2010

On Hopf algebras of dimension 4p

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by

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A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

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Iowa State University

Ames, Iowa

2010

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ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to my advisor, Dr. Siu-Hung Ng, for his guidance and patience throughout graduate study. I appreciate his assistance of tackling difficult problems, and writing up our joint paper which constitutes the main body of my Ph.D. thesis.

I specially thank Yi-Chih Chiu for being a friend, who had inspired, supported and assisted me a lot, and for being the impetus of my Ph.D. study. He always offered profoundly insightful suggestions and helped me through hard times. I am deeply appreciative of his companionship, both intellectual and personal, for the past decade.

I am grateful to all of my committee members for their efforts and contributions to this work: Dr. Leslie Hogben, Dr. Yiu-Tung Poon, Dr. Sung-Yell Song, Dr. Ling Long and Dr. Jonathan Smith, who asked me good questions and gave insightful comments. I wish to thank those who helped me and gave me chances to finish my study. I would additionally like to thank the Department of Mathematics at Iowa State University for providing me with the studying opportunity.

I would like to thank my friends who shared their life with me every Friday over the past years at Iowa State for the joy of friendship. Finally, and most importantly, I want to express my deep gratitude to my family for their unconditional love and support.

CHAPTER 1. Introduction

We consider the problem of classification of finite dimensional non-semisimple Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic zero. This classification problem is to determine the isomorphism classes of Hopf algebras over $\Bbbk$ for a given dimension. Before discussing the main problems and the results of this thesis, we provide some related backgrounds of finite dimensional Hopf algebras.

1.1 Backgrounds

A Hopf algebra over $\Bbbk$ is a $\Bbbk$-algebra H endowed with a coalgebra structure and a $\Bbbk$-linear endomorphism S called antipode which satisfies some compatibility conditions. Hopf algebras were introduced in the context of algebraic topology by Heinz Hopf in 1941. Starting with the late 1960s, Hopf algebras became a subject of study from an algebraic viewpoint. The research of this area increased extensively in the 1980s because of its connection to the quantum groups and deformed enveloping algebras which were introduced by Drinfeld and Jimbo. They systematically provide solutions of the quantum Yang-Baxter equation, and interesting examples of non-commutative and non-cocommutative Hopf algebras. The module categories of these quantum groups can be used to obtain invariants of knots and 3-manifolds. In particular, the Jones-Conway polynomials of a knot can be derived from the point of view of quantum groups.

In 1975, Kaplansky conjectured that there are only finitely many finite-dimensional Hopf algebras over $\mathbb{C}$, up to isomorphism, for a given dimension. The conjecture was disproved by a family of Hopf algebras of dimension p^4 with an odd prime p. However, the conjecture appears to be true when the dimension of the Hopf algebra has less than four prime factors including multiplicity. It remains unclear how the dimension of a Hopf algebra determines finiteness of isomorphism classes or the structure of the Hopf algebra. Some progress on the classification of finite dimensional Hopf algebras has been made recently, and it provides a window for a deeper understanding of these questions.

The classification of finite dimensional Hopf algebras basically consists of two branches - semisimple and non-semisimple Hopf algebras. Non-semisimple Hopf algebras can also be divided into two types - pointed and non-pointed. A Hopf algebra is called pointed if every simple subcoalgebra is one-dimensional, or equivalently, every simple comodule is one-dimensional. Therefore, duals of finite dimensional pointed Hopf algebras are basic algebras. Examples of "small" dimensional non-semisimple and non-pointed Hopf algebras with non-pointed duals are very rare.

There has been some progress on the study of pointed non-semisimple Hopf algebras, in particular the case when the group of group-like elements is abelian (cf. [4, 5, 6]). However, very little is known about the general classification of finite dimensional non-semisimple Hopf algebras, and even those small dimensions less than 32 are not completed yet. It has been proved in [14] that there are infinitely many isomorphism classes of 32-dimensional Hopf algebras.

It follows from some recent result, the remaining incomplete classification of Hopf algebras with dimensions less than 32 are dimensions 20, 24, 27, 28 and 30. All these integers are products of at least three primes. For the dimension p^3 with p an odd prime, semisimple Hopf algebras were classified completely in [26] and the classification of pointed Hopf algebras of dimension p^3 have been completed independently by different techniques in [4], [9] and [43]. Also, partial classification of quasitriangular Hopf algebras of dimension p^3 is given in [19].

Some partial results on the classification of pq^2-dimensional Hopf algebras have been obtained recently, where p, q are distinct primes. Those semisimple Hopf algebras of dimension ≤ 100 were classified in [29], [30] and [32]. The recent preprint [16] proved that every semisimple Hopf algebra H of dimension pq^2 is of Frobenius type, i.e. the dimension every simple module of H divides the dimension of H. The semisimple Hopf algebras of dimension pq^2

which are of Frobenius type have been studied in [29] and [30]. For $q = 2$, it is shown in [30] that there are exactly two isomorphism classes of non-trivial semisimple Hopf algebras over $\Bbbk$. These are self-dual Hopf algebras constructed by Gelaki in [20].

Non-semisimple pointed Hopf algebras of dimension pq^2 were classified completely by Andruskiewitsch and Natale in the appendix of [1]. There are $4(q-1)$ isomorphism classes and these Hopf algebras can be described as extensions of Taft algebras and group algebras. Moreover, the set of its group-like elements is a cyclic group of order pq. However, the classification of general non-semisimple Hopf algebras of dimension pq^2 still remains open. The most recently classified dimension is $2q^2$. It has been shown in [21] that every non-semisimple Hopf algebra of dimension $2q^2$ is either pointed or dual-pointed, i.e. the dual of the Hopf algebra is pointed. More details of the history about classification of finite dimensional Hopf algebras can be found in [7].

In this thesis, we will look at non-semisimple Hopf algebras of dimension $4p$, where p is an odd prime. The only classified dimension of $4p$ is 12 which was completed by Natale in [31]. Every non-semisimple Hopf algebra of dimension 12 is either pointed or dual-pointed. In some of the papers mentioned above, all known examples of non-semisimple Hopf algebras of dimension pq^2 are pointed or dual-pointed Hopf algebras. We conjecture that all Hopf algebras of dimension $4p$ are pointed or dual-pointed. It is natural to speculate that one may use the analogous techniques for classifying Hopf algebras of dimensions $2p^2$ and $4p$ as these two dimensions look quite similar and both have a factor $2p$. Besides, it is also known that every Hopf algebra of dimension $2p$ is semisimple (cf. [35]). Nevertheless, we found that the latter is more subtle.

The goal of this thesis is to investigate the classification of non-semisimple Hopf algebras of dimension $4p$ for odd prime p. The content of Chapter 3 has been submitted to Journal of Algebra [12]. We show that every non-semisimple $4p$-dimensional Hopf algebra with more than two grouplike elements is pointed. It remains unclear whether there exists a non-semisimple Hopf algebra H of dimension $4p$ with at most two grouplike elements such that neither H nor H^* is pointed. For any odd prime $p \leq 11$, we show by counting arguments that every non-semisimple Hopf algebra of dimension $4p$ admits a non-trivial skew primitive element. This implies that non-semisimple Hopf algebras of dimensions 20, 28 and 44 are pointed or dual-pointed.

1.2 Thesis Organization

The thesis is organized as follows: Chapter 1 introduces related background and development of finite dimensional Hopf algebras. In Chapter 2, we give some basic definitions, important theorems and useful lemmas of Hopf algebras for remainder discussion. The main part of the thesis is Chapter 3 which discusses the classification of non-semisimple Hopf algebras of dimension $4p$. This chapter includes several cases by studying the possible number of group-like elements of H and H^*, and p-dimensional braided Hopf algebras in the Yetter-Drinfeld category. In particular, we complete the classification of Hopf algebras with dimensions 20, 28 and 44.

CHAPTER 2. Preliminaries

Throughout this thesis, we assume $\Bbbk$ to be an algebraically closed field of characteristic zero. All vector spaces, algebras, coalgebras, bialgebras and Hopf algebras are over the same base field $\Bbbk$ in the remaining discussion. In this chapter, we will collect some basic definitions and facts on Hopf algebras. The notation and convention introduced in this section will be used in the sequel.

2.1 Basic Definitions

In this section, we collect some elementary facts and introduce the definitions of bialgebras, Hopf algebras, and some related terminologies for our work. The readers are referred to [8], [10], [42] and [28] for more details.

2.1.1 Algebras

Now we recall some definitions about an $\Bbbk$-algebra A and its modules.

Definition 2.1.1. An $\Bbbk$-algebra A is a $\Bbbk$-linear space together with linear maps $m : A \otimes A \to A$ and $u : \Bbbk \to A$ satisfying the conditions:

$$\begin{aligned} &\text{Associativity:} && m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m) \\ &\text{Unit Property:} && m(u \otimes \mathrm{id}_A) = \mathrm{id}_A = m(\mathrm{id}_A \otimes u). \end{aligned}$$

The maps m and u are respectively called the multiplication and the unit of A. The unit property gives the existence of a unit $u(1) = 1_A$ in A.

Example 2.1.1. The set of all square matrices of order n with entries from $\Bbbk$ is a n^2-dimensional $\Bbbk$-algebra with respect to the ordinary operations on matrices. Denote by $M_n(\Bbbk)$.

Example 2.1.2. If V is a $\Bbbk$-space, then the space of the linear transformations of V forms an algebra $\mathrm{End}(V)$. The algebra $\mathrm{End}(V)$ is finite dimensional if and only if V is finite dimensional.

Example 2.1.3. Consider the elements of a group G as basis elements of a vector space. i.e. $\Bbbk G = \{\sum_{g\in G} \alpha_g g \,|\, \alpha_g \in \Bbbk \text{ all but finitely many are zero}\}$. The group multiplication defines the algebra structure over the space $\Bbbk G$. This algebra is called the group algebra of the group G over the field $\Bbbk$.

To investigate the structure and properties of algebras such as how to determine whether the structures of two $\Bbbk$-algebras are similar, we begin with introducing the morphisms between them.

Definition 2.1.2. An algebra map $\phi : A \to B$ is a $\Bbbk$-linear map such that

$$m_B(\phi \otimes \phi) = \phi m_A \quad \text{and} \quad \phi u_A = u_B$$

i.e. $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in A$ and $\phi(1_A) = 1_B$.
ϕ is an algebra isomorphism if it is injective and surjective.

From the representation theory of finite dimensional algebras, it is known that each element of an algebra can be viewed as a linear operator on a space V and this gives us a concept about modules over an algebra. In what follows, we assume that A is a finite dimensional $\Bbbk$-algebra and focus on A-modules with finite dimension.

Definition 2.1.3. A $\Bbbk$-linear space M is a left A-module if there is a module structure map $\gamma : A \otimes M \to M$ such that

$$\gamma(\mathrm{id}_A \otimes \gamma) = \gamma(m_A \otimes \mathrm{id}_M),$$

i.e. $a(b \cdot m) = (ab) \cdot m$ for any $a, b \in A$ and $m \in M$.

There is a correspondence between representations of an algebra A and A-modules. In particular, a regular representation of A corresponds to a regular A-module. i.e. let A act on itself with the action that is the multiplication of A.

Let M be an A-module. A subspace $N \subset M$ is a submodule of M if $a \cdot n \in N$ for any $a \in A$, $n \in N$. M is called an irreducible or simple A-module if M is the only non-zero submodule of

M. M is called decomposable if $M \cong M_1 \oplus M_2$ where M_1, M_2 are non-zero modules and called direct summands of M; otherwise M is indecomposable. Each finite dimensional A-module M admits a descending chain of A-submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_s = 0$$

such that all the factor modules M_i/M_{i+1} are simple. Such a chain is called a composition series of M and each simple factor module is called a composition factor of M. A submodule N of M is called maximal if $N \neq M$ and there is no submodule $L \subset M$ such that $M \supsetneq L \supsetneq N$. The left(resp. right) ideals of an algebra A are submodules of the left(resp. right) regular A-modules. A subspace I is an ideal of A if it is both a left and right ideal of A. An (two-sided) ideal I is indecomposable if it cannot be expressed as the direct sum of two non-zero ideals. A left(right or two sided) ideal I of A is called nilpotent if $I^n = 0$ for some $n > 0$.

Definition 2.1.4. Let M and N be left A-modules with structure maps γ and ν, respectively. A map $\phi : M \to N$ is a module morphism if $\nu(\mathrm{id}_A \otimes \phi) = \phi\gamma$.

The set of all A-module morphisms from M to N forms a $\Bbbk$-space and is denoted by $\mathrm{Hom}_A(M, N)$. When $M = N$, the space $\mathrm{Hom}_A(M, M)$ is a $\Bbbk$-algebra which is denoted by $\mathrm{End}_A(M)$. One can prove that A and $\mathrm{End}_A(A)$ are isomorphic as $\Bbbk$-algebras.

We say that $e \in A$ is an idempotent if $e^2 = e$. Two idempotents e and f are called orthogonal if $ef = fe = 0$. An idempotent is called minimal or primitive if e cannot be expressed as a sum of two non-zero idempotents. An idempotent e is called central if it commutes with all the elements of A.

For each A-module M, there is a bijective correspondence between: the decomposition into a direct sum of submodules, the decomposition of identity of the algebra $\mathrm{End}_A(M)$, and the decomposition of the regular module over the algebra $\mathrm{End}_A(M)$. In particular, when $M = A$, we have the following theorem:

Theorem 2.1.1. *There is a bijective correspondence between*

- *the decomposition into a direct product of algebras:* $A \cong A_1 \times \cdots \times A_s$;

- *the decomposition of the identity into central idempotents:* $1_A = e_1 + \cdots + e_s$*;*

- *the decomposition into a direct sum of ideals:* $A = I_1 \oplus \cdots \oplus I_s$*.*

A module M isomorphic to a direct sum of simple modules is called semisimple. An algebra A is called semisimple if its regular module is semisimple. The simple submodules of the regular left A-module are called minimal left ideals of A. An algebra that has no non-zero ideal different from zero and itself is called simple. Therefore, a simple algebra over $\Bbbk$ is isomorphic to an n^2-dimensional matrix algebra for some integer $n > 0$. Every semisimple algebra is a direct product of simple subalgebras and has a decomposition into minimal ideals. Moreover, the identity of a semisimple algebra A can be decomposed into the sum of primitive central idempotents. A commutative semisimple algebra over an algebraically closed field $\Bbbk$ is isomorphic to $\Bbbk^n$.

The annihilator of an element $m \in M$ is the set of all elements $a \in A$ such that $a \cdot m = 0$. It is a left ideal of A, which is maximal if and only if the submodule generated by m is simple. Define $\mathrm{ann} M := \{a \in A \big| a \cdot m = 0 \text{ for all } m \in M\}$, the intersection of the annihilators of the elements of M. It is a maximal ideal if and only if M is simple. Define $J = J(A)$, the Jacobson radical of A, to be the intersection of annihilators of the simple A-modules, i.e. the intersection of the maximal ideals. J is the maximal nilpotent ideal of A. Moreover, A is semisimple if and only if $J = 0$. For any A-module M, JM is the radical of M which is equal to the intersection of all maximal proper submodules of M. The socle of an A-module M is the sum of all simple submodules of M. The head of M is Head(M)= M/JM. Thus M/JM and $J^iM/J^{i+1}M$ are semisimple for $i > 0$. In particular, M is semisimple if $JM = 0$. An algebra A is called primary if A/J is a simple algebra. It is known that A is primary if and only if A has a single simple module.

An A-module P is called projective if for any surjective A-module map $p : M \to N$ and $f \in \mathrm{Hom}_A(P, N)$, there exists $\widetilde{f} \in \mathrm{Hom}_A(P, M)$ such that $f = p\widetilde{f}$. Or equivalently, for any exact sequence of A-modules:

$$0 \to N \to X \xrightarrow{\pi} P \to 0,$$

the A-module map π splits, i.e. $X \cong P \oplus N$. A projective module P is a projective cover of a module V if there is an epimorphism $p : P \to V$ which induces $P/JP \cong V/JV$. An A-module I is injective if for any injective A-module map $\iota : N \to M$ and $g \in \mathrm{Hom}_A(N, I)$, there exists $\widetilde{g} \in \mathrm{Hom}_A(I, M)$ such that $g = \widetilde{g}\iota$. An injective hull I of an A-module V means I is a minimal injective A-module containing V. A left A-module M is an extension of V by W if there exists an epimorphism $f \in \mathrm{Hom}_A(M, V)$ such that $\ker f = W$. Thus M fits into an exact sequence

$$0 \to W \to M \xrightarrow{f} V \to 0.$$

If this sequence splits, or equivalently, W is a summand of M, then the M is called a split extension of V.

For each finite dimensional $\Bbbk$-algebra A, A itself is a left and right regular A-module, and its $\Bbbk$-linear dual $A^* = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ has a structure of (A, A)-bimodule, namely,

$$(x \cdot f)(a) = f(ax) \quad \text{and} \quad (f \cdot y)(a) = f(ya)$$

for any $x, y, a \in A$ and $f \in A^*$. A finite dimensional $\Bbbk$-algebra A is a Frobenius algebra if the left A-modules A and A^* are isomorphic.

