

INTERACTION BETWEEN AN INCIDENT WAVE AND A DYNAMICALLY TRANSFORMING INHOMOGENEITY

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INTRODUCTION

Transformation-toughening of ceramics has attracted considerable attention [1,2,3] in recent years. The key mechanism in this toughening is the stress-induced phase transformation of the partially stabilized zirconia (PSZ) inhomogeneities, which accompanies volumetric expansion. Due to this expansion, the composite material consisting of PSZ inhomogeneities in a brittle matrix becomes more resistant to fracturing. While this problem has been studied for quasi-static loadings [4,5], the corresponding dynamic case has remained relatively unexplored.

As a first step towards understanding the effect of a traveling stress pulse on the phase transformation of the PSZ inhomogeneity, the elastic field due to a dynamically transforming spherical inclusion has been investigated in a recent paper [6] by the authors. In the present paper, a phenomenological model is proposed for the interaction between an incident wave and a transforming inhomogeneity. The solution for this model is obtained by combining solutions to a scattering problem, a dynamic inhomogeneity problem, and a static inhomogeneity problem. Scattering by a spherical particle has been treated by Pao and Mow [7]. An exact closed form solution is obtained for the dynamic inhomogeneity problem by following their approach. Numerical results are shown for the PSZ toughened alumina ceramics. These numerical results suggest that, under the high frequency dynamic loading, the transformation-toughened ceramics might lose some of its toughness due to a relatively large tension field caused by the dynamically transforming PSZ particle.

STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Fig.1. A longitudinal wave is incident upon the spherical inhomogeneity Ω of radius a . Both the matrix (material 1) and the inhomogeneity (material 2) are isotropic elastic media. A phase transformation of the inhomogeneity Ω is induced by the incident longitudinal stress wave. We model this phase transformation by the eigenstrain $e_{ij}^*(t)$,

$$e_{ij}^*(t) = \frac{1}{2} e_{ij}^* [e^{-2i\omega t} + 1], \quad (1)$$

where ω is the angular frequency of the incident wave. The doubled frequency, 2ω , for the inhomogeneity models transformation in both tension and compression (see Fig.2). Only the real part of $e_{ij}^*(t)$ is taken as physically meaningful. The assumed reversibility of the eigenstrain $e_{ij}^*(t)$ is partially supported by the experimental observations [3]. The governing equations are

$$\begin{aligned} \sigma_{ij,j}^1 &= \rho_1 \ddot{u}_i^1, & \sigma_{ij}^1 &= C_{ijkl}^1 e_{kl}^1, \\ e_{kl}^1 &= \frac{1}{2} (u_{k,1}^1 + u_{1,k}^1), & & \text{in } R^3 - \Omega, \\ \sigma_{ij,j}^2 &= \rho_2 \ddot{u}_i^2, & \sigma_{ij}^2 &= C_{ijkl}^2 (e_{kl}^2 - e_{kl}^*), \\ e_{kl}^2 &= \frac{1}{2} (u_{k,1}^2 + u_{1,k}^2), & & \text{in } \Omega, \end{aligned} \quad (2)$$

$$\quad (3)$$

where ρ_m and C_{ijkl}^m are the mass density and the elasticity tensor of material m ($=1$ or 2), respectively. The boundary conditions are

$$u_i^1 = u_i^2, \quad \sigma_{ij}^1 n_j = \sigma_{ij}^2 n_j, \quad \text{at } r = a, \quad (4)$$

where n_j is the outward unit normal of Ω . The incident longitudinal wave is

$$u^{in} = -\frac{i}{k_{L1}} i_z e^{i(k_{L1}z - \omega t)}, \quad (5)$$

where i_z is a unit vector in the z direction, and

$$k_{L1} = \frac{\omega}{c_{L1}}, \quad c_{L1} = \sqrt{\frac{2\mu_1 + \lambda_1}{\rho_1}}. \quad (6)$$

The constant factor in (5) is so chosen as to make the strain amplitude derived from (5) unity.

