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Propagation of error applied to linear vehicle dynamics

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Propagation of error applied to linear vehicle dynamics

by

Daniel Jay Buttars

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Mechanical Engineering
Major Professor: James E. Bernard

Iowa State University
Ames, Iowa
2001
This is to certify that the Master's thesis of

Daniel Jay Buttars

has met the thesis requirements of Iowa State University

Signatures have been redacted for privacy
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IV

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CHAPTER 1

INTRODUCTION

The study of automobile vehicle dynamics is usually concerned with the directional response of vehicles as a function of steering and braking. There is a rich history of literature in this area dating back to the nineteen thirties.

Beginning in the fifties, computer simulation became an often-used tool in the study of vehicle dynamics. Simulation has become valuable both in aiding in the general understanding of how vehicles perform and in the study of particular vehicles. When particular vehicles are at issue, a key matter is the correspondence between the performance called for by the simulation and the performance that might be expected for that particular vehicle. This has led to thinking about the validity of the computer simulation, which is often construed to mean the expected correspondence between calculated results and results one might measure on the test track or expect in practice.

An important paper in this regard is Heydinger et al. [1], which attempts to lay the groundwork for determining, based on a comparison between tests and calculations, whether a particular vehicle simulation is adequate to draw conclusions about particular vehicles. The paper says: “A computerized, mathematical model of a physical system, such as a vehicle stability and control simulation, will be considered to be valid if, within some specified operating range of a system, a simulation’s predictions of a system’s responses of interest to specified input(s) agree with the actual physical system’s responses to the same input(s) to within some specified level of accuracy.” The paper continues to say that every experimental measurement contains random error and that there are sources of experimental
non-repeatability including variability in the inputs. "Since simulations cannot predict random error, if a simulation’s predictions agree with experimental measurements to within the experimental random error level, the simulation should be considered valid." Figure 1.1 from this paper shows the scatter from the steady state lateral acceleration from six repeat vehicle test runs. Figure 1.2 also from [1] shows the variation between experimentally measured data and two simulations for yaw rate.

![Graph of steady state lateral acceleration](image)

**Figure 1.1** Scatter of six repeat vehicle test runs of steady state lateral acceleration.

Another useful paper on this topic was presented by Bernard and Clover [2]. They claim there are three questions that bear on the utility of a computer simulation for aiding in the understanding of a particular vehicle.

1. Is the model appropriate for the vehicle and maneuver of interest?
2. Is the simulation based on equations that faithfully replicate the model?
3. Are the input parameters reasonable?
This thesis again addresses the utility of computer-based calculations of vehicle directional response. Here the focus is this question: Based on a statistical representation of vehicle parameters, as indicated by a mean and a normal distribution, what differences should be expected in calculations of vehicle performance?

The immediate goal for this work is to supplement our understanding of the confidence we might hold in simulated results as a function of our confidence in the vehicle parameters that we use as input data. Furthermore, the groundwork laid here may enable follow-up efforts to shed some light on the scatter in test data one might expect on the track as a result of on-the-track variations in vehicle components and vehicle control inputs, and perhaps degradation of vehicle components due to continued use on the highway.

To facilitate the presentation, this thesis illustrates the methodology using very simple linear simulations. The assumption is that the techniques presented here can easily...
be extended into more complex linear models. Nonlinear simulations, which are a promising extension of this work, are not discussed in the body of the thesis. The conclusions and future work section of this thesis comment briefly on this topic.

The next chapter presents a very brief explanation of the linear model, including definition of parameters, which will later be the basis for statistical considerations. Chapter 3 considers variations of calculated steady state measures of vehicle performance that may be expected as a result of variations in input parameters. Chapter 4 proceeds along similar lines, this time focusing on transient measures in the time domain. Chapter 5 proceeds to consideration of the frequency domain. Chapter 6 presents the Summary and Future Work followed by the Appendices and References.
CHAPTER 2
BACKGROUND

This chapter presents a brief overview of vehicle directional response. We use SAE nomenclature, as indicated in Figure 2.1 from Gillespie [3].

As indicated by the figure, the positive x axis is along the vehicle’s longitudinal direction, out the right door to the side is the positive y axis, and z, which is down when the vehicle is at curb orientation, follows from x and y in a right hand system. Presuming very small roll and pitch, the yaw angle is about the z-axis and is positive (in a right-hand turn).