Theorem 2.1.2. *Let A be a finite dimensional $\Bbbk$-algebra. The following are equivalent:*

(1) *A is a Frobenius algebra*

(2) *There is an associative non-degenerate bilinear form $\lambda : A \times A \to \Bbbk$ such that $\lambda(ab, c) = \lambda(a, bc)$ for any $a, b, c \in A$.*

(3) *There is a linear functional $f : A \to \Bbbk$ such that $\ker f$ does not contain any non-zero left or right ideals of A.*

Let $\{a_i\}$ be a basis of a Frobenius algebra A. As $A \cong A^*$, there exists $\{b_i\}$ in A such that $\lambda(a_i, b_j) = \delta_{ij}$. The pair $\{a_i\}$, $\{b_i\}$ is called a pair of dual bases for A with respect to λ. Thus we have $a_i a = \sum_j \lambda(a_i a, b_j) a_j$ and $ab_i = \sum_j b_j \lambda(a_j, ab_i)$ for any $a \in A$.

A non-semisimple finite dimensional $\Bbbk$-algebra A has a unique decomposition into a direct sum of (principle) indecomposable A-modules up to isomorphism. Each principle indecomposable A-module has the form Ae with a primitive idempotent e. Moreover, every principle indecomposable A-module is a projective cover $P(V)$ for some simple A-module V. From a property of Frobenius algebras, an indecomposable A-module is projective if, and only if, it is injective.

2.1.2 Coalgebras

The notion of a coalgebra is dual to that of an algebra. It is an important ingredient in the definition of a Hopf algebra.

Definition 2.1.5. A coalgebra is a vector space C together with linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$, called the comultiplication and the counit respectively, such that

$$\begin{aligned} &\text{Coassociativity:} && (\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta, \\ &\text{Counit Property:} && (\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes \varepsilon)\Delta\,. \end{aligned}$$

Example 2.1.4. Let S be a finite set and $\Bbbk S$ be the $\Bbbk$-space with basis S. Then $\Bbbk S$ is a coalgebra with comultiplication Δ and the counite ε defined by $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1$ for any $s \in S$. In particular, the field $\Bbbk$ is a $\Bbbk$-coalgebra with the comultiplication $\Delta : \Bbbk \to \Bbbk \otimes \Bbbk$ defined by $\Delta(c) = c \otimes 1$ for any $c \in \Bbbk$ and the counit $\varepsilon : \Bbbk \to \Bbbk$ the identity map.

Example 2.1.5. Assume that $M_n(\Bbbk)$ is a matrix algebra of order n with basis $(e_{ij})_{1\leq i,k\leq n}$, where $e_{ij}e_{pq} = \delta_{jp}e_{iq}$ for any i, j, p, q. Let $(e^*_{ij})_{1\leq i,j\leq n}$ be the dual basis of the dual space $M_n(\Bbbk)^*$. Then $M_n(\Bbbk)^*$ is a coalgebra with the comultiplication Δ and the counit ε

$$\Delta(e^*_{ij}) = \sum_{1\leq q\leq n} e^*_{iq} \otimes e^*_{qj}, \quad \varepsilon(e^*_{ij}) = \delta_{ij}.$$

Let D be a coalgebra. A coalgebra map $\phi : C \to D$ is a linear map such that

$$(\phi \otimes \phi)\Delta_C = \Delta_D\phi \quad \text{and} \quad \varepsilon_D\phi = \varepsilon_C.$$

To perform calculation in a coalgebra, we use the Sweedler notation for the comultiplication of $c \in C$ with the summation notation suppressed:

$$\Delta(c) = c_1 \otimes c_2.$$

The coassociativity axiom allows us to write

$$\Delta_n(c) = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \cdots \Delta(c) = c_1 \otimes c_2 \otimes \cdots \otimes c_{n+1}$$

by applying the comultiplication n times with $n \geq 1$.

A linear map $\tau : V \otimes W \to W \otimes V$ of spaces defined by $\tau(v \otimes w) = w \otimes v$ is called a twist map. A coalgebra C is cocommutative if $\Delta(c) = \tau \circ \Delta(c)$ for all $c \in C$.

An element $g \in C$ is called group-like if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Denote by $G(C)$ the set of group-like elements, which is always independent. For $g, h \in G(C)$, c is called a (g, h)-primitive element if $\Delta(c) = c \otimes g + h \otimes c$. Denote by $\mathcal{P}_{g,h} = \{c \in C | \Delta(c) = c \otimes g + h \otimes c\}$ the space of all (g, h)-primitive elements. Note that the space $\Bbbk(g - h) \subseteq \mathcal{P}_{g,h}$ for any $g, h \in G(C)$. A (g, h)-skew primitive x is called non-trivial if $x \notin \Bbbk(g - h)$.

A subspace $I \subseteq C$ is a left coideal if $\Delta(I) \subseteq C \otimes I$. A space $I \subseteq C$ is a coideal if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and if $\varepsilon(I) = 0$. A coalgebra is simple if it has no non-trivial subcoalgebras. A coalgebra is called cosemisimple if it is a sum of simple subcoalgebras. A coalgebra C is pointed if every simple subcoalgebra is one-dimensional, i.e. C is spanned by a group-like element. The coradical C_0 is defined to be the sum of all simple subcoalgebras. The coalgebra C is called irreducible if C_0 is a simple coalgebra and it is called connected if C_0 is one-dimensional.

As the notion of coalgebras is dual to that of algebras, we are also interested in the dual notion of modules of an algebra.

Definition 2.1.6. Let C be a coalgebra. A left comodule over C is a vector space M with a linear map $\rho : M \to C \otimes M$, called the C-coaction of M, such that

$$(\mathrm{id}_C \otimes \rho)\rho = (\Delta \otimes \mathrm{id}_M)\rho \quad \text{ and } (\varepsilon_C \otimes \mathrm{id}_M)\rho = \mathrm{id}_M \ .$$

For a left comodule M with the structure map $\rho : M \to C \otimes M$, we use Sweedler's notation with the summation suppressed: $\rho(m) = m_{(-1)} \otimes m_{(0)}$. The coassociativity axiom gives

$$(m_{(-1)})_{(1)} \otimes (m_{(-1)})_{(2)} \otimes m_{(0)} = m_{(-1)} \otimes (m_{(0)})_{(-1)} \otimes (m_{(0)})_{(0)},$$

We write the expression as

$$(\Delta \otimes \mathrm{id})\rho(m) = (\mathrm{id} \otimes \rho)\rho(m) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}.$$

A map $\phi : M \to N$ between C-comodules is a comodule morphism if

$$(\mathrm{id}_C \otimes \phi) \circ \rho_M = \rho_N \circ \phi.$$

A left C-comodule is simple if it has no proper subcomodules.

Similarly, a right C-comodule M is a linear space with the structure map $\rho : M \to M \otimes C$ given by $\rho(m) = m_{(0)} \otimes m_{(1)}$ satisfying

$$(\rho \otimes \mathrm{id}_C)\rho = (\mathrm{id}_M \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id}_M \otimes \varepsilon_C)\rho = \mathrm{id}_M .$$

A coalgebra C is a right and a left comodule of C using $\rho = \Delta$.

Lemma 2.1.3. (1) *If A is a finite dimensional algebra, then the dual $\Bbbk$-space A^* is a coalgebra, with the comultiplication $\Delta = m^*$ and counit $\varepsilon = u^*$. If A is commutative, then A^* is cocommutative.*

(2) *If a coalgebra C is finite dimensional, then it is cosemisimple if, and only if, C^* is cosemisimple.*

(3) *If M is a right C-comodule with the structure map $\rho : M \to M \otimes C$ given by $\rho(m) = m_{(0)} \otimes m_{(1)}$, then M is a left C^*-module via $f \cdot m = f(m_{(1)})m_{(0)}$ for any $f \in C^*$.*

2.1.3 Hopf algebras

Let (C, Δ, ε) be a $\Bbbk$-coalgebra and (A, m, u) be an $\Bbbk$-algebra. Then the space $\mathrm{Hom}_{\Bbbk}(C, A)$ becomes an associative algebra with multiplication defined by the so-called convolution product

$$f * g = m \circ (f \otimes g) \circ \Delta \quad \text{for any } f, g \in \mathrm{Hom}_{\Bbbk}(C, A),$$

i.e. $(f * g)(c) = f(c_1)g(c_2)$ for all $c \in C$.

Definition 2.1.7. A 5-tuple is called a bialgebra $(H, m, u, \Delta, \varepsilon)$ if (H, m, u) is an algebra and (H, Δ, ε) is a coalgebra such that one of the following holds

- Δ and ε are algebra morphisms,
- m amd u are coalgebra morphisms.

The bialgebra H is called a Hopf algebra if the identity map id_H is invertible in the convolution algebra $\mathrm{End}_{\Bbbk}(H)$. That is, there exists an element $S \in \mathrm{End}_{\Bbbk}(H)$, called an antipode of H, such that $S * \mathrm{id}_H = u \circ \varepsilon = \mathrm{id}_H * S$, i.e. for all $h \in H$, S satisfies $S(h_1)h_2 = \varepsilon(h)1_H = h_1 S(h_2)$. The antipode S is an anti-algebra and anti-coalgebra morphism, that is, for any $h, k \in H$,

$$S(hk) = S(k)S(h), \quad S(1_H) = 1_H$$

$$S(h)_{(1)} \otimes S(h)_{(2)} = S(h_{(2)}) \otimes S(h_{(1)}), \quad \varepsilon \circ S(h) = \varepsilon(h).$$

A morphism $\phi : H \to H'$ of bialgebras is a morphism of algebra and coalgebra. A Hopf algebra morphism is a bialgebra morphism and $\phi \circ S_H(h) = S_{H'} \circ \phi(h)$ for all $h \in H$. A subspace $I \subseteq H$ is a Hopf ideal if it is an ideal, a coideal and $S(I) \subseteq I$. For a Hopf ideal I, the quotient H/I is a Hopf algebra with its structure induced from H.

A $\Bbbk$-space M is called a left Hopf module if M is a left H-module and a left H-comodule with the H-coaction $\rho : M \to H \otimes M$ such that ρ is a left H-module map, i.e.

$$(h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} = h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}.$$

Every Hopf algebra H is an H-Hopf module using $\rho = \Delta$.

If H is a finite dimensional Hopf algebra over $\Bbbk$ with the antipode S, then its linear dual H^* is also a Hopf algebra by reversing all structure maps. The coalgebra structure of H^* is given by $\Delta_{H^*} f(h \otimes k) = m^* f(h \otimes k) = f(hk)$ for any $f \in H^*$, $h, k \in H$. The antipode of H^*, S^*, is the transpose of S.

An element $\Lambda \in H$ is called a right (resp. left) integral if $\Lambda h = \varepsilon(h)\Lambda$ (resp. $h\Lambda = \varepsilon(h)\Lambda$) for all $h \in H$. Denote by $\int_H^r$ and $\int_H^\ell$ the corresponding space of right and left integrals. Both $\int_H^r$ and $\int_H^\ell$ are ideals of H. The Hopf algebra H is called unimodular if $\int_H^r = \int_H^\ell$. Similarly, a nonzero element $\lambda \in H^*$ is a right integral if $\lambda * f = \lambda(1) f$ for any $f \in H^*$. The distinguished group-like element $a \in H$ is defined by $f * \lambda = f(a)\lambda$.

The set $G(H)$ of group-like elements of H forms a group under the multiplication of H. The distinguished group-like element $\alpha \in H^*$ is defined by $\Lambda h = \alpha(h)\Lambda$ for any $h \in H$. Recall that a Frobenius algebra is an algebra such that there exists an associative non-degenerated bilinear form on it. The non-zero right integral λ defines a non-degenerate associative bilinear form on H, and hence H is a Frobenius algebra.

Example 2.1.6. Given a finite group G of order n. The group algebra $\Bbbk G$ has a coalgebra structure defined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for any $g \in G$. The group algebra $\Bbbk G$ is a Hopf algebra with the antipode $S : \Bbbk G \to \Bbbk G$ defined by $S(g) = g^{-1}$. The group algebra $\Bbbk G$ and its dual are semisimple Hopf algebras.

Example 2.1.7. Let n be a positive integer, and $\xi \in \Bbbk$ a primitive n-th root of unity. Consider a Hopf algebra $T(\xi)$ generated by the elements g and x satisfying the relations

$$g^n = 1, \quad x^n = 0, \quad xg = \xi gx.$$

Its coalgebra structure and the antipode S are defined by:

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$

$$S(g) = g^{-1}, \qquad S(x) = -xg^{-1}.$$

This Hopf algebra is called a Taft algebra of dimension n^2 with basis $\{g^i x^j | 0 \leq i, j \leq n-1\}$. The Taft algebras are neither commutative nor cocommutative, and the square of their antipodes are not the identity. In fact, the order of S is $2n$. It is known that Taft algebras are non-semisimple self-dual Hopf algebras.

2.2 General results

In this section, we give some known results for finite dimensional Hopf algebras over $\Bbbk$. Those properties will be repeatedly used in the remainder of the thesis. The last result of the following lemma is the generalization of Lagrange's theorem, due to Nichols and Zoeller. Our references for those results are [23], [24], [28], [37] and [40].

Lemma 2.2.1. *Let H be a finite dimensional Hopf algebra over $\Bbbk$ with antipode S. Let $\alpha \in G(H^*)$ and $a \in G(H)$ be the distinguished group-like elements. Let K be a Hopf subalgebra of H. Then*

(1) *Both $\int_H^r$ and $\int_H^\ell$ are one-dimensional.*

(2) *The antipode of S is bijective, and $S(\int_H^\ell) = \int_H^r$.*

(3) *The order of the antipode S is finite.*

(4) *H is a Frobenius algebra.*

(5) *H is free as a left (right) K-module. In particular,* $\dim K$ *divides* $\dim H$.

Besides the results above, the following important formula was discovered by Radford:

$$S^4(h) = a(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})a^{-1} \quad \text{for all } h \in H. \tag{2.2.1}$$

There are a couple of nice results which describe some relation between the semisimplicity and the order of the square of the antipode. Moreover, there is a Maschke Theorem for Hopf algebras.

Lemma 2.2.2. *Let H be a finite dimensional Hopf algebra over $\Bbbk$ with antipode S. The following are equivalent:*

(1) *H is semisimple;*

(2) *H^* is semisimple;*

(3) *$\varepsilon(\int_H^\ell) \neq 0$ ($\Leftrightarrow \varepsilon(\int_H^r) \neq 0$);*

(4) *$S^2 = \mathrm{id}_H$;*

(5) *$\mathrm{Tr}(S^2) \neq 0$;*

(6) *$\mathrm{Tr}(S^2 \circ r(b)) = 0$ for all $b \in H$, where $r(b)$ is the linear operator defined by $r(b)(a) = ab$ for any $a \in H$.*

The following result is very useful to the classification of general finite dimensional non-semisimple Hopf algebras. Let ξ be a primitive m-th root of unity. The Hopf algebra $K = K_\mu(mn, \xi)$ is generated by x, g subject to the relations

$$x^m = \mu(1 - g^m), \quad g^{mn} = 1, \quad gx = \xi xg,$$

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \varepsilon(x) = 0, \quad \varepsilon(g) = 1,$$

$$S(g) = g^{-1}, \quad S(x) = -xg^{-1},$$

where $\mu = 0$ if $n = 1$ and $\mu \in \{0, 1\}$ if $n \neq 1$.

Lemma 2.2.3 ([1]). *Let H be a non-semisimple finite dimensional Hopf algebra over $\Bbbk$. If H contains a non-trivial skew primitive element, then H contains a pointed Hopf subalgebra $K = K_\mu(mn, \xi)$ of dimension $m^2 n$ where m, n are nonnegative integers with $m > 1$. Moreover, when $n = 1$, K is isomorphic to a Taft algebra of dimension m^2.*

We list a few results about Hopf algebras of particular dimensions which are related to our work.

Lemma 2.2.4 ([44]). *Every Hopf algebra of dimension p is a group algebra.*

Lemma 2.2.5 ([27],[33]). *Let H be a Hopf algebras of dimension p^2, p is a prime.*

(1) *If H is semisimple, then H is isomorphic to a group algebra either $\Bbbk[\mathbb{Z}_{p^2}]$ or $\Bbbk[\mathbb{Z}_p \times \mathbb{Z}_p]$.*

(2) *If H is non-semisimple, then H is isomorphic to a Taft algebra.*

Lemma 2.2.6 ([35]). *Let p be an odd prime. A $2p$-dimensional Hopf algebra over $\Bbbk$ is semisimple and isomorphic to $\Bbbk[\mathbb{Z}_{2p}]$, $\Bbbk[D_{2p}]$ or $\Bbbk[D_{2p}]^*$, where $\mathbb{Z}_{2p}$ is a cyclic group of order $2p$, and D_{2p} is a dihedral group of order $2p$.*

2.3 Notation and useful lemmas

Let H be a finite dimensional Hopf algebra over $\Bbbk$ with the comultiplication Δ, the counit ε and the antipode S. Tensor product $\otimes$, $\mathrm{End}(H)$ and $\mathrm{Hom}(H, \Bbbk)$ are taken over $\Bbbk$ when not specified.

