Virtual mass of a deformable body

Wallace Albert Raab

Iowa State College

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VIRTUAL MASS OF A DEFORMABLE BODY

by

Wallace Albert Raab

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Signature was redacted for privacy.

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Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

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TABLE OF CONTENTS

I. INTRODUCTION ........................................... 1

II. LINEAR PERTURBATION FLOW PAST A DEFORMABLE CYLINDER .... 6

III. KINETIC ENERGY OF A DEFORMABLE ELLIPSOID .... 24

   A. Prolate Ellipsoidal Coordinates ...................... 25

   B. Two Node Flexural Vibration with Uniform Flow in the x-direction ....................... 35

   C. Two Node Flexural Vibration with Uniform Flow in the y-direction ...................... 43

   D. Three Node Flexural Vibration with Uniform Flow in the x-direction .................... 47

IV. SUMMARY .................................................. 53

V. LITERATURE CITED ........................................ 55

VI. ACKNOWLEDGMENT ......................................... 56
I. INTRODUCTION

The concept of the virtual mass of a body moving in a fluid was first noted by Chevalier DuBuat while experimenting with spheres oscillating in water. A historical background and a comprehensive survey of the literature regarding this topic may be found in [1].

The fundamental idea of virtual mass may be approached through the concept of kinetic energy. The kinetic energy of an infinite body of fluid, \( T_f \), through which a body moves in rectilinear motion at a constant velocity is given by

\[
T_f = \frac{1}{2} Cv^2
\]

where \( C \) is a constant and \( v \) is the velocity of the body. The total kinetic energy, \( T \), is the sum of the kinetic energies of the body and the fluid. Thus,

\[
T = \frac{1}{2} mv^2 + \frac{1}{2} Cv^2
\]

where \( m \) is the mass of the body. If the velocity changes, the time rate of change of the kinetic energy of the system is

\[
\frac{d}{dt} (T) = \frac{d}{dt} \left( \frac{1}{2} mv^2 + \frac{1}{2} Cv^2 \right)
\]

or

\[
\frac{dT}{dt} = (m + C)v \frac{dv}{dt}.
\]

*Numbers in brackets refer to the bibliography at the end of the paper.
Since the time rate of change of kinetic energy is equal to the product of the resultant external force $F$ and the velocity,

$$Fv = (m + C)v \frac{dv}{dt}$$

or

$$F = (m + C) \frac{dv}{dt}.$$

Thus, the presence of the fluid increases the effective mass of the body from $m$ to $(m + C)$. The above equation may be written in the form

$$F = (m + Km') a$$

where $m'$ is the mass of the displaced fluid and $a$ is the acceleration. The constant $K$ is called the added mass constant and $Km'$ is the added mass. The sum of the actual mass and the added mass is the virtual mass. The added mass constant for any given body is a function of the shape and size of the body, the direction of motion, and the density of the surrounding fluid.

Qualitatively, the idea of virtual mass is a familiar one. For example, let a light paddle be dipped into still water and then suddenly given a rapid acceleration broadside. It is a matter of common experience that the apparent inertia of the paddle is greatly increased by the presence of the water around it. This increased inertia is what is called the virtual mass of the paddle and the difference between the real mass and the virtual mass is called the added or "induced" mass.
In [2], Hayes considered the problem of the virtual mass of a deformable circular cylinder. The important fact pointed out by Hayes is that in all cases with circular cross-sections, the local virtual mass for area elements contained between two planes an infinitesimal distance apart is, apart from the external reduction factor, the mass of the fluid displaced by the cylindrical volume elements. This reduction factor which is always less than unity, indicates the ratio in which the energy is reduced as compared with the two-dimensional case. This reduction factor for the deformable cylinder may be obtained by comparing the ratio of the kinetic energy of the fluid with the kinetic energy of the cylinder. In this problem it is assumed that the fluid is incompressible.

The flow past an infinitely long deformable cylinder is examined in more detail in Chapter II. The linear perturbation flow past an infinitely long deformable cylinder, for both compressible and incompressible fluids, is obtained by solving Laplace's equation in cylindrical coordinates. A solution is obtained which contains as a limiting case the Prandtl-Glauert correction factor for two-dimensional flow. To indicate the manner in which the transition to two-dimensional flow takes place, one specific case is plotted in Figure 3. As the radius of the basic cylinder becomes large, it is shown that the perturbation velocity at the surface of the cylinder is of the same general form as the perturbation velocity along
a wave-shaped wall.

This paper is particularly concerned with the calculation of the virtual mass of a vibrating ellipsoid. Since the virtual mass of a body moving in a fluid may be computed once the kinetic energy of the fluid is known, the kinetic energy for an ellipsoid vibrating in two and three nodes will be determined. Using the idea set forth by Hayes, the kinetic energy of the fluid will be compared with the kinetic energy of the ellipsoid itself. This ratio is always less than unity and will be plotted versus the ratio of the semi-minor to the semi-major axis of the ellipsoid to indicate the manner in which the energy is reduced as compared with the two-dimensional case.

The problem of the two-node vibration of the ellipsoid was first solved by Lewis [3] and Taylor [4]. However, both of these authors used boundary conditions which are different from those used in this paper. In the calculation of the kinetic energy for a vibrating ellipsoid, a term involving $\frac{\partial \phi}{\partial \xi}$ is required where $\phi$ is the potential function and $\xi$ is one of the coordinates in the prolate ellipsoidal system of coordinates. It is assumed that $\phi = \phi(x,y)$ so that

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \xi}.$$ 

In the paper by Lewis the second term in the right-hand member of the above equation was omitted and in the paper by Taylor
the term was retained but was preceded by a minus sign. By
the proper choice of this boundary condition, a solution of
a vibrating ellipsoid of two or three nodes is obtained. A
comparison of the results obtained by Lewis and Taylor is
made with the results of this paper for the two node vibra-
tion in Chapter III. This comparison shows the effect of the
boundary condition discussed above.

In many practical applications an elongated body of revo-
lution may be approximated by an ellipsoid. In certain types
of problems the concept of virtual mass is very important.
For example, one of the important parameters in determining
the natural frequencies and vertical nodes of vibrations of
ships is the virtual mass. In reference [5] , a comprehen-
sive bibliography is included regarding this particular prob-
lem.
II. LINEAR PERTURBATION FLOW PAST A DEFORMABLE CYLINDER

The linear perturbation flow past a deformable cylinder for both compressible and incompressible fluids, is obtained by solving Laplace's equation in cylindrical coordinates. A solution is obtained which contains as a limiting case the Prandtl-Glauert correction for two-dimensional flow.

The origin of the coordinate system used in this problem will lie on the axis of the cylinder where the x and y axes lie in a plane which is perpendicular to the z-axis which is the axis of the cylinder, Figure 1. It will be convenient to use cylindrical coordinates \((r, \theta, z)\) which are related to the Cartesian coordinates by the usual relationships.

Consider a long circular cylinder (infinite in length) of radius \(a\) which is being deformed in such a manner that the cross-section always remains circular; the lateral motion in the \((x,z)\) plane is not uniform but is in the form of a sine curve of wave length \(L\) and amplitude \(b\), the z-axis is the axis of the cylinder, Figure 1.

For steady, reversible, adiabatic, irrotational, subsonic, compressible flow, a velocity potential \(\Phi(r, \theta, z)\) will exist such that the definitions

\[
\begin{align*}
    u &= -\frac{\partial \Phi}{\partial r} \quad \text{(radial component of the velocity)}, \\
    v &= -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \quad \text{(tangential component of the velocity)}, \quad (1) \\
    w &= -\frac{\partial \Phi}{\partial z} \quad \text{(axial component of the velocity)},
\end{align*}
\]

Fig. 1 Cylindrical Coordinate System and the Meridian Plane xoz for the Deformable Cylinder
are valid. The subscript notation is used in this section to indicate partial differentiation, i.e., $\Phi_r = \frac{\partial \Phi}{\partial r}$. The proper partial differential equation in cylindrical coordinates to be solved is then [6]

$$
(a^2 - u^2) \Phi_{rr} + \frac{d^2 - v^2}{r^2} \Phi_{\theta\theta} + (d^2 - w^2) \Phi_{zz} + \frac{u(d^2 + v^2)}{r}
- \frac{2uv}{r} \Phi_{r\theta} - \frac{2vw}{r} \Phi_{\theta z} = 0,
$$

where $d$ is the velocity of sound in the medium.

