a = 1 \quad \text{and} \quad a; a_j = 0' \cdot \overline{a}_j \quad \text{for} \quad j > 2 \]

Now choose \( k > n + 1 \) so that \( E_{k+1}^{(1,3)} \) is representable. Let \( b_1, \ldots, b_k \) be the diversity atoms of \( E_{k+1}^{(1,3)} \). Define \( b = b_1 + \ldots + b_{k-n+1} \). Then we have \( b; b = 1 \) and \( b; b_j = 0' \cdot \overline{b}_j \) for \( j > k-n+1 \):

\[
\begin{align*}
  b; b &= (b_1 + \ldots + b_{k-n+1}); (b_1 + \ldots + b_{k-n+1}) \\
  &= \sum_{i,j \leq k-n+1} b_i; b_j \\
  &\geq b_1; b_1 + b_1; b_2 + b_2; b_2 \\
  &= (b_1 + 1') + (0' \cdot \overline{b}_1 + \overline{b}_2) + (b_2 + 1') \\
  &= 1' + b_1 + b_2 + 0' \cdot \overline{b}_1 + \overline{b}_2 \\
  &= 1
\end{align*}
\]

Then also we have for \( j > k - n + 1 \)

\[
\begin{align*}
  b; b_j &= \sum_{i \leq k-n+1} b_i; b_j \\
  &= \sum_{i,j \leq k-n+1} 0' \cdot \overline{b}_i + \overline{b}_j \\
  &= 0' \cdot \overline{b}_j
\end{align*}
\]

Thus the behavior of \( a \in E_{n+1}^{(1,3)} \) and \( b \in E_{k+1}^{(1,3)} \) in their respective algebras is the same.

Thus the mapping

\[
\begin{align*}
  a &\mapsto b \\
  a_j &\mapsto b_{j+k-n} \quad \text{for} \quad j > k - n + 1
\end{align*}
\]
on the atoms of \( E_{n+1}^{(1,3)} \) establishes an isomorphism from \( A \) to the subalgebra of \( E_{k+1}^{(1,3)} \) with atoms \( b, b_{k-n+1}, \ldots, b_{k} \). Since \( E_{k+1}^{(1,3)} \) is representable, so also are its subalgebras, and hence \( A \) is representable also.

Now if we take \( I = \{ n \in \omega : n \geq 5 \text{ and there is no projective plane of order } n - 1 \} \), then \( \{ E_{n+1}^{(1,3)} \}_{n \in I} \) is a set of arbitrarily large finite non-representable relation algebras all of whose proper subalgebras are representable. The existence of such a set gives us the following.

**Theorem 4.3.2 (Jónsson '91; Tarski '74).** RRA has no \( n \)-variable equational basis for \( n < \omega \).

A proof of this theorem first appeared in print in [Jón91], but was known to Tarski previously—he mentioned it in a taped lecture in 1974.

**Proof.** Let \( \Sigma \) be a set of equations with no more than \( n \) variables. Suppose \( \text{RRA} \models \Sigma \). Choose \( k \in I \) so large that \( 2^{2^n} < 2^{k+1} \). Then \( E_{k+1}^{(1,3)} \notin \text{RRA} \), \( |E_{k+1}^{(1,3)}| = 2^{k+1} \). Take \( \epsilon \in \Sigma \). Choose \( x_1, \ldots, x_n \in E_{k+1}^{(1,3)} \). Let \( B = \text{Sg}_{E_{k+1}^{(1,3)}} (x_1, \ldots, x_n) \). Since \( B \) is generated by \( n \) elements, we know from boolean-algebraic considerations that \( |B| \leq 2^{2^2} < 2^{k+1} = |E_{k+1}^{(1,3)}| \), and so \( B \subseteq E_{k+1}^{(1,3)} \).