The linear dual H^* of H has natural Hopf algebra structures and from Lemma 2.1.3, the comultiplication Δ determines the natural actions $\rightharpoonup$ and $\leftharpoonup$ of H^* on H given by

$$f \rightharpoonup h = h_1 f(h_2) \quad \text{and} \quad h \leftharpoonup f = f(h_1)h_2, \text{ for any } h \in H, f \in H^*.$$

For $\beta \in G(H^*)$, β is an algebra epimorphism from H onto $\Bbbk$ and an irreducible character of H with degree 1. We write $\Bbbk_\beta$ as the 1-dimensional simple module which has character β and $\Bbbk$ as the trivial H-module $\Bbbk_\varepsilon$.

A semisimple Hopf algebra is called trivial if it is isomorphic to a group algebra or the dual of a group algebra.

A Hopf algebra H is pointed if every simple left or right H-comodule is one-dimensional, i.e. the coradical H_0 is a group algebra. Equivalently, H^* is a basic algebra.

For a left H-module V, the antipode S of H can be used to construct a left H-module structure on the linear dual $V^* = \mathrm{Hom}(V, \Bbbk)$. The left dual $V^\vee$ is the left H-module with the underlying space $V^* = \mathrm{Hom}(V, \Bbbk)$ and the H-action defined by

$$(hf)(v) = f(S(h)v) \quad \text{for any } f \in V^*, h \in H, v \in V.$$

The right dual ${}^\vee V$ is defined similarly with the different H-action given by

$$(hf)(v) = f(S^{-1}(h)v) \quad \text{for any } f \in V^*, h \in H, v \in V.$$

If V is a finite dimensional simple H-module, then $V^\vee$ and ${}^\vee V$ are simple H-modules.

Given an algebra automorphism σ of H, ${}_\sigma V$ is an H-module with the underlying space V and the action given by $h \cdot_\sigma v = \sigma(h)v$ for $h \in H, v \in V$. Since S^2 is an algebra automorphism of H, we have the H-module isomorphisms

$$V^{\vee\vee} \cong {}_{S^2}V \quad \text{and} \quad {}^{\vee\vee}V \cong {}_{S^{-2}}V.$$

Moreover, by Radford's formula, we have

$$\Bbbk_\alpha \otimes V^{\vee\vee\vee\vee} \otimes \Bbbk_{\alpha^{-1}} \cong \Bbbk_\alpha \otimes {}_{S^4}V \otimes \Bbbk_{\alpha^{-1}} \cong {}_\sigma V \cong V, \tag{2.3.2}$$

where $\sigma(h) = aha^{-1}$ for the distinguished group-like element $a \in G(H)$.

Let H be a non-semisimple finite dimensional Hopf algebra. Denote $P(V)$ and $I(V)$ the projective cover and the injective hull of a finite dimensional H-module V respectively. Denote $\mathrm{Irr}(H)$ a complete set of non-isomorphic simple H-modules. For any $V \in \mathrm{Irr}(H)$ and a distinguished group-like element $\alpha \in G(H^*)$, it is shown in Lemma 1.1 of [36] that

$$\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V \cong \mathrm{Soc}(P(V)) \text{ and } V \cong \mathrm{Soc}(P(\Bbbk_\alpha \otimes V^{\vee\vee})).$$

Therefore, we get

$$P(V) \cong I(\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V) \quad \text{and} \quad I(V) \cong P(\Bbbk_\alpha \otimes V^{\vee\vee}). \tag{2.3.3}$$

For any finite dimensional H-modules U, V, W, we have the following natural isomorphisms:

$$\mathrm{Hom}_H(V^\vee \otimes U, W) \cong \mathrm{Hom}_H(U, V \otimes W) \cong \mathrm{Hom}_H(U \otimes {}^\vee W, V) \tag{2.3.4}$$

Moreover, if P is projective, then $V \otimes P$, $P \otimes V$ are projective.(cf. [22])

Let M, V be finite dimensional H-modules. Define $[M : V] := \dim \mathrm{Hom}_H(P(V), M)$. Then $[M : V]$ is equal to the multiplicity of V appearing as a composition factor of M for any simple H-module V. By duality, we also have

$$[M : V] = \dim \mathrm{Hom}_H(M, I(V)). \tag{2.3.5}$$

Lemma 2.3.1. *Let V, W be simple H-modules. Then*

$$[P(V) : W] = [P(W) : \Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V].$$

In addition, if $\mathrm{dim}V \neq \mathrm{dim}W$ *and* $[P(V) : W] > 0$, *then*

$$\mathrm{dim}P(W) \geq 2\mathrm{dim}W + [P(V) : W]\mathrm{dim}V \quad \textit{and} \tag{2.3.6}$$

$$\mathrm{dim}P(V) \geq 2\mathrm{dim}V + [P(V) : W]\mathrm{dim}W. \tag{2.3.7}$$

Proof. By (2.3.3) and (2.3.5), we have

$$\begin{aligned}[P(V) : W] &= \mathrm{dimHom}_H(P(W), P(V)) \\ &= \mathrm{dimHom}_H(P(W), I(\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V)) = [P(W) : \Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V].\end{aligned}$$

If $[P(V):W]>0$ and $\dim V\neq\dim W$, then $[P(W):\Bbbk_{\alpha^{-1}}\otimes{}^{\vee\vee}V]>0$ and $P(W)\ncong W$. Thus, besides the socle and the head, $P(W)$ has the composition factor $\Bbbk_{\alpha^{-1}}\otimes{}^{\vee\vee}V$ with multiplicity $[P(V):W]$. Therefore,

$$\dim P(W)\geq 2\dim W+[P(V):W]\dim V\,.$$

The second inequality can be obtained by the same argument. □

Remark 2.3.1. For $V\in \mathrm{Irr}(H)$, the H-module $V^\vee\otimes P(V)$ is projective and $\dim\mathrm{Hom}_H(V^\vee\otimes P(V),\Bbbk_\beta)$ is equal to the multiplicity of $P(\Bbbk_\beta)$ in an indecomposable decomposition of $V^\vee\otimes P(V)$ for all $\beta\in G(H^*)$. From (2.3.4), we have

$$\dim\mathrm{Hom}_H(V^\vee\otimes P(V),\Bbbk_\beta)=[V\otimes\Bbbk_\beta:V]=\begin{cases}1, & \text{if } V\cong V\otimes\Bbbk_\beta;\\ 0, & \text{otherwise.}\end{cases}\tag{2.3.8}$$

The first of the following lemma comes from Lemma 2.1 and the second has been presented in the proof of Lemma 2.3 in [15].

Lemma 2.3.2 ([15])**.** *Let H be a non-semisimple Hopf algebra over $\Bbbk$ of finite dimension.*

(1) *The one-dimensional simple H-module $\Bbbk$ is not projective.*

(2) *If there exists an indecomposable 2-dimensional H-module which has an 1-dimensional simple quotient, then H^* must contain a non-trivial skew primitive element.*

Lemma 2.3.3. *Let H be a finite-dimensional non-semisimple Hopf algebra with Jacobson radical J. For any simple H-modules V,W, we have*

$$[JP(V)/J^2P(V):W]=\dim\mathrm{Ext}(V,W)=[\mathrm{Soc}(I(W)/W):V].$$

In addition, if H^* has no non-trivial skew primitive element, then all the simple H-submodules of $JP(\Bbbk)/J^2P(\Bbbk)$ or $\mathrm{Soc}(I(\Bbbk)/\Bbbk)$ are not 1-dimensional.

Proof. Let us abbreviate $P(V)$ as P, and consider the natural exact sequence

$$0\to JP\to P\to V\to 0\,.$$

For any simple H-module W, we have the associated long exact sequence

$$0 \to \mathrm{Hom}_H(V,W) \to \mathrm{Hom}_H(P,W) \to \mathrm{Hom}_H(JP,W) \to \mathrm{Ext}(V,W) \to \mathrm{Ext}(P,W)\,.$$

Since $\mathrm{Ext}(P,W)=0$ and $\mathrm{Hom}_H(V,W) \to \mathrm{Hom}_H(P,W)$ is an isomorphism, we find

$$\mathrm{Hom}_H(JP/J^2P,W) \cong \mathrm{Hom}_H(JP,W) \xrightarrow{\cong} \mathrm{Ext}(V,W)\,.$$

Therefore, $[JP/J^2P : W] = \dim \mathrm{Ext}(V,W)$. The second equality can be obtained similarly by considering the exact sequence

$$0 \to W \to I(W) \to I(W)/W \to 0\,.$$

If H^* does not have any non-trivial skew primitive element, then $\mathrm{Ext}(V,W)=0$ for any 1-dimensional H-modules V,W. Therefore,

$$[JP(\Bbbk)/J^2P(\Bbbk) : W] = [\mathrm{Soc}(I(\Bbbk)/\Bbbk) : W] = 0$$

for all 1-dimensional H-modules W. □

The dimensions of projective modules over a finite-dimensional Hopf algebra are of particular importance to the remaining chapters. The following fact on linear algebra is quite useful to study these dimensions.

Lemma 2.3.4 ([35, Lemma 1.4])**.** *Let V be a finite-dimensional vector space over the field $\Bbbk$, p a prime, and T a linear automorphism on V such that $\mathrm{Tr}(T)=0$. If $T^{p^n}=\mathrm{id}_V$ for some positive integer n, then p divides the dimension of V.*

The following corollary is an application of the preceding lemma.

Corollary 2.3.5. *Let H be a non-semisimple Hopf algebra over $\Bbbk$ with antipode S. If $\mathrm{ord}(S^2)=p^n$ for some prime p, then every indecomposable projective H-module P such that $P \cong P^{\vee\vee}$ as H-modules has dimension divisible by p.*

Proof. Let P be an indecomposable projective H-module such that $P \cong P^{\vee\vee}$. In view of (2.3.2), $P \cong {}_{S^2}P$. By Lemma A.2 in [21], there exists an H-module isomorphism $\varphi : P \to {}_{S^2}P$ such that $\varphi^{p^n} = \mathrm{id}$. It follows from Lemma 1.3 of [36], $\mathrm{Tr}(\varphi)=0$. Therefore, Lemma 2.3.4 implies that p divides $\dim P$. □

2.4 Braided Hopf algebras

In this section, we introduce a specific type of Hopf algebras – Bosonization or Radford biproduct. A large part of chapter 3 in this thesis shall involve this kind of Hopf algebras. We begin by recalling the notion of braided monoidal category, Yetter-Drinfeld category, braided Hopf algebras and bosonization. Our main references for this section are [2], [5] and [39].

2.4.1 Yetter-Drinfeld category

Braided Hopf algebras play an important role in the structure theory of pointed Hopf algebras. We start with the definition of strict monoidal categories.

Definition 2.4.1. A strict monoidal category is a 3-tuple $(\mathcal{C}, \otimes, \mathbb{I})$ where $\mathcal{C}$ is a category, the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, the unit $\mathbb{I}$ is an object of $\mathcal{C}$ such that

$$(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad U \otimes \mathbb{I} = U = \mathbb{I} \otimes U$$

for all $U, V, W \in \mathcal{C}$

Definition 2.4.2. A strict braided monoidal category $\mathcal{C}$ is a collection $(\mathcal{C}, \otimes, \mathbb{I}, c)$, where $(\mathcal{C}, \otimes, \mathbb{I})$ is a strict monoidal category and $c_{V,W} : V \otimes W \to W \otimes V$ is a natural isomorphism subject to the conditions

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W), \quad c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}).$$

In this thesis, our main example of strict braided monoidal category is the Yetter-Drinfeld category over a Hopf algebra.

Definition 2.4.3. Let B be a Hopf algebra over $\Bbbk$ with bijective antipode S. A (left) Yetter-Drinfeld module M over B is a left B-module, a left B-comodule and satisfies the compatibility condition

$$(bm)_{(-1)} \otimes (bm)_{(0)} = b_{(1)} m_{(-1)} S(b_{(3)}) \otimes b_{(2)} m_{(0)}, \quad \forall b \in B, m \in M.$$

We denote by ${}^B_B\mathcal{YD}$ the category of left Yetter-Drinfeld modules over B.

The category ${}^B_B\mathcal{YD}$ is strict monoidal with $\mathbb{I} = \Bbbk$ and the usual tensor product $\otimes$ of vector spaces. The tensor product of two Yetter-Drinfeld modules M, N is again a Yetter-Drinfeld module with structure

$$b(m \otimes n) = b_1 m \otimes b_2 n, \quad (m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} = m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}.$$

It is also a braided monoidal category with the braiding:

$$c_{M,N} : M \otimes N \to N \otimes M, \quad c(m \otimes n) = m_{(-1)} n \otimes m_{(0)},$$

which is an isomorphism with inverse

$$(c_{N,M})^{-1} = (c^{-1})_{M,N} : m \otimes n \mapsto n_{(0)} \otimes S^{-1}(n_{(-1)}) m.$$

If $M \in {}^B_B\mathcal{YD}$ is finite dimensional, the linear dual $M^* = \mathrm{Hom}(M, \Bbbk)$ is a Yetter-Drinfeld module with action and coaction

- $(b \cdot f)(m) = f(S(b)m)$, for all $b \in B$, $f \in M^*$ and $m \in M$.
- $\rho(f) = f_{(-1)} \otimes f_{(0)} := \sum\limits_{1 \leq i \leq \ell} S^{-1}(m_{i(-1)}) \otimes f(m_{i(0)}) m_i{}^*$, where $\{m_i\}_{1 \leq i \leq \ell}$ is a basis of M with dual basis $\{m_i^*\}_{1 \leq i \leq \ell}$.

It is natural to define algebras and coalgebras in a strict monoidal category $\mathcal{C}$. An algebra in $\mathcal{C}$ is a triple (A, m, u), where A is an object of $\mathcal{C}$, $m : A \otimes A \to A$ and $u : \mathbb{I} \to A$ are morphisms in $\mathcal{C}$ such that

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \quad \text{and} \quad m \circ (u \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes u).$$

A coalgebra in $\mathcal{C}$ is a triple (C, Δ, ε), where C is an object of $\mathcal{C}$, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{I}$ are morphisms in $\mathcal{C}$ such that

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes \varepsilon) \circ \Delta.$$

Definition 2.4.4. A bialgebra in a braided category $\mathcal{C}$ is a 5-tuple $(R, m, u, \Delta, \varepsilon)$, where (R, m, u) is an algebra in $\mathcal{C}$ and (R, Δ, ε) is a coalgebra in $\mathcal{C}$, $m, u, \Delta, \varepsilon$ are morphisms of $\mathcal{C}$ satisfying the compatibility

$$\Delta m = (m \otimes m) \circ (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R) \circ (\Delta \otimes \Delta).$$

A braided Hopf algebra R is a bialgebra with a morphism $S : R \to R$ (the antipode), which is the convolution inverse of the identity in the monoid $\mathrm{Hom}_{\mathcal{C}}(R, R)$, satisfying the usual axioms

$$m(S \otimes \mathrm{id})\Delta = u\varepsilon = m(\mathrm{id} \otimes S)\Delta.$$

Notice that the multiplication of $R \otimes R$ and the coproduct involve the braiding $c_{R,R}$ which is not a usual twisting map,

$$m_{R\otimes R} = (m_R \otimes m_R) \circ (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R), \quad \Delta_{R\otimes R} = (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R) \circ (\Delta_R \otimes \Delta_R).$$

2.4.2 Bosonization or Biproduct

There are a few ways to obtain a Hopf algebra. We shall introduce one kind by extending a finite dimensional Hopf algebra, called bosonization or Radford biproduct.

An algebra A is called a left H-module algebra if A is a left H-module and the multiplication and unit map m_A, u_A are H-module maps. An algebra A is called a left H-comodule algebra if A is a left H-comodule and m_A and u_A are left H-comodule maps.

Recall that a finite dimensional braided Hopf algebra R in the Yetter-Drinfeld category ${}^B_B\mathcal{YD}$ is a Yetter-Drinfeld module in ${}^B_B\mathcal{YD}$, a bialgebra $(R, m_R, u_R, \Delta_R, \varepsilon_R)$ in the braided category ${}^B_B\mathcal{YD}$, and the identity id_R has a convolution inverse S_R, the antipode.

We now let H and B be finite dimensional Hopf algebras over $\Bbbk$ with antipodes S, S_B.

Assume that the injection $\iota : B \to H$ and the surjection $\pi : H \to B$ are Hopf algebra maps such that $\pi\iota = \mathrm{id}_B$. By analogy with group theory, one may reconstruct H as an extension from B. Let

$$R = H^{\mathrm{co}\,\pi} = \{h \in H : (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1_B\},$$

the coinvariant of π in H. It is immediate to see that R is a subalgebra of H, with the same unit. However, R is not a Hopf algebra in general, but it is a braided Hopf algebra in the Yetter-Drinfeld category ${}^B_B\mathcal{YD}$ with the following structures:

comultiplication $\Delta_R(r) = r_{(1)}\iota\pi S(r_{(2)}) \otimes r_{(3)}$, antipode $S_R(r) = \pi(r_{(1)})S(r_{(2)})$,

action $b \triangleright r = b_{(1)} r S_B(b_{(2)})$, coaction $\rho_R(r) = (\pi \otimes \mathrm{id})\Delta(r)$.

In order to distinguish comultiplications in usual Hopf algebras from those in braided Hopf algebras, we use Sweedler notation with upper indices for braided Hopf algebras $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$.