Since equation (2) is a non-linear partial differential equation for which exact mathematical solutions are difficult to obtain, it will be linearized by assuming that the velocity in the $z$-direction can be expressed as a sum of the free stream velocity and a perturbation velocity. Specifically the following assumptions and definitions are made:

(a) let the free stream velocity be in the direction of the positive $z$-axis;

(b) $u$, $v$, and $w$ are the $r$, $\theta$, $z$ components of velocity, respectively, and

$$
V = (u^2 + v^2 + w^2)^{1/2}
$$
is the speed at any point;

(c) $w = w' + U$

where

$$
U = \text{free stream velocity},
$$

$$
w' = \text{perturbation velocity in the } z\text{-direction};
$$

(d) $M = \text{Mach number}

$$
= \frac{V}{d}$$. 


where
\[ d = \text{speed of sound in the flow at the same point at which } V \text{ is measured.} \]

(e) \( M_s = \text{free stream Mach number} \)
\[ = \frac{U}{d_s} \]

where
\[ d_s = \text{free stream velocity of sound.} \]

(f) \( \frac{U^2}{d_s^2} \ll 1, \frac{V^2}{U^2} \ll 1, \frac{W'^2}{U^2} \ll 1, \frac{2W'}{U} \ll 1; \)

(g) \( 2M_s^2 \left[ \frac{1}{r} \frac{U}{U} \frac{V}{U} \frac{\phi}{r} \phi + \frac{U}{U} \frac{W}{U} \frac{\phi}{r} \phi \right] \)
\[ \text{is negligibly small compared with the largest term of (2);} \]

(h) \( (\gamma - 1) M_s^2 \frac{W'}{U^2} \) is small compared with unity, where \( \gamma = \) ratio of specific heat at constant pressure to specific heat at constant volume.

Bernoulli's equation for compressible flow may be written as

(i) \[ \frac{d^2}{d_s^2} = 1 + \frac{\gamma - 1}{2} \frac{U^2}{d_s^2} (1 - \frac{V^2}{U^2}). \]

It is then evident that
\[ \frac{d^2}{d_s^2} = 1 - (\gamma - 1) M_s^2 \frac{W'}{U} \]
\[ = 1. \]

(j) \[ \frac{U^2}{d^2} = \frac{U^2}{d_s^2} \frac{d_s^2}{d^2} \]
\[ = M_s^2. \]
To determine what the above assumptions mean physically, consider the uniform steady flow $U$ in the direction of the positive $x$-axis. The velocity field for this basic flow is given by

$$u = U,$$
$$v = 0,$$
$$w = 0,$$

where $u$, $v$, and $w$ are the velocity components in the $x$, $y$, and $z$-directions, respectively. Now assume that a thin body such as an airfoil be placed in the uniform stream. The body disturbs the uniform flow and, hence, the velocity field of the uniform flow is altered. The new velocity field created by the presence of the body may be expressed as

$$u = U + u',$$
$$v = v',$$
$$w = w',$$

where $u'$, $v'$ and $w'$ are called the "induced" or "perturbation" velocity components. In the study of small perturbation theory, these perturbation velocities are assumed to be small compared with the free stream velocity, $U$, i.e.,

$$\frac{u'}{U} \ll 1,$$
$$\frac{v'}{U} \ll 1,$$
$$\frac{w'}{U} \ll 1.$$ 

Using the above assumptions, it is possible in many cases to arrive at equations, which may or may not be linear, but are
still simpler to solve than the original complete equation. It is these assumptions that permits the analytic solutions for such problems as the flow past slender bodies and for a large number of problems in airfoil theory.

Equation (2) may now be expressed in the following form by using the assumptions listed in (a) through (j)

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} + B^2 \frac{\partial^2 \Phi}{\partial z^2} = 0,
\]

where \( B^2 = (1 - M_s^2) \). This linear partial differential equation may be expressed in terms of a perturbation velocity potential in the following manner. Let

\[
\Phi = \phi(r, \theta, z) - Uz
\]

where

\( \phi = \) perturbation velocity potential.

When equation (4) is substituted into equation (3), it is evident that

\[
\frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} + B^2 \frac{\partial^2 \phi}{\partial z^2} = 0.
\]

Equation (2) has thus been reduced to the problem of solving a linear partial differential in the perturbation velocity potential. From equations (1) and (4), it follows that \( u, v \) and \( w' \) are perturbation velocities.

The boundary conditions appropriate for the flow past a deformable cylinder will now be determined. Assume \( b << a \), where
b = amplitude of the sine wave,
a = radius of the undeformed cylinder.

Since all cross-sections remain circular after deformation, it may be seen from Figure 1 that

\[(x - b \sin kz)^2 + y^2 = a^2\]

or

\[a^2 = x^2 - 2bx \sin kz + b^2 \sin^2 kz + y^2. \quad (6)\]

Since \(x^2 + y^2 = r^2\), equation (6) becomes

\[a^2 = r^2 - 2br \cos \theta \sin kz + b^2 \sin^2 kz \]

\[= r^2 - 2b \cos \theta \sin kz + b^2 \sin^2 kz \quad (7)\]

where \(x\) has been replaced by \(r \cos \theta\). This equation may now be solved for \(r\):

\[r = b \cos \theta \sin kz \pm \sqrt{b^2 \cos^2 \theta \sin^2 kz + (a^2-b^2 \sin^2 kz)} \]

\[= b \cos \theta \sin kz \pm \sqrt{a^2(1 - \frac{b^2}{a^2} \sin^2 kz \sin^2 \theta)}. \quad (8)\]

Since \(b << a\) and \(r > 0\), \(r\) may be written approximately as

\[r \approx a + b \cos \theta \sin kz. \quad (9)\]

The differential equation of the streamlines is given by

\[\frac{dz}{w' + U} = \frac{dr}{u} \quad (10)\]

where

\[w' = \text{perturbation velocity in the } z\text{-direction},\]
\[u = \text{perturbation velocity in the radial direction},\]
\[U = \text{free stream velocity}.\]

Hence,
$$\frac{dr}{dz} = \frac{u}{w^i + U}. \quad (11)$$

The right-hand member of equation (11) may be written as

$$\frac{u}{w^i + U} = \frac{u}{U} \left( 1 + \frac{w^i}{U} \right)^{-1},$$

and

$$\frac{u}{w^i + U} = \frac{u}{U} \left[ 1 - \frac{w^i}{U} + O\left(\frac{w^i^2}{U^2}\right) \right] = \frac{u}{U}. \quad (12)$$

When equation (12) is substituted into equation (11), it follows that

$$u = U \frac{\partial r}{\partial z}. \quad (13)$$

From equation (9), \( \frac{\partial r}{\partial z} \) may be calculated and substituted into the right-hand side of (13). This provides the following boundary condition which must be satisfied at the surface of the deformable cylinder, defined by \( r = a \):

$$u = -\phi_r = -U b k \cos \theta \cos kz. \quad (14)$$

The differential equation (5) must now be solved using the boundary condition (14). The solution of the differential equation (5) may be obtained using the technique of separation of variables. Assume a solution of the form

$$\phi = F_1(\theta) F_2(r) F_3(z). \quad (15)$$

Then

$$F_1 \frac{2}{r} \frac{\partial F_2}{\partial r} F_3 + \frac{1}{r} F_1 \frac{\partial F_2}{\partial r} F_3 + \frac{1}{r^2} \frac{\partial F_1}{\partial r} F_2 F_3 + B^2 F_1 F_2 \frac{\partial^2 F_3}{\partial z^2} = 0.$$
or
\[
\frac{1}{B^2} \left[ \frac{1}{F_2} \frac{\partial^2 F_2}{\partial r^2} + \frac{1}{F_2} \frac{1}{r} \frac{\partial F_2}{\partial r} + \frac{1}{F_1} \frac{1}{r^2} \frac{\partial^2 F_1}{\partial \phi^2} \right] = -\frac{\partial^2 F_3}{\partial z^2} F_3. \quad (16)
\]

As the right-hand side of the above equation is a function of \( z \) alone and the left-hand side is independent of \( z \), each side must be a constant. Equate each side of equation (16) to a constant \( p^2 \). To separate the terms containing \( r \) and \( \phi \), multiply the resulting equation for \( r \) and \( \phi \) by \( B^2 r^2 \), then
\[
\frac{r^2}{F_2} \frac{\partial^2 F_2}{\partial r^2} + \frac{r}{F_2} \frac{\partial F_2}{\partial r} - B^2 r^2 p^2 = -\frac{\partial^2 F_1}{\partial \phi^2} \frac{1}{F_1}. \]

Since the right-hand side is a function of \( \phi \) alone and the left-hand side is independent of \( \phi \), each side must be a constant. By setting the right-hand side of the above equation equal to a constant, \( n^2 \), the partial differential equation (16) may be reduced to the following three ordinary differential equations:

\[
F_3'' + p^2 F_3 = 0, \quad (17)
\]
\[
F_1'' + n^2 F_1 = 0, \quad (18)
\]
\[
r^2 F_2'' + r F_2' - (n^2 + r^2 p^2 B^2) F_2 = 0, \quad (19)
\]

where the primes indicate total derivatives.