This thesis uses linear analysis. As we will see, this will call for non-severe maneuvering at constant speed. Furthermore, we will restrict the motion to maneuvers in which roll is unimportant. This is quite a severe restriction in practice, but leads to no loss

Figure 2.2 taken from Lund and Bernard [4] (with $\delta_r$ added) is a free body diagram of the vehicle as seen looking down the z-axis. $\delta_r$ is the front steer angle, $\beta$ is the side slip angle, and $a$ and $b$ are the distances from the center of gravity to the front and rear wheels. The lateral forces shown in the figure derive from the tires operating at a slip angle, that is, the velocity of the hub of the tire having a velocity vector at an angle to its longitudinal orientation. The angle between its direction of heading and its direction of travel is known as the slip angle, $\alpha$. The lateral force, $F_y$, is called the “cornering force” which grows with the slip angle.

\[
F_y = -C_\alpha \alpha
\]  

(1)
where the minus sign derives from SAE nomenclature. Positive $F_y$ is to the right, but positive $\alpha$ leads to negative $F_y$.

The proportionality constant, $C_\alpha$, is known as the “cornering stiffness” and is defined as the slope of the curve for $F_y$ versus $\alpha$.

Figure 2.2 presents a classical two-degree of freedom model for directional control. Summing forces and moments yields

$$\Sigma F_y = m(\dot{\beta} + r)u \quad \text{(2)}$$

and

$$\Sigma M_z = I\dot{r} \quad \text{(3)}$$

Figure 2.4 Tire cornering force properties.
The sum of the lateral forces $F_y$ takes into account the lateral forces due to the front and rear tires. Assuming a small steer angle $\delta_f$ and a linear relationship between $F_y$ and slip angle $\alpha$ yields

$$F_{yf} = -C_{a_f}\alpha_f$$

(4)

and

$$F_{yr} = -C_{a_r}\alpha_r$$

(5)

where the $C_{a}$ terms are the tire cornering stiffnesses and the $\alpha$ terms are the slip angles. The minus signs derive from SAE nomenclature, positive lateral forces are to the right, and positive slip angles are clockwise as seen from above.

Presuming angles are small and rewriting the slip angle $\alpha$ using sideslip $\beta$, and yaw rate $r$, the lateral forces become

$$F_{yf} = -C_{a_f}(\beta + ar/u)$$

(6)

and

$$F_{yr} = -C_{a_r}(\beta + br/u)$$

(7)

The yaw moment $M_z$ in equation (3) includes the effects of moments due to front and rear lateral forces.

$$\Sigma M_z = F_{yf} a - F_{yr} b$$

(8)

Substituting these results into equations (2) and (3) yields

$$m(\dot{\beta} + r)u = -C_{a_f}\alpha_f - C_{a_r}\alpha_r$$

(9)

and

$$I_r = -C_{a_f}a_\alpha + C_{a_r}b\alpha_r$$

(10)

Writing the slip angles in terms of the sideslip, yaw rate, and steer angle the two equations (9) and (10) become

$$mu\dot{\beta} + (C_{a_f} + C_{a_r})\beta + (C_{a_f}a - C_{a_r}b + mu^2)r/u = C_{a_f}\delta_f$$

(11)
and

$$I_f + (C_{\alpha a} - C_{\alpha b})\beta + (C_{\alpha a}^2 - C_{\alpha b}^2)n/u = C_{\alpha a}\delta_f$$ (12)

The simple model formed by these two equations will be used to model the effect of variance in vehicle components on expected vehicle performance.

The next two sections deal with steady state measures of vehicle performance, which derive from the steady state solutions to equations (11) and (12), and transient measures, as derived from time varying steer angle $\delta_f$. 
CHAPTER 3
ANALYSIS OF LINEAR STEADY STATE

Consider the response to a steady state steer angle, $\delta_f$. Setting the derivatives to zero in equations (11) and (12) yields the steady state solution

$$r_{ss} = u \frac{\delta_f}{(L + (Ku^2) / g)}$$

(13)

where $r_{ss}$ is the steady state yaw rate and

$$K = W_f / C_{af} - W_r / C_{ar}$$

(14)

The Understeer Gradient, $K$, is a key measure of steady state directional response.

In equation (13) for the steady state yaw rate $r_{ss}$, $u$ is the linear velocity, $\delta_f$ is the steering angle, $L$ is the wheelbase, and $g$ is the acceleration due to gravity. Dividing by $\delta_f$ in equation (13) gives

$$\bar{r} = r_{ss}/\delta_f = u / (L + (Ku^2) / g)$$

(15)

$\bar{r}$, the yaw rate gain, gives the steady state yaw rate response per unit input of steer $\delta_f$. Our goal here is to consider the variation we should expect in steady state measures of vehicle performance as a function of variation in parameters describing the vehicle. We base our work on the well-known relationship:

$$\sigma_{f(x,y,\ldots)} \approx \left[ (\sigma_x (\partial f/\partial x))^2 + (\sigma_y (\partial f/\partial y))^2 + \ldots \right]^{1/2}$$

(16)

where $\sigma_x, \sigma_y, \ldots$ are the standard deviation of the independent variables $x, y, \ldots$ in the function $f(x,y,\ldots)$, and $\sigma_{f(x,y,\ldots)}$ is the standard deviation of that function. See, for example, Section 5.5 of Vardeman and Jobe [5].

