LONGWAVE SPEEDS IN MATERIALS WITH CRACKS AND CAVITIES OF VARIOUS SHAPES

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INTRODUCTION

Propagation of plane elastic waves in materials with cavities of various shapes and cracks (or mixtures of cavities of diverse shapes) is discussed in the longwave limit. In this limit, the wavespeeds are determined by the effective elastostatic moduli, i.e. the material is modeled by the homogeneous elastic solid having effective elastic constants.

A particular attention is paid to anisotropies due to preferential orientations of cavities of various shapes and to the number of independent constants and wavespeeds.

The analysis is based on recently obtained results of Kachanov [1] and Kachanov et al [2] for the effective elastic properties of materials with cavities of various shapes.

GENERAL RELATIONS FOR THE EFFECTIVE PROPERTIES

This section briefly overviews the general approach to the problem of effective moduli of solids with cavities and cracks. For details, see [1], [2].

An Infinite Solid with One Cavity

The starting point is the observation that the total strain in a solid subjected to a remotely applied stress $\boldsymbol{\sigma}$ and containing a cavity is given by a sum

$$\boldsymbol{\varepsilon} = \boldsymbol{M}^0 : \boldsymbol{\sigma} + \Delta \boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{M}^0$ is the compliance tensor of the matrix; a colon denotes contraction over two indices. The additional strain due to introduction of a cavity is

$$\Delta \boldsymbol{\varepsilon} = -\frac{1}{2V} \int_{\Gamma} (\boldsymbol{un} + \boldsymbol{nu})\, d\Gamma \tag{2}$$

where $\boldsymbol{u}$ and $\boldsymbol{n}$ denote displacements of the cavity boundary Γ and a unit normal to Γ (directed inwards the cavity), V is the (total, including the cavity) reference volume and $\boldsymbol{un}$, $\boldsymbol{nu}$ denote dyadic (tensor) product of two vectors.

The representation (2) directly follows from application of the divergence theorem to a strained solid when the solid contains a cavity. The strain $\Delta\boldsymbol{\varepsilon}$ is a linear function of the applied stress:

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{H} : \boldsymbol{\sigma} \tag{3}$$

where the fourth rank tensor $\boldsymbol{H}$ can be called a cavity compliance tensor.

Our analysis is based on the elastic potential in stresses (complementary energy density) of a representative volume element with many cavities, i.e. such a function $f(\boldsymbol{\sigma})$ that the effective stress-strain relations are given by $\varepsilon_{ij} = \partial f / \partial \sigma_{ij}$. The starting point is the representation of $f(\boldsymbol{\sigma})$ for a solid with one cavity as a sum of two terms:

$$f(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{M}^0 : \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{H} : \boldsymbol{\sigma} \equiv f_0 + \Delta f \tag{4}$$

where f_0 is the potential in the absence of a cavity and Δf is the change in potential due to introduction of the cavity.

Many Non-Interacting Cavities

In the approximation of non-interacting cavities, each cavity is placed in the externally applied stress field $\boldsymbol{\sigma}$ and is not influenced by the neighboring cavities. Then,

$$\Delta f = \sum \Delta f^{(k)} = \frac{1}{2}\boldsymbol{\sigma} : \sum \boldsymbol{H}^{(k)} : \boldsymbol{\sigma} \tag{5}$$

so that Δf and the effective moduli follow from the above obtained results for one cavity. Summation over cavities can be replaced by integration over orientations.

Interacting Cavities

The approximation of non-interacting cavities is the simplest approach to the problem. In the case of cracks, this approximation remains accurate up to high crack densities, if the mutual positions of cracks are random [1].

We represent, in a usual way, the problem of a solid with traction free cavities as a superposition of several problems containing one defect each. Traction on a defect in a given subproblem consists of $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ and the interaction traction $\Delta \boldsymbol{t}$ generated by defects in other sub-problems in a continuous material along the site of the considered defect.

Intuitively, it appears reasonable to assume that, for randomly located cavities, the interaction tractions $\Delta \boldsymbol{t}$ reflect the average stress environment $\boldsymbol{\sigma}^S$ in the solid phase, so that $\Delta \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^S$. This assumption constitutes Mori-Tanaka's scheme, MTS (formulated by them for a more general case of inclusions).