Since \( \phi \) must be non-singular as \( z \to \infty \), \( p^2 \) must be positive or zero in equation (17) and, hence, the solution for \( F_3 \) is
\[
F_3 = A \cos pz + B' \sin pz, \quad (p \neq 0)
\]
\[
= A_0 + B_0' z, \quad (p = 0)
\]
where $A$ and $B$ are arbitrary constants. Once again to avoid a singularity as $z \to \infty$, $B_0'$ must be zero. Then

$$F_3 = A \cos pz + B \sin pz, \quad (p \neq 0)$$

$$= A_0. \quad (p = 0)$$

Since the solution must be single-valued, it must be periodic of period $2\pi$ in $\theta$; i.e., $F_1(\theta) = F_1(\theta + 2\pi)$. Hence, $n$ must be an integer (including zero). The solution of (18) is then

$$F_1 = C \sin n\theta + D \cos n\theta, \quad (n \neq 0)$$

$$= C_0, \quad (n = 0)$$

where $C$ and $D$ are arbitrary constants.

Equation (19) may be written in the following form by the transformation $t = \sqrt{p}rB$:

$$t^2 \frac{d^2 F_2}{dt^2} + t \frac{d F_2}{dt} + (t^2 - n^2)F_2 = 0 \quad (22)$$

where $i = \sqrt{-1}$. The above equation is now in the form of Bessel's equation, and the solution of (22) written in terms of the original variables is\[ 7 \]

$$F_2 = E I_n(prB) + G K_n(prB) \quad (23)$$

where $I_n$ and $K_n$ are modified Bessel's functions of order $n$ of the first and second kind, respectively, and $E$ and $G$ are arbitrary constants. As $r \to \infty$, $I_n(prB) \to \infty$. Thus, for a non-singular solution $E$ must equal zero. Hence,
\[ F_2 = G K_n(prB). \]  

(24)

From equation (15),

\[
\phi = \left[ G_0 K_0(prB) + G_n K_n(prB) \right] \left[ (A_0 C_0 + C_0 (A_p \cos \theta + B_p \sin \theta) \right.
\]

\[
+ (A_p \cos \theta + B_p \sin \theta) (C_n \sin n\theta + D_n \cos n\theta)
\]

\[
+ A_0 (C_n \sin n\theta + D_n \cos n\theta) \right) \right]
\]

where

\[ n = 1, 2, 3, \ldots \]

To satisfy the boundary condition (14), it is evident that \( n \) must be equal to unity and \( p = k \). It also follows that

\[ A_0 = 0, \]
\[ C_0 = 0, \]
\[ B_p = 0, \]
\[ C_n = 0, \]
\[ G_0 = 0, \]
\[ A_p = 0 \quad \text{except for } p = k. \]

Therefore, the solution of the differential equation (5) is

\[ \phi = C' \cos \theta \cos kz K_1(krB) \]

(25)

where \( C' \) is an arbitrary constant to be determined by the boundary conditions. From equation (25) the radial component of the velocity at \( r = a \) is

\[ \frac{\partial \phi}{\partial r} \bigg|_{r=a} = C' B k K_1'(kaB) \cos \theta \cos kz. \]

(26)
Using the boundary condition (14), the arbitrary constant in equation (25) is found to be

$$C' = \frac{b U}{B K_1(kaB)}.$$  

Hence,

$$\phi = \frac{U b K_1(Bkr) \cos \theta \cos kz}{B K_1(Bka)}. \tag{27}$$

For incompressible flow, the velocity of sound is infinite and, hence, the Mach number is zero. Therefore, the perturbation velocity potential, $\phi_1$, for incompressible flow may be obtained from equation (27) by setting $M_s^2 = 0$ which is equivalent to replacing $B$ by unity. Thus,

$$\phi_1 = \frac{U b K_1(kr) \cos \theta \cos kz}{K_1(ka)}. \tag{28}$$

The ratio of the perturbation velocities in the z-direction for compressible and incompressible flow will now be compared at $r = a$. From equations (27) and (28) it is readily seen that

$$w'_c = -\frac{\partial \phi}{\partial z} = \frac{U b K_1(Bkr) \cos \theta \sin kz}{B K_1(Bka)} \tag{29}$$

and

$$w'_i = -\frac{\partial \phi_1}{\partial z} = \frac{U b K_1(kr) \cos \theta \sin kz}{K_1(ka)}, \tag{30}$$

where $w'_c$ and $w'_i$ are the compressible and incompressible perturbation velocities, respectively.
Therefore,
\[
\frac{w'_c}{w'_i} \bigg|_{r=a} = \frac{K_1(Bka) K'_1(ka)}{B K'_1(Bka) K_1(ka)}.
\] (31)

For large values of the argument, the modified Bessel functions have the following asymptotic behavior [7]:
\[
K_1(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x},
\] (32)
and
\[
K_1'(x) \approx -K_1(x)
= -\sqrt{\frac{\pi}{2x}} e^{-x}.
\] (33)

Therefore, if \(kaB \gg 1\), or alternatively if \(2\pi\frac{a}{L} B \gg 1\), since \(k = 2\frac{\pi}{L}\), equation (31) becomes
\[
\frac{w'_c}{w'_i} \approx \frac{1}{B}
= \frac{1}{(1 - M_s^2)^{\frac{1}{2}}}
\] (34)

Equation (34) is precisely the Prandtl-Glauert correction for two-dimensional flow. As the radius of the basic cylinder becomes large, and if the amplitude of the deformation is small compared with the radius (\(b \ll a\)), the deformation becomes practically a cylindrical surface which corresponds physically to a two-dimensional flow problem.
Furthermore, as the radius of the basic cylinder becomes large, the perturbation velocity in the z-direction at the surface of the cylinder is shown to be of the same general form as the two-dimensional problem of the perturbation velocity along a wave-shaped wall.

Consider a uniform flow, $U$, in the direction of the positive $z$-axis along a wave-shaped wall. Let the shape of the wall be a sine wave of amplitude $b$ and period $L$, Figure 2. It can be shown \[8\] that the perturbation velocity in the $z$-direction is given by

$$w_s = - \frac{2\pi U}{B}\frac{b}{L} \sin \frac{2\pi z}{L}$$

(35)

where

$$B = (1 - M_s^2)^{1/2}.$$ 

From equation (29), the perturbation velocity in the $z$-direction on the surface of the deformable cylinder is

$$w_c' = \frac{U b k (B kr)}{B K'(B ka)} \cos \theta \sin kz$$

(36)

Since $r = a + b \cos \theta \sin kz$, as $a$ tends to infinity so does $r$. Therefore, as the radius of the basic cylinder becomes large

$$\lim_{a \to \infty} \lim_{r \to \infty} w_c' = - \frac{U b k \cos \theta \sin kz}{B}$$

$$= - \frac{2\pi U}{B}\frac{b}{L} \sin kz \cos \theta$$

(37)
Fig. 2 Flow Past a Wave-Shaped Wall
where \( K_1^{(Ekr)} \) and \( K_1^{(Eka)} \) have been replaced in equation (36) by their asymptotic values as given by equations (32) and (33), and \( k \) has been replaced by \( \frac{2\pi}{L} \).

Equations (35) and (37) are now of the same form except for the term \( \cos \theta \). By referring to Figure (1) and equation (9), it is seen that the term \( \cos \theta \) is the appropriate reduction factor for the deformation around the surface of the cylinder. For the special case where \( \theta = 0 \), the two velocities are identical. For this particular case, the limiting form of the deformation becomes a cylindrical surface which is a wave-shaped wall. When \( \theta = \frac{\pi}{2} \), \( w_0' \) is equal to zero as there is no disturbance of the flow and the perturbation velocity is zero as it should be.