If we assume that R is a braided Hopf algebra in ${}^B_B\mathcal{YD}$, then the space $R \otimes B$ is turned to be a bialgebra $R \times B$ by using the cross product

$$(r \times b)(r \times b') = (rb_1 \triangleright r') \times b_2 b'$$

with unit $1_R \times 1_B$ and the cross coproduct

$$\Delta_{R\times B}(r \times b) = r^{(1)} \times r^{(2)}{}_{(-1)} b_1 \otimes r^{(2)}{}_{(0)} \times b_2$$

with counit $\varepsilon_R \otimes \varepsilon_B$. This bialgebra is an ordinary Hopf algebra with antipode

$$S_{R\times B}(r \times b) = \big(1_R \times S_B(r_{(-1)}b)\big)\big(S_R(r_{(0)}) \times 1_B\big).$$

This Hopf algebra is called the bosonization or biproduct of R by B. We have a Hopf algebra projection

$$\pi : R \times B \to B, \quad \pi(r \times b) = \varepsilon_R(r)b,$$

and a Hopf algebra injection

$$\iota : B \to R \times B, \quad \iota(b) = 1_R \times b.$$

so that $\pi\iota = \mathrm{id}_B$. Therefore, R is a subalgebra of $R \times B$ and B is a Hopf subalgebra of $R \times B$.

We have seen that the Radford biproduct has a projection onto a Hopf subalgebra. Define a linear map $\theta : H \to R$ by $\theta(h) = h_1 \iota S_B \pi(h_2)$. Then θ is a coalgebra epimorphism and by [5], [39], we have the following theorem which explains that these constructions are inverse to each other.

Theorem 2.4.1. *The maps*

$$H \to R \times B, \quad h \mapsto \theta(h_1) \times \pi(h_2)$$
$$R \times B \to H, \quad r \times b \mapsto r\iota(b) \qquad \textit{for any } h \in H, r \in R, b \in B,$$

are mutually inverse isomorphisms.

Now we have a mapping system

$$R \underset{\pi_R}{\overset{\iota_R}{\rightleftarrows}} H \cong R \times B \underset{\pi}{\overset{\iota}{\leftrightarrows}} B,$$

where ι_R is an algebra injection and π_R is a coalgebra surjection such that $\pi_R \iota_R = \mathrm{id}_R$. From the definition of R, this algebra is stable under S^2.

It is natural to consider R^*, the dual of a finite dimensional braided Hopf algebra $R \in {}_B^B\mathcal{YD}$, as a Hopf algebra in ${}_{B^*}^{B^*}\mathcal{YD}$ by the construction:

assume

$$B \underset{\pi}{\overset{\iota}{\rightleftarrows}} R \times B \cong H,$$

we can dualize it to get

$$B^* \underset{\pi^*}{\overset{\iota^*}{\leftrightarrows}} (R \times B)^* \cong R^* \otimes B^*,$$

where the second is an isomorphism of vector spaces and $\iota^* \pi^* = \mathrm{id}_{B^*}$. It follows that $B^{*\mathrm{co}\,\iota^*} = R^* \otimes \varepsilon_B \subseteq R^* \otimes B^*$ and hence R^* is a Hopf algebra in ${}_{B^*}^{B^*}\mathcal{YD}$.

Besides, we can make the linear dual $R^* = \mathrm{Hom}(R, \Bbbk)$ into a Hopf algebra in ${}_B^B\mathcal{YD}$. By using the antipode S_B, R^* has a left B-module structure given by

$$bf(r) = f(S_B(b) \triangleright r).$$

Since R is a finite dimensional left B-comodule, R^* has a transposed right B-comodule structure, and it becomes a left B-comodule via S_B^{-1}, the inverse of S_B:

$$\rho_{R^*} : R^* \to B \otimes R^*, \; \rho_{R^*}(f) = f_{(-1)} \otimes f_{(0)} \quad \text{where } f_{(0)}(r) f_{(-1)} = f(r_{(-1)}) S_B^{-1}(r_{(0)}),$$

for any $f \in R^*$, $r \in R$, $b \in B$.

Here we list some important properties of braided Hopf algebras in ${}_B^B\mathcal{YD}$ for our work obtained in [3] and [39].

Lemma 2.4.2. *If H, B are finite dimensional Hopf algebras such that $H = R \times B$ is a biproduct with antipodes S, S_B, where R is a finite dimensional braided Hopf algebra in ${}_B^B\mathcal{YD}$ with antipode S_R. Then*

(1) *g is a group-like element of H if and only if $g = r \times b$ where $b \in G(B)$ and $r \in G(R)$ satisfies $\rho(r) = 1 \otimes r$.*

(2) *If λ_R, λ_B are right integrals, then $\lambda_R \times \lambda_B$ is a right integral of $R \times B$.*

(3) *If the trace of S^2 restricted to R is not zero, then both R and R^* are semisimple.* □

CHAPTER 3. Non-semisimple Hopf Algebras Of Dimension $4p$

3.1 Introduction

This chapter is devoted to the classification of Hopf algebras of dimension $4p$ where p is an odd prime. Semisimple Hopf algebras of dimension $4p$ have been classified in [16], [29], [30], and [32]. We mainly study $4p$-dimensional non-semisimple Hopf algebras in this chapter. Currently, dimension 12 is the only known case that was completely classified in [18] and [31]. The former classified the semisimple case and the latter showed that any non-semisimple Hopf algebra of dimension 12 is either pointed or dual-pointed.

Our goal is to show the following conjecture:

Conjecture. Let p be an odd prime. Any non-semisimple Hopf algebra over $\Bbbk$ of dimension $4p$ is either pointed or dual-pointed.

Let us briefly recall the results of classification of pointed Hopf algebras of dimension pq^2 with distinct primes p, q that were completed by Andruskiewitsch and Natale in Appendix of [1]. There are $4(q-1)$ types of pointed Hopf algebras that can be generated by Taft algebras and a particular kind of algebras described in the following Lemma:

Lemma 3.1.1 ([1]). *Let H be a pointed non-semisimple Hopf algebra of dimension pq^2 where p, q are distinct primes.*

(i) *$G(H)$ is a cyclic group of order pq.*

(ii) *Let $\mu \in \{0, 1\}$ and $\tau \in \Bbbk$ be a primitive q-th root of unity. Then $\mathcal{A}(\tau, \mu)$ is the $\Bbbk$-algebra generated by the elements g and y subject to the relations*

$$g^{pq} = 1, \quad y^q = \mu(1 - g^q), \quad gy = \tau yg.$$

The comultiplication is defined by

$$\Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes 1 + g \otimes 1.$$

There are exactly $4(q-1)$ *pointed Hopf algebras:*

$$\mathcal{A}(\tau, 0), \quad \mathcal{A}(\tau, 1), \quad \mathcal{A}(\tau, 0)^*, \quad T(\tau) \otimes \Bbbk[\mathbb{Z}_p]$$

where $T(\tau)$ *is a Taft algebra of dimension* q^2*. Here* $T(\tau) \otimes \Bbbk[\mathbb{Z}_p]$ *is a self-dual Hopf algebra.*

Remark 3.1.1. It has been shown in [38] that the Hopf algebra $\mathcal{A}(\tau, 1)^*$ is not pointed and its coalgebra structure is the direct sum of the Taft algebra $T(\tau)$ and simple coalgebras $M_q(\Bbbk)^*$ with multiplicity $p-1$. Therefore,

$$\mathcal{A}(\tau, 0), \quad \mathcal{A}(\tau, 0)^*, \quad \mathcal{A}(\tau, 1), \quad \mathcal{A}(\tau, 1)^* \text{ and } T(\tau) \otimes \Bbbk[\mathbb{Z}_p]$$

account for all non-semisimple pointed or dual-pointed Hopf algebras of dimension pq^2.

The following lemma from [1] also provides us the necessary and sufficient condition for a pointed Hopf algebra of dimension pq^2:

Lemma 3.1.2. *Let* H *be a non-semisimple Hopf algebra of dimension* pq^2*, where* p, q *are two distinct primes. If the coradical of* H *is a Hopf algebra, then* H *is pointed.*

Therefore, any pointed Hopf algebra of dimension $4p$ must be one of the types described in the above lemma. Indeed, we shall justify the conjecture with the following main results of this chapter.

- A non-semisimple Hopf algebra H over $\Bbbk$ of dimension $4p$ with an odd prime p is pointed if, and only if, $|G(H)| > 2$.
- Any non-semisimple Hopf algebra H of dimension $4p$ with an odd prime p, $2 < p < 12$, is pointed or dual-pointed.

Since the dimension of the group algebra generated by group-like elements divides the dimension of Hopf algebra, we start by considering all possible orders of the group of group-like elements. This chapter is organized as follows: In Section 2, we show that if H is a non-semisimple Hopf algebra of dimension $4p$ with an odd prime p such that $|G(H)| > 2$, then H is pointed. In Section 3, we study semisimple braided Hopf algebras of dimension p in the Yetter-Drinfeld category over a 4-dimensional Taft algebra T and conclude the commutativity for dimension p =3, 7 and 11. The case of semisimple braided Hopf algebra of dimension 5 over T will be discussed in Section 4. In the last section of this chapter, Section 5, we show that Hopf algebras of dimension 20, 28, and 44 must be pointed or dual-pointed.

3.2 Non-semisimple Hopf algebras with dimension $4p$

In this section, we will focus on $4p$-dimensional non-semisimple Hopf algebras H, where p is an odd prime. In particular, we prove in Theorem 3.2.5 that if there are more than two group-like elements in such a Hopf algebra, then H or H^* must be pointed. This completes the classification of non-semisimple Hopf algebras of dimension $4p$ with $|G(H)|$ or $|G(H^*)| > 2$ from Lemma 3.1.1. We continue to study the case both $G(H)$ and $G(H^*)$ with at most two elements in the next few sections and show that this case does not exist for any non-semisimple Hopf algebras of dimensions 20, 28 and 44. Hence, they are pointed or their duals are pointed.

Throughout this section, we assume that p is an odd prime, and H is a non-semisimple Hopf algebra of dimension $4p$. We begin with

Lemma 3.2.1. (1) *If H contains a Hopf subalgebra K of dimension $2p$, then H is pointed and K must be a cyclic group algebra.*

(2) *If H contains a Hopf ideal I of dimension $2p$, then H^* contains a Hopf subalgebra isomorphic to H/I, which is a cyclic group algebra of dimension $2p$.*

Proof. (1) By Lemma 2.2.6, K is semisimple and $S_K^2 = \mathrm{id}_K$. Hence, at least $2p$ eigenvalues of S^2 are 1. Since H is non-semisimple, $\mathrm{Tr}(S^2) = 0$. This forces the rest of eigenvalues of S^2 are all -1. Therefore, we have $S^4 = \mathrm{id}_H$. From Proposition 5.1 of [3], if H contains a semisimple Hopf subalgebra K such that $\dim H = 2\dim K$, then K is equal to the coradical of H. It follows from Lemma 3.1.2 that H is pointed. Also, from Lemma 3.1.1, $K = \Bbbk[G(H)]$ and $G(H)$ is a cyclic group of order $2p$.

(2) If I is a Hopf ideal, then H/I is a Hopf algebra of dimension $2p$ and hence $H \to H/I$ is a Hopf algebra epimorphism. By dualizing it, H^* contains a Hopf subalgebra of dimension $2p$. The result follows from (1).

□

Remark 3.2.1. Assume that $G(H)$ is an abelian group of order n. Then the group algebra $K = \Bbbk[G(H)]$ is a commutative semisimple Hopf algebra of dimension n. Let $E = \{e_i\}_{1\leq i\leq n}$ be the complete set of orthogonal primitive idempotents of $G(H)$. By Nichols-Zoeller Theorem, H is a free K-module and hence $\dim K$ divides $\dim H$. Therefore, $H = \bigoplus_{1\leq i\leq n} He_i$ is the decomposition into right K-modules and $\dim He_i = \dim H/\dim K$ for all e_i.

We use the above remark to proceed the following lemma:

Lemma 3.2.2. *If p divides $|G(H)|$, then H is pointed.*

Proof. As $p\big||G(H)|$, we may assume that $g \in G(H)$ has order p. First we show that all the composition factors of $P(\Bbbk)$ are 1-dimensional.

Let $\{e_1, \dots, e_p\}$ be the complete set of orthogonal primitive idempotents of $\Bbbk[g]$. By the preceding remark, $H = \bigoplus_i He_i$ is a decomposition of left H-modules, and $\dim He_i = 4$. By Krull-Schmidt Theorem, $\dim P(V) \leq 4$ for all $V \in \mathrm{Irr}(H)$. Since H is not semisimple, $\Bbbk$ is not projective. Therefore, $2 \leq \dim P(\Bbbk) \leq 4$.

If $\dim P(\Bbbk) = 2, 3$, then all its composition factors are 1-dimensional. Suppose $\dim P(\Bbbk) = 4$ and it has a composition factor V of dimension greater than 1. Then $\dim V = 2$, $[P(\Bbbk) : V] = 1$ and V is not projective. By Lemma 2.3.1, $\dim P(V) \geq 2\dim V + 1 > 4$, a contradiction.

Since $P(\Bbbk)$ is not simple and all its composition factors are 1-dimensional, it follows from Lemma 2.3.3 that there exists a non-trivial extension of 1-dimensional H-module. Therefore, by Lemma 2.3.2, H^* admits a non-trivial skew primitive element. Now, we apply Lemma 2.2.3

to conclude that H^* contains a non-semisimple pointed Hopf subalgebra K of dimension m^2n with $m > 1$. By Nichols-Zoeller Theorem, $m^2n \big| 4p$ and this implies $m = 2$ and $n = 1$ or p.

If $n = p$, then $\dim K = \dim H^*$ implies that $K = H^*$ is pointed. From the classification of pointed Hopf algebras of dimension $4p$ in Remark 3.1.1, either H is pointed or $|G(H)| = 2$. Since $p \big| |G(H)|$, H is pointed.

If $n = 1$, then K is isomorphic to the Taft algebra T of dimension 4. By dualizing the inclusion map $T \to H^*$, we find a Hopf algebra surjection $\pi : H \to T$. It is known that the coinvariant $R = H^{\mathrm{co}\,\pi} = \{r \in H \mid r_1 \otimes \pi(r_2) = r \otimes 1\}$ is a left coideal subalgebra of dimension p. Since $\pi(g)$ is a group-like element of T, $\pi(g) = 1$ and hence $R = \Bbbk[g]$. As R is a Hopf subalgebra and hence $H^{\mathrm{co}\,\pi} = {}^{\mathrm{co}\,\pi}H = \{r \in H \mid \pi(r_1) \otimes r_2 = 1 \otimes r\}$, this implies that $R^+H = HR^+$ and R is normal. Therefore, we have the exact sequence of Hopf algebras:

$$1 \to \Bbbk[g] \to H \to T \to 1\,.$$

Since T and $\Bbbk[g] \cong \Bbbk[\mathbb{Z}_p]$ are self-dual, we can dualize it to get

$$1 \to T \to H^* \to \Bbbk[g] \to 1.$$

It is well known that the Jacobson radical $J(T)$ is a Hopf ideal and $T/J(T) \cong \Bbbk[\mathbb{Z}_2]$. Applying [21, Proposition 1.7], we obtain the exact sequence

$$1 \to \Bbbk[\mathbb{Z}_2] \to H^*/H^*J(T) \to \Bbbk[g] \to 1,$$

and that $H^*/H^*J(T)$ is a semisimple $2p$-dimensional Hopf algebra. After dualizing this sequence, we get a Hopf subalgebra of dimension $2p$ in H and hence by Lemma 3.2.1, H is pointed. $\square$

Lemma 3.2.3. *Let S be the antipode of H. Then $S^{16} = \mathrm{id}_H$.*

Proof. Let $a \in G(H)$ and $\alpha \in G(H^*)$ be distinguished group-like elements. If $p \big| \mathrm{ord}(\alpha)\mathrm{ord}(a)$, then $p \big| |G(H)||G(H^*)|$. In view of Lemma 3.2.2, H or H^* is pointed. Note that the pointedness of H implies that its coradical is a cyclic group algebra of dimension $2p$. Therefore, it follows from Lemma 3.1.1 and the structure for dual-pointed H described in Remark 3.1.1, $S^4 = \mathrm{id}_H$.

If neither H nor H^* is pointed, then p does not divide $|G(H)||G(H^*)|$. In particular, p does not divide $\mathrm{ord}(a)\mathrm{ord}(\alpha)$. Therefore, $\mathrm{lcm}(|G(H)|, |G(H^*)|) \leq 4$. From Radford's formula (2.2.1), we have $S^{16} = \mathrm{id}$. □

We prove that the order of S^2 is 4 in the next lemma.

Lemma 3.2.4. *The order of $G(H)$ is not divisible by 4 and $S^8 = \mathrm{id}$. Moreover, if $V \in \mathrm{Irr}(H)$ such that $V \cong V^{\vee\vee}$, then $\dim P(V)$ is even. In particular, $\dim P(\Bbbk)$ is an even integer.*

Proof. Suppose $4 \mid |G(H)|$. Since H is non-semisimple, $|G(H)|$ cannot be $4p$ and so $|G(H)| = 4$. Hence, the Hopf subalgebra $K = \Bbbk[G(H)]$ is commutative and semisimple. Let $\{e_1, e_2, e_3, e_4\}$ be the complete set of orthogonal primitive idempotents of K. In view of Remark 3.2.1,

$$\dim He_i = p\,.$$

As S^2 is an algebra automorphism of H and the restriction of S^2 to K is id_K, S^2 stabilizes He_i. Moreover, $\mathrm{Tr}(S^2|_{He_i}) = \mathrm{Tr}(S^2 \circ r(e_i)) = 0$. It follows from Lemma 2.3.5 and $S^{16} = \mathrm{id}$ that $\dim He_i$ is even, a contradiction. Therefore, $4 \nmid |G(H)|$.