The average stress in the solid phase $\boldsymbol{\sigma}^S$ is given exactly, in terms of porosity p:

$$\boldsymbol{\sigma}^S = \frac{1}{1-p}\boldsymbol{\sigma} \tag{6}$$

Thus, porosity raises all the stress components σ_{ij} by the same ratio.

For cracks, p=0 and MTS coincides with the approximation of non-interacting cracks. (Physically, this reflects cancellation of the competing interaction effects of shielding and amplification if mutual positions of cracks are random). Extensive computer experiments on large arrays of interacting cracks confirm the accuracy of the approximation of non-interacting cracks at high crack densities, both isotropic and anisotropic matrices.

We now apply MTS to cavities, placing them into the average stress environment. Then, replacing $\boldsymbol{\sigma}$ by $(1-p)^{-1}\boldsymbol{\sigma}$ in the compliance relation $\Delta\boldsymbol{\varepsilon}=\boldsymbol{H}:\boldsymbol{\sigma}$ for each cavity, we obtain Δf from Δf for non-interacting cavities by a simple adjustment

$$\Delta f=\frac{1}{1-p}\Delta f_{\text{non-int}} \tag{7}$$

so that the effective moduli immediately follow from the results for non-interacting cavities.

EFFECTIVE MODULI OF SOLIDS WITH PENNY-SHAPED CRACKS

In the framework of the method described above, the potential for a solid with many circular cracks of radii $a^{(i)}$ with crack surface normals $\boldsymbol{n}^{(i)}$ has the following structure:

$$\Delta f=\frac{8\left(1-\nu_0^2\right)}{3(1-\nu_0/2)E_0}\left\{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}):\boldsymbol{\alpha}-\frac{\nu_0}{2}\boldsymbol{\sigma}:\frac{1}{V}\sum_i\left(a^3\boldsymbol{nnnn}\right)^{(i)}:\boldsymbol{\sigma}\right\} \tag{8}$$

where $\boldsymbol{\alpha}=(1/V)\sum_i\left(a^3\boldsymbol{nn}\right)^{(i)}$ is the 3-D crack density tensor. Since $\boldsymbol{\alpha}$ is a symmetric second rank tensor, the first term in the braces of (8) describes the orthotropic response, the orthotropy axes coinciding with the principal axes of $\boldsymbol{\alpha}$. The second term, involving the fourth rank tensor $(1/V)\sum_i\left(a^3\boldsymbol{nnnn}\right)^{(i)}$, causes deviations from orthotropy (see [1] for details). However, deviations from orthotropy are small, due to the following two factors.

1. The second term has a relatively small multiplier $\nu_0/2$. If its relative weight is evaluated by the ratio $\kappa=\left(\Delta M''_{ijkl}\Delta M''_{ijkl}\right)^{1/2}/\left(\Delta M'_{ijkl}\Delta M'_{ijkl}\right)^{1/2}$ of the Euclidean norms of the compliances $\Delta\boldsymbol{M}', \Delta\boldsymbol{M}''$ corresponding to this term and to the $\boldsymbol{\alpha}$ - term, then $\kappa<\nu_0/\sqrt{6}$.

2. Only a part of this term causes deviations from orthotropy; a substantial part of $\Delta\boldsymbol{M}''$ has the structure of orthotropy coaxial to $\boldsymbol{\alpha}$ and simply adds to the moduli $\Delta\boldsymbol{M}'$.

These deviations from orthotropy can be further reduced if the normal and shear "compliances of a crack" (average relative displacements of the crack faces under uniform unit loading in the normal and shear modes) are enforced to be the same and equal to their average, then

$$\Delta f\approx\frac{8\left(1-\nu_0^2\right)(1-\nu_0/4)}{3(1-\nu_0/2)E_0}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}):\boldsymbol{\alpha} \tag{9}$$

and the fourth rank tensor is eliminated from the potential. Thus, characterization of a crack array by the crack density tensor $\boldsymbol{\alpha}$ only constitutes a good approximation. The effective properties are then orthotropic.

Below, we consider three different orientational distributions of cracks as examples. We then give results for wavespeeds in materials with cracks. Wavespeeds in materials with various cavities are obtained in a similar way, utilizing the results of [2].