Since the modified Bessel functions are tabulated functions, equation (31) may be evaluated numerically. However, equation (31) may be represented as a series expansion by using the following asymptotic expansion for \( K_n(x) \):

\[
K_n(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{4n^2-1^2}{1! \cdot 8x} + \frac{(4n^2-1^2)(4n^2-3^2)}{2! \cdot (8x)^2} + \ldots \right] .
\]

When the modified Bessel functions in equation (31) are replaced by their asymptotic series expansions defined by (38),
and the identity

\[ K_1(x) = - K_0(x) - \frac{K_1(x)}{x} \]

is used, equation (31) may be expressed in the following form:

\[
\frac{w_c}{w_1} \bigg|_{r=a} = \frac{1}{(1-M_s^2)^{1/3}} \left[ 1 + \frac{(1-M_s^2)^{1/3} - 1}{2(2\pi) \frac{a}{L}(1-M_s^2)^{1/3}} - \frac{2(1-M_s^2)^{1/3} - 2 + 3M_s^2}{8(2\pi)^2 \left[ \frac{a}{L} \right]^2} \frac{1}{(1-M_s^2)} \right]
\]

\[
+ \frac{-6(1-M_s^2)^{3/2} + 3M_s^2 - (1-M_s^2)^{1/3} + 7}{16(2\pi) \left[ \frac{a}{L} \right]^3} \frac{1}{(1 - M_s^2)^{3/2}} + \ldots \right] . \quad (39)
\]

Equation (39) is plotted in Figure 3 for \( M_s = 0.8 \), to illustrate how the transition to two-dimensional flow takes place.
Fig. 3 Compressibility Correction Factor for Subsonic Flow
III. KINETIC ENERGY OF A DEFORMABLE ELLIPSOID

The kinetic energy of a perfect incompressible fluid in irrotational motion may be determined by use of Green's theorem. The kinetic energy per unit volume is equal to $\frac{1}{2} mq^2$ where $m$ is the mass per unit volume and $q$ is the velocity. Hence, the kinetic energy of the fluid is given by

$$T = \frac{1}{2} \rho \int q^2 \, dv$$

(40)

where $\rho$ is the density of the fluid and $V$ is the finite volume occupied by the fluid. Since the fluid is irrotational

$$\mathbf{q} = -\nabla \phi$$

and

$$q^2 = \mathbf{q} \cdot \mathbf{q} = (\nabla \phi)^2$$

(41)

Therefore,

$$T = \frac{1}{2} \int V \rho (\nabla \phi)^2 \, dv$$

$$= -\frac{1}{2} \int V \rho \, \nabla^2 \phi \, dv + \frac{1}{2} \int S \rho \, \frac{\partial \phi}{\partial n} \, dS$$

(42)

by the application of Green's theorem. Here $V$ is bounded by the closed surface $S$, and $n$ denotes a unit vector normal to the surface, taken as positive when drawn exterior to the volume. Since the flow is irrotational, $\nabla^2 \phi = 0$ and hence, the kinetic energy for an incompressible fluid is

$$T = \frac{1}{2} \rho S \int \phi \, \frac{\partial \phi}{\partial n} \, dS.$$  

(43)
The above formula is also valid for an infinite region, where the velocity is zero at infinity, when bounded internally by a solid. This may be proven in the following manner. Let \( S \) be an impermeable body completely enclosed by a large surface \( \Sigma \). Then the kinetic energy of the fluid lying in the region between \( \Sigma \) and \( S \) may be obtained by applying equation (43) and is found to be

\[
T = \frac{1}{2} \rho \int_{S} \phi \frac{\partial \phi}{\partial n} \, dS + \frac{1}{2} \rho \int_{\Sigma} \phi \frac{\partial \phi}{\partial n} \, d\Sigma. \tag{44}
\]

As the velocity is zero at infinity, \( \phi \) becomes a constant at infinity since \( \phi \) is a harmonic function. If the surface increases indefinitely in all directions, then

\[
\int_{\Sigma} \phi \, \frac{\partial \phi}{\partial n} \, d\Sigma \to 0,
\]

since \( \phi \) is a constant at infinity. Therefore,

\[
T = \frac{1}{2} \rho \int_{S} \phi \frac{\partial \phi}{\partial n} \, dS. \tag{45}
\]

A. Prolate Ellipsoidal Coordinates

The prolate ellipsoid is generated by revolving an ellipse about its major axis. This type of ellipsoid may be used to approximate elongated bodies of revolution in many
practical applications. Only prolate ellipsoids will be con­sidered in this paper.

To determine the coordinate system appropriate for a pro­late ellipsoid, it is convenient to do so by first expressing the equation of an ellipse in parametric form, Figure 4, [9]. Draw two concentric circles of radii \( f \) and \( g \) respectively. Draw any convenient radius \( OQ_1Q \). Now draw \( QE \) perpendicular to the \( x \)-axis and through \( Q_1 \), draw \( FG \) perpendicular to the \( y \)-axis. These two lines will intersect at a point \( P(x,y) \). The point \( P \) will now lie on an ellipse whose semi-major and semi-minor axes are \( f \) and \( g \), respectively. From Figure 4, it is evident that

\[
\begin{align*}
x &= f \cos \varphi, \\
y &= g \sin \varphi,
\end{align*}
\]

where \( \varphi \) is the angle between \( OQ \) and the \( x \)-axis.

Now, define

\[
\begin{align*}
f &= c \cosh \xi, \\
g &= c \sinh \xi.
\end{align*}
\]

From the definitions of the hyperbolic functions in terms of the exponential function, it follows that

\[
e^{\xi} = \frac{f + g}{c}
\]
From equation (47), it follows that

\[ e^{-\xi} = \frac{f - g}{c}. \]

From equation (47), it follows that

\[ \frac{f^2 - g^2}{c^2} = 1 \]

or

\[ c = \pm \sqrt{f^2 - g^2} \]  \hspace{1cm} (49)

where by definition \((\pm c, 0)\) are the foci of the ellipse.
From equations (47) and (48),

\[ S = \tanh^{-1} \frac{g}{f} \]
\[ = \ln \frac{r + g}{r - g} \]  

Now place the values of \( f \) and \( g \) from equation (47) into equation (46); then

\[ x = c \cosh S \cos \eta, \]
\[ y = c \sinh S \sin \eta. \]  

Thus, the \( x \) and \( y \) coordinates of points on the ellipse are expressed in terms of the parameter \( c \) defined by (49) and the two variables \( S \) and \( \eta \), where \( S \) is defined by (50), and \( 0 \leq \eta \leq 2\pi \). From equation (51) it is evident that

\[ \frac{x^2}{c^2 \cosh^2 S} + \frac{y^2}{c^2 \sinh^2 S} = 1. \]  

If \( c \) is held constant, by varying \( S \), a series of confocal ellipses are generated, over the boundary of each of which \( S \) will be a constant.

The relationship between the Cartesian coordinates \((x,y,z)\) and the prolate ellipsoidal coordinates \((r,x,\theta)\) of a point in space may be found as follows. Let \( x \) be the axis of symmetry of the prolate ellipsoid; \( r \) the radius of any circular cross-section of the ellipsoid, perpendicular to the \( x \)-axis; and \( \theta \), the angle between the \((r,x)\) plane and the \((x,y)\) plane. From
Figure 5, it follows that

\[ y = r \cos \theta \]
\[ z = r \sin \theta . \]  

In the \((r,x)\) plane, \(\theta = \text{constant},\)

\[ x = c \cosh \xi \cos \eta \]
\[ r = c \sinh \xi \sin \eta , \]  

where \(0 \leq \xi < \infty\), and \(0 \leq \eta \leq 2\pi\).

Thus,

\[ x = c \xi \mu , \]
\[ y = c (\xi^2 - 1)^{\frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}} \cos \theta , \]
\[ z = c (\xi^2 - 1)^{\frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}} \sin \theta , \]  

where \(\xi = \cosh \xi\) and \(\mu = \cos \eta \).

Fig. 5 Rectangular and Prolate Elliptic Coordinate System

The following relationships pertinent to the prolate ellipsoidal coordinates will be found useful [10]:
\[
\begin{align*}
\text{d}s_\mu &= \sqrt{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2} d\mu; \\
&= c \frac{(s^2 - \mu^2)^{\frac{1}{2}}}{(1 - \mu^2)^{\frac{1}{2}}} d\mu; \quad (56) \\
\text{d}s_\phi &= c \frac{(s^2 - \mu^2)^{\frac{1}{2}}}{(s^2 - 1)^{\frac{1}{2}}} ds; \quad (57) \\
\text{d}s_\theta &= c (1 - \mu^2)^{\frac{1}{2}} (s^2 - 1)^{\frac{1}{2}} d\theta; \quad (58)
\end{align*}
\]

\[
\begin{align*}
\text{u}_\mu &= -\frac{\partial \phi}{\partial s_\mu} = -\frac{1}{c} \frac{(s^2 - 1)^{\frac{1}{2}}}{(s^2 - \mu^2)^{\frac{1}{2}}} \frac{\partial \phi}{\partial s}; \\
\text{u}_\phi &= -\frac{\partial \phi}{\partial s_\phi} = -\frac{1}{c} \frac{1}{(1 - \mu^2)^{\frac{1}{2}} (s^2 - 1)^{\frac{1}{2}}} \frac{\partial \phi}{\partial \phi}; \quad (59) \\
\text{u}_\theta &= -\frac{\partial \phi}{\partial s_\theta} = -\frac{1}{c} \frac{1}{(1 - \mu^2)^{\frac{1}{2}} (s^2 - 1)^{\frac{1}{2}}} \frac{\partial \phi}{\partial \theta}. \quad (60)
\end{align*}
\]

where \(d s_\mu, d s_\phi,\) and \(d s_\theta\) are the length of the line elements and \(u_\mu, u_\phi,\) and \(u_\theta\) are the velocity components in the direction of the coordinate lines \(\mu, \phi,\) and \(\theta,\) respectively which form an orthogonal coordinate system.

