THE ROLE OF NONLINEARITIES IN PROPAGATION, REFLECTION, AND TRANSMISSION OF
STRESS WAVES AND POSSIBLE APPLICATIONS TO NDE

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INTRODUCTION

In an infinite elastic medium dilatational waves exhibit much more nonlinear distortion than do shear waves, but such behavior cannot be assumed when a boundary, such as a free surface, is present. Because of the process of mode conversion, a dilatational (P) wave that has already undergone significant distortion prior to its incidence at the boundary must impart its distortion to both the reflected P and vertically polarized shear (SV) wave. The problem of oblique incidence at a free surface was previously considered by us [1,2]. The present paper generalizes that analysis by permitting the boundary conditions at the planar interface to be arbitrary, and also by considering critical and supercritical angles of incidence.

Cases of normal incidence, for which mode conversion is not a factor, have received the greatest attention in the past. Theoretical results for liquids have been reported by Blackstock [3], while Breazeale and Lester [4] used experiments to gain insight into nonlinear generation of higher harmonics. Normal incidence in solids was analyzed by Buck and Thompson [5], in order to study phase reversal resulting from reflection at a free surface, which could result in reflecting a signal whose waveform is in opposition to the nonlinear effect. Oblique incidence at a planar interface between liquids was considered by VanBuren and Breazeale [6] in an ad hoc manner, while the analyses by Feng [7], Qian [8], and Coteras and Blackstock [9] all involved perturbation series that breakdown with increasing distance from the boundary. In order to correct this failure, Coteras and Blackstock hypothesized a nonlinear version of Snell's law. The modified Snell's law was subsequently proven in Refs. [1] and [2].

The present work outlines a general analysis of the dominant nonlinear effects encountered when an incident planar wave of either dilatational or shear type, arrives at a planar boundary. The effects of nonlinearity are described in a consistent manner that yields a solution that is not range-limited.

SUPERPOSITION USING THE METHOD OF CHARACTERISTICS

Our earlier analysis of incidence at a free surface [1,2] used a perturbation analysis to prove that the dominant nonlinear effect in the
propagation of the various waves is for P waves to distort based on their own signal, while SV waves are unaltered away from the boundary. It also showed that the various waves do not interact as they propagate. The lack of interaction enables us to form a solution as a superposition of nonlinear incident and reflected waves of the type associated with the corresponding linear problem.

We consider two semi-infinite half spaces whose interface is defined as the plane \( z = 0 \), with \( z > 0 \) taken to be the medium in which a planar source is situated. Nonlinearity represents a second order effect for a dilatational wave in an infinite homogeneous, isotropic medium, while it is a third order effect for shear waves. Correct to second order, the particle velocity in either type of wave at position \( r \) having coordinates \((x,z)\) may be written as

\[
\psi_i = \frac{1}{2} A_i d_i \exp(i\omega t) + c.c.
\]

\[
\psi_i = t - \frac{(n_i \cdot r + L_i)}{(c_i + \beta_i |\psi_i|)}
\]

where subscript \( i \) later will be used to identify the various incident, reflected, and refracted waves. Further, \( d_i \) and \( n_i \) are unit vectors denoting the directions of the displacement and propagation, respectively, so that \( d_i \cdot n_i = 1 \) for dilatational waves and \( |d_i \times n_i| = 1 \) for shear waves. The corresponding strains referenced to the \( n_i \) and \( d_i \) directions are

\[
\begin{align*}
E_{nn} & = -v_i \cdot n_i / c_i: \text{ dilatation} \\
E_{nd} & = -v_i \cdot d_i / c_i: \text{ shear}
\end{align*}
\]

Note that these relations between stress and particle velocity have been linearized, consistent with an analysis that seeks a first order approximation. (The stress-strain relations may also be linearized for the same reason.) Other parameters appearing in Eqs. (1)-(3) are \( c_i \), which is the phase speed of the wave when nonlinear effects are neglected (\( c_d \) or \( c_s \)), and \( \beta_i \), which is the coefficient of nonlinearity. For a dilatational wave, \( \beta_i \) depends on the cubic coefficients in the strain energy density, while \( \beta_i = 0 \) for a shear wave. In addition, \( L_i \) is a phase lag.

For a true planar wave, \( n_i \) and \( d_i \) are constants, but solution of the equations of motion by the method of characteristics shows this to be overly restrictive. Consider a two-dimensional signal passing through a point on the interface at \( x = \xi \) at \( t = \tau \) and having phase \( \psi_i \). In the characteristic \((x,z,t)\) space the characteristic surface for such a signal consists of a cone whose axis is parallel to the \( t \) axis, whose apex is at \((\xi,0,\tau)\), and whose apex angle is \( \chi = \tan^{-1} \left( \frac{c_i}{\beta_i |\psi_i|} \right) \). Any straight line generator of this cone represents a possible solution along which \( \psi_i \) and hence \( v_i \) is constant. These straight lines are the characteristics, and \( n_i \) is the projection of a characteristic onto the physical \( x-z \) plane.