Isotropic (Random) Orientational Distribution

Both the crack density tensor $\boldsymbol{\alpha}$ and the fourth rank tensor $(1/V)\sum_i \left(a^3 \boldsymbol{nnnn}\right)^{(i)}$ are isotropic. Since $tr\,\boldsymbol{\alpha} = \rho \equiv (1/V)\sum \left(a^3\right)^{(i)}$ (scalar crack density),

$$\boldsymbol{\alpha} = (\rho/3)\boldsymbol{I} \tag{10}$$

where $\boldsymbol{I}$ is a 3-D unit tensor. Since tensor $(1/V)\sum_i \left(a^3 \boldsymbol{nnnn}\right)^{(i)}$, in addition to being isotropic, is symmetric with respect to all rearrangements of indices, its $ijkl$-components are: $(\rho/15)\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$, see [1], [2] for details. Inserting into (8) yields

$$\Delta f = \frac{8\left(1-\nu_0^2\right)}{9(1-\nu_0/2)E_0}\rho\left\{\left(1-\frac{\nu_0}{5}\right)tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) - \frac{\nu_0}{10}(tr\boldsymbol{\sigma})^2\right\} \tag{11}$$

and the effective moduli readily follow:

$$E = E_0\left[1+\frac{16\left(1-\nu_0^2\right)(1-3\nu_0/10)}{9(1-\nu_0/2)}\rho\right]^{-1}, \quad G = G_0\left[1+\frac{16(1-\nu_0)(1-\nu_0/5)}{9(1-\nu_0/2)}\rho\right]^{-1} \tag{12}$$

If the simplified potential (9) is used, then the moduli are close to the exact result (12):

$$E = E_0\left[1+\frac{16\left(1-\nu_0^2\right)(1-\nu_0/4)}{9(1-\nu_0/2)}\rho\right]^{-1}, \quad G = G_0\left[1+\frac{16(1-\nu_0)(1-\nu_0/4)}{9(1-\nu_0/2)}\rho\right]^{-1} \tag{13}$$

Parallel Cracks (Normal to the x_1 Axis)

The fourth rank tensor $(1/V)\sum_i \left(a^3 \boldsymbol{nnnn}\right)^{(i)} = \rho\, \boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_1$ and the crack density tensor $\boldsymbol{\alpha} = \rho\, \boldsymbol{e}_1\boldsymbol{e}_1$. Thus,

$$\Delta f = \frac{8\left(1-\nu_0^2\right)}{3(1-\nu_0/2)E_0}\rho\left\{\sigma_{1j}\sigma_{j1} - \frac{\nu_0}{2}\sigma_{11}{}^2\right\} \tag{14}$$

so that

$$E_1 = E_0\left[1+\frac{16\left(1-\nu_0^2\right)}{3}\rho\right]^{-1}, \quad G_{12} = G_0\left[1+\frac{8(1-\nu_0)}{3(1-\nu_0/2)}\rho\right]^{-1} \tag{15}$$

If the simplified potential (9) is used,

$$E_1 = E_0\left[1+\frac{16\left(1-\nu_0^2\right)(1-\nu_0/4)}{3(1-\nu_0/2)}\rho\right]^{-1}, \quad G_{12} = G_0\left[1+\frac{8(1-\nu_0)(1-\nu_0/4)}{3(1-\nu_0/2)^2}\rho\right]^{-1} \tag{16}$$

The expressions (16) are close to the exact moduli (15).

Symmetric Exponential Statistics of Crack Orientation

Let us consider the orientational distribution of the crack normals $\boldsymbol{n}^{(i)}(\varphi)$ given by the exponential probability density:

$$P_\lambda(\varphi) = \frac{1}{2\pi}\left[\left(\lambda^2+1\right)e^{-\lambda\varphi} + \lambda e^{-\lambda\pi/2}\right] \tag{17}$$

where $\varphi \in [0, \pi/2]$ is the angle between $\boldsymbol{n}^{(i)}$ and x_1-axis (symmetry axis) and $\lambda \geq 0$. This case is important, since it covers, as limiting cases, the random (isotropic) distribution ($\lambda = 0$) and parallel cracks ($\lambda = \infty$, normals $\boldsymbol{n}^{(i)}$ aligned with x_1-axis). Thus, it can represent the case of slightly perturbed parallel orientations (large λ) as well as weakly expressed preferential orientation (small λ).