The kinetic energy of a fluid, bounded internally by a prolate ellipsoid defined by \(S = S_0,\) may now be calculated. From equation (45)

\[
\begin{align*}
2T &= \rho \int_S \phi \frac{\partial \phi}{\partial n} dS. \quad (62)
\end{align*}
\]

From Figure 5, it is evident that

\[
\begin{align*}
ds &= r d\theta ds.
\end{align*}
\]
\[ 31 \]

\[ = c(s^2 - 1)^{\frac{3}{2}}(1 - \mu^2)^{\frac{3}{2}} \, d\theta \, ds \]  

(63)

where

\[ r = \sqrt{y^2 + z^2} \]

\[ = c(s^2 - 1)^{\frac{3}{2}}(1 - \mu^2)^{\frac{3}{2}} \]

and \( ds \) denotes the element of length along a meridian profile.

The velocity component normal to the surface defined by \( S = S_o \) is equal to \( -\frac{\partial \phi}{\partial S} \) and, hence,

\[ \frac{\partial \phi}{\delta n} \, ds = -\frac{\partial \phi}{\partial S} \, ds_n . \]  

(64)

When equations (56) and (59) are substituted into equation (64),

\[ \frac{\partial \phi}{\delta n} \, ds = -\frac{1}{c} \frac{(s^2 - 1)^{\frac{3}{2}}}{(s^2 - \mu^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial S} \frac{c}{(1 - \mu^2)^{\frac{3}{2}}} \, d\mu \]

\[ = -\frac{(s^2 - 1)^{\frac{3}{2}}}{(1 - \mu^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial S} \, d\mu . \]  

(65)

The expression for the kinetic energy is then

\[ 2T = -\rho \, c \int_0^{2\pi} \int_{-1}^1 (s_0^2 - 1) \phi \frac{\partial \phi}{\partial S} \, d\mu \, d\theta . \]  

(66)

The Laplacian in prolate elliptic coordinates may be obtained directly by using the equations of transformation defined by equation (55). However, it is easier to calculate the flux through an element of volume bounded by three pairs of coordinate surfaces. For example, the flux through the faces normal to \( \mu = \) constant is given by
\[
\frac{2}{\Delta \mu} \left( \frac{\partial \phi}{\partial s} \ ds \ s_0 \right) \ d\mu
\]

where \( ds \ s_0 \) is the area of the surface element. The flux through all three pairs of surface elements is then

\[
\frac{2}{\Delta \mu} \left( \frac{\partial \phi}{\partial s} \ ds \ s_0 \right) \ d\mu + \frac{2}{\Delta \mu} \left( \frac{\partial \phi}{\partial s} \ ds \ s_0 \right) \ d\mu
\]

\[
+ \frac{\partial}{\partial \theta} \left( \frac{2}{\Delta \mu} \ ds \ s_0 \right) \ d\theta = 0,
\]

as the total flux through the three surfaces must be zero so that the equation of continuity is satisfied. By replacing the expressions in equation (67) by their equivalent values defined by equations (56) through (61), the appropriate partial differential equation for prolate ellipsoidal coordinates is thus obtained:

\[
\frac{\partial}{\partial s} \left( s^2 - 1 \right) \frac{\partial \phi}{\partial s} + \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial \phi}{\partial \mu}
\]

\[
+ \frac{1}{s^2 - 1} \frac{1}{1 - \mu^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{68}
\]

The differential equation (68) may be solved by the technique of separation of variables in the following manner.

Assume a solution of the form

\[
\phi = F_1(s) F_2(\mu) F_3(\theta); \tag{69}
\]

equation (68) then becomes

\[
\frac{\partial}{\partial s} \left( s^2 - 1 \right) \frac{\partial F_1}{\partial s} F_2 F_3 + \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial F_2}{\partial \mu} F_1 F_3
\]

\[
+ \frac{(s^2 - \mu^2)}{(s^2 - 1)} \frac{\partial^2 F_3}{\partial \theta^2} = 0. \tag{70}
\]
The above equation may be rewritten in the form
\[ \left( \frac{s^2 - 1}{s^2 - \mu^2} \right) \left[ \frac{1}{F_1} \frac{\partial}{\partial s} (s^2 - 1) \frac{\partial F_1}{\partial s} + \frac{1}{F_2} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial F_2}{\partial \mu} \right] = -\frac{1}{F_3} \frac{\partial^2 F_3}{\partial \theta^2}. \] (71)

Since the left-hand member is independent of \( \theta \) and the right-hand member is a function of \( \theta \) alone, each side must be a constant, say \( m^2 \). Hence,
\[ \frac{d^2 F_3}{d \theta^2} + m^2 F_3 = 0, \] (72)
and
\[ \frac{1}{F_1} \frac{\partial}{\partial s} (s^2 - 1) \frac{\partial F_1}{\partial s} - \frac{m^2}{(s^2 - 1)} = -\frac{1}{F_2} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial F_2}{\partial \mu} + \frac{m^2}{1 - \mu^2}. \] (73)

Since the left-hand side of the above expression is a function of \( s \) alone and the right-hand side is a function of \( \mu \) alone, each member of equation (73) must be equal to a constant. For the sake of convenience write this constant in the form \( \alpha (\alpha + 1) \). Thus, the original partial differential equation has been reduced to the following three ordinary differential equations
\[ \frac{d^2 F_2}{d \theta^2} + m^2 F_3 = 0, \] (74)
\[
\frac{d}{ds} \left( s^2 - 1 \right) \frac{dF_1}{ds} - \left[ \alpha(\alpha + 1) + \frac{m^2}{s^2 - 1} \right] F_1 = 0 \quad (75)
\]
\[
\frac{d}{d\mu} \left( 1 - \mu^2 \right) \frac{dF_2}{d\mu} + \left[ \alpha(\alpha + 1) - \frac{m^2}{1 - \mu^2} \right] F_2 = 0. \quad (76)
\]

The solution of equation (74) is
\[
F_3 = A \cos m\theta + B' \sin m\theta \quad \text{ (m \neq 0)}
\]
\[
= A_0 + B'_0 \theta \quad \text{ (m = 0)}
\]

where \(m^2\) must be non-negative so that \(F_3\) will be single-valued and \(A\) and \(B'\) are arbitrary constants. The single-valuedness of \(F_3\) implies that \(F_3(\theta) = F_3(\theta + 2\pi)\), and, hence, \(m\) must be an integer (including zero) and \(B'_0 = 0\). Therefore,
\[
F_3 = A \cos m\theta + B' \sin m\theta \quad \text{ (m \neq 0)}
\]
\[
= A_0. \quad \text{ (m = 0)}
\]

The general solution of equation (76) [11] is
\[
F_2 = C P^m_\alpha(\mu) + D Q^m_\alpha(\mu)
\]
where \(P^m_\alpha\) and \(Q^m_\alpha\) are the associated Legendre functions of order \(m\) and degree \(\alpha\) and \(C\) and \(D\) are arbitrary constants. Since \(Q^m_\alpha\) is not finite when \(\mu = 1, D = 0\). \(P^m_\alpha(-1)\) is finite only when \(\alpha\) is an integer; therefore, \(\alpha\) may take on only integral values, say \(n\). Hence,
\[
F_2 = C P^m_n(\mu) \quad (78)
\]

where \(m \leq n\). If \(m > n\), \(F_2 = 0\).

The general solution of equation (75) is [11]
\[
F_1 = E P^m_\alpha(s) + F Q^m_\alpha(s)
\]
where \( P_\alpha^m(z) \) and \( Q_\alpha^m(z) \) are the associated Legendre functions and \( E \) and \( F \) are arbitrary constants. Since \( \alpha \) here is the same as in equation (76), \( \alpha = n \). Since \( P_n^m(z) \) becomes infinite for large \( z \), set \( E = 0 \). Thus,

\[
F_1 = F Q_n^m(z). \tag{79}
\]

Therefore, from equations (69), (77), (78) and (79) it follows that the general solution for flow exterior to a prolate ellipsoid is given by

\[
\phi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(z) Q_n^m(z) (A_n^m \cos m\theta + B_n^m \sin m\theta) \tag{80}
\]

where \( A_n^m \) and \( B_n^m \) are arbitrary constants.