Hence \( B \in \text{RRA} \). Thus \( B \models \epsilon [x_1, \ldots, x_n] \). But since \( B \subseteq E_{k+1}^{(1,3)} \), \( E_{k+1}^{(1,3)} \models \epsilon [x_1, \ldots, x_n] \), \( x_1, \ldots, x_n \) were arbitrary, so we get \( E_{k+1}^{(1,3)} \models \epsilon \), but \( E_{k+1}^{(1,3)} \notin \text{RRA} \). Hence \( \Sigma \) does not axiomatize RRA.

**Open Question:** Is there an \( n \)-variable first-order axiomatization of RRA?

**Guess:** This seems unlikely.

For more on this problem see [HirHod], chapter 21.

---

\(^2\) \( B \models \epsilon [x_1, \ldots, x_n] \) means that the equation is satisfied when \( x_1 \) through \( x_n \) are substituted into it.
Appendix: A proof of Birkhoff’s Theorem

Birkhoff’s Theorem says that varieties are defined by equations and conversely. (See [Bir35])

First we must say what exactly we mean by “equations.” An equation is a pair of terms: so \( f(x_1, x_2) = g(x_3, h(x_4)) \) is associated with \( \langle f(x_1, x_2), g(x_3, h(x_4)) \rangle \). A term will be an element of the absolutely free algebra of type \( \rho \) with generating set \( \omega \), denoted by \( \text{Fr}_\omega^\rho \). The elements of \( \omega \) are the variables. An assignment of variables is a function \( f : \omega \to A \) for some algebra \( A \) with type \( \rho \). Any such \( f \) extends to a homomorphism \( \hat{f} : \text{Fr}_\omega^\rho \to A \) which is an assignment of terms to elements of the algebra \( A \). To say that an equation \( \epsilon \) is valid in an algebra is to say that given any assignment of the variables \( f : \omega \to A \), \( \hat{f} \) sends both sides of the equation \( \epsilon \) to the same element of the algebra. More precisely, for \( \epsilon = \langle \epsilon_0, \epsilon_1 \rangle \in \text{Fr}_\omega^\rho \times \text{Fr}_\omega^\rho \), \( A \models \epsilon \) if for all homomorphisms \( h : \text{Fr}_\omega^\rho \to A \), \( h(\epsilon_0) = h(\epsilon_1) \).

**Theorem 4.3.3.** Let \( \Sigma \) be a set of equations \( (\Sigma \subset \text{Fr}_\omega^\rho \times \text{Fr}_\omega^\rho) \). Let \( K = \{ A : A \models \Sigma \} \). Then \( K = \text{H}K = \text{S}K = \text{P}K \).

**Proof.** Let \( \epsilon = \langle \epsilon_0, \epsilon_1 \rangle \in \Sigma \).

i. First we show \( K = \text{S}K \). (It is only necessary to show \( K \supseteq \text{S}K \).)

Let \( A \in K \), and \( B \subseteq A \). Let \( \psi : \text{Fr}_\omega^\rho \to B \) be a homomorphism. Let \( \phi \) be the inclusion \( B \hookrightarrow A \).
Then $\varphi \circ \psi$ is a homomorphism from $\text{Fr}_\omega^\rho$ to $A$. Since $A \models \varepsilon$, $\varphi \circ \psi(\varepsilon_0) = \varphi \circ \psi(\varepsilon_1)$. But $\varphi$ is the inclusion map, so $\psi(\varepsilon_0) = \psi(\varepsilon_1)$, and $B \models \varepsilon$. So $B \in K$.

ii. $PK = K$.

$\forall j \in J$, let $A_j \in K$. Consider $\prod A_j = \{ \varphi : J \to \bigcup A_j \mid \varphi(j) \in A_j \}$. Let $\psi : \text{Fr}_\omega^\rho \to \prod A_j$.

$\forall j \in J \exists \psi_j = \pi_j \circ \psi$, where $\pi_j$ is the projection homomorphism onto $A_j$.