At a later instant, the phase \( \psi_i \) would change, corresponding to a change in \( v_i \), which in turn would alter the associated apex angle \( \chi \). If the characteristic line for this signal has the same projection \( n_i \) onto the \( x-z \) plane as did the previous characteristic, regardless of the corresponding phase, then the wave is truly one-dimensional. However, \( n_i \) may be vary with the phase without violating any of the fundamental principles. Recall that \( n_i \) is the direction of the ray for this signal. A ray represents the locus of physical points at which the same phase \( \psi_i \), and corresponding particle velocity \( v_i \), is located at different instants. The orientation of rays emanating from a specified point on the boundary therefore may be a function of the phase, which corresponds to an implicit dependence on time. The earlier perturbation analysis enables us
to superpose these quasi-one-dimensional waves without concern for the manner in which they interact.

SATISFACTION OF THE BOUNDARY CONDITIONS

We define the subscript $i$ as follows: $i = 0$ for the incident wave, $i = 1$ and 2 for the reflected dilatational and shear waves, and $i = 3$ and 4 for the transmitted dilatational and shear waves, respectively. Incidence at a free surface may be accommodated by setting the transmitted amplitudes $A_3$ and $A_4$ to zero, while cases where a medium is an ideal liquid is treated by setting the amplitude of the shear wave in that region to zero.

For a given source, the amplitude $A_0$ and type of incident wave is known, and the phase angle $L_0$ is the distance from the origin to the source plane. The boundary conditions on stress and displacement are like those for the corresponding linear analysis. Hence, the number of such conditions matches the number of unknown amplitudes $A_i$, as well as the number of unknown phase lags $L_i$, and the number of unknown direction angles describing the direction $n_i$. Satisfaction of all boundary conditions at all locations is only possible if the waves arriving and departing from a specified boundary point have the same phase. Matching the position dependent and independent parts of Eq. (2) leads to

$$\sin \theta_i/b_i = \sin \theta_0/b_0, \quad L_i/b_i = L_0/b_0 ; \quad j = 1, \ldots, 4$$

(4)

where $b_i$ denotes the phase speed at this boundary point,

$$b_i = c_i + \beta_i v \bigg|_{a_1=0}$$

(5)

The first of Eqs. (4) is the nonlinear modification of Snell's law hypothesized by Coteras and Blackstock.

Since the phases of all waves have been equated at $a_3 = 0$, these parameters may be factored out, thereby reducing the boundary conditions to a set of simultaneous equations that are linear in the amplitudes $A_i$,

$$[D(\theta_1)] (A) = (F(\theta_1)) A_0$$

(6)

These equations are exactly like those obtained in a linearized analysis of the problem prior to the application of Snell's law to simplify the equations, except that the angles $\theta_i$ for reflected and transmitted waves now oscillate in accord with the fluctuation of $b_i$ in Eq. (5).

WAVEFORM EVALUATION

In principle, Eqs. (1)-(6) fully describe all waves, once the values of $\omega$, $A_0$, $\theta_0$, and $L_0$ are specified. However, the task of evaluating waveforms at a specified field point is quite complicated, due to the variability of the reflection and refraction angles and the complicated dependency on $a_1, a_3$, and $t$. This difficulty is illustrated in Fig. 1, which shows that the reflected rays arriving at a specified field point at different instants emanate from a range of locations on the boundary, depending on the particle velocity associated with that ray. The alternative procedure we follow for evaluating a waveform is an iterative one in which $\psi_i$, rather than $t$, is taken as the independent time variable for a field point C at $(\xi, \eta)$. For any $\psi_i$ in the range 0 to $2\pi$, the incident wave is known from Eqs. (1) and (3). Equations (1)-(6) may then be solved iteratively for the properties of the other waves, based on using linear theory as an initial approximation of the angles $\theta_i$. Once
the particle velocity in a wave has been determined, the corresponding value of \( t \), which represents the time at which a signal having phase \( \psi_1 \) arrives at the specified field point may be determined from Eq. (2).