B. Two Node Flexural Vibration with Uniform Flow in x-direction

The solution of the partial differential equation (68) is given by equation (80). By the proper choice and combination of the associated Legendre functions the flow corresponding to a vertical vibration of the ellipsoid of two or more nodes can be obtained. Referring to the rectangular coordinate system \((x,y,z)\), let the deformation of the ellipsoid be such that a point in the \((x,y)\) plane before deformation will still be in the \((x,y)\) plane after deformation. This corresponds to a pure bending motion of the ellipsoid.

The solution of the partial differential equation (68) will satisfy the following boundary condition:
\[ \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial s} \]

\[ = (h^2 - x^2) \frac{\partial y}{\partial s} - 2xy \frac{\partial x}{\partial s} \quad (81) \]

at \( s = s_0 \) where \( h \) is an arbitrary constant. In the above equation, \( \frac{\partial \phi}{\partial y} \) and \( \frac{\partial \phi}{\partial x} \) are the velocities in the y and x directions, respectively. If the \( \frac{\partial \phi}{\partial x} = 0 \), equation (81) would represent a boundary condition in which the major axis of the ellipsoid is deformed into the arc of a parabola. When this term is included, the deformation will depart from the true parabolic form. However, the nodes of vibration will still be given by \( x = \pm h \).

For a uniform flow \( U \) in the positive x-direction, a solution of equation (68) which will satisfy the boundary condition (81) is obtained by using equation (80) and is found to be

\[ \phi = C_1 \phi_1 + C_2 \phi_2 - Ux \quad (82) \]

where

\[ \phi_1 = P^1_3(\mu) Q^1_3(s) \cos \theta, \]

\[ \phi_2 = P^0_1(\mu) Q^0_1(s); \]

\[ P^1_3(\mu) = \frac{(1 - \mu^2)^{\frac{1}{2}}}{2} (15\mu^2 - 3), \]

\[ Q^1_3(s) = (s^2 - 1)^{\frac{1}{2}} \left[ \frac{1}{8}(15s^2 - 3) \ln \frac{s + 1}{s - 1} - \frac{1}{2}(15s + \frac{1}{s^2 - 1}) \right], \]

\[ P^0_1(\mu) = \mu, \]

\[ Q^0_1(s) = \frac{1}{8} s \ln \frac{s + 1}{s - 1} - 1, \]
where $P_{\frac{m}{n}}$ and $Q_{\frac{m}{n}}$ are associated Legendre functions; and $C_1$ and $C_2$ are arbitrary constants. Hence,

$$\phi = C_1 (1 - \mu^2)^{\frac{1}{2}} (15\mu^2 - 3) Q_{\frac{1}{2}}(\xi) \cos \theta + C_2 Q_0(\xi) - Uc \mu \xi. \quad (83)$$

Now

$$\frac{\partial \phi}{\partial \xi} = C_1 (1 - \mu^2)^{\frac{1}{2}} (15\mu^2 - 3) \frac{\partial Q_{\frac{1}{2}}(\xi)}{\partial \xi} \cos \theta + C_2 \mu \frac{\partial Q_0(\xi)}{\partial \xi} - Uc \mu. \quad (84)$$

where (84) is evaluated at $\xi = \xi_0$. The boundary condition (81) may now be expressed in the following form using (55):

$$\frac{\partial \phi}{\partial \xi} = (h^2 - x^2) \frac{\partial y}{\partial \xi} - 2xy \frac{\partial x}{\partial \xi} \quad (85)$$

Equations (83) and (84) may now be used to evaluate the arbitrary constants. From these two equations it follows that

$$(1 - \mu^2)^\frac{1}{2} \cos \theta \left[ C_1 \left( \frac{15\mu^2 - 3}{2} \right) \frac{\partial Q_{\frac{1}{2}}(\xi)}{\partial \xi} - \left[ C_0 (\xi^2 - 1)^{-\frac{1}{2}} \right] \right]$$

$$X \left[ h^2 - 2c^2 \mu^2 \xi^2 - 2c^2 \mu^2 (\xi^2 - 1) \right] + \left[ C_2 \frac{\partial Q_0(\xi)}{\partial \xi} - Uc \right] \mu = 0. \quad (86)$$

As equation (86) must be true for all values of $\theta$ and $\mu$, it follows that

$$C_2 = \frac{Uc}{\frac{\partial Q_0(\xi)}{\partial \xi}} \quad (87)$$

and
\[ C_1(15\mu^2 - 3) = \frac{2c\mathcal{S}_o(s_o^2 - 1)^{\frac{1}{2}}}{\partial Q_3(s)/\partial \mathcal{S}} \left[ h^2 - \alpha^2 \mu^2 s_o^2 - 2\alpha^2 \mu^2 (s_o^2 - 1) \right] \] (88)

at \( s = s_o \). When coefficients of like powers of \( \mu \) in (88) are equated,

\[ 15C_1 = \frac{2c^3 \mathcal{S}_o(s_o^2 - 1)^{\frac{1}{2}}(2 - 3s_o^2)}{\partial Q_3(s)/\partial \mathcal{S}} \]

and

\[ -3C_1 = 2c \mathcal{S}_o(s_o^2 - 1)^{\frac{1}{2}} h^2; \]

or

\[ C_1 = -\frac{2}{3} c \mathcal{S}_o(s_o^2 - 1)^{\frac{1}{2}} h^2, \] (89)

and

\[ h^2 = c^2 \frac{3s_o^2 - 2}{5}. \] (90)

Hence, substituting \( C_1 \) and \( C_2 \) as defined by equations (89) and (87) into equation (82), \( \phi \) is completely determined.

The nodes of vibration are given by

\[ x = \pm h \]
\[ = \pm c \sqrt{\frac{3s_o^2 - 2}{5}}. \] (91)

In the paper by J. L. Taylor [4], the position of the nodes were given by
\[ x = \pm h = \pm c \sqrt{2 - \frac{S_o^2}{5}} \]  \hspace{1cm} (92)

In order for the nodes of vibration to be real, Taylor found that \( S_o \leq \sqrt{2} \). However, the range of \( S \) is such that \( 1 \leq S < \infty \). Thus, his solution would only be valid for \( 1 \leq S_o \leq \sqrt{2} \). By definition

\[ S = \cosh \xi \]
\[ = \frac{r}{c} \]
\[ = \frac{1}{e} \]

from equation (47) where \( e \) is the eccentricity of the ellipsoidal meridian section. Therefore, the solution by Taylor would only be valid for those ellipsoids where \( 1 \leq \frac{1}{e} \leq \sqrt{2} \).

From a physical point of view there is no reason to believe that the nodes of vibration should occur in the region restricted by equation (92).

The nodes of vibration as given by equation (91) will exist if \( S_o^2 \geq 2/3 \). Since by definition, \( S_o \geq 1 \), the solution obtained in this paper will be valid for any prolate ellipsoid.

Upon examination of equation (91), it will be noted that the position of the nodes is independent of the free stream velocity. Therefore, the motion of the prolate ellipsoid
parallel to the axis of symmetry does not affect the position of the nodes of vibration.

The kinetic energy for a prolate ellipsoid undergoing a two node vibration will now be calculated. The following properties of Legendre functions will be found useful [10]:

\[
\int P_m(\mu) \, d\mu = 0,
\]
\[
\int P_m(\mu) P_n(\mu) \, d\mu = 0; \quad m \neq n
\]
\[
\int P_m^s(\mu) P_n^s(\mu) \, d\mu = 0;
\]
\[
\int [P_n(\mu)]^2 \, d\mu = \frac{2}{2n + 1}; \quad m=n
\]
\[
\int [P_n^s(\mu)]^2 \, d\mu = \frac{(n+s)!}{(n-s)!} \frac{2}{2n + 1}
\]

From equation (66);

\[
2T = -\rho \cdot c \int_{0}^{2\pi} \int_{0}^{\pi} (\xi^2 - 1) \frac{\partial \phi}{\partial \xi} \, d\mu \cdot d\theta,
\]

where \( \phi \) is defined by equation (82). Therefore,
The second, third, fourth and seventh terms in the right-hand member of the above equation will vanish when the integration is performed since
\[ \int_0^{2\pi} \cos \theta \, d\theta = 0. \]

The fifth and eighth and the sixth and ninth terms cancel out when \( C_2 \) is replaced by its value determined by (87), and \( P_1(\mu) \) is replaced by its value, \( \mu \). Hence,
\[
2T = \rho c (s_0^2 - 1) \int_0^{2\pi} \frac{c_1}{c_1} \frac{Q_1^2(s)}{Q_3^2(s)} \cos^2 \theta \frac{\partial Q_3^2(s)}{\partial s} \, d\mu \, d\theta
\]
\[
= \rho c (s_0^2 - 1) \frac{2\pi}{c_1} \frac{Q_1^2(s)}{Q_3^2(s)} \frac{\partial Q_3^2(s)}{\partial s}
\]
\[
= -\frac{32}{31} \rho c^3 T \frac{Q_1^2(s)}{Q_3^2(s)} \frac{\partial Q_3^2(s)}{\partial s}
\]
at \( s = s_0 \).