If H or H^* is pointed, then $S^4 = \mathrm{id}$. If neither H nor H^* is pointed, then Lemma 3.2.2 forces $p \nmid |G(H)||G(H^*)|$. Thus, $\mathrm{lcm}(|G(H)|, |G(H^*)|) \leq 2$ and $S^8 = \mathrm{id}$ follow immediately from Radford's formula (2.2.1).

Consider the natural exact sequence of H-modules

$$0 \to JP(V) \to P(V) \to V \to 0,$$

we can dualize it to get

$$0 \to V^{\vee} \to P(V)^{\vee} \to JP(V)^{\vee} \to 0.$$

Since H is a Frobenius algebra and we also have the exact sequence

$$0 \to V^{\vee} \to I(V^{\vee}) \to I(V^{\vee})/V^{\vee} \to 0,$$

it follows that $P(V)^{\vee} \cong I(V^{\vee})$. If $V \in \mathrm{Irr}(H)$ is isomorphic to $V^{\vee\vee}$, then we have $P(V)^{\vee\vee} \cong P(V^{\vee\vee}) \cong P(V)$. From Lemma 2.3.5, $\dim P(V)$ is even. As $\Bbbk \cong \Bbbk^{\vee}$, $P(\Bbbk)$ is self-dual and hence $\dim P(\Bbbk)$ is always even.

□

If the order of $G(H)$ or $G(H^*)$ is greater than two, in view of Lemmas 3.2.1, 3.2.2 and 3.2.4, then H or H^* must be pointed. Therefore, we have the following theorem:

Theorem 3.2.5. *Let H be a non-semisimple Hopf algebra of dimension $4p$ with an odd prime p. Then H is pointed if, and only if, $|G(H)| > 2$.*

If neither H nor H^* is pointed, the last lemma of this section shows the non-existence of a 2-dimensional self-dual simple H-module:

Lemma 3.2.6. *Every 2-dimensional simple H-module is not self-dual.*

Proof. Suppose that V is a 2-dimensional simple H-module such that $V \cong V^\vee$ as H-modules. Note that H^* cannot be pointed, otherwise every simple H-module is one-dimensional. Since S is bijective and the H-module $V^\vee$ is given by $h \cdot f(v) = f(S(h)v)$ for any $v \in V$, $f \in V^*$, we have $S(\operatorname{ann} V) = \operatorname{ann} V^\vee = \operatorname{ann} V$. Therefore, S induces an isomorphism $\overline{S}$ on $H/\operatorname{ann} V$. Since V is simple, $H/\operatorname{ann} V$ is a simple algebra of dimension 4. Let $\pi : H \to H/\operatorname{ann} V$ be the natural algebra surjection. Then $\pi^* : (H/\operatorname{ann} V)^* \to H^*$ is an injective coalgebra map which satisfies the commutative diagram:

$$\begin{array}{ccc} (H/\operatorname{ann} V)^* & \xrightarrow{\pi^*} & H^* \\ \downarrow{\scriptstyle \overline{S}^*} & & \downarrow{\scriptstyle S^*} \\ (H/\operatorname{ann} V)^* & \xrightarrow{\pi^*} & H^* . \end{array}$$

In particular, $C = \pi^*((H/\operatorname{ann} V)^*)$ is a simple subcoalgebra of H^* stabilized by S^*.

Let K be the subalgebra of H^* generated by C. Then K is a Hopf subalgebra of H^* and hence $\dim K = 4, p, 2p$ or $4p$. Since K contains the simple subcoalgebra C, $\dim K$ cannot be 4 or p, otherwise K would be a group algebra, whose simple subcoalgebras are all one-dimensional. If $\dim K = 2p$, then H^* is pointed by Lemma 3.2.1. So we may now assume $\dim K = 4p$, and hence $H^* = K$. It follows from Proposition 1.3 of [31] that there exists an exact sequence of Hopf algebras

$$1 \to \Bbbk^G \to H^* \to B \to 1,$$

where B^* is a pointed non-semisimple Hopf algebra and G is a finite group. Note that $\dim B$ can only be 4 or $4p$ since Hopf algebras of dimension $2, 2p, p$ are semisimple. If $\dim B = 4p$, then $B = H^*$ and H is pointed. By the classification of pointed Hopf algebras in Remark 3.1.1, H has no 2-dimensional self-dual simple module and hence this is impossible. Therefore, $\dim B = 4$. Then B is the Taft algebra of dimension 4 and G is a cyclic group of order p. Since $\Bbbk^G \cong \Bbbk[G]$ as Hopf algebras, H^* contains a subgroup of order p. It follows from Lemma 3.2.2 that H^* is pointed, a contradiction. □

3.3 p-dimensional Hopf algebras in the Yetter-Drinfeld category

Assume that H is a non-semisimple Hopf algebra of dimension $4p$ with an odd prime p. In this section, we will investigate the p-dimensional braided Hopf algebra in the category of Yetter-Drinfeld modules over a four dimensional Taft algebra T. Also, we determine the necessary and sufficient conditions that H or H^* is pointed provided that they are bosonization of a braided Hopf algebra of dimension p and a Taft algebra of dimension 4.

3.3.1 Semisimple Braided Hopf algebras in ${}^T_T\mathcal{YD}$

Recall that a four dimensional Taft algebra T over $\Bbbk$ is generated by x, g subject to the relations

$$g^2 = 1, \quad x^2 = 0 \quad \text{and} \quad gx = -xg\,.$$

The comultiplication Δ, the counit ε and the antipode S of T are given by

$$\begin{aligned} \Delta(x) = x \otimes g + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -xg, \\ \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g. \end{aligned} \tag{3.3.1}$$

A Yetter-Drinfeld module R over the Taft algebra T is a $\Bbbk$-space with the T-action $\triangleright$, the T-coaction $\rho : R \to T \otimes R$, $\rho(r) = r_{(-1)} \otimes r_{(0)}$, satisfying the compatibility condition:

$$\rho(t \triangleright r) = t_1 r_{(-1)} S_T(t_3) \otimes t_2 \triangleright r_{(0)}.$$

Let R be a finite dimensional braided Hopf algebra in the Yetter-Drinfeld category ${}^T_T\mathcal{YD}$ with the comultiplication Δ_R, denote by $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$, the counit ε_R and antipode S_R. The bosonization $H = R \times T$ is an ordinary Hopf algebra with the underlying space $R \otimes T$ and its comultiplication Δ, the counit ε, the antipode S are given by

$$(r \times t)(r' \times t') := rt_1 \triangleright r' \times t_2 t', \quad \Delta(r \times t) := r^{(1)}(r^{(2)})_{(-1)} \times t_1 \otimes (r^{(2)})_{(0)} \times t_2$$

$$\varepsilon(r \times t) := \varepsilon_R(r)\varepsilon_T(t), \quad S(r \times t) := S_T(t_2) \triangleright S_R(r) \times S_T(t_1)$$

for any $r, s \in R$, $t, t' \in T$. Note that R^* is a braided Hopf algebra in ${}^T_T\mathcal{YD}$ and we have an isomorphism

$$(R \times T)^* \cong R^* \times T^*$$

as Hopf algebras.

In this section, we focus on p-dimensional braided Hopf algebras in ${}^T_T\mathcal{YD}$. They are always semisimple and cosemisimple.

Theorem 3.3.1. *If R is a p-dimensional braided Hopf algebra in ${}^T_T\mathcal{YD}$, the category of Yetter-Drinfeld modules over T, then R and R^* are semisimple and cosemisimple.*

Proof. As R is stable under S^2, by Lemma 2.4.2, it suffices to show that $\mathrm{Tr}(S^2\big|_R) \neq 0$. From Lemma 3.2.4, $S^8 = \mathrm{id}$. In particular, the order of $S^2\big|_R$ is a power of 2. Since $\dim R = p$ is an odd prime, $\mathrm{Tr}(S^2\big|_R) \neq 0$ from Lemma 2.3.4. □

We showed in the previous section that if neither H nor H^* is pointed, then both $|G(H)|$ and $|G(H^*)|$ cannot be greater than 2. We would like to know the role of non-trivial skew primitive elements in H and when it can be isomorphic to a bosonization of the form $R \times T$. This can be explained in the next proposition and we shall prove it by using the following lemma.

Recall that a left coideal subalgebra K of H is an algebra satisfying $\Delta(K) \subseteq H \otimes K$.

Lemma 3.3.2 ([25])**.** *Let H be a finite dimensional Hopf algebra and K is a left coideal subalgebra of H. The following are equivalent:*

(1) *H is left and right free over K.*

(2) *K is a Frobenius algebra.*

(3) *every object in ${}^H_K\mathcal{M}$ or in ${}^H\mathcal{M}_K$ is a free K-module.*

Proposition 3.3.3. *Let H be a non-semisimple Hopf algebra of dimension $4p$ such that neither H nor H^* is pointed. Then $H \cong R \times T$ for some p-dimensional braided Hopf algebra in ${}^T_T\mathcal{YD}$ if, and only if, both H and H^* admit a non-trivial skew primitive element.*

Proof. Since $R \times T$ and $(R \times T)^*$ contain a Hopf subalgebra T, both of them contain a non-trivial skew primitive element. Conversely, assume both H and H^* have non-trivial skew primitive elements, hence $|G(H)| = 2 = |G(H^*)|$. By Lemma 2.2.3, there exists a pointed non-semisimple Hopf subalgebra K of H such that $\dim K = m^2 n$ with $m > 1$ and $|G(K)| = mn$. As $|G(H)| = 2$, we have $m = 2$ and $n = 1$. Therefore, K is isomorphic to the Taft algebra T. By the same reason, H^* contains a pointed Hopf subalgebra isomorphic to T. Then there exists a Hopf algebra surjection $\pi : H \to T$.

The coinvariants $R = H^{\mathrm{co}\,\pi} := \{h \in H \big| (\mathrm{id}_H \otimes \pi)\Delta(h) = h \otimes 1_T\}$ and $L = T^{\mathrm{co}\,\pi} := \{h \in T \big| (\mathrm{id}_H \otimes \pi)\Delta(h) = h \otimes 1_T\}$ are left coideal subalgebras in H, and $\dim R = p$. Since T is a Hopf subalgebra of H and $L \subseteq R$, R is a Hopf module in ${}^H_L\mathcal{M}$. As π is a Hopf algebra map, $\pi(T)$ is a Hopf subalgebra of T. Hence $\dim L = 1$, 2 or 4. More precisely, $L = 1_T$, $\Bbbk[1_T, xg]$ or T and they are Frobenius algebras. By Lemma 3.3.2, R is a free L-module and this implies $\dim L = 1$. Therefore, $\pi(T) = T$ or $\pi|_T$ is an isomorphism of T. It follows from Section 2.4.2 that H is a Radford biproduct $H \cong R \times T$, where R is a p-dimensional braided Hopf algebra in the category of Yetter-Drinfeld modules over T. □

It has been shown in Remark 5.9 of [17] that the braided Hopf algebra R is a Frobenius algebra with the Frobenius map given by a right integral λ_R in R^*, i.e. $\lambda_R(r^{(1)})r^{(2)} = \lambda_R(r)1_R$ for all $r \in R$. Moreover, there exists a right integral $\Lambda_R \in R$ such that $\lambda_R(\Lambda_R) = 1$, and an algebra map $\chi \in G(T^*)$ such that

$$\lambda_R(t \triangleright r) = \chi(t)\lambda_R(r) \quad \text{for any } t \in T, r \in R. \tag{3.3.2}$$

This implies

$$t \triangleright \Lambda_R = \chi(t)\Lambda_R \quad \text{for any } t \in T.$$

From Corollary 5.8 in [17], the semisimplicity of R implies $\varepsilon_R(\Lambda_R) \neq 0$. Therefore, $\chi = \varepsilon_T$ and

$$t \triangleright \Lambda_R = \varepsilon_T(t)\Lambda_R \quad \text{and} \quad \lambda_R(t \triangleright r) = \varepsilon_T(t)\lambda_R(r) \tag{3.3.3}$$

for any $t \in T$.

Lemma 3.3.4. *Let R be a semisimple braided Hopf algebra in ${}^T_T\mathcal{YD}$. Then*

$$\rho(\Lambda_R) = 1_T \otimes \Lambda_R, \quad r_{(-1)}\lambda_R(r_{(0)}) = 1_T\lambda_R(r), \quad S_R(\Lambda_R) = \Lambda_R. \tag{3.3.4}$$

Proof. From [13, (18),(19)], there exists a group-like element $g' \in T$ such that

$$\rho(\Lambda_R) = S_T(g') \otimes \Lambda_R \text{ and } \lambda_R(r)g' = S^{-1}(r_{(-1)})\lambda_R(r_{(0)}).$$

Since $\varepsilon_R(\Lambda_R) \neq 0$ and ε_R is a T-comodule map, $\varepsilon_T(\Lambda_R)1_T = S_T(g')\varepsilon_R(\Lambda_R)$. Hence, $S_T(g') = 1_T$ and $\rho(\Lambda_R) = 1_T \otimes \Lambda_R$. By applying S_T on both sides of the second equality, we have $r_{(-1)}\lambda_R(r_{(0)}) = 1_T\lambda_R(r)$.

Note that S_R is an antialgebra map satisfying $S_R(rs) = (r_{(-1)} \triangleright S_R(s))S_R(r_{(0)})$ for any $r, s \in R$. In particular,

$$\varepsilon_R(r)S_R(\Lambda_R) = \varepsilon_R(S_R^{-1}(r))S_R(\Lambda_R) = S_R(\Lambda_R S_R^{-1}(r)) = rS_R(\Lambda_R)$$

since $\varepsilon_R \circ S_R = \varepsilon_R$. Hence $S_R(\Lambda_R)$ is a left integral. The semsimplicity of R implies $S_R(\Lambda_R) = \gamma\Lambda_R$ for some $\gamma \in \Bbbk$. Since $\varepsilon_R(S_R(\Lambda_R)) = \varepsilon_R(\Lambda_R) \neq 0$, we have $\gamma = 1$. i.e. $S_R(\Lambda_R) = \Lambda_R$. □

Lemma 3.3.5. *Let A be a finite dimensional left (resp. right) T-module algebra. If A is semisimple and e is a central idempotent of A such that the ideal $I = Ae$ is closed under the action of g, then I is a T-submodule of A, and*

$$g \triangleright e = e \quad \text{and} \quad x \triangleright e = 0.$$

In particular, I is also a T-module algebra.

Proof. If e is not a primitive idempotent, write $e = e_1 + \cdots + e_k$ where $e_i's$ are primitive central idempotents of A, then $e_ie = e_i = ee_i \in I$ for $1 \leq i \leq k$. Since I is an ideal closed under g-action, $g \triangleright e_i \in I$ is a primitive central idempotent of R. Moreover, $g \triangleright e_i \neq -e_i$, otherwise $-e_i = g \triangleright e_i = (g \triangleright e_i)(g \triangleright e_i) = e_i$ results a contradiction. Hence $g \triangleright e_i = e_i$ or e_j for some $j \neq i$ and

$$g \triangleright e = \sum_{g \triangleright e_i = e_i} e_i + \sum_{g \triangleright e_i \neq e_i} g \triangleright e_i = \sum_{g \triangleright e_i = e_i} e_i + \sum_{g \triangleright e_i = e_j} e_j = e.$$

Since x is a g-derivation, $x \triangleright e = x \triangleright e^2 = (x \triangleright e)(g \triangleright e) + e(x \triangleright e) = 2e(x \triangleright e) \in I$. Write $x \triangleright e = ae$ for some $a \in A$, then $ae = 2eae = 2ae$ implies $ae = 0$. i.e. $x \triangleright e = 0$. Now, for any $a \in A$, $x \triangleright ae = (x \triangleright a)e \in I$. Thus, I is a T-submodule of A. □

Lemma 3.3.6. *Let R be a finite dimensional semisimple braided Hopf algebra in ${}^T_T\mathcal{YD}$. If I is an one-dimensional ideal of R, then $x \triangleright I = 0$.*

Proof. Since R is semisimple, there exists a primitive central idempotent e_1 of R such that $I = Re_1$.

If $g \triangleright e_1 = e_1$, then $x \triangleright e_1 = 0$ from Lemma 3.3.5. If $g \triangleright e_1 = e_2$ where $e_2 \neq e_1$, then e_2 is also a primitive central idempotent. The ideal $\widehat{I} = R(e_1 + e_2) = Re_1 + Re_2$ is closed under g action and hence closed under T action from Lemma 3.3.5. Assume $x \triangleright e_i = \alpha_{i1}e_1 + \alpha_{i2}e_2$ for some $\alpha_{ij} \in \Bbbk$ and $i, j \in \{1, 2\}$. As $x \triangleright (e_1 + e_2) = 0$, we have $\alpha_{11} = -\alpha_{21}$ and $\alpha_{12} = -\alpha_{22}$. By the equation $gx \triangleright e_1 = -x \triangleright e_2$, we find $\alpha_{12} = -\alpha_{21}$ and $\alpha_{11} = -\alpha_{22}$. Write $\alpha = \alpha_{11}$, we have $x \triangleright e_1 = \alpha(e_1 + e_2)$ and $x \triangleright e_2 = -\alpha(e_1 + e_2)$. Applying (3.3.2) with $t = g$ and $r = e_1$, then $\lambda(e_1) = \lambda(e_2) \neq 0$ since $\ker\lambda$ cannot contain a nonzero ideal. Applying (3.3.2) with $t = x$ and $r = e_1$, then $2\alpha\lambda(e_1) = 0$ implies $\alpha = 0$. Therefore, $x \triangleright I = 0$. □

3.3.2 A few important properties

Before we prove the main theorem of this section, we need the following lemma, which states that if the semisimple braided Hopf algebra R has more than one group-like element, then there exist more then two group-like elements in H^*.