Consider first the understeer gradient $K$. Applying relationship (16) yields
Appendix A presents parameters for an '88 Mercury Sable. With the parameters given, the understeer gradient $K$ is 0.06 radians.

Now consider the standard deviation of $K$ as a function of the standard deviation of the front and rear cornering stiffnesses. Assuming these are 5% of the nominal value, equation (17) yields

$$\sigma_K = 0.007 \text{ radians}$$

Thus, the standard deviation of the understeer gradient, as a result of a five-percent standard deviation in the front and rear cornering stiffnesses, is 12.2% of the nominal understeer gradient.

Now consider the variation in the yaw rate gain to be expected from variation in the cornering stiffnesses. Applying (16) to equation (15) results in:

$$\sigma_\tau = \left[ (\sigma_{C_{af}} (\partial \tau / \partial C_{af}))^2 + (\sigma_{C_{ar}} (\partial \tau / \partial C_{ar}))^2 \right]^{1/2}$$

where

$$\partial \tau / \partial C_{af} = \partial \tau / \partial K \partial K / \partial C_{af}$$

and

$$\partial \tau / \partial C_{ar} = \partial \tau / \partial K \partial K / \partial C_{ar}$$

Let

$$B = u^2 / g$$

so

$$\tau = u / (L+BK) = u(L+BK)^{-1}$$

and

$$\partial \tau / \partial K = -u(L+BK)^{-2} B = -uB/(L+BK)^2$$

$$= -u^3 / (g(L+Ku^2/g)^2)$$

If

$$K = W_t C_{af}^{-1} - W_r C_{ar}^{-1}$$

then

$$\partial K / \partial C_{af} = -W_t C_{af}^{-2} = -W_t / C_{af}^2$$

and

$$\partial K / \partial C_{ar} = W_r C_{ar}^{-2} = W_r / C_{ar}^2$$
Bringing everything together gives

\[
\frac{\partial \hat{r}}{\partial C_{af}} = -\frac{u^3}{(g(L+Ku^2/g)^2)}(-W_f/C_{af})
\]

\[
= \frac{(W_f u^3)}{(C_{af}^2 g(L+Ku^2/g)^2)}
\]

(27)

A similar derivation gives

\[
\frac{\partial \hat{r}}{\partial C_{ar}} = \frac{(-W_r u^3)}{(C_{ar}^2 g(L+Ku^2/g)^2)}
\]

(28)

\(\sigma_{\hat{r}}\) then becomes

\[
[(\sigma_{C_{af}}(W_f u^3)/(C_{af}^2 g(L+Ku^2/g)^2')))^2 + (\sigma_{C_{af}}(W_r u^3)/(C_{ar}^2 g(L+Ku^2/g)^2))']^{\frac{1}{2}}
\]

(29)

With the parameters shown in appendix A, \(\hat{r}\) is 3.93 rad/sec/rad. Using five percent of the nominal value of \(C_{af}\) and \(C_{ar}\) for \(\sigma_{C_{af}}\) and \(\sigma_{C_{ar}}\)

\[\sigma_{\hat{r}} = .29\text{ rad/sec/rad}\]

This gives a standard deviation of 7.4% of the nominal yaw rate gain, \(\hat{r}\), with a 5.0% standard deviation of the front and rear cornering stiffnesses.

We have shown here a technique for determining the variation in two key measures of vehicle performance, understeer gradient and yaw rate gain, as a function of variation in front and rear cornering stiffnesses. While the analysis is based on a very simple linear model, the procedure can easily be applied to more complex linear models. In the next chapter, we apply these techniques to transient measures of performance.
To facilitate the consideration of transient maneuvers, we use numerical integration facilitated by MATLAB [6]. For our purposes here, we choose a simple example, a ramp steer input at 60 mph. See Figure 4.1. Appendix B provides vehicle parameters. Also in this chapter $\dot{\delta}$ is represented as $r/\delta_f$ due to the fact that the input steer, $\delta_f$, is not constant.

Figure 4.2 has 3 curves. The center curve represents the yaw rate per input steer, $r/\delta_f$, under the conditions stated above. The top curve shows $r/\delta_f$ under the same conditions when $C_{af}$ is increased by 2%. The lower curve shows $r/\delta_f$, again under the same conditions as the center curve, only this time with $C_{ar}$ increased by 2%.