Integration over the spherical surface yields the crack density tensor

$$\begin{gathered}\boldsymbol{\alpha} = \frac{1}{V}\sum_i \left(a^3\boldsymbol{nn}\right)^{(i)} \equiv \alpha_1\boldsymbol{e}_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_3\boldsymbol{e}_3 = \\ \rho\frac{\left(\lambda^2+3\right)\left(3+\lambda e^{-\lambda\pi/2}\right)}{3\left(\lambda^2+9\right)}\boldsymbol{e}_1\boldsymbol{e}_1 + \rho\frac{18-\lambda e^{-\lambda\pi/2}\left(\lambda^2+3\right)}{3\left(\lambda^2+9\right)}\left(\boldsymbol{e}_2\boldsymbol{e}_2+\boldsymbol{e}_3\boldsymbol{e}_3\right)\end{gathered} \tag{18}$$

so that x_1, x_2, x_3 constitute the principal axes of $\boldsymbol{\alpha}$.

Using the approximate (orthotropic) potential (9) we find the effective stiffnesses C_{ij} by taking an inverse of the compliance matrix (that is obtained from (9), (18)):

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & \text{symm} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} \frac{\left(A_2A_3-\nu_0^2\right)}{E_0D} & \frac{\left(A_3+\nu_0\right)\nu_0}{E_0D} & \frac{\left(A_2+\nu_0\right)\nu_0}{E_0D} & 0 & 0 & 0 \\ & \frac{\left(A_3A_1-\nu_0^2\right)}{E_0D} & \frac{\left(A_1+\nu_0\right)\nu_0}{E_0D} & 0 & 0 & 0 \\ & & \frac{\left(A_1A_2-\nu_0^2\right)}{E_0D} & 0 & 0 & 0 \\ & \text{symm} & & G_{23}^{\text{eff}} & 0 & 0 \\ & & & & G_{31}^{\text{eff}} & 0 \\ & & & & & G_{12}^{\text{eff}} \end{bmatrix} \tag{19}$$

where $D = A_1A_2A_3 - \nu_0^2(A_1+A_2+A_3) - 2\nu_0^3$ and

$$A_i = \left[1+\frac{16\left(1-\nu_0^2\right)\left(1-\nu_0/4\right)}{3\left(1-\nu_0/2\right)}\alpha_i\right], \quad G_{ij}^{\text{eff}} = G_0\left[1+\frac{8\left(1-\nu_0\right)\left(1-\nu_0/4\right)}{3\left(1-\nu_0/2\right)}\left(\alpha_i+\alpha_j\right)\right]^{-1}.$$

Simplified Character of Orthotropy and the Number of Independent Constants

The effective elastic properties of 3-D cracked solids are approximately orthotropic (orthotropy axes being coaxial with $\boldsymbol{\alpha}$). The orthotropy is of a simplified type:

(1) Directional variation of the compliances is described by ellipsoids (rather than surfaces of the fourth order, as in the case of general orthotropy).

(2) The number of independent constants is drastically reduced, from nine (general orthotropy) to only four, due to the following simplifications:

(a) shear moduli can be expressed in terms of Young's moduli and Poisson's ratios:

$$1/G_{ij} = 1/E_i + 1/E_j + 2\nu_0/E_0 \tag{20}$$

(b) under uniaxial loading in one of the principal directions of $\boldsymbol{\alpha}$, the presence of cracks does not cause any additional lateral strain:

$$\nu_{12}/E_1 = \nu_{23}/E_2 = \nu_{31}/E_3 = \nu_0/E_0 \tag{21}$$

This fact and the general symmetry relations $\nu_{ij}/E_i = \nu_{ji}/E_j$, imply that all six Poisson's ratios can be expressed in terms of Young's moduli E_i and the matrix constant ν_0/E_0.

Thus, the entire set of elastic constants can be expressed in terms of three Young's moduli E_i (in the principal directions of $\boldsymbol{\alpha}$) and one matrix constant ν_0/E_0.

EFFECTIVE MODULI OF SOLIDS WITH ELLIPSOIDAL CAVITIES

One Ellipsoidal Cavity

We consider an ellipsoidal cavity with the surface Γ and axes $2a_1, 2a_2, 2a_3$ oriented along unit vectors $\boldsymbol{\ell},\boldsymbol{m},\boldsymbol{n}$, correspondingly; $\boldsymbol{\sigma}$ is the uniform stress field at infinity. Using the solution of Eshelby's problem [5] we find the strain $\boldsymbol{\varepsilon}^{\Gamma}$ at Γ in terms of Eshelby's tensor $\boldsymbol{S}$. Expressing the displacement $\boldsymbol{u}$ on Γ as $\boldsymbol{u} = \boldsymbol{\varepsilon}^{\Gamma}\cdot\boldsymbol{x}$ where $\boldsymbol{x}$ is the position vector originating at the ellipsoid center, we obtain the integral (2) in terms of $\boldsymbol{\sigma}$ and, thus, find $\boldsymbol{H}$:

$$\begin{aligned}\boldsymbol{H} = \frac{V_{cav}}{V}\frac{1}{E_0}\{&H_{1111}\boldsymbol{\ell\ell\ell\ell} + H_{2222}\boldsymbol{mmmm} + H_{3333}\boldsymbol{nnnn}\\ &+H_{1122}(\boldsymbol{\ell\ell mm} + \boldsymbol{mm\ell\ell}) + H_{2233}(\boldsymbol{mmnn} + \boldsymbol{nnmm}) + H_{3311}(\boldsymbol{nn\ell\ell} + \boldsymbol{\ell\ell nn})\\ &+H_{1212}(\boldsymbol{\ell m} + \boldsymbol{m\ell})(\boldsymbol{\ell m} + \boldsymbol{m\ell}) + H_{2323}(\boldsymbol{mn} + \boldsymbol{nm})(\boldsymbol{mn} + \boldsymbol{nm}) + H_{3131}(\boldsymbol{n\ell} + \boldsymbol{\ell n})(\boldsymbol{n\ell} + \boldsymbol{\ell n})\}\end{aligned} \tag{22}$$

where indices 1,2,3 correspond to the axes $2a_1, 2a_2, 2a_3$ of the ellipsoid and V_{cav} is the cavity volume. Components H_{ijkl} are given in terms of Eshelby's tensor $\boldsymbol{S}$ (see [2]). For a general ellipsoid, S_{ijkl} are expressed in terms of elliptic integrals; they reduce to elementary functions for an ellipsoid of revolution (spheroid).

For a spheroid ($a_1 = a_2 \equiv a$ and $\boldsymbol{n}$ is the axis of symmetry), $\Delta f = \frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{H}:\boldsymbol{\sigma}$ is

$$\Delta f = \frac{V_{cav}}{V}\frac{1}{2E_0}\left\{A_1(tr\boldsymbol{\sigma})^2 + A_2 tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \left[A_3(tr\boldsymbol{\sigma})\boldsymbol{\sigma} + A_4(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\right]:\boldsymbol{nn} + A_5\boldsymbol{\sigma}:\boldsymbol{nnnn}:\boldsymbol{\sigma}\right\} \tag{23}$$

where the coefficients A_i are given in terms of Eshelby's tensor $\boldsymbol{S}$ (see [2]).

Many Ellipsoidal Cavities

The effective moduli were derived in [2] for the general ellipsoidal cavities, spheroids and for several special cases when the expressions are simplified: spheres, needles, slightly deformed spheres and slightly inflated penny-shaped cracks. All these results can be directly used for calculating the wavespeeds.

Here, we consider only one example of randomly oriented non-interacting spheroidal cavities. Using the result (23) for one cavity we obtain the isotropic potential

$$\Delta f = \frac{p}{2E_0}\left\{(A_1 + A_3/3 + A_5/15)(tr\boldsymbol{\sigma})^2 + (A_2 + A_4/3 + 2A_5/15)tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\right\} \quad (24)$$

MIXTURES OF CAVITIES OF DIVERSE SHAPES

The approach described above allows analysis of mixtures of cavities of diverse shapes. For non-interacting cavities, the potential is a sum:

$$\Delta f_{non-int} = \sum_J \Delta f^{(J)}_{non-int} \quad (25)$$

where $\Delta f^{(J)}$ denotes Δf for the cavities of the type (J) considered separately. For interacting cavities, considered in the approximation of Mori-Tanaka's scheme,

$$\Delta f_{int} = \frac{1}{1-p}\sum_J \Delta f^{(J)}_{non-int} \quad (26)$$

where p is the overall porosity (due to cavities of all types).