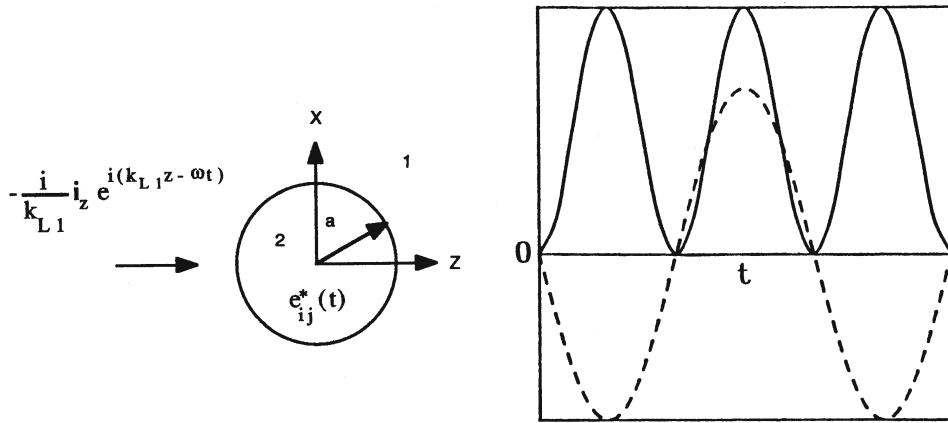


Fig. 1. Plane longitudinal wave incident on a dynamically transforming spherical inhomogeneity.

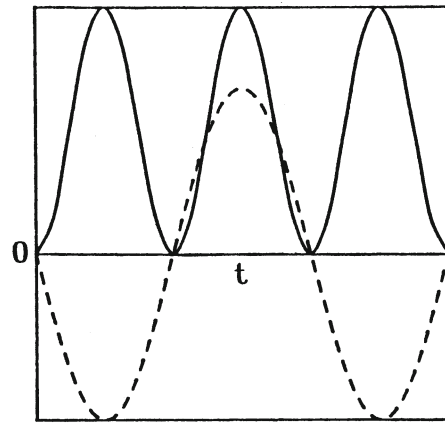


Fig. 2. Phase transformation (solid line) induced by an incident stress wave (dashed line)

Due to the linearity, the problem is decomposed into three sub-problems: scattering, dynamic inhomogeneity, and static inhomogeneity. In the following sections, scattering and dynamic inhomogeneity will be treated in detail. The solution of the static inhomogeneity is standard [8,9,10].

SCATTERING

Scattering of a plane longitudinal wave by a spherical inhomogeneity has been treated by Pao and Mow [7]. In this section, we summarize their results.

Since the incident longitudinal wave has the time factor $e^{-i\omega t}$, the elastic field due to scattering has also the same time factor. In the following analysis, we omit this time factor, and concentrate on the spatial part of the solution. Because of the axisymmetry with respect to the z -axis, the displacement field in spherical coordinates (r, θ, ϕ) is expressed in terms of potentials Φ and Ψ as

$$\mathbf{u}^m = \nabla\Phi^m + \nabla \times (\mathbf{e}_\phi \frac{\partial\Psi^m}{\partial\theta}), \quad m = 1 \text{ or } 2, \quad (7)$$

where \mathbf{e}_ϕ is the base vector for the ϕ -axis, and

$$\nabla^2\Phi^m + k_{Lm}^2\Phi^m = 0, \quad \nabla^2\Psi^m + k_{Tm}^2\Psi^m = 0, \quad (8)$$

where

$$\begin{aligned} k_{Lm} &= \frac{\omega}{c_{Lm}}, & c_{Lm} &= \sqrt{\frac{2\mu_m + \lambda_m}{\rho_m}}, \\ k_{Tm} &= \frac{\omega}{c_{Tm}}, & c_{Tm} &= \sqrt{\frac{\mu_m}{\rho_m}}. \end{aligned} \quad (9)$$

In material 1, the potentials are decomposed into the incident and reflected waves. Thus the expansions of the potentials in terms of spherical harmonics are given by

$$\begin{aligned} \Phi^1 &= \Phi^i + \Phi^r, & \Psi^1 &= \Psi^r, \\ \Phi^i &= \Phi_0 \sum_{n=0}^{\infty} (2n+1) i^n j_n(k_{L1}r) P_n(\cos\theta), \\ \Phi^r &= \sum_{n=0}^{\infty} A_n h_n(k_{L1}r) P_n(\cos\theta), & \Psi^r &= \sum_{n=0}^{\infty} B_n h_n(k_{T1}r) P_n(\cos\theta), \\ \Phi^2 &= - \sum_{n=0}^{\infty} C_n j_n(k_{L2}r) P_n(\cos\theta), & \Psi^2 &= - \sum_{n=0}^{\infty} D_n j_n(k_{T2}r) P_n(\cos\theta), \end{aligned} \quad (10)$$

where j_n is the spherical Bessel function of the first kind, h_n is the spherical Hankel function of the first kind, and P_n is the Legendre polynomial. Here, A_n, B_n, C_n , and D_n are the expansion coefficients to be determined by the boundary conditions (4). From (7) and (10), the displacement and stress fields in each material are obtained [7]. By applying the boundary conditions (4), an infinite set of linear algebraic equations for the coefficients is obtained.