Taking the difference, at any one given time, between the upper and center curves, and dividing by 2% of $C_{af}$ gives the finite difference $\partial(r/\delta_f)/\partial C_{af}$. Doing this calculation at all the time values and plotting the results as $\partial(r/\delta_f)/\partial C_{af}$ vs. time gives the upper curve of Figure 4.3. Doing the same procedure with the center and lower curves of Figure 4.2 and using $C_{ar}$ will give the lower curve of Figure 4.3.

Substituting the finite differences vs. time of both curves of Figure 4.3 for the partial derivatives in equation (18), $\sigma(r/\delta_f)$ can be calculated and plotted vs. time. This is done in Figure 4.4. The values for $\sigma C_{af}$ and $\sigma C_{ar}$ used in the equation (18) calculation were chosen as 5% of the nominal corresponding cornering stiffnesses.

Figure 4.5 shows the results of taking $\sigma(r/\delta_f)$ from Figure 4.4, and, adding it to, and subtracting it from $r/\delta_f$. The center curve shows the nominal value of $r/\delta_f$ with the
Figure 4.1 Front steer angle $\delta_f$ vs. time.
Figure 4.2 Three curves for $r/\delta_f$ used to calculate $\partial(r/\delta_f)$ values for the numerators of $\partial(r/\delta_f)/\partial C_{af}$ and $\partial(r/\delta_f)/\partial C_{ar}$ plotted in the next figure.
Figure 4.3 Finite difference approximations to partial derivatives.
Figure 4.4 Standard deviation of $r/\delta_r$. 
Figure 4.5 Variation of $r/\delta_f$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 
upper and lower curves representing the mean plus or minus a standard deviation of \( \frac{r}{\delta_f} \). In short, if the standard deviation of the front and rear cornering stiffnesses \( C_{ar} \) and \( C_{ar} \) are 5% of their nominal values, Figure 4.5 presents the variation of \( \frac{r}{\delta_f} \). This simulation results in a standard deviation of 7.4% of the nominal value of \( \frac{r}{\delta_f} \) in the steady state portion and a maximum of 8.0% of the value of \( \frac{r}{\delta_f} \) at time = 1.54 sec. Assuming a normal distribution, this range created by plus and minus one standard deviation will include 68% of all cases with 32% still falling outside of this band. Notice that the steady state values of these curves match the values calculated in Chapter 3 for \( \bar{r} \).

Some other measures of vehicle directional response are Lateral (or Side) Velocity \( v \), Lateral Acceleration \( A_y \), and the Side Slip Angle \( \beta \). Figures 4.6, 4.7, and 4.8 show the results of these three measures per input steer \( \delta_f \). Each one was derived under the same manner and conditions as that of Figure 4.5. The contents of Figures 4.5, 4.6, 4.7, and 4.8 are summarized in Table 4.1.

<table>
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<th>Inputs varied by 5.0% of nominal value</th>
<th>Output parameter/( \delta_f )</th>
<th>See Figure</th>
<th>Steady state resultant standard deviation (% of nominal)</th>
<th>Maximum resultant standard Deviation (% of nominal)</th>
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<tr>
<td>( C_{afs}, C_{ar} )</td>
<td>( r )</td>
<td>4.5</td>
<td>7.4</td>
<td>8.0</td>
</tr>
<tr>
<td>( C_{afs}, C_{ar} )</td>
<td>( v )</td>
<td>4.6</td>
<td>12.5</td>
<td>13.0</td>
</tr>
<tr>
<td>( C_{afs}, C_{ar} )</td>
<td>( A_y )</td>
<td>4.7</td>
<td>7.4</td>
<td>7.6</td>
</tr>
<tr>
<td>( C_{afs}, C_{ar} )</td>
<td>( \beta )</td>
<td>4.8</td>
<td>12.5</td>
<td>13.0</td>
</tr>
</tbody>
</table>
Figure 4.6 Variation of Lateral Velocity $v/\delta_r$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 

Variation of Calpha --> variation of side velocity

$\sigma(v/\delta f)=13.0\%$ $v/\delta f$ at time=1.80 sec

$\sigma(v/\delta f)=12.5\%$ $v/\delta f$ at time=3.00 sec

$\sigma Calphaf=5.0\%$ $Calphaf$

$\sigma Calphar=5.0\%$ $Calphar$

$v/\delta f$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 

Figure 4.6 Variation of Lateral Velocity $v/\delta_r$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 

Variation of Calpha --> variation of side velocity

$\sigma(v/\delta f)=13.0\%$ $v/\delta f$ at time=1.80 sec

$\sigma(v/\delta f)=12.5\%$ $v/\delta f$ at time=3.00 sec

$\sigma Calphaf=5.0\%$ $Calphaf$

$\sigma Calphar=5.0\%$ $Calphar$
Figure 4.7 Variation of Lateral Acceleration $A_y/\delta_t$ with that of $C_{af}$ and $C_{ar}$. 
Figure 4.8 Variation of Side Slip Angle $\beta/\delta_f$ with that of $C_{af}$ and $C_{ar}$.
CHAPTER 5
FREQUENCY RESPONSE