The kinetic energy is independent of the uniform flow \( U \) in the direction of the axis of symmetry. This shows that as far as the kinetic energy or position of the nodes of vibration are concerned the superposition of the uniform flow in
the direction of the axis of symmetry is immaterial. In the paper by Lewis it was assumed that the frequency and, consequently, the inertia of the fluid is the same whether the ellipsoid is moving or stationary. His assumption was based upon experimental evidence. Equation (96) shows that the kinetic energy and, hence, the inertia of the water is independent of the uniform velocity \( U \); thus this shows analytically that Lewis's assumption is valid.

The inertia of the water may be compared with the inertia of the ellipsoid itself by calculating the kinetic energy of the ellipsoid for a vibration in which the amplitude is proportional to \( (h^2 - x^2) \) as the basis for comparison. The area of a circular section of the ellipsoid is given by

\[
\pi(y^2 + z^2) = \pi c^2 \left( S_o^2 - 1 \right) \left( 1 - \mu^2 \right),
\]

\[
= \pi c^2 \left( S_o^2 - 1 \right) \left( 1 - \frac{x^2}{c^2 S_o^2} \right),
\]

\[
= \frac{\pi}{S_o^2} \left( S_o^2 - 1 \right) \left( c^2 S_o^2 - x^2 \right). \quad (97)
\]

The kinetic energy of the ellipsoid, assuming a velocity equal to \( (h^2 - x^2) \) is given by

\[
2T_e = \pi \left( S_o^2 - 1 \right) \int_{S_o}^{c S_o} (h^2 - x^2)^2 \left( c^2 S_o^2 - x^2 \right) \, dx,
\]

\[
= \frac{16}{525} \pi \rho S_o \left( S_o^2 - 1 \right)c^7 \left( 7 - 14 S_o^2 + 9 S_o^4 \right). \quad (98)
\]
The ratio of $T$ may now be found by using equations (96) and (98), and is equal to

$$\frac{T}{T_e} = \left[ \frac{-2 S_0^4(9 S_0^4 - 12 S_0^2 + 4)}{7 - 14 S_0^2 + 9 S_0^4} \right]$$

(99)

$$\times \left[ \frac{(15 S_0^2 - 3) \ln \frac{S_0 + 1}{S_0 - 1} - 2(15 S_0 + \frac{2}{S_0^2 - 1})}{\frac{3 S_0(15 S_0^2 - 11) \ln \frac{S_0 + 1}{S_0 - 1} - 90 S_0^2 + 36 + \frac{4}{S_0^2 - 1}}{S_0^2 - 1}} \right].$$

Figure (6) shows a plot of $T/T_e$ versus the ratio of the semi-major axis to the semi-minor axis of the ellipsoid. This ratio approaches unity as $S_0 \rightarrow 1$, i.e., as the diameter of the ellipsoid becomes small and the length remains fixed, the flow is then practically two-dimensional. Included in this graph are the values obtained by Lewis in [3] and Taylor in [4]. It is noted that the ratio $T/T_e$ found by both Lewis and Taylor is lower than the analytical values found here.

C. Two Node Vibration with Uniform Flow in the y-direction

For uniform flow $U$ in the positive $y$-direction, a solution of equation (68) which will satisfy the boundary condition (81) is given by

$$\phi = \phi_1 - Uy$$

(100)

where
Fig. 6 Ratio of Kinetic Energy of Fluid to Kinetic Energy of the Ellipsoid Versus the Ratio Semi-major to Semi-minor Axis of the Ellipsoid for a Two-node Vibration
\[ F_{\frac{3}{2}}(\mu) = (1 - \mu^2)^{\frac{3}{2}} (15 \mu^2 - 3) \]

\[ Q_{\frac{3}{2}}(s) = (s^2 - 1)^{\frac{3}{2}} \frac{4}{s} (15 s^2 - 3) \ln \frac{s + 1}{s - 1} - \frac{1}{2} (15 s + \frac{2}{s^2 - 1}) \]

\[ \theta_1 = C_3 F_{\frac{3}{2}}(\mu) Q_{\frac{3}{2}}(s) \cos \theta, \]

and \( C_3 \) is an arbitrary constant. From equations (100) and (55), it follows that

\[ \frac{\partial \theta}{\partial \xi} = \frac{\partial \theta_1}{\partial \xi} - U \frac{\partial v}{\partial \xi}, \]

at \( \xi = \xi_0 \).

The boundary condition given by equation (85) and equation (101) may be used to evaluate the arbitrary constants \( C_3 \) and \( a \), as follows:

\[ C_3 (15 \mu^2 - 3) \frac{\partial Q_{\frac{3}{2}}(s)}{\partial \xi} = a \xi_0 \left[ h^2 + U + \mu^2 (2c^2 - 3c^2 \xi_0^2) \right] (\xi_0^2 - 1)^{-\frac{1}{2}} \]

Then

\[ 15C_3 = 2c \xi_0 (2c^2 - 3c^2 \xi_0^2) \left( \xi_0^2 - 1 \right)^{-\frac{1}{2}}, \]

and

\[ -3C_3 = 2c \xi_0 (h^2 - U) \left( \xi_0^2 - 1 \right)^{-\frac{1}{2}} \]

at \( \xi = \xi_0 \).
Now substitute equation (104) into equation (103).

\[-10c \mathcal{S}_o (h^2 + U) \frac{(\mathcal{S}_o^2 - 1)^{-\frac{3}{2}}}{\partial \mathcal{Q}_3 \partial (\mathcal{S})} = 2c \mathcal{S}_o (2c^2 - 3c^2 \mathcal{S}_o^2) \frac{(\mathcal{S}_o^2 - 1)^{-\frac{3}{2}}}{\partial \mathcal{Q}_3 \partial (\mathcal{S})} \]

or

\[h^2 = c^2 \frac{3 \mathcal{S}_o^2 - 2}{5} - U. \quad (105)\]

From equation (104)

\[c_3 = -\frac{2c \mathcal{S}_o (U + h^2)}{3} \frac{(\mathcal{S}_o^2 - 1)^{-\frac{3}{2}}}{\partial \mathcal{Q}_3 \partial (\mathcal{S})} \quad (106)\]

at \(S = \mathcal{S}_o\).

The kinetic energy may now be obtained by using equation (66), and is found to be

\[2T = -\rho c \left[ \frac{96\pi}{1575} \frac{\mathcal{Q}_3^1 (\mathcal{S})}{\partial \mathcal{S}_3 \partial (\mathcal{S})} \mathcal{S}_o^2 (3 \mathcal{S}_o^2 - 2)^2 + \frac{4}{3} \frac{v^2 \pi c^2 \mathcal{S}_o (\mathcal{S}_o^2 - 1)}{\partial \mathcal{S}} \right]. \quad (107)\]

In contrast to the preceding problem with the uniform flow in the direction of the positive x-axis, both the position of the nodes and the kinetic energy are now functions of the uniform flow in the direction of the positive y-axis. However, for most practical applications the uniform flow would be in the direction of the axis of symmetry. As a result the kinetic energy of the fluid will not be compared with the kinetic energy of the ellipsoid itself, in this case.
D. Three Node Flexural Vibration with Uniform Flow in the x-direction

For three node flexural vibration the solution of the partial differential equation (68) will satisfy the following boundary condition which corresponds to an approximate type of flexural vibration in which the amplitude is proportional to \( x(s^2 - x^2) \) where \( s \) is an arbitrary constant

\[
\begin{align*}
\frac{\partial \varphi}{\partial s} &= x \left( s^2 - x^2 \right) \frac{\partial \psi}{\partial s} + s^2 \frac{\partial^2 \psi}{\partial s \partial y} - 3x^2 y \frac{\partial^2 \psi}{\partial s \partial y} \\
&= c^2 \left( s^2 - c^2 \mu^2 \right) \left( s^2 - 1 \right)^{-\frac{1}{2}} \left( 1 - \mu^2 \right)^{\frac{3}{2}} \cos \theta \\
&+ s^2 c^2 \mu \left( s^2 - 1 \right)^{\frac{3}{2}} \left( 1 - \mu^2 \right)^{\frac{3}{2}} \cos \theta - 3c^4 \mu^3 \left( s^2 - 1 \right)^{\frac{3}{2}} \left( 1 - \mu^2 \right)^{\frac{3}{2}} \cos \theta. 
\end{align*}
\]  