PLANE ELASTIC WAVES IN ANISOTROPIC MEDIA

As is well known (see, for example, [3],[4],[6]), the wavespeeds and displacement vectors for propagation (along the direction $\boldsymbol{p}$) of the plane wave in the homogeneous anisotropic solid can be found as a solution of the eigenvalue problem for the matrix of Christoffel stiffnesses $\Gamma_{ik} = C_{ijkl}p_j p_l$

$$\left[\boldsymbol{\Gamma} - \rho_d c^2 \boldsymbol{I}\right]\boldsymbol{d} = 0 \quad (27)$$

Since the matrix $[\boldsymbol{\Gamma}]$ is positive definite, all the eigenvalues $c_{\mathrm{I}}^2, c_{\mathrm{II}}^2, c_{\mathrm{III}}^2$ are positive and the corresponding eigenvectors $\boldsymbol{d}_{\mathrm{I}}, \boldsymbol{d}_{\mathrm{II}}, \boldsymbol{d}_{\mathrm{III}}$ are orthogonal, where c_i is the speed of the wave with polarization vector $\boldsymbol{d}_i$ and ρ_d is the material density. In contrast with the isotropic case the displacements are neither longitudinal nor transverse (with the exception of pure modes).

EXAMPLE: WAVE PROPAGATION IN A SOLID WITH CIRCULAR CRACKS

We constructed slowness curves for wave propagation in x_1, x_2-plane for the case of cracks with the orientational statistics (17). Since x_1, x_2, x_3 are the axes of orthotropy, the wavespeeds and three types of wave polarization are given by (28)-(29) (see [4]).
I. Pure shear wave (polarized along x_3-axis):

$$\frac{1}{c_{\mathrm{I}}} = \sqrt{\rho_d}\left\{C_{44}\sin^2\phi + C_{55}\cos^2\phi\right\}^{-1/2} \quad (28)$$

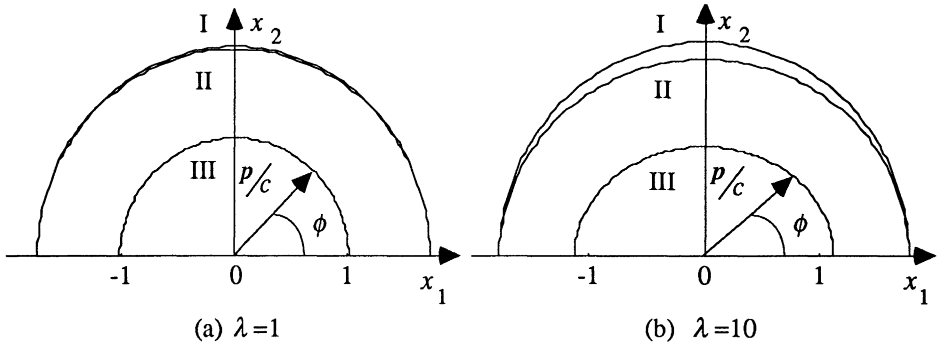

Fig. 1 Slowness curves for the principal plane of orthotropy in the cracked solid.

II. Quasishear wave (pure shear modes for $\phi = k\pi/2$):

$$\frac{1}{c_{II}} = \sqrt{2\rho_d}\left\{C_{66} + C_{11}\cos^2\phi + C_{22}\sin^2\phi - \sqrt{\left(C_{66} + C_{11}\cos^2\phi + C_{22}\sin^2\phi\right)^2 - 4B}\right\}^{-\frac{1}{2}} \quad (29)$$

III. Quasilongitudinal wave (pure longitudinal modes for $\phi = k\pi/2$):

$$\frac{1}{c_{III}} = \sqrt{2\rho_d}\left\{C_{66} + C_{11}\cos^2\phi + C_{22}\sin^2\phi + \sqrt{\left(C_{66} + C_{11}\cos^2\phi + C_{22}\sin^2\phi\right)^2 - 4B}\right\}^{-\frac{1}{2}} \quad (30)$$

where $B = \left(C_{11}\cos^2\phi + C_{66}\sin^2\phi\right)\left(C_{66}\cos^2\phi + C_{22}\sin^2\phi\right) - \left(C_{12} + C_{66}\right)^2\cos^2\phi\sin^2\phi$ and ϕ is the angle between the direction of propagation and x_1-axis.

We consider (fig. 1) the case of crack density $\rho = 0.1$ and two values of the orientational parameter: (a) λ=1 - almost isotropic case and (b) λ=10 - slightly disturbed parallel cracks. The slowness curves are obtained by substitution of (18), (19) into (28)-(30). Due to the simplified character of orthotropy mentioned above, we obtain second order curves (ellipses), which become circles for the random orientations; the eccentricity increases as the preferential direction of crack orientation becomes more noticeable.

For solids with other cavity shapes, the longwave speeds can be derived in exactly the same way (by utilizing the results of [2] for the effective moduli and solving (27)).

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