Another method of analyzing transient maneuvers is with frequency response. Going back to equations (11) and (12), and substituting in a sinusoidal input for the steer angle $\delta_f$, results in

$$m u \dot{\beta} + (C_{af} + C_{ar}) \beta + (C_{af} a - C_{ar} b + mu^2) r/u = C_{af} A \sin(\omega t)$$  \hspace{1cm} (31)

and

$$I r + (C_{af} a - C_{ar} b) \beta + (C_{af} a^2 - C_{ar} b^2) r/u = C_{af} a A \sin(\omega t)$$  \hspace{1cm} (32)

Rearranging these equations into the forms

$$\dot{\beta} + C_1 \beta + C_2 r = C_3 A \sin(\omega t)$$  \hspace{1cm} (33)

and

$$\dot{r} + C_4 \beta + C_5 r = C_6 A \sin(\omega t)$$  \hspace{1cm} (34)

then combining them into matrix form produces

$$\dot{q} = [A]q + [B]d$$  \hspace{1cm} (35)

where $q = [\beta; r]$,

$d = A \sin(\omega t)$,  \hspace{1cm} $B = \left[ \frac{C_{af}}{mu}; a C_{af} / I \right]$  

and

$$A_{11} = -(C_{af} + C_{ar}) / (mu)$$

$$A_{12} = -(a C_{af} - b C_{ar} + mu^2) / (mu^2)$$

$$A_{21} = -(a C_{af} - b C_{ar}) / I$$

$$A_{22} = -(a^2 C_{af} + b^2 C_{ar}) / (lu)$$

Using MATLAB [6], equations (33) and (34) can be solved while in the matrix form (35) and plotted as Bode diagrams. See appendix C for the source code, which performs the calculations and does the plots. The Bode diagrams plot the ratio of the magnitude of the
response to the magnitude of the input against the frequency of the input. Standard Bode plots use units of decibels for the ratio of the magnitudes. The figures in this chapter are plots of the ratio of the magnitudes vs. frequency with and without units of decibels as well as a plot of the phase shift of the input to the output vs. the frequency.

In order to obtain a standard deviation on the frequency response, with input deviations on the cornering stiffnesses, finite difference approximations to derivatives with respect to the cornering stiffnesses were calculated point by point across the frequency range in a similar manner to the method used in Chapter 4 for the transients.

For each point frequency, three values were calculated for the ratio of the magnitudes. One was calculated using the given $C_{sf}$ and $C_{ar}$, one was calculated with a 2% increase in $C_{sf}$, and one with a 2% increase in $C_{ar}$. The plots of these three curves are shown in Figure 5.1.

The differences between the latter two values and the first divided by 2% of the corresponding $C_a$ give the derivative values $\partial(r/\delta_t)/\partial C_a$ needed for

$$\sigma(r/\delta_t) = \left[ (\partial(r/\delta_t)/\partial C_{sf} \sigma C_{sf})^2 + (\partial(r/\delta_t)/\partial C_{ar} \sigma C_{ar})^2 \right]^{1/2} \tag{36}$$

which comes from (15). These derivatives plotted vs. frequency are shown in Figure 5.2.

Equation (36) is used to calculate the approximate standard deviation $\sigma(r/\delta_t)$ from the values of Figure 5.2. This is then plotted in Figure 5.3.

The final object is to find and plot $(r/\delta_t) \pm \sigma(r/\delta_t)$. This is done in Figure 5.4. Figures 5.1, 5.2, 5.3, and 5.4 were calculated with $\sigma C_{sf} = 5\% C_{sf}$, and $\sigma C_{ar} = 5\% C_{ar}$ as in the previous chapter. Notice that the left-hand sides of these figures are approaching steady state, as the frequency of 1 rad/sec is slow. The values from the left-hand sides match up
Figure 5.1 Three curves for $r/\delta_t$ to calculate $\partial (r/\delta_t)/\partial C_{\alpha t}$ and $\partial (r/\delta_t)/\partial C_{\alpha e}$. 
Figure 5.2 Finite difference approximations to partial derivatives.
Figure 5.3 Standard deviation of \((r/\delta_f)\).
Figure 5.4 Variation of \( \frac{r}{\delta_f} \) with that of \( C_{\alpha f} \) and \( C_{\alpha r} \).
very well to the previously shown steady state values.