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & pE_{33} & pE_{34} \\ E_{41} & E_{42} & pE_{43} & pE_{44} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}, \quad (11)$$

where E_{ij} and E_i are given in [7], and

$$p = \frac{\mu_2}{\mu_1}, \quad \Phi_0 = -\frac{1}{k_{L1}}. \quad (12)$$

DYNAMIC INHOMOGENEITY

The exact closed form solution can be obtained for the inhomogeneity problem with the dynamic part of the eigenstrain (1), if the following conditions are met.

$$e_{11}^* = e_{22}^* \neq e_{33}^*, \quad e_{ij}^* = 0 \quad (i \neq j). \quad (13)$$

Equations (13) ensure the axisymmetry with respect to the z-axis. Thus this problem can be solved in exactly the same manner as the scattering problem of the previous section, except for a few changes.

The elastic field due to the dynamic part of the eigenstrain (1) has the same time factor $e^{-2i\omega t}$. In the following, we omit this time factor, and concentrate on the spatial part of the solution. The solution procedure is exactly the same as that of the scattering problem except that now, instead of the incident wave, the dynamic eigenstrain is present. The resulting equation for the expansion coefficients is given by (11) with $\Phi_0 = a^2/2\mu_1$, $E_1=E_2=0$, $E_3=f_{3n}$, and $E_4=f_{4n}$, where E_{ij} are exactly the same as those in [7] except for k_{Lm} and k_{Tm} , which now are

$$k_{Lm} = \frac{2\omega}{c_{Lm}}, \quad k_{Tm} = \frac{2\omega}{c_{Tm}}, \quad (14)$$

and

$$\begin{aligned} f_{30} &= -\frac{1}{3} (2\sigma_{11}^* + \sigma_{33}^*), & f_{32} &= \frac{2}{3} (\sigma_{11}^* + \sigma_{33}^*), \\ f_{3n} &= 0, & & \text{if } n \neq 0, 2, \end{aligned} \quad (15)$$

$$\begin{aligned} f_{42} &= \frac{1}{3} (\sigma_{11}^* - \sigma_{33}^*), & f_{4n} &= 0, & \text{if } n \neq 2, \\ \sigma_{ij}^* &= 2\mu a_{ij}^* + \lambda a_{kk}^* \delta_{ij}, & a_{ij}^* &= \frac{1}{2} e_{ij}^*. \end{aligned} \quad (16)$$

It is seen from (11) and (15) that A_n , B_n , C_n , and D_n ($n=0$ or 2) are the only non-vanishing coefficients. It has been confirmed numerically that the solution of the equations (11) and (15) reduces to that of a dynamic inclusion problem, solved recently by Mikata and Nemat-Nasser [6], when materials 1 and 2 are the same.

RESULTS AND DISCUSSION

From the solutions of the three sub-problems, we now obtain the combined solution. For example, the combined stress field is

$$\sigma_{ij}(x, t) = \sigma_{ij}^{st}(x) + \sigma_{ij}^{sc}(x) e^{-i\omega t} + \sigma_{ij}^d(x) e^{-2i\omega t}, \quad (17)$$

where σ_{ij}^{st} , σ_{ij}^{sc} , and σ_{ij}^d are the solutions of static inhomogeneity, scattering, and dynamic inhomogeneity, respectively. Only the real part of $\sigma_{ij}(x,t)$ is physically meaningful.

The material properties used for the model are given in Table 1. The transformation strain is modelled as

$$e_{11}^* = e_{22}^* = e_{33}^* (= e^*), \quad e_{ij}^* = 0 \quad (i \neq j), \quad (18)$$

which gives a uniform expansion of the PSZ particle. The volumetric expansion of the PSZ particle is usually expected to be 3-5% [1], which gives the transformation strain, e^* , of 1-1.7%. The strain component e_{33} of the incident longitudinal wave may be, in most cases, 0.2-0.4%. Thus in the following computations, the strain ratio e^*/e_{33} is taken as 5.