Lemma 3.3.7. *Let e be a central primitive idempotent of a Hopf algebra R in ${}^T_T\mathcal{YD}$ such that $g \triangleright e = e$. Then the 1-dimensional character χ of R associated with the idempotent e satisfies*

$$\chi(t \triangleright r) = \varepsilon(t)\chi(r) \quad \text{for all } t \in T,\ r \in R.$$

Moreover, $\chi^n \times \beta$ is a group-like element of $(R \times T)^*$ for all $\beta \in G(T^*)$.

Proof. Since e is a central primitive idempotent such that $g \triangleright e = e$, by Lemma 3.3.5 $x \triangleright e = 0$. Hence $t \triangleright e = \varepsilon_T(t)e$. χ is a 1-dimensional character associated with e, so $re = \chi(r)e$ and

$\chi(re) = \chi(r)$. Thus

$$\chi(t \triangleright r) = \chi((t \triangleright r)e) = \chi(t \triangleright re) = \chi(r)\chi(t \triangleright e) = \varepsilon_T(t)\chi(r).$$

We show that $\Bbbk[\chi]$ is a coalgebra. Since χ is linear,

$$\Delta_{R^*}(\chi)(a \otimes b) = \chi(ab) = \chi(a)\chi(b) = \chi \otimes \chi(a \otimes b),$$

i.e. χ is a group-like element in R^*. By induction, one can show that χ^n is a group-like element in R^* for any n. Moreover, $\chi^n(t \triangleright r) = \varepsilon_T(t)\chi^n(r)$ for all $n \geq 1$. Therefore, for any $r, r' \in R, t, t' \in T$, we have

$$\begin{aligned}(\chi^n \times \beta)((r \times t)(r' \times t')) &= (\chi^n \times \beta)(rt_1 \triangleright r' \times t_2 t') = \chi^n(r)\varepsilon_H(t_1)\chi^n(r')\beta(t_2)\beta(t') \\ &= \chi^n(r)\chi^n(r')\beta(t)\beta(t') = (\chi^n \times \beta)(r \times t)(\chi^n \times \beta)(r' \times t').\end{aligned}$$

This shows that $\chi^n \times \beta$ is a group-like element in H^* since $G(H^*) = \mathrm{Alg}(H, \Bbbk)$.

□

Proposition 3.3.8. *Let R be a finite dimensional semisimple braided Hopf algebra in ${}^T_T\mathcal{YD}$. The following are equivalent:*

(1) *R is commutative*

(2) *R is cocommutative*

(3) *$R \times T$ is pointed*

(4) *$(R \times T)^*$ is pointed*

(5) *There exists a g-invariant 1-dimensional ideal which is not generated by a non-zero integral in R.*

(6) *The action of x on R is trivial.*

Proof. (1)⇒(4), (5), (6) If R is commutative and semisimple, then every simple ideal of R is one-dimensional. From Lemma 3.3.5, $x \triangleright R = 0$. Moreover, $I = xH = Hx$ is a Hopf ideal of dimension $2p$ and H/I is a commutative Hopf algebra of dimension $2p$. Therefore, H^* contains a Hopf subalgebra of dimension $2p$. It follows from Lemma 3.2.1 that H^* is pointed.

(3)⇒(1), (4)⇒(2) Assume $R \times T$ is pointed. Then $R \times T$ contains a cyclic group algebra K of dimension $2p$. This implies that there exists a non-trivial element $r \in R$ such that $r \times 1_T$ is a group-like element of $R \times T$. Moreover, r is a group-like element of R by Lemma 2.4.2 of order p. Therefore, R is commutative. Similarly, if $(R \times T)^*$ is pointed, then this would imply the commutativity of R^*. Therefore, R is cocommutative.

(2)⇒(3) If R is cocommutative, then R^* is a commutative braided Hopf algebra in ${}^{T^*}_{T^*}\mathcal{YD}$. Since $T \cong T^*$, by a similar argument of (1)⇒(4), $R \times T$ is pointed.

(5)⇒(3), (4) If there exists a g-invariant 1-dimensional ideal which is not generated by a non-zero integral in R, then by Lemma 3.3.7, H^* contains more than two group-like elements. By Theorem 3.2.5 , H^* or H is pointed and the result follows from (3) ⇔ (4).

(6)⇒(4) If the x action to R is trivial, then $xH = Hx$ is a Hopf ideal of dimension $2p$. By Lemma 3.2.1, H^* is pointed. □

To further study the structure of $R \times T$, our motivation comes from the following observations of the decomposition of $A \times T$ into ideals:

- If e is a central primitive idempotent of R such that $g \triangleright e \neq e$, then $I \times T$ is a block of $H = R \times T$ where I is the 2-dimensional ideal generated by $e, g \triangleright e$. As an H-module, $I \times T \cong 2P$ where P is the indecompsable H-module with an exact sequence

$$0 \to V \to P \to V \to 0$$

  for some simple H-module V of dimension 2. In particular, $\mathrm{Soc}(P) \cong V \cong \mathrm{Head}(P)$.

- If I is a 4-dimensional simple ideal of R and $x \triangleright I = 0$, then $I \times T$ has a decomposition into four 4-dimensional principal indecomposable H-modules whose socle is not isomorphic to its head.

- If I is a 4-dimensional simple ideal of R and $x \triangleright I \neq 0$, then any simple quotient module of $I \times T$ has dimension 4.

- If I_1, I_2 are 4-dimensional simple ideals of R such that $g \triangleright (I_1 + I_2) = I_1 + I_2$, then $g \triangleright I_i = I_i$ for $i = 1, 2$.

Recall that a semisimple algebra can always be decomposed into a direct sum of simple ideals. The group-like element g of T acts on R as an algebra automorphism, so it permutes all the simple ideals of R as well. Since the order of g is 2, the ideal $A_I = g \triangleright I + I$ is closed under the g action for any simple ideal I in R. From Lemma 3.3.5, A_I is an ideal closed under T action, called the T-simple ideal. The semisimplicity of R implies that R has a decomposition of T-simple ideals, namely

$$R = A_1 \oplus A_2 \oplus \cdots \oplus A_\ell, \quad A_i's \text{ are } T\text{-simple ideals of } R.$$

Therefore, H has an ideal decomposition

$$H = A_1 \times T \oplus A_2 \times T \oplus \cdots \oplus A_\ell \times T.$$

We will show that this is a decomposition of indecomposable ideals of H.

Let A be a T-simple ideal in R. Then A has no proper T-simple ideals of R. Let $e_0 = (1+g)/2$ and $e_1 = (1-g)/2$. Then $\{e_0, e_1, xe_0, xe_1\}$ forms a basis of T. For simplicity, we use the identification $t := (1_R \times t)$ and $r := r \times 1_T$ for all $r \in R$ and $t \in T$. We may assume that a central idempotent of $A \times T$ has the form

$$E = r_1 e_0 + r_2 e_1 + r_3 x e_0 + r_4 x e_1, \tag{3.3.5}$$

where r_1, r_2, r_3, r_4 are elements in A.
Since E commutes with g, this implies that

$$g \triangleright r_1 = r_1 \quad g \triangleright r_2 = r_2, \quad g \triangleright r_3 = -r_3, \quad g \triangleright r_4 = -r_4. \tag{3.3.6}$$

Moreover, one can get

$$x \triangleright r_1 = 0, \quad x \triangleright r_2 = 0, \quad x \triangleright r_3 = x \triangleright r_4 = r_1 - r_2. \tag{3.3.7}$$

from the equation $Ex = xE$. For any $r \in A$, $Er = rE$ results

$$(r_1 + r_2)r + (r_3 - r_4)xg \triangleright r = r(r_1 + r_2), \quad (r_1 - r_2)g \triangleright r + (r_3 + r_4)x \triangleright r = r(r_1 - r_2),$$
$$(r_3 + r_4)r = r(r_3 + r_4), \quad (r_3 - r_4)g \triangleright r = r(r_3 - r_4). \tag{3.3.8}$$

If $g \triangleright r = r$, then one has $x \triangleright r = 0$ due to $gx = -xg$, and from (3.3.8) we have

$$rr_i = r_i r \quad \text{for any } i = 1, 2, 3, 4. \tag{3.3.9}$$

Note that both Ee_0 and Ee_1 are idempotents. It follows from $(Ee_0)^2 = Ee_0$ and $(Ee_1)^2 = Ee_1$,

$$r_1^2 = r_1, \quad r_2^2 = r_2, \quad r_3 = r_1 r_3 + r_3 r_2, \quad r_4 = r_2 r_4 + r_4 r_1. \tag{3.3.10}$$

In view of (3.3.6) and (3.3.9), (3.3.10) becomes

$$r_1^2 = r_1, \quad r_2^2 = r_2, \quad r_3 = (r_1 + r_2)r_3 = r_3(r_1 + r_2), \quad r_4 = (r_1 + r_2)r_4 = r_4(r_1 + r_2). \tag{3.3.11}$$

We analyze some properties of T-simple ideals in the next lemma.

Lemma 3.3.9. *Let I be a simple ideal with dimension n^2 in the semisimple braided Hopf algebra R in ${}^T_T\mathcal{YD}$.*

(1) *If A_I is a simple ideal of R or the action of x on A_I is trivial, then $A_I \times T$ is an indecomposable ideal in $R \times T$.*

(2) *If $A_I = I \oplus g \triangleright I$, then every simple quotient algebra of $A_I \times T$ has dimension $4n^2$. Moreover, $A_I \times T$ is either a direct sum of two simple ideals of dimension $4n^2$ or a primary algebra.*

(3) *If $A_I = I \oplus g \triangleright I$ and x acts on A_I trivially, then $A_I \times T$ is a primary algebra and $A_I \times T/J(A_I \times T)$ is a simple algebra of dimension $4n^2$.*

In particular, $A_I \times T$ is either an indecomposable ideal or a direct sum of two simple ideals of dimension $4n^2$ in $R \times T$.

Proof. (1) Suppose E given in (3.3.5) is a non-trivial central idempotent of $A_I \times T$. If A_I is a simple ideal in R, then (3.3.8) implies $r_3 + r_4 = \gamma 1_{A_I}$. Since 1_{A_I} is stabilized by the g action, $x \triangleright 1_{A_I} = 0$ and so $\gamma = 0$. Hence, from (3.3.7), $r_1 = r_2$. If the action of x on A_I is trivial, then $x \triangleright (r_3 + r_4) = 0$ and $r_1 = r_2$ by (3.3.7). It follows from (3.3.11) that $r_i = 2r_2 r_i = 0$ for $i = 3, 4$ and so $r_3 = r_4 = 0$. Therefore, $E = r_1$. In particular, r_1 is a nonzero central idempotent in A_I. As r_1 is invariant under the g action and A_I is a

T-simple ideal, this implies $r_1 = 1_{A_I}$. Therefore, there is no non-trivial central idempotent in $A_I \times T$.

(2) For the first statement, it is equivalent to show that each indecomposable $A_I \times T$-module has dimension $2n$. Note that A_I and T are subalgebras of $A_I \times T$, so a simple $A_I \times T$-module V is a both A_I and T module as well. Assume $V = m_1V_1 + m_2V_2$ as an A_I-module, where V_1 and V_2 are corresponding simple I-module and simple $g \triangleright I$-module. We claim $m_1 = m_2 = 1$. Let $z_1, z_2, \ldots, z_{m_1} \in V$ such that $Iz_1, Iz_2, \ldots, Iz_{m_1}$ are simple I-modules and $IV = \bigoplus_{i=1}^{m_1} Iz_i$ as an I-module. Since $g(Iz_i) = (g \triangleright I)(gz_i)$ is a $g \triangleright I$-module and $g(IV) = \bigoplus_{i=1}^{m_1} g(Iz_i)$ as a vector space, we have $m_1 \leq m_2$. Similarly, $m_2 \leq m_1$ and so $m_1 = m_2 = m$. Note that $g(IV_1)$ is a $g \triangleright I$-module, hence $\dim V_1 = \dim V_2 = n$. Therefore, $\dim V = 2mn$ and so $A_I \times T/J(A \times T)$ has a simple ideal of dimension $4m^2n^2$. This implies $m = 1$.

If $A_I \times T$ is not a primary algebra, then there is more than one simple quotient algebra over $A_I \times T$ and

$$8n^2 = \dim(A_I \times T) \geq \dim(A_I \times T/J(A_I \times T)) \geq 8n^2.$$

This equality implies $J(A_I \times T) = 0$, so $A_I \times T$ is a direct sum of two ideals.

(3) If x acts on A_I trivially, then A_Ix is a nilpotent ideal of $A_I \times T$. Hence $A_I \times T$ is not semisimple and it follows from (2) that $A_I \times T$ is a primary algebra.

□

Lemma 3.3.10. *If neither $R \times T$ nor $(R \times T)^*$ is pointed, then $|G(R)|$ and $|G(R^*)|$ are integers congruent to 1 modulo 4.*

Proof. By Proposition 3.3.8, R and R^* must be non-commutative. Since the ideal generated by the integral element Λ_R of R is invariant under the action of g, there exists at least one 1-dimensional ideal of R. Moreover, there can be only one in $\mathfrak{D}$, the set of all 1-dimensional ideals of R, invariant under the g action by Proposition 3.3.8. Therefore, $|\mathfrak{D}|$ must be odd.

Suppose $|\mathfrak{D}|$ is congruent to 3 modulo 4. Let I be a 1-dimensional ideal in $\mathfrak{D}$ which is not generated by Λ_R. From Lemma 3.3.9, $A_I = I \oplus g \triangleright I$ is a T-simple ideal of R and $A_I \times T$ is a primary algebra of dimension 8. Moreover, if a simple ideal $\widehat{I}$ of R has dimension ≥ 2, then the dimension of $(\widehat{I} + g \triangleright \widehat{I}) \times T$ must be at least 16. Therefore, $\widehat{\mathfrak{D}} = \{A_T \times T | I \in \mathfrak{D}, I \text{ is not generated by } \Lambda_R\}$ is the set of all 8-dimensional indecomposable ideal summands of $R \times T$. Besides, each of these indecomposable ideals is a primary algebra whose simple quotient algebra is 4-dimensional. Since $A_I \times I$ is indecomposable, S permutes all of ideals of $\widehat{\mathfrak{D}}$. As $|\widehat{\mathfrak{D}}|$ is odd and the order of S is even from Lemma 3.2.4, one of these 8-dimensional ideal is stable under S. Hence, there exists a 2-dimensional self-dual H-module. However, this contradicts Lemma 3.2.6. Thus, $|\mathfrak{D}|$ must be congruent to 1 modulo 4. □

Now, we make the conclusion in the following corollary:

Corollary 3.3.11. *Let R be a semisimple braided Hopf algebra of dimension 3, 7 or 11 in ${}^T_T\mathcal{YD}$. Then R must be a trivial Yetter-Drinfeld module and a group algebra.*

Proof. By Proposition 3.3.8, the statements that $R \times T$ is pointed or dual-pointed, R in ${}^T_T\mathcal{YD}$ is commutative and R is a p-dimensional group algebra, are equivalent. Suppose that neither $R \times T$ nor $(R \times T)^*$ is pointed. Therefore, R is not commutative. In view of Lemma 3.3.10, $\dim R = 3$ implies that R is commutative, a contradiction. For $\dim R = 7$, R contains at least one simple algebra of dimension 4. So $|G(R)|$ can only be congruent to 3 modulo 4, which is impossible by Lemma 3.3.10. If $\dim R = 11$, then R must contain a simple algebra of dimension 4 or 9. Since the number of one-dimensional ideal of R must be odd, this simple algebra is 4-dimensional. So $|G(R)|$ is congruent to 3 modulo 4, which is again impossible by Lemma 3.3.10. Thus, for $\dim R = 3, 7$ or 11, R is a trivial Yetter-Drinfeld module and a group algebras. □

3.4 Semisimple Braided Hopf algebras of dimension 5 in ${}^T_T\mathcal{YD}$

In the previous section we have shown that if neither $R \times T$ nor $(R \times T)^*$ is pointed, then $|G(R)|$ and $|G(R^*)|$ are integers congruent to 1 modulo 4. However, Lemma 3.3.10 cannot be used to determine what if the dimension of a semisimple algebra R is congruent to 1 modulo 4. In this section, we will use the result of Chen and Zhang to study a semisimple braided Hopf algebra of dimension 5 in ${}^T_T\mathcal{YD}$.