Following the same approach as used above to derive the plots in Figure 5.4; three more sets of plots are made for Lateral Velocity \( v \), Lateral Acceleration \( A_y \), and Side Slip Angle \( \beta \), which are Figures 5.5, 5.6, and 5.7.

Visual inspection of Figures 5.4, 5.5, 5.6, and 5.7 shows that the areas with the largest imprecision are near the steady state portions. This would indicate that the maximum resultant percentage deviations from the nominal values are the same as the steady state results for the transients found in Table 4.1.
Figure 5.5 Variation of $v/\delta_f$ with that of $C_{af}$ and $C_{ar}$. 
Figure 5.6 Variation of $A_y/\delta_f$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 
Figure 5.7 Variation of $\beta/\delta_r$ with that of $C_{\alpha f}$ and $C_{\alpha r}$. 
CHAPTER 6
SUMMARY AND FUTURE WORK

This thesis presents a way to relate variations in input parameters to variation in simulated results. Very simple linear models were used. The extension of this work to more complex linear models will be very straightforward.

The methods here were applied to a) steady state measures of vehicle performance, b) transient measures of vehicle performance, and c) frequency response. Examples were presented illustrating the methodology in each case.

Extension to more complex linear models will provide the basis for interesting and informative analysis from two points of view. First, it seems reasonable to use these methods to aid in the understanding of expected variation of vehicle performance based on knowledge of the expected variation of new vehicle components. Secondly, considering it from the parameter measurement point of view, one might use the methodology to provide expectations for discrepancies between on-the-track measurements of vehicle performance and simulated results deriving from errors in measuring vehicle parameters.

This work can further be solidified by generating large sets of randomly simulated input parameters and using them in multiple runs to find the mean and standard deviation values for the desired output.

Success of this work should embolden researchers to expand the work in two areas. First of all, it will be reasonable to consider larger variations in the parameters, perhaps well beyond the limits of the linear assumptions in the statistics underlying the methods presented here. Second, it seems that important results will derive from extending these results into
nonlinear simulations. While this will not, in itself, call for non-linearities in the statistical underpinnings of the work, the lack of linearity in the simulation will force consideration of the computed results on a scenario by scenario basis, a severe limitation. Still, the results may be rich in possibilities, particularly if the statistics can be applied to measures of the maximum friction at the tire road interface. This could yield exciting results in the assessment of the simulation of very severe maneuvers, including rollover-inducing maneuvers.
APPENDIX A
PARAMETERS FOR THE STEADY STATE CALCULATIONS

\[ C_{ar} = 350 \text{ lb/deg} = 20053.5 \text{ lb/rad} \]
\[ C_{ar} = 280 \text{ lb/deg} = 16042.8 \text{ lb/rad} \text{ (this has steering compliance of 80% worked in)} \]
\[ L = 105.5 \text{ in} = 8.7917 \text{ ft} \]
\[ T = 60 \text{ in} \]
\[ W = 3270 \text{ lb} \text{ (3100 lb empty car + 170 lb driver)} \]
\[ \text{cg of empty car} = .6 \]
\[ \text{cg of car and driver} = .5986 \text{ giving} \]
\[ a = 42.3456 \text{ in} \]
\[ b = 63.1544 \text{ in} \]
\[ h_1 = 8 \text{ in} \]
\[ h_2 = 12 \text{ in} \]
\[ m = 101.6349 \text{ slugs} \]
\[ \mu = .8 \]
\[ u = 60 \text{ mph} = 88 \text{ ft/sec} \]
\[ \delta_r = 1 \text{ deg} \]
\[ W_f = 1957.5 \text{ lb} \]
\[ W_r = 1312.5 \text{ lb} \]
APPENDIX B

PARAMETERS FOR THE TRANSIENT CALCULATIONS

% This vehicle is an '88 Mercury Sable

% Vehicle parameters
We=3100;  % weight of empty car (lbs)
Wf=170;   % weight of front seat occupants (lbs)
Wr=0;     % weight of rear seat occupants (lbs)
Wtr=0;    % weight of trunk contents (lbs)

cge=.6;   % fraction of weight on the front of empty car
L=105.5/12;  % wheelbase (in) converted to (ft)
C1=45/12;  % distance from front tire to front seat (in)
converted to (ft)
C2=84/12;  % distance from front tire to rear seat (in)
converted to (ft)
C3=128/12; % distance from front tire to trunk (in)
converted to (ft)
l=7.5/12;  % front contact patch (in) converted to (ft)
Calphaf=.8*350*180/pi; % front cornering stiffness per tire (with compliance added)
Calphar=350*180/pi; % (lbs/deg) converted to (lbs/rad)