(108)

For a uniform flow \( U \) in the direction of the positive x-axis, a solution of equation (68) which will satisfy the boundary condition (108) is given by

\[
\varphi = C_1 \varphi_1 + C_2 \varphi_2 - Ux 
\]  

(109)

where

\[
\varphi_1 = P_4^1(\mu) Q_4^1(s) \cos \theta, \\
\varphi_2 = P_1^0(\mu) Q_1^0(s); \\
P_4^1(\mu) = (1 - \mu^2)^{\frac{3}{2}} \left( \frac{35 \mu^3 - 15}{2} \right), \\
Q_4^1(s) = (s^2 - 1)^{\frac{3}{2}} \left[ \frac{1}{3} (35 s^3 - 15s) \ln \frac{s + 1}{s - 1} \\
- \frac{1}{8(s^2 - 1)} (35 s^4 - 30 s^2 + 3) - \frac{105}{6} s^2 + \frac{55}{24} \right].
\]
and \( C_1 \) and \( C_2 \) are arbitrary constants. Hence,

\[
\phi = C_1 (1 - \mu^2) \frac{35}{2} (35\mu^3 - 15\mu) Q^1_4(s) \cos \theta \\
+ C_2 \mu Q^0_1(s) - \text{Uc} \mu s.
\]

(110)

Now

\[
\frac{\partial \phi}{\partial s} = \frac{C_1}{2} (1 - \mu^2)^{\frac{3}{2}} (35\mu^3 - 15\mu) \frac{\partial Q^1_4(s)}{\partial s} \cos \theta \\
+ C_2 \mu \frac{\partial Q^0_1(s)}{\partial s} - \text{Uc} \mu s, \\
\]

(111)

which is to be evaluated at \( s = s_0 \).

The arbitrary constants \( C_1 \), \( C_2 \) and \( s \) may now be evaluated by equating equations (108) and (111).

\[
\mu \left[ C_2 \frac{\partial Q^0_1(s)}{\partial s} - \text{Uc} \right] + (1 - \mu^2)^{\frac{3}{2}} \cos \theta \left[ \mu^3 \left( \frac{35}{2} C_1 \frac{\partial Q^1_4(s)}{\partial s} + s_0^2 (s_0^2 - 1)^{\frac{3}{2}} c^4 \right. \\
+ 3 c^4 s_0^2 (s_0^2 - 1)^{\frac{3}{2}} \right] - \mu \left[ s^2 c^2 (s_0^2 - 1)^{\frac{3}{2}} + s^2 a^2 s_0^2 (s_0^2 - 1)^{-\frac{1}{2}} \\
+ \frac{15}{2} C_1 \frac{\partial Q^1_4(s)}{\partial s} \right] = 0.
\]

(112)

As the above expression must be true for all values of \( \mu \).
and $\theta$, then

$$C_2 = \frac{U_0}{\partial Q_4(\xi)}$$

(113)

$$C_1 = \frac{2c^4}{35} \xi_0^2 \left[ 1 - 4\xi_0^2 \right] \frac{(\xi_0^2 - 1)^{-\frac{3}{2}}}{\partial Q_4(\xi)}$$

and

$$C_1 = -\frac{2}{15} s^2 c^2 \left( 2\xi_0^2 - 1 \right) \frac{(\xi_0^2 - 1)^{-\frac{3}{2}}}{\partial Q_4(\xi)}$$

(114)

Hence,

$$s^2 = \frac{3}{7} c^2 \xi_0^2 \frac{(4\xi_0^2 - 3)}{2\xi_0^2 - 1}$$

(115)

The nodes of vibration are given by $x = 0$ and

$$x = \pm s = \pm c\xi_0 \sqrt{\frac{3(4\xi_0^2 - 3)}{7(2\xi_0^2 - 1)}}$$

The kinetic energy may be determined from equation (66) and is found to be

$$2T = -\rho c \int_0^\frac{a}{c} \int_0^{\xi_0^2 - 1} \frac{3}{7} c^2 \xi_0^2 \frac{(4\xi_0^2 - 3)}{2\xi_0^2 - 1} \frac{\partial Q_4(\xi)}{\partial \xi} \cos^2 \theta \ d\mu \ d\theta$$

$$= -\rho c \frac{\pi}{2} (\xi_0^2 - 1) \frac{4\xi_0}{9} \frac{c_1 Q_4(\xi)}{Q_4(\xi)} \frac{\partial Q_4(\xi)}{\partial \xi}$$
The kinetic energy and the positions of the nodes of vibration are independent of the free stream velocity $U$ which is parallel to the axis of symmetry. The same phenomena was also observed in the case of the two-node vibration.

The inertia of the water may be compared with the inertia of the ellipsoid itself by calculating the kinetic energy of the ellipsoid for a vibration in which the amplitude is proportional to $x(s^2 - x^2)$ as the basis for comparison. The area of a circular section of the ellipsoid is given by equation (97) as

$$
\frac{\pi (s_0^2 - 1)}{s_0^2} (c^2 s_0^2 - x^2).
$$

The kinetic energy of the ellipsoid, assuming a velocity equal to $x(s^2 - x^2)$ is given by

$$
2T_e = \pi \rho \frac{(s_0^2 - 1)}{s_0^2} \int_{-c}^{c} x^2(s^2 - x^2)(c^2 s_0^2 - x^2)^2 \, dx
$$

$$
= 2 \pi \rho (s_0^2 - 1) c^9 s_0^4 \left[ \frac{2}{15} \frac{s_0^4}{c^4 s_0^4} - \frac{4}{35} \frac{s_0^2}{c^2 s_0^2} + \frac{2}{63} \right] \quad (117)
$$

The ratio of $T/T_e$ may now be found by using equations (116) and (117) and is found to be
\[
\frac{T}{T_e} = -\frac{16}{405} s^4 \frac{(2 s_0^2 - 1)^2}{s_0} \frac{Q_4(s)}{c^4 s_0^2 (s_0^2 - 1)} \left[ \frac{2}{15} s_0^4 - \frac{4}{35} s_0^2 + \frac{2}{63} \right] \tag{118}
\]

Figure 7 shows a plot of \( \frac{T}{T_e} \) versus the ratio of the semi-major axis to the semi-minor axis of the ellipsoid. This ratio approaches unity as \( s_0 \to 1 \). This graph indicates the ratio in which the energy is reduced as compared with the two-dimensional case.
Fig. 7  Ratio of Kinetic Energy of Fluid to Kinetic Energy of the Ellipsoid Versus the Ratio Semi-major to Semi-minor Axis of the Ellipsoid for a Three-node Vibration
IV. SUMMARY

The linear perturbation flow past an infinitely long deformable cylinder, for both compressible and incompressible fluids, was obtained by solving Laplace's equation in cylindrical coordinates. A solution was obtained which contains as a limiting case the Prandtl-Glauert correction factor for two-dimensional flow. To indicate the manner in which the transition to two-dimensional flow takes place, one specific case was plotted in the form of a graph. As the radius of the basic cylinder becomes large, it was shown that the perturbation velocity at the surface of the cylinder is of the same general form as the perturbation velocity along a wave-shaped wall.

This paper is particularly concerned with the calculation of the virtual mass of a vibrating ellipsoid. Since the virtual mass of a body moving in a fluid may be computed once the kinetic energy of the fluid is known, the kinetic energy for an ellipsoid vibrating in two and three nodes was determined. The kinetic energy of the fluid was compared with the kinetic energy of the ellipsoid itself. This ratio is always less than unity and was plotted versus the ratio of the semi-minor to the semi-major axis of the ellipsoid to indicate the manner in which the energy is reduced as compared with the two-dimensional case. It was shown that if a uniform flow is superimposed in the direction of the major axes of the ellipsoid,
the positions of the nodes of vibration and the kinetic energy of the fluid are independent of this uniform flow.

Future study regarding the problem of the virtual mass of a deformable body could consider bodies other than the cylinder and the prolate ellipsoid. For example, an ellipsoid with three unequal axes would be more appropriate for application to the study of the vibration of ships. Further study could be made on the type of boundary conditions to be used in the solution of this problem. For example, the restriction that the body will deform in such a manner that a point in the \((x,y)\) plane before deformation will remain in the \((x,y)\) plane after deformation could be removed.
V. LITERATURE CITED


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