We begin with assuming that R is a 5-dimensional non-commutative semisimple braided Hopf algebra in ${}^T_T\mathcal{YD}$. By Proposition 3.3.8, R has a unique ideal decomposition $R = A \oplus I$ where A is a 4-dimensional isomorphic to a matrix algebra and I is a 1-dimensional ideal generated by the normalized integral $e = \Lambda_R/\varepsilon_R(\Lambda_R)$. Then $\varepsilon_R(e) = 1$ and $\varepsilon_R(A) = 0$. In particular, e is a central idempotent of R. It follows from Lemma 3.3.4 that $t \triangleright e = \varepsilon_T(t)e$ for $t \in T$ and $\rho_R(e) = 1_T \otimes e$. Since the action of g is an algebra automorphism of R, $g \triangleright A = A$ and hence g fixes the central idempotent ι of A. By Lemma 3.3.6, A is a T-submodule of R and $x \triangleright \iota = 0$. Thus, A is also a T-module algebra.

The left T-comodule algebra structure of R induces a right T^*-module algebra on R. Since $T^* \cong T$ as Hopf algebras, by the same arguments, A is also a right T^*-submodule of R and hence a left T-comodule algebra. Thus, A is a 4-dimensional matrix algebra in the category ${}^T_T\mathcal{YD}$.

3.4.1 Four dimensional simple algebras

Since R is not commutative, it follows from Proposition 3.3.8 that the T-action on R is non-trivial and so is on A. This kind of four dimensional matrix algebras in ${}^T_T\mathcal{YD}$ has been classified in [11]. Although those algebras were considered as Yetter-Drinfeld module algebras over $D(T)$, where $D(T)$ is the Drinfeld double $T^{*cop} \bowtie T$, it has been proved that the left-right Yetter-Drinfeld category ${}_T\mathcal{YD}^T$ and the category ${}_{D(T)}\mathcal{M}$ of left $D(T)$-modules are equivalent (cf. [28]).

A finite dimensional Yetter-Drinfeld T-module M in the left-right Yetter-Drinfeld category over a Taft algebra T, is a left T-module and a right T-comodule satisfying the following compatibility

$$(tm)_{(0)} \otimes (tm)_{(1)} = t_2 \triangleright m_{(0)} \otimes t_3 m_{(1)} S^{-1}(t_1). \tag{3.4.12}$$

The category of left-right Yetter Drinfeld modules and left T-linear, right T-colinear maps is denoted by ${}_T\mathcal{YD}^T$. For any two Yetter-Drinfeld-modules $M, N \in {}_T\mathcal{YD}^T$, $M \otimes N$ again is a Yetter-Drinfeld-module in ${}_T\mathcal{YD}^T$. This category is a braided monoidal category with the braiding $c_{M,N} : M \otimes N \to N \otimes M$ given by $c_r = c_{r\,M,N}(m \otimes n) = n_{(0)} \otimes n_{(1)} m$. Braided algebras, coalgebras, bialgebras and Hopf algebras in ${}_T\mathcal{YD}^T$ are defined with the same rationale as ${}^T_T\mathcal{YD}$.

We claim that the functor $F : {}_T\mathcal{YD}^T \to {}^T_T\mathcal{YD}$ is an isomorphism of monoidal categories. Let M be an object in ${}_T\mathcal{YD}^T$ with the right comodule structure ${\rho_M}^r(m) = m_{(0)} \otimes m_{(1)}$. We define $F(M)$ in ${}^T_T\mathcal{YD}$ to be M as a vector space with the same left H-module action and the coaction is defined by

$$ {\rho_M}^\ell(m) = m_{(-1)} \otimes m_{(0)} = S(m_{(1)}) \otimes m_{(0)}. $$

We show that this map makes $F(M)$ into a Yetter Drinfeld module in ${}^T_T\mathcal{YD}$. The coassociativity axiom of ${\rho_M}^\ell$ follows from that of ${\rho_M}^r$ and the anti-comultiplicativity of S. i.e.

$$ S(m_{(1)}) \otimes S((m_{(0)})_{(1)}) \otimes (m_{(0)})_{(0)} = S(m_{(2)}) \otimes S(m_{(1)}) \otimes m_{(0)}. $$

Also, $\varepsilon(m_{(-1)})m_{(0)} = \varepsilon(S(m_{(1)}))m_{(0)} = \varepsilon(m_{(1)})m_{(0)} = m$. Moreover, it follows from (3.4.12) that $(M, {\rho_M}^\ell)$ satisfies the compatibility

$$ (tm)_{(-1)} \otimes (tm)_{(0)} = t_{(1)} m_{(-1)} S(t_{(3)}) \otimes t_{(2)} \triangleright m_{(0)}, $$

and hence is a Yetter Drinfeld module in ${}^T_T\mathcal{YD}$. Conversely, for each Yetter Drinfeld module M with coaction ${\rho_M}^\ell$ in ${}^T_T\mathcal{YD}$, one can analogously define $G(M)$ in ${}_T\mathcal{YD}^T$ as a vector space M with the same module structure and the coaction is given by

$$ {\rho_M}^r(m) = m_{(0)} \otimes m_{(1)} = m_{(0)} \otimes S^{-1}(m_{(-1)}). $$

If a morphism $f : M \to N$ in ${}_T\mathcal{YD}^T$ is a T-linear map, then $F(f)$ is a left T-linear map. Let f be a T-comodule map, i.e. $f(m)_{(0)} \otimes f(m)_{(1)} = f(m_{(0)}) \otimes m_{(1)}$, then $F(f) := f$ is a left T-colinear map satisfying $S(m_{(1)}) \otimes f(m_{(0)}) = S(f(m)_{(1)}) \otimes f(m_{(0)})$. This shows that the categories ${}^T_T\mathcal{YD}$ and ${}_T\mathcal{YD}^T$ are isomorphic as monoidal categories. Moreover, A is a finite dimensional $\Bbbk$-algebra in ${}_T\mathcal{YD}^T$ if, and only if, A is a $\Bbbk$-algebra in ${}^T_T\mathcal{YD}$.

Now we turn to consider those four-dimensional simple algebras in [11] which have the same left T-module structure as ours. In [11], six kinds of such simple algebras corresponding to our cases are described in Theorem 4.10 (c), (h), (i), (k), (m), (n) and their structures are detailed in Lemmas 2.19, 2.27 and 2.31. There exist $u, v \in A$ such that $\{\iota, u, v, uv\}$ forms a basis for A and satisfy the relations

$$u^2 = \alpha\iota, \quad v^2 = \beta\iota, \quad uv + vu = \gamma\iota \quad \text{for some } \alpha, \beta, \gamma \in \Bbbk\,. \tag{3.4.13}$$

The action $\triangleright$ of T is given by

$$\begin{aligned} x \triangleright u &= 0, \quad x \triangleright v = \iota, \\ g \triangleright u &= -u, \quad g \triangleright v = -v\,. \end{aligned} \tag{3.4.14}$$

We have seen in Chapter 2 that A is a right T-comodule $\Leftrightarrow$ A is a left T^*-module, and if $\rho(a) = a_{(0)} \otimes a_{(1)}$ is the comodule map, then $f \cdot a = f(a_{(1)})a_{(0)}$ for all $f \in T^*$. Let $\{\overline{1}, \overline{g}, \overline{x}, \overline{xg}\}$ be the dual basis of $\{1, g, x, xg\}$ and assume $\rho(a) = 1 \otimes f_1 + u \otimes f_u + v \otimes f_v + uv \otimes f_{uv}$. We shall apply the corresponding $D(T)$-actions in [11] that are listed below

(I) $x \cdot u = 0, g \cdot u = -u, \overline{x} \cdot u = 0, \overline{g} \cdot u = 0, \overline{xg} \cdot u = uv, \overline{1} \cdot u = u$

$x \cdot v = 1, g \cdot v = -v, \overline{x} \cdot v = 2\beta, \overline{g} \cdot v = v, \overline{xg} \cdot v = 0, \overline{1} \cdot v = 0$

(II) $x \cdot u = 0, g \cdot u = -u, \overline{x} \cdot u = 0, \overline{g} \cdot u = 0, \overline{xg} \cdot u = 0, \overline{1} \cdot u = 0$

$x \cdot v = 1, g \cdot v = -v, \overline{x} \cdot v = -\eta, \overline{g} \cdot v = v, \overline{xg} \cdot v = 0, \overline{1} \cdot v = 0$

(III) $x \cdot u = 0, g \cdot u = -u, \overline{x} \cdot u = -1, \overline{g} \cdot u = u, \overline{xg} \cdot u = 0, \overline{1} \cdot u = 0$

$x \cdot v = 1, g \cdot v = -v, \overline{x} \cdot v = 0, \overline{g} \cdot v = v, \overline{xg} \cdot v = 0, \overline{1} \cdot v = 0.$

For case (I), by letting a be u and v respectively, we found $\rho(u) = u \otimes 1 + uv \otimes hg$ and $\rho(v) = 1 \otimes 2\beta h + v \otimes g$. Likewise, we can get the right comodule maps for cases (II) and (III). Since the correspondence of the coaction for each object transformed from ${}_T\mathcal{YD}^T$ to ${}^T_T\mathcal{YD}$ is given by $\rho(a) = S(a_{(1)}) \otimes a_{(0)}$, together with (3.3.1), the left T-comodule structures on A are

$$\begin{aligned} &\text{(I)} \quad \rho(u) = \iota \otimes u + 2x \otimes uv, \quad \rho(v) = \iota \otimes v - 2\beta xg \otimes \iota \text{ and } \gamma = 0; \\ &\text{(II)} \quad \rho(u) = g \otimes u, \quad \rho(v) = \eta xg \otimes \iota + g \otimes v \text{ where } \eta \in \Bbbk; \\ &\text{(III)} \quad \rho(u) = xg \otimes \iota + g \otimes u, \quad \rho(v) = g \otimes v. \end{aligned} \tag{3.4.15}$$

3.4.2 Commutativity of R

By (3.4.14), A can be decomposed into the direct sum $A = P_0 \oplus P_1$ as T-modules, where P_0 is spanned by uv, u and P_1 by v, ι. The T-modules P_0 and P_1 are respectively the projective covers of the simple T-modules $\Bbbk$ and $\Bbbk_\alpha$. Therefore, $R = P_0 \oplus P_1 \oplus I$ is an indecomposable T-module decomposition of R. Note that

$$\mathrm{Hom}_T(P_i, P_j) \cong \mathrm{Hom}_T(P_0, I) \cong \mathrm{Hom}_T(I, P_1) \cong \mathrm{Hom}_T(I, I) \cong \Bbbk$$

as $\Bbbk$-linear spaces for all $i, j = 0, 1$. Since the antipode of R is a left T-module map, we have

$$\begin{array}{rclllll}
S_R(\iota) & = & \zeta_1\iota & & & & \\
S_R(u) & = & & \zeta_2 u & & & \\
S_R(v) & = & & \zeta_3 u & +\zeta_1 v & & \\
S_R(uv) & = & \zeta_4\iota & & & +\zeta_2 uv & +\zeta_5 e \\
S_R(e) & = & \zeta_6\iota & & & & +\zeta_7 e
\end{array}$$

for some $\zeta_1, \ldots, \zeta_7 \in \Bbbk$. Since $S_R(1_R) = 1_R$ and $\varepsilon_R \circ S_R = \varepsilon_R$, it follows immediately that

$$\zeta_5 = \zeta_6 = 0, \quad \text{and} \quad \zeta_1 = \zeta_7 = 1.$$

Let λ_R be a right integral of R^* such that $\lambda_R(e) = 1_R$. Since R^* is also a semisimple $\Bbbk$-algebra, λ_R is a both left and right integral. In view of (3.3.2) and (3.4.14), $\lambda_R(\iota) = \lambda_R(u) = \lambda_R(v) = 0$. Since λ_R is a Frobenius map, $\ker \lambda_R$ cannot contain any non-zero ideal of R and hence $\lambda_R(uv) \neq 0$. It follows from (3.3.4) of Lemma 3.3.4 that case (I) is impossible. One may normalize u, v and assume

$$\lambda_R(uv) = 1 \quad \text{and} \quad \lambda_R(vu) = -1\,.$$

The last equality is an immediate consequence of (3.4.13). Thus, with respect to the form $\lambda_R(\cdot, \cdot)$, the dual basis of $\{\iota, u, v, uv, e\}$ is $\{uv, -v, u, \iota, e\}$, i.e.

$$r = \iota\lambda_R(uvr) - u\lambda_R(vr) + v\lambda_R(ur) + uv\lambda_R(\iota r) + e\lambda_R(er) \quad \text{for } r \in R.$$

From [17], dual bases for R with respect to λ_R are given by

$$\{S_R^{-1}(({\Lambda_R}^{(2)})_{(0)}), \quad \chi_T(({\Lambda_R}^{(2)})_{(-1)})S_T^{-1}(({\Lambda_R}^2)_{(-2)}) \triangleright {\Lambda_R}^{(1)}\}.$$

Therefore,

$$\iota \otimes uv - u \otimes v + v \otimes u + uv \otimes \iota + e \otimes e$$
$$= S_R^{-1}\big((e^{(2)})_{(0)}\big) \otimes \varepsilon_T\big((e^{(2)})_{(-1)}\big) S_T^{-1}\big((e^{2})_{(-2)}\big) \triangleright e^{(1)}$$
$$= S_R^{-1}\big((e^{(2)})_{(0)}\big) \otimes S_T^{-1}\big((e^{(2)})_{(-1)}\big) \triangleright e^{(1)}. \quad (3.4.16)$$

Hence, by applying $S_R \otimes \mathrm{id}_R$,

$$S_R(\iota) \otimes uv - S_R(u) \otimes v + S_R(v) \otimes u + S_R(uv) \otimes \iota + S_R(e) \otimes e$$
$$= (e^{(2)})_{(0)} \otimes S_T^{-1}\big((e^{(2)})_{(-1)}\big) \triangleright e^{(1)}. \quad (3.4.17)$$

Applying $\lambda_R \otimes \mathrm{id}_R$ to the equation and by (3.3.4), we then have

$$\zeta_2\iota + e = \lambda_R\big((e^{(2)})_{(0)}\big) S_T^{-1}\big((e^{(2)})_{(-1)}\big) \triangleright e^{(1)}$$
$$= S_T^{-1}\big(\lambda_R(e^{(2)})1_T\big) \triangleright e^{(1)} = 1_T \triangleright \lambda_R(e^{(2)})e^{(1)} = \lambda_R(e)1_R = 1_R. \quad (3.4.18)$$

Therefore, $\zeta_2 = 1$.

From [13, (1.6)], we apply the equality

$$S_R^{-1}((e^{(2)})_{(0)}) S_T^{-1}((e^{(2)})_{(-1)}) \triangleright e^{(1)} = \varepsilon_R(e)1_R$$

to (3.4.16). We find $\gamma\iota + e = 1_R$ and hence $\gamma = 1$. However, this implies S_R cannot be an antialgebra homomorphism in the braided sense, i.e.

$$S_R(rs) = (r_{(-1)} \triangleright S_R(s))S(r_{(0)})\,. \quad (3.4.19)$$

Suppose S_R satisfies the condition (3.4.19). Also from (3.4.13), we find in both cases (II) and (III)

$$S(u^2) = \alpha\iota, \quad S(u^2) = (u_{(-1)} \triangleright S_R(u))S(u_{(0)}) = -\alpha\iota,$$
$$S(vu) = -\zeta_3\alpha\iota - uv, \quad S(vu) = (1 - \zeta_4)\iota - uv.$$

These equalities imply $\alpha = 0$ and $\zeta_4 = 1$. Thus, $S_R(uv) = uv + \iota$. However,

$$(u_{(-1)} \triangleright S_R(v))S(u_{(0)}) = \begin{cases} uv - \iota, & \text{in case (II)} \\ uv - 2\iota, & \text{in case (III)}, \end{cases}$$

do not satisfy (3.4.19). Therefore, there does not exist any non-commutative semisimple braided Hopf algebra of dimension 5 in ${}^{T}_{T}\mathcal{YD}$. By Proposition 3.3.8, this can be concluded in the following corollary:

Corollary 3.4.1. *A 5-dimensional semisimple braided Hopf algebra in ${}^{T}_{T}\mathcal{YD}$ is a group algebra.*

3.5 Non-semisimple Hopf algebras of dimension 20, 28 and 44

We have seen from Proposition 3.3.3 that if a non-semisimple Hopf algebra H of dimension $4p$ with an odd prime p is neither pointed nor dual-pointed, then H is the biproduct $R\times T$ if, and only if, both H and H^* contain a non-trivial skew primitive element, i.e. $|G(H)| = 2 = |G(H^*)|$. In this section, we shall discuss the existence of non-semisimple Hopf algebras that are neither pointed nor dual-pointed with either $(|G(H)|, |G(H^*)|) = (1,1), (1,2)$ or $(2,1)$t. We will prove the main result: non-semisimple Hopf algebras of dimension 20, 28 and 44 are pointed or dual-pointed.

3.5.1 Dimension counting

Assume that H is a non-semisimple Hopf algebra of dimension $4p$ with an odd prime ≤ 11 p. The following technical lemma shall be used to complete the proof of our main theorem:

Lemma 3.5.1. *A non-semisimple Hopf algebra over $\Bbbk$ of dimension $4p$ with an odd prime $p \leq 11$ must contain a non-trivial skew primitive element.*

To prove this lemma, we first make some observations and establish a few lemmas below.