Now for $\varepsilon = (\varepsilon_0, \varepsilon_1)$, let $\psi(\varepsilon_0) = \varphi_0 : J \to \bigcup A_j$. By the diagram, $\varphi_0$ is given by $j \mapsto \psi_j(\varepsilon_0)$. Similarly, $\psi(\varepsilon_1) = j \mapsto \psi_j(\varepsilon_1)$.

Now we know that for all $j \in J$, $\psi_j(\varepsilon_0) = \psi_j(\varepsilon_1)$, since $\varepsilon$ holds in each $A_j$. Thus $\varphi_0$ and $\varphi_1$ agree at each $j \in J$, and so $\varphi_0 = \varphi_1$, which means that $\psi(\varepsilon_0) = \psi(\varepsilon_1)$, and so $\prod A_j \models \varepsilon$.

iii. $HK = K$. 
Let $A \in K$, and let $\varphi : A \xrightarrow{\text{onto}} B$, so $B \in HK$. Let $\varepsilon \in \Sigma$, $A \models \varepsilon$. Thus for all hom's $\psi : Fr_\omega^\rho \rightarrow A$, $\psi(\varepsilon_0) = \psi(\varepsilon_1)$, $\varepsilon = (\varepsilon_0, \varepsilon_1)$.

Let $\psi : Fr_\omega^\rho \rightarrow B$. We show $\psi(\varepsilon_0) = \psi(\varepsilon_1)$.

$$
\begin{array}{c}
Fr_\omega^\rho \\
\downarrow \psi \\
B \xrightarrow{\varphi} A
\end{array}
$$

Now we define an assignment of variables $f : \omega \rightarrow A$ by

$$
n \in \omega \quad \mapsto \quad \text{some } a \in \varphi^{-1}[\psi(n)]
$$

(So $f$ is a choice function from $\omega$ to $\bigcup_{n \in \omega} \varphi^{-1}[\psi(n)] \subseteq A$.) Then $f$ extends to a hom $\hat{f} : Fr_\omega^\rho \rightarrow A$, and $\varphi \circ f(n) = \psi(n)$ for $n \in \omega$, so $\varphi \circ f = \psi|_\omega$.

$$
\begin{array}{c}
Fr_\omega^\rho \\
\downarrow \hat{f} \\
\omega \xrightarrow{f} A
\end{array}
$$

Therefore $\varphi \circ \hat{f} = \psi$ since they agree on the generating set of $Fr_\omega^\rho$. Thus the following diagram commutes:

$$
\begin{array}{c}
Fr_\omega^\rho \\
\downarrow \psi \\
B \xrightarrow{\varphi} A
\end{array}
$$
Then \( \psi(\varepsilon_0) = \phi \circ \hat{f}(\varepsilon_0) = \phi \circ \hat{f}(\varepsilon_1) = \psi(\varepsilon_1) \). Then \( \varepsilon \) is valid in \( \mathcal{B} \) as well, and \( \mathcal{B} \in \mathcal{K} \).

\[ \square \]

**Theorem 4.3.4 (Birkhoff, ’35).** Let \( \mathcal{K} \) be a class of similar algebras (with similarity type \( \rho : I \to \omega \)). Suppose that \( \mathcal{K} = \mathcal{H} \mathcal{K} = \mathcal{S} \mathcal{K} = \mathcal{P} \mathcal{K} \). Then \( \mathcal{K} = \{ \mathcal{A} : \mathcal{A} \models \text{Eq}(\mathcal{K}) \} \), where Eq(\( \mathcal{K} \)) is the set of equations true in \( \mathcal{K} \).