pbar=.6;  % rear cornering stiffness per tire
front
Hzphi=2;  % (lbs/deg) converted to (lbs/rad)
rollcrit=.2;  % fraction of side to side load transfer on
Hztheta=1;  % roll frequency (Hz)
pitchcrit=.2;  % roll critical damping ratio
h1=8/12;   % pitch frequency (Hz)
converted to (ft)
h2=12/12;  % distance from road to suspension pivot (in)
converted to (ft)
phi=4;     % distance from suspension pivot to cg (in)
T=60/12;   % roll gain (deg/g)
rollsteer=0;  % track (in) converted to (ft)

negative)
mu=.8;     % rear roll understeer (deg/g) (oversteer if

% Driver input
u=60*5280/3600;  % initial vehicle speed (mph) converted to
(ft/sec)
A2=1.0*pi/180;   % step steer amplitude (deg) converted to (rad)

% Calculated values
W=We+Wf+Wr+Wtr;  % total weight of loaded car (lbs)
g=32.174;  % acceleration due to gravity (ft/sec^2)
me=We/g;     % mass of empty car (slugs)
m=W/g;       % mass of loaded car (slugs)
\[ h = h_1 + h_2; \] % vehicle cg height (ft)
\[ NR = \frac{(W_e \times L \times (1-cge)+W_f \times C_1 + W_r \times C_2 + W_tr \times C_3)}{(2 \times L)}; \] % static normal load on each rear tire (lbs)
\[ N_f = \frac{(W-2 \times NR)}{2}; \] % static normal load on each front tire (lbs)
\[ cg = \frac{2 \times N_f}{W}; \] % fraction of weight on the front of loaded car (lbs)
\[ a = L \times (1-cg); \] % distance from front tires to center of gravity (ft)
\[ b = L \times cg; \] % distance from rear tires to center of gravity (ft)
\[ I_{zze} = m_e L^2/4; \] % yaw moment of inertia about z axis for empty car (slug ft^2)
\[ I_{zz} = m L^2/4; \] % yaw moment of inertia about z axis for loaded car (slug ft^2)
\[ k_{phi} = \frac{W_e h_2 \times 180}{(\phi_{ig} \times \pi)} + W_e h_2; \] % roll suspension stiffness (ft-lb/rad)
\[ k_{phi2} = \frac{W h_2 \times 180}{(\phi_{ig} \times \pi)} + W h_2; \] % roll suspension stiffness for loaded calculation below (ft-lb/rad)
\[ I_{xxe} = \frac{(k_{phi} - W e h_2)}{(H_{zphi} \times 2 \times \pi)}; \] % roll moment of inertia about x axis of empty car (slug ft^2)
\[ I_{xx} = \frac{(k_{phi2} - W h_2)}{(H_{zphi} \times 2 \times \pi)}; \] % roll moment of inertia about x axis of loaded car (slug ft^2)
\[ C_{phi} = \text{rollcrit} \times 2 \times ((k_{phi} - W e h_2) \times I_{xxe})^{.5}; \] % roll damping constant (ft-lb-sec)
\[ I_{yye} = 8 \times I_{zze}; \] % pitch moment of inertia about y axis for empty car (slug ft^2)
\[ I_{yy} = 8 \times I_{zz}; \] % pitch moment of inertia about y axis for loaded car (slug ft^2)
\[ k_{theta} = \frac{I_{yye} \times (2 \times \pi \times H_{theta})^2}{H_{theta} \times 2 \times \pi}; \] % pitch suspension stiffness (ft-lb/rad)
\[ C_{theta} = \text{pitchcrit} \times 2 \times (k_{theta} \times I_{yye})^{.5}; \] % pitch damping constant (ft-lb-sec)
\[ k_{bar} = k_{theta} / L; \] % normal load change constant for pitch (lb/rad)
APPENDIX C