Lemma 3.5.2. *Let V be a simple H-module with* $\dim V > 1$*. Assume that V is a composition factor of $P(\Bbbk)$. Then*

(1) $\dim P(V) \geq 2\dim V + 1$*,* $\dim P(\Bbbk) \geq \dim V + 2$ *and* $\dim P(\Bbbk)$ *is even.*

(2) *If $V \cong V^{\vee\vee}$, then* $\dim P(V) \geq 2\dim V + 2$ *and*

$$\sum_{W} [P(V) : W] \text{ is a positive even integer,}$$

where W runs through all isomorphism classes of odd dimensional simple H-modules.

Proof. (1) The results follow immediately from Lemma 2.3.1 and Lemma 3.2.4.

(2) Note that the order of S^2 is a two power by Lemma 3.2.3. If $V \cong V^{\vee\vee}$, then $P(V) \cong P(V)^{\vee\vee}$ and $\dim P(V)$ is even from Lemma 3.2.4. Thus, the number of odd dimensional simple constituent of $P(V)$ including multiplicities must be even. Moreover, by Lemma 2.3.1, $[P(\Bbbk) : V] = [P(V) : \Bbbk_{\alpha^{-1}}]$ where α is a distinguished group-like element. Hence, $\sum_W [P(V) : W]$ is a positive even integer, where W runs through all isomorphism classes of odd dimensional simple H-modules.

□

The cyclic group generated by the antipode S acts on the set $\mathrm{Irr}(H)$ by duality. That is, for any simple H-module U, $S \cdot U = U^{\vee}$. The orbit of U is given by

$$\mathfrak{D}_U = \{U, U^{\vee}, U^{\vee\vee}, U^{\vee\vee\vee}, \cdots\}$$

up to isomorphism. Moreover, $|\mathfrak{D}_U| = 1, 2, 4$ or 8 for any $U \in \mathrm{Irr}(H)$ because $S^8 = \mathrm{id}$ from Lemma 3.2.4. By taking the dual of the exact sequence of H-modules

$$0 \to JP(U) \to P(U) \to U \to 0$$

and by (2.3.3), we have the isomorphism

$$P(U)^{\vee} \cong I(U^{\vee}) \cong P(\Bbbk_\alpha \otimes U^{\vee\vee\vee}).$$

Moreover, together with (2.3.2) and the isomorphisms $P(\Bbbk_\alpha \otimes V) \cong \Bbbk_\alpha \otimes P(V)$, $P(V \otimes \Bbbk_\alpha) \cong P(V) \otimes \Bbbk_\alpha$, one can get

$$P(U)^{\vee\vee\vee} \otimes \Bbbk_\alpha \cong P(U^{\vee})$$

for any $U \in \mathrm{Irr}(H)$. Therefore,

$$\dim P(W) = \dim P(U) \quad \text{for any } W \in \mathfrak{D}_U. \tag{3.5.20}$$

Since H is a Frobenius algebra, we have the equation

$$\dim H = \sum_{U \in \mathrm{Irr}(H)} \dim U \cdot \dim P(U) = \sum_{U \in \mathcal{R}} |\mathfrak{D}_U| \cdot \dim U \cdot \dim P(U), \tag{3.5.21}$$

where $\mathcal{R}$ is a complete set of representatives of $\mathfrak{D}$-orbits.

Since $\mathrm{Hom}_H(U^\vee \otimes P(U), \Bbbk) \cong \mathrm{Hom}_H(P(U), U \otimes \Bbbk)$ and $U^\vee \otimes P(U)$ is projective, $P(\Bbbk)$ is a summand of $U^\vee \otimes P(U)$. So

$$\dim U \dim P(U) \geq \dim P(\Bbbk) \tag{3.5.22}$$

for all simple H-module U. Hence by (3.5.21), we have

$$|\operatorname{Irr}(H)| \leq \dim H / \dim P(\Bbbk)\,. \tag{3.5.23}$$

Lemma 3.5.3. *For any simple H-module U such that* $\dim U > 1$ *and* $|\mathfrak{D}_U| = 1$,

$$\dim U \dim P(U) \geq 16.$$

Proof. If $|\mathfrak{D}_U| = 1$, then $U \cong U^\vee$ and so $\dim U \geq 3$. Since $U \cong U^{\vee\vee}$, $\dim P(U)$ is even. Thus, $\dim P(U) > \dim U$ for $\dim U$ odd. Therefore,

$$\dim U \dim P(U) \geq \dim U \cdot 2\dim U \geq 18$$

for odd dimensional U. If U is even dimensional, then $\dim P(U) \geq \dim U \geq 4$ and so

$$\dim U \dim P(U) \geq 16.$$

□

Lemma 3.5.4. *Let H be a non-semisimple Hopf algebra of dimension $4p$. If H^* does not have any non-trivial skew primitive element, then there exist two simple H-modules V, W of dimension greater than 1 such that $\mathfrak{D}_V \neq \mathfrak{D}_W$.*

Proof. By the assumption, H^* does not have any non-trivial skew primitive element, so H^* is not pointed. Hence, by Lemma 2.3.2, $\dim P(\Bbbk) \geq 2$ and $P(\Bbbk)$ contains a simple constituent V of dimension greater than one. From Theorem 3.2.5, $|G(H^*)| \leq 2$.

We proceed the proof by contradiction. Suppose that V is a simple H-module such that $\mathfrak{D}_V$ is the only one complete set of non-isomorphic simple H-modules of dimension greater than 1. For simplicity, we let

$$D_0 = \dim P(\Bbbk), \quad d = \dim V, \quad D = \dim P(V), \quad m = |\mathfrak{D}_V|.$$

By Lemma 3.5.2, we have $D_0 \geq 4$ and $D \geq 5$. From (3.5.21), we have the equation

$$4p = D_0|G(H^*)| + mdD. \tag{3.5.24}$$

From Remark 2.3.1, by decomposing of $V^\vee \otimes P(V)$ into indecomposable H-modules and (3.5.20), we have

$$dD = \dim(V^\vee \otimes P(V)) = n_0 D_0 + n_1 D \tag{3.5.25}$$

for some non-negative integer n_0, n_1 such that $1 \leq n_0 \leq |G(H^*)|$ and $0 \leq n_1 < d$. Thus, one can obtain from these equations that

$$(d - n_1)D = n_0 D_0 \quad \text{and} \quad 4p = (md + k(d - n_1))D$$

where $k = \frac{|G(H^*)|}{n_0}$. Since $m, k \geq 1$, $d \geq 2$, and $d - n_1 \geq 1$, hence $md + k(d - n_1) \geq 3$. Therefore, (3.5.25) implies that $D = p$ and $md + k(d - n_1) = 4$. However, since D is odd, we have $V \not\cong V^{\vee\vee}$ and hence $m \geq 4$. Thus, $md + k(d - n_1) \geq 8$, a contradiction. □

3.5.2 Main result

Now we can prove our key Lemma:

proof of Lemma 3.5.1. By duality, it suffices to show that H^* has a non-trivial skew primitive element. Assume that H^* does not have any non-trivial skew primitive element, then H^* is not pointed. From Lemma 2.3.2 and Theorem 3.2.5, $|G(H^*)| \leq 2$ and $P(\Bbbk)$ contains a simple constituent V of dimension ≥ 2. By Lemma 3.5.4, there exists another simple H-module W such that $\dim W \geq 2$ and $\mathfrak{D}_V \neq \mathfrak{D}_W$.

(1) $|\mathfrak{D}_V| \leq 2$ and $\dim P(V) \geq 2\dim V + 2$.

First we claim that for $\dim W \geq 2$, $\dim W \dim P(W) \geq 8$. If $\dim W = 2$, $W \not\cong W^\vee$ and hence $\dim P(W) \geq 4$. For $\dim W \geq 3$, $\dim P(W) \geq \dim W \geq 3$. This proves our claim.

From (3.5.21), we have

$$\dim H \geq \dim P(\Bbbk) + |\mathfrak{D}_V| \dim V \dim P(V) + \dim W \dim P(W) \geq 4 + |\mathfrak{D}_V| \cdot 2 \cdot 5 + 8.$$

This implies that $|\mathfrak{D}_V|$ must be less than 4 and hence $|\mathfrak{D}_V| \leq 2$. Therefore, $V \cong V^{\vee\vee}$ and $\dim P(V) \geq 2\dim V + 2$ by Lemma 3.5.2.

(2) $\dim V = 2$ and $\dim P(V) \geq 6$.

Assume that $\dim V \geq 3$. Then $\dim P(V) \geq 8$ by (1) and $\dim P(\Bbbk) \geq 6$ by Lemma 3.5.2. From the proof of (1), $\dim W \dim P(W) \geq 8$ and therefore

$$\dim H \geq \dim P(\Bbbk) + \dim V \dim P(V) + |\mathfrak{D}_W| \dim W \dim P(W) \geq 6 + 3 \cdot 8 + |\mathfrak{D}_W| 8.$$

This implies that $|\mathfrak{D}_W| = 1$. Hence, by Lemma 3.5.3,

$$\dim W \dim P(W) \geq 16$$

and hence

$$\dim H \geq 6 + 3 \cdot 8 + 16 > 44,$$

which is a contradiction. Thus, $\dim V = 2$ and so $\dim P(V) \geq 6$ form (1).

(3) $|\mathfrak{D}_V| = 2$ by (1) and Lemma 3.2.6.

(4) Every composition factor of $P(V)$ has dimension 1 or 2.

If U is a composition factor of $P(V)$ and $\dim U \geq 3$, then by (1) $\dim P(V) \geq 6$ and by Lemma 2.3.1

$$\dim P(U) \geq 2\dim U + \dim V \geq 8\,.$$

Thus,

$$\dim H \geq \dim P(\Bbbk) + |\mathfrak{D}_V| \dim V \dim P(V) + \dim W \dim P(W) \geq 4 + 2 \cdot 2 \cdot 6 + 3 \cdot 8 > 44\,,$$

a contradiction. Therefore, all the composition factors of $P(V)$ can only have dimension 1 or 2.

(5) Consider the H-module $M = \bigoplus_{\beta \in G(H^*)} P(\Bbbk_\beta)$. From Lemma 2.3.1, number of multiplicities of V in M is

$$[M : V] = \sum_{\beta \in G(H^*)} [P(\Bbbk_\beta) : V] = \sum_{\beta \in G(H^*)} [P(V) : \Bbbk_\beta].$$

By (4) and Lemma 3.5.2, $[M : V] \geq 2$. Since M is self-dual, we have $[M : V] = [M : V^\vee]$. Therefore, we have

$$\dim M \geq 2|G(H^*)| + 2[M : V]\dim V \geq 2|G(H^*)| + 8.$$

If $|G(H^*)| = 2$, then $\dim P(\Bbbk) = \dim M/2 \geq 6$ since $\dim P(\Bbbk)$ is even. Hence, $\dim W \dim P(W) \geq \dim P(\Bbbk) \geq 6$. Now,

$$\dim H \geq \dim M + 2\dim V \dim P(V) + |\mathfrak{D}_W|\dim W \dim P(W) \geq 12 + 2 \cdot 2 \cdot 6 + |\mathfrak{D}_W| \cdot 6.$$

This implies that $|\mathfrak{D}_W| = 1$ and hence $\dim W \dim P(W) \geq 16$ from Lemma 3.5.3. However, we get

$$\dim H \geq 12 + 2 \cdot 2 \cdot 6 + 16 > 44,$$

a contradiction. Therefore, $|G(H^*)| = 1$. In this case, $M = P(\Bbbk)$ and so $\dim P(\Bbbk) \geq 10$. Then $\dim W \dim P(W) \geq 10$ and

$$\dim H \geq \dim P(\Bbbk) + 2\dim V \dim P(V) + |\mathfrak{D}_W|\dim W \dim P(W) \geq 10 + 2 \cdot 2 \cdot 6 + |\mathfrak{D}_W| \cdot 10,$$

which implies $|\mathfrak{D}_W| = 1$. Then from Lemma 3.5.3, $\dim W \dim P(W) \geq 16$ but

$$\dim H \geq 10 + 2 \cdot 2 \cdot 6 + 16 > 44$$

leads to another contradiction again. Thus, H^* must contain a non-trivial skew primitive element.

□

We have shown that if H is a non-semisimple Hopf algebra of dimension $4p$ with p an odd prime ≤ 11 such that neither H nor H^* is pointed, then both H and H^* have a non-trivial skew primitive element. We shall use Lemma 3.5.1 to prove the following theorem:

Theorem 3.5.5. *Non-semisimple Hopf algebras of dimension 28 and 44 are either pointed or dual-pointed.*

Proof. Suppose that a non-semisimple $4p$-dimensional Hopf algebra H is neither pointed nor dual-pointed. By Lemma 3.5.1 and Proposition 3.3.3, H is a biproduct $H \cong R \times T$, where R is a semisimple braided Hopf algebra of dimension $3, 7$ or 11 in ${}^T_T\mathcal{YD}$. By Corollary 3.3.11, R is a commutative group algebra. Thus by Proposition 3.3.8, $R \times T$ is a pointed Hopf algebra, a contradiction. $\square$

CHAPTER 4. General Conclusion

4.1 Summary

This thesis deals with the classification of $4p$-dimensional non-semisimple Hopf algebras over an algebraically closed field of characteristic zero where p is an odd prime.

Our first main result, Theorem 3.2.5 states that the necessary and sufficient condition for the pointedness of a non-semisimple Hopf algebras of dimension $4p$ with an odd prime p is the order of $G(H)$ greater than 2. Therefore, a $4p$-dimensional non-semisimple Hopf algebra with more than two group-like elements is either pointed or dual pointed. Together with Lemma 3.1.1 listing all pointed Hopf algebras of dimension pq^2, the structure of a $4p$-dimensional non-semisimple Hopf algebra or its dual with more than two group-like elements is clear.

Proposition 3.3.3 implies that the Hopf algebras of dimension $4p$ with exactly two group-like elements and $p \leq 11$ must be isomorphic to a Radford biproduct of the form $R \times T$. The existence of skew primitive elements plays an important role to this case. Related propositions or equivalent statements of pointedness for such a biproduct were studied in Lemmas 3.3.9 and 3.3.10 and used to show that a p-dimensional braided Hopf algebra must be a group algebra for dimension $p = 3, 7$ and 11. For $p = 5$, we use the result from [11], classification of 4-dimensional algebras in the Yetter Drinfeld category over T, to show the commutativity of 5-dimensional braided Hopf algebras.

Another main result of this thesis is Theorem 3.5.5, which completes the classification of Hopf algebras of dimensions 20, 28 and 44. We considered the decomposition into indecomposable modules and used counting arguments to show the existence of non-trivial skew primitive elements for non-semisimple Hopf algebras of dimension 12, 20, 28 and 44. Together with the previous results, this theorem follows.

4.2 Future works

For this classification problem of non-semisimple Hopf algebras of dimension $4p$, where p is an odd prime, there are two remaining questions. First, we don't know yet whether for any general prime p, there exists a non-trivial skew primitive element. If yes, we can focus on the study of Hopf algebras which are the form of biproduct. Secondly, it is still not clear about the existence of non-commutative semisimple p-dimensional braided Hopf algebras in ${}_T^T\mathcal{YD}$. Moreover, as stated in Proposition 3.3.8, we believe that pointed Hopf algebras which admit a commutative braided Hopf algebra in ${}_T^T\mathcal{YD}$, are self-dual.

Though some results are known about braided Hopf algebras over group algebras. There are still many open problems about braided Hopf algebras over general semisimple or non-semisimple Hopf algebras, even the non-semisimple Hopf algebra of least dimension – 4-dimensional Taft algebra. It appears difficult to understand the structure of braided semisimple Hopf algebras R in ${}_T^T\mathcal{YD}$. It may be possible to extend the case of biproduct by a 4-dimensional Taft algebra to a Taft algebra of general dimension q^2. If this is a case, we may use it to obtain another necessary and sufficient condition for the pq^2-dimensional non-semisimple Hopf algebras of the form as biproduct.

As we introduced in Chapter 1, semisimple or pointed Hopf algebras of dimension p^3 with a prime p are known. The general classification for non-semisimple Hopf algebras with lower dimension less than 32 is not complete yet. There are eight types of non-semisimple pointed Hopf algebras of dimension p^3. Non-semisimple quasitriangular Hopf algebras of dimension 27 have been classified completely in [19], which shows that it is isomorphic to $\mathbf{u}_q(\mathfrak{sl}_2)$ where $q \neq 1$ is a p-th root of unity. We are interested in the classification of non-semisimple Hopf algebras of dimension 27.

Let H be a non-semisimple Hopf algebra of dimension 27. As the dimension is small, the method of counting dimensions used in Section 3.4 may be applied to classify such Hopf algebras. The method of counting dimensions was pretty much used in the classification of pq-dimensional Hopf algebras with distinct primes p, q (cf. [36]), although it didn't work very well. To begin with this problem, we use Nichols-Zoeller's theorem and consider possible pairs

$(|G(H)|, |G(H^*)|)$. By the non-semisimplicity of H and the duality, we only need to consider

$$(|G(H)|, |G(H^*)|) = (9,9),\ (9,3),\ (9,1),\ (3,3),\ (3,1) \text{ or } (1,1) \quad \text{for} \quad \dim H = 27.$$

If we apply the equation (3.5.21) and analyze the dimension of each principal indecomposable module of H that are projective cover of simple modules with certain dimensions at most 3, this approach could work for a few cases listed above. Different methods may be required for the rest of them.

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