**Proof.** Let \( \Sigma = \text{Eq}(\mathcal{K}) \). Let \( \mathcal{A} \models \Sigma \). Show \( \mathcal{A} \in \mathcal{K} \). Thus \( \mathcal{K} \supseteq \{ \mathcal{A} : \mathcal{A} \models \Sigma \} \), and \( \mathcal{K} = \{ \mathcal{A} : \mathcal{A} \models \Sigma \} \), since clearly \( \mathcal{K} \subseteq \{ \mathcal{A} : \mathcal{A} \models \Sigma \} \).

i. Let \( \mathcal{A} \models \Sigma \).

ii. Construct \( \mathcal{F}_{\mathcal{A}}^\rho \), the absolutely free algebra generated by \( \mathcal{A} \).

iii. Let \( I = \{ C \in \text{Con}(\mathcal{F}_{\mathcal{A}}^\rho) : \mathcal{F}_{\mathcal{A}}^\rho / C \in \mathcal{K} \} \). \( I \neq \emptyset \), since \( \mathcal{F}_{\mathcal{A}}^\rho \times \mathcal{F}_{\mathcal{A}}^\rho \) is a congruence, and so \( \mathcal{F}_{\mathcal{A}}^\rho / (\mathcal{F}_{\mathcal{A}}^\rho \times \mathcal{F}_{\mathcal{A}}^\rho) \cong \mathbb{1} \in \mathcal{K} \), since \( \mathcal{K} \) is closed under \( \mathcal{H} \).

iv. Let \( \mathcal{B} = \prod_{C \in I} \mathcal{F}_{\mathcal{A}}^\rho / C \in \mathcal{P} \mathcal{K} = \mathcal{K} \).

v. Let \( f : \mathcal{A} \to \mathcal{B}, a \mapsto (a/C : C \in I) \). Let \( \mathcal{S} = \text{Sg}^\mathcal{B}\{ (a/C : C \in I) \in \mathcal{B} : a \in \mathcal{A} \} \). So \( \mathcal{S} \) is the subalgebra of \( \mathcal{B} \) that is generated by the range of \( f \). We can also write \( f : \mathcal{A} \to \mathcal{S} \).

vi. \( f \) extends to a homomorphism \( h : \mathcal{F}_{\mathcal{A}}^\rho \to \mathcal{S}, f \subseteq h \).

vii. Note: \( \mathcal{S} \in \mathcal{S} \mathcal{P} \mathcal{K} = \mathcal{K} \). We want a homomorphism from \( \mathcal{S} \) onto \( \mathcal{A} \), so that \( \mathcal{A} \in \mathcal{H} \mathcal{S} \mathcal{P} \mathcal{K} = \mathcal{K} \).

viii. Let \( g : \mathcal{F}_{\mathcal{A}}^\rho \to \mathcal{A} \) be the homomorphism that extends \( \text{Id}_{\mathcal{A}} \). Then consider

\[
\begin{array}{ccc}
\mathcal{F}_{\mathcal{A}}^\rho & \xrightarrow{h} & \mathcal{S} \\
\downarrow g & & \downarrow h^{-1} g \\
\mathcal{A} & \xrightarrow{h^{-1} g} & \mathcal{A}
\end{array}
\]

\[ \text{For any algebra} \mathcal{A}, \text{Con}(\mathcal{A}) \text{ is the set of congruences on} \mathcal{A}. \]
Note that $h$ is in fact onto $S$: $f$ maps onto the generating set for $S$, so $h$ maps onto the generated subalgebra.

ix. $h^{-1}|g$ is a function.

First we prove a little lemma: $\forall x \in \mathcal{F}^p_A, h(x) = \langle x/C : C \in I \rangle$.

Proof: show set of elements with this property is a subalgebra. $\supseteq A$.

Base Case: $x \in A \Rightarrow h(x) = f(x) = \langle x/C : C \in I \rangle$.

Inductive Case: Assume for $x, y \in \mathcal{F}^p_A$, $h(x) = \langle x/C : C \in I \rangle$ and $h(y) = \langle y/C : C \in I \rangle$.

Without loss of generality we consider a binary function symbol $\beta$.

\[ h(\beta(x, y)) = \beta(h(x), h(y)) \quad \text{(h is a hom.)} \]
\[ = \beta(\langle x/C : C \in I \rangle, \langle y/C : C \in I \rangle) \quad \text{(by inductive hyp.)} \]
\[ = \langle \beta(x/C, y/C) : C \in I \rangle \quad \text{(by def. of op's in direct prod.)} \]
\[ = \langle \beta(x, y)/C : C \in I \rangle \quad \text{(by def. of op's in quotient alg.)} \]

Thus $h(\beta(x, y)) = \langle \beta(x, y)/C : C \in I \rangle$.