SOURCE CODE FOR THE YAW RATE GAIN FREQUENCY PLOTS

```matlab
% boder.m

qdot = [A]q + [B]u
y = [C]q + [D]u
q = [beta; x]
u = deltaf

clear
vehicle1
dCalphaf = .02*Calphaf;
dCalphaR = .02*CalphaR;
sigmaCalphaf = .05*Calphaf;
sigmaCalphaR = .05*CalphaR;

A(1,1) = -(Calphaf+CalphaR)/(m*u);
A(1,2) = -(a*Calphaf-b*CalphaR+m*u^2)/(m*u^2);
A(2,1) = -(a*Calphaf-b*CalphaR)/Izz;
A(2,2) = -(a^2*Calphaf+b^2*CalphaR)/(Izz*u);
B = [Calphaf/(m*u); a*Calphaf/Izz];
C = [0 1];
D= [0];
[num,den] = ss2tf(A,B,C,D,1);
sys = tf(num,den);
[rmag, rphase, f] = bode(sys);

A(1,1) = -(Calphaf*1.02+CalphaR)/(m*u);
A(1,2) = -(a*Calphaf-b*CalphaR*1.02+m*u^2)/(m*u^2);
A(2,1) = -(a*Calphaf-b*CalphaR*1.02)/Izz;
A(2,2) = -(a^2*Calphaf+b^2*CalphaR*1.02)/(Izz*u);
B = [Calphaf*1.02/(m*u); a*Calphaf*1.02/Izz];
C = [0 1];
D= [0];
[num,den] = ss2tf(A,B,C,D,1);
sys = tf(num,den);
[rmagpR, rphasepR, f] = bode(sys);

A(1,1) = -(Calphaf+CalphaR*1.02)/(m*u);
A(1,2) = -(a*Calphaf-b*CalphaR*1.02+m*u^2)/(m*u^2);
A(2,1) = -(a*Calphaf-b*CalphaR*1.02)/Izz;
A(2,2) = -(a^2*Calphaf+b^2*CalphaR*1.02)/(Izz*u);
B = [Calphaf/(m*u); a*Calphaf/Izz];
C = [0 1];
D= [0];
[num,den] = ss2tf(A,B,C,D,1);
sys = tf(num,den);
[rmagpR, rphasepR, f] = bode(sys);
```

max=0;
for i=1:length(f)
    drf(i) = rmagpf(i)-rmag(i);
    dphf(i) = rphasepf(i)-rphase(i);
    drR(i) = rmagpR(i)-rmag(i);
    dphR(i) = rphasepR(i)-rphase(i);
    sigmarmag(i) = 
        ((drf(i)/dCalphaf*sigmaCalphaf)^2+(drR(i)/dCalphaR*sigmaCalphaR)^2)^.5;
    if sigmarmag(i) > max
        max = sigmarmag(i);
        flag = (i);
    end
    sigmarphase(i) = 
        ((dphf(i)/dCalphaf*sigmaCalphaf)^2+(dphR(i)/dCalphaR*sigmaCalphaR)^2)^.5;
    rmaghigh(i) = rmag(i)+sigmarmag(i);
    rmaglow(i) = rmag(i)-sigmarmag(i);
    rphasehigh(i) = rphase(i)+sigmarphase(i);
    rphaselow(i) = rphase(i)-sigrnarphase(i);
end

subplot(3,1,1);
semilogx(f,rmag,f,rmagpf,f,rmagpR);
grid on;
title('Frequency response for r/deltaf(Calphaf+2%), r/deltaf,
    r/deltaf(Calphar+2%)');
ylabel('r/deltaf (rad/sec/rad)');
subplot(3,1,2);
semilogx(f,20*log10(rmag),f,20*log10(rmagpf),f,20*log10(rmagpR));
grid on;
ylabel('r/deltaf (dB)');
subplot(3,1,3);
semilogx(f,rphase,f,rphasepf,f,rphasepR);
grid on;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
orient tall

figure
subplot(3,1,1);
semilogx(f,drf/dCalphaf,f,drR/dCalphaR);
grid on;
title('Frequency response for derivatives d(r/deltaf)/dCalphaf,
    d(r/deltaf)/dCalphaR');
ylabel('d(r/deltaf)/dCalpha (rad^2/lb-sec/rad)');
subplot(3,1,2);
semilogx(f,20*log10(drf/dCalphaf),f,20*log10(drR/dCalphaR));
grid on;
ylabel('d(r/deltaf)/dCalpha (dB)');
subplot(3,1,3);
semilogx(f,dphf/dCalphaf,f,dphR/dCalphaR);
grid on;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
orient tall
figure
subplot(3,1,1);
semilogx(f,sigmag);
grid on;
title('Frequency response for sigma(r/deltaf)');
ylabel('sigma(r/deltaf) (rad/sec/rad)');
subplot(3,1,2);
semilogx(f,20*log10(sigmag));
grid on;
ylabel('sigma(r/deltaf) (dB)');
subplot(3,1,3);
semilogx(f,sigma);
grid on;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
orient tall

figure
subplot(3,1,1);
semilogx(f,rmag,f,rmaghigh,f,rmaglow);
grid on;
title('Frequency response for standard deviation of r/deltaf given sigmaCalphaf, sigmaCalphar');
ylabel('r/deltaf (rad/sec/rad)');
subplot(3,1,2);
semilogx(f,20*log10(rmag),f,20*log10(rmaghigh),f,20*log10(rmaglow));
grid on;
ylabel('r/deltaf (dB)');
subplot(3,1,3);
semilogx(f,rphase,f,rphasehigh,f,rphaselow);
grid on;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
orient tall
REFERENCES