In Fig.3, the stress (σ_{rr}) amplitude variations at (0,0,a) with a non-dimensional wave number ka for both scattering and dynamic inhomogeneity are shown. Here, k means k_{L1} given by (6), and the stress is normalized by μ_1 . It is seen from Fig.3 that, in the high frequency range, the stress amplitude for dynamic inhomogeneity becomes large. In Fig.4, the maximum and minimum of the combined stress (σ_{rr}) at (0,0,a) are shown for various non-dimensional wave numbers ka . The solid circle at $ka=0$ corresponds to the stress (σ_{rr}) for the static case. As a reference, the stress amplitude for scattering is also shown by the dashed line. It is found from Fig.4 that, in the high frequency range, the maximum combined stress is larger than the maximum stress caused by scattering alone. Similar results are obtained in Figs. 5 and 6 for point (0,0,a/2), Figs. 7 and 8 for point (0,0,-a/2), and Figs. 9 and 10 for point (0,0,-a).

Table 1. Material properties

	E (MPa)	ρ (g / cm ³)	ν
Al ₂ O ₃	3.8 x 10 ⁵	3.7	0.22
PSZ	2.3 x 10 ⁵	5.8	0.32

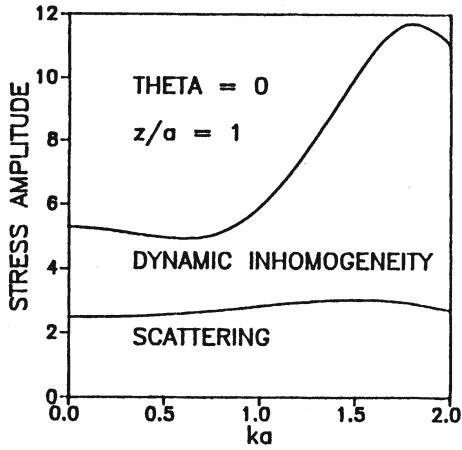


Fig. 3 Stress (σ_{rr}) amplitudes at $(0,0,a)$ v.s. ka .

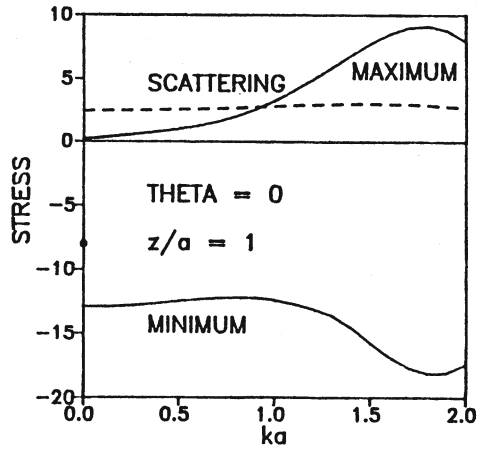


Fig. 4. Combined stress (σ_{rr}) at $(0,0,a)$ v.s. ka .

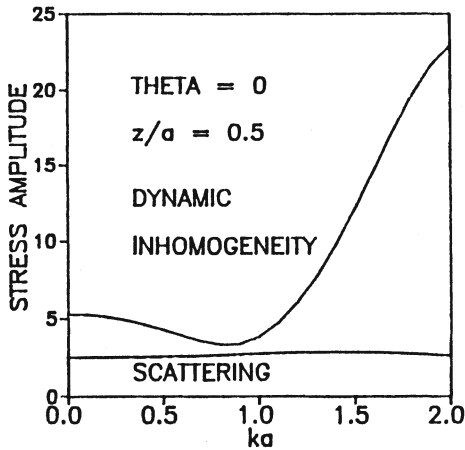


Fig. 5 Stress (σ_{rr}) amplitudes at $(0,0,a/2)$ v.s. ka .

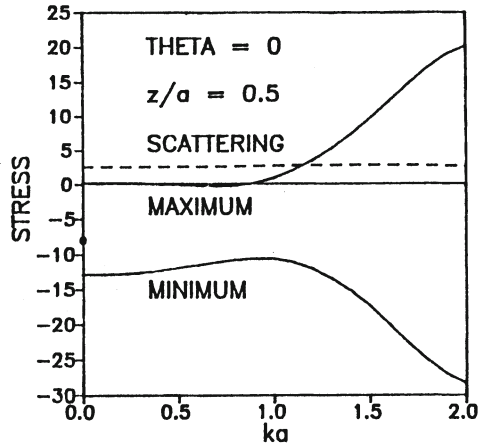


Fig. 6. Combined stress (σ_{rr}) at $(0,0,a/2)$ v.s. ka .