Now we proceed to prove that $h^{-1}|g$ is a function. We want $x = y \Rightarrow h^{-1}|g(x) = h^{-1}|g(y)$, i.e. $h(x) = h(y) \Rightarrow g(x) = g(y)$.

So suppose $h(x) = h(y)$. $x$ and $y$ can be regarded as terms in an equational language by associating $x = t_{\mathcal{F}^p_A}(a_1, \ldots, a_k), a_1, \ldots, a_n \in A$ with $t_{\mathcal{F}^p}(n_1, \ldots, n_k), n_1, \ldots, n_k \in \omega$. So we can say that $\langle x, y \rangle$ is an equation.
By the previous lemma, we have \( \langle x/C : C \in I \rangle = h(x) = h(y) = \langle y/C : C \in I \rangle \), and so \( \forall C \in I, x/C = y/C \). Thus for any congruence \( C \in I \) on \( \mathfrak{F}_A \), \( x \sim_C y \). So then suppose we have a homomorphism \( \varphi : \mathfrak{F}_A^\rho \to Z \) for \( Z \in K \). \( \varphi \) induces a congruence \( C = \varphi|\varphi^{-1} \) on \( \mathfrak{F}_A^\rho \) such that \( \mathfrak{F}_A^\rho /C \cong Z \in K \). Thus \( C \in I \), and hence \( \varphi(x) = \varphi(y) \), since \( x \sim_C y \). Thus \( K \models \langle x, y \rangle \), and so \( \langle x, y \rangle \in \Sigma \). Since \( \langle x, y \rangle \in \Sigma \), \( A \models \langle x, y \rangle \). Thus for any homomorphism from \( \mathfrak{F}_A^\rho \) to \( A \), \( x \) and \( y \) map to the same element of \( A \). Now \( g : \mathfrak{F}_A^\rho \to A \), so \( g(x) = g(y) \). Hence \( h^{-1}|g \) is functional.

x. \( h^{-1}|g \) is a homomorphism.

Again, we work only with a binary function symbol \( \beta \). We show \( h^{-1}|g(\beta(x, y)) = \beta(h^{-1}|g(x), h^{-1}|g(y)), x, y \in S \).

Consider the following diagram:

\[
\begin{array}{ccc}
\mathfrak{F}_A^\rho & \xrightarrow{h} & S \\
\downarrow g & & \downarrow h^{-1}|g \\
A & \xrightarrow{h^{-1}|g} & S
\end{array}
\]

Note that \( h \) is surjective. Then

\[ \exists t \in \mathfrak{F}_A^\rho \quad h(t) = \beta(x, y) \]

\[
\begin{array}{ccc}
t & \xrightarrow{h} & \beta(x, y) \\
\downarrow g & & \\
h^{-1}|g(\beta(x, y))
\end{array}
\]
Now note that $h(\beta(t_x, t_y)) = \beta(h(t_x), h(t_y)) = \beta(x, y)$.

So then

\[
h^{-1}|g(\beta(x, y)) = g(h^{-1}|g(x)), h^{-1}|g(y))
\]

(holds since $h(\beta(t_x, t_y)) = \beta(x, y)$)

\[
= \beta(g(t_x), g(t_y))
\]

($g$ is a hom.)

\[
= \beta(h^{-1}|g(x), h^{-1}|g(y))
\]

($h(t_x) = x, h(t_y) = y$)

Therefore $h^{-1}|g$ is a homomorphism.

Thus $A = h^{-1}|g[\mathcal{S}] \in \text{SPK}$, so $A \in \text{HSPK} = K$. So $A$ is in $K$ and $K$ is defined by equations. \qed
Bibliography