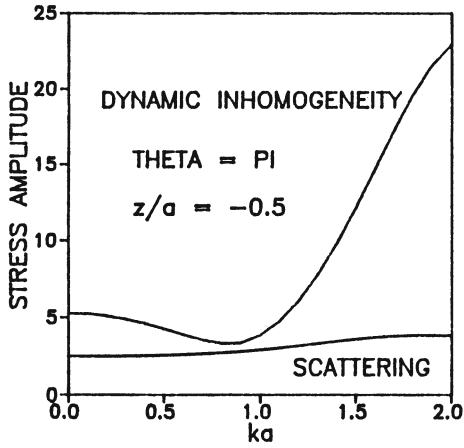


Fig. 7 Stress (σ_{rr}) amplitudes at $(0,0,-a/2)$ v.s. ka .

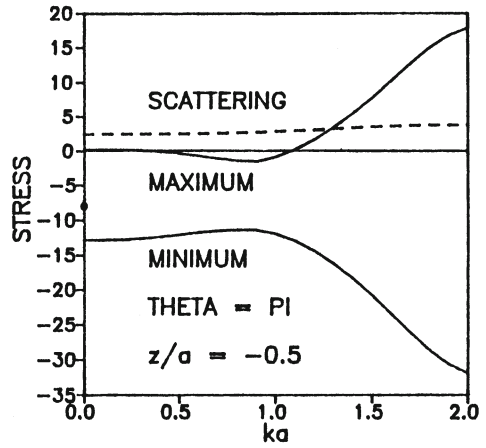


Fig. 8. Combined stress (σ_{rr}) at $(0,0,-a/2)$ v.s. ka .

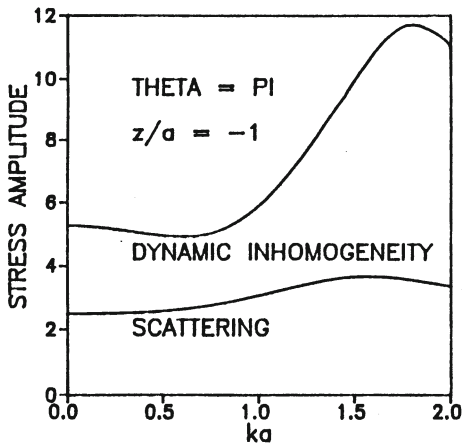


Fig. 9 Stress (σ_{rr}) amplitudes at $(0,0,-a)$ v.s. ka .

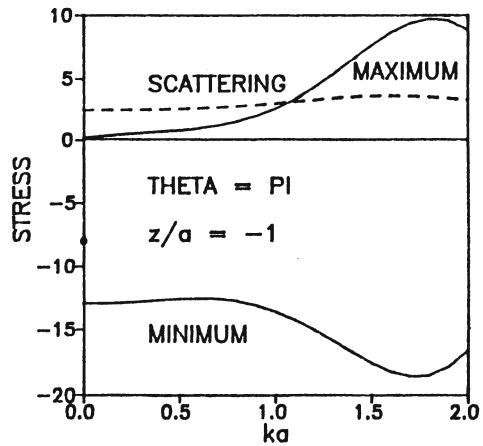


Fig. 10. Combined stress (σ_{rr}) at $(0,0,-a)$ v.s. ka .

CONCLUSION

An exact closed form solution has been obtained for a dynamic inhomogeneity problem. By using this exact solution, the interaction between the incident harmonic stress wave and the dynamically transforming spherical inhomogeneity, which serves as a model to investigate the elastodynamic behavior of a transformation toughened ceramics, has been studied. Maximum and minimum stress variation with the incident wave frequency have been obtained within the transforming particle. It has been found that, in the high frequency range, the dynamic transformation makes the maximum combined stress larger than the tensile stress which would have been obtained by the incident harmonic stress wave alone.

It should be emphasized here that, in principle, by using the present results, any form of an incident stress pulse in time and any form of a transformation strain in time can be handled by way of Fourier decomposition.

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