The evaluation of arrival time has an interesting graphical interpretation. In Fig. 2, transmitted rays associated with a specific phase \( \psi_1 \) are shown. (A similar description applies for reflection). For the incident wave, all signals having the same phase have the same particle velocity, and consequently the same phase speed \( c_0 + \beta_0 |v_0| \). Hence, any line perpendicular to the rays of the incident wave, such as AA', represents a wavefront of that wave. Further, because the incident rays are associated with identical phases, the particle speeds associated with the transmitted rays are identical, \( |v_0| = |v_1| \), from which it follows that the transmission angles for the transmitted rays are the same. Consider line C'C'CC" perpendicular to a transmitted ray. The time required for a signal to propagate from A to B on the boundary and then to field point C is \( L_{AB}/(c_0 + \beta_0 |v_0|) + L_{BC}/(c_0 + \beta_1 |v_1|) \). Trigonometry and the nonlinear Snell's law, eq. (4), shows this time to be identical to the elapsed time from A' to B' to C' or from A" to B" to C". Thus, line C'C" represents a wavefront, along which all signals are in-phase. In other words, the wave emanating from the boundary at any instant is planar, with parallel rays and orthogonal wavefronts, but the direction in which those planar waves propagate varies slightly within one period due to the dependence of the nonlinear Snell's law on the particle speed.

It is possible that some of the waves emanating from the boundary evanescence. If \( \sin \theta_0 \) is sufficiently large compared to \( c_0/c_1 \), then Eq. (4) indicates that \( \sin \theta_1 > 1 \), regardless of the value of \( |v_1| \), in which case that wave is completely evanescent. A more interesting possibility is that \( \sin \theta_0 \) is close to \( c_0/c_1 \), so that the issue depends on the value of the particle velocity associated with a ray. In that case, the signal at a specified phase will be associated with (propagating) rays for part of the period and evanescence for the remainder. The iterative algorithm employed to evaluate wavefronts in propagation is also suitable for evanescence. Using it requires that one regard the trigonometric functions to be complex variables. Correspondingly, the wavefronts in the evanescent wave are perpendicular to the boundary and the signal decays exponentially along a wavefront at a rate that is slightly dependent on the associated incident particle velocity.
RESULTS AND DISCUSSION

Although regular incidence at a stress free boundary is the simplest case from the standpoint of the number of waves, the effects one observes are qualitatively identical to those for regular incidence at other types of boundaries. The material that we choose for numerical evaluation is steel, for which $\beta_0 \approx 3.1$. Since $\beta_0$ is a small value, the nonlinear phenomena are not appreciable for realistic values of $\epsilon$. In order to magnify the nonlinear effects, we consider here $\epsilon = 0.01$.

Figure 3 shows temporal waveforms for x-z stress components in the reflected dilatational wave, while Fig. 4 gives the corresponding information for the reflected shear wave. In both cases, the source is operating at 2 MHz, and the angle of incidence of the incoming P wave is 60°. For this calculation, the field point was selected such that, measured along the reflected dilatational rays according to linear theory ($\theta_1 = \theta_0$), the distance from the field point to the origin equals the distance $L_0$ from the origin to the source. Further, $L_0$ is taken to be one half of the shock formation distance for the incident P wave, $r = c_0/2\epsilon \beta_0 \omega$. Hence, if the incident wave were to propagate along the ray path without undergoing a decrease in amplitude and a change of phase due to the reflection process, it would first form a shock at the field point. We see in both figures that using the nonlinear Snell's law to evaluate the reflected signals leads to relatively small modifications, in which the peak tension is reduced and the peak compression is increased. Since the zero stress states are unaltered, the corollary of this observation is that a small DC bias is introduced.

It is interesting to note that the SV wave at the field point shows more distortion than the P wave, even though the effect of nonlinearity in the propagation of shear waves was neglected as a lower order effect. This phenomenon arises from a combination of factors. In general, the deviation of the reflected shear wave from a pure sinusoid is attributable to the fact that the incident wave arrives at the boundary in a distorted state. The absence of nonlinearity merely means that the waveform along any ray of the reflected SV wave is constant.

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Fig. 3. Waveforms - dilatational wave reflected from a free surface.

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Fig. 4. Waveforms - shear wave reflected from a free surface.
The primary cause for the greater strength of the distortion of the SV wave is the phase reversal of the P wave upon reflection, which causes that waveform to tend to distort back to a sinusoidal shape. This is similar to the effect noted by Breazeale and Lester [4] for reflection at a pressure release boundary. In addition, the rays of the incident P wave that generate the SV wave travel through a greater distance to the boundary, and therefore arrive at the boundary with more distortion than their counterparts that generate the reflected P wave. A third factor influencing the distortion level is associated with the mode conversion process, which lowers the overall amplitude of the P wave. In other words, the distance from the boundary to the field point, which was selected to be half the shock formation distance for the incident P wave, is much less than half the distance for shock formation for the reflected P wave.

We conclude from this that mode conversion in reflection extends the distance along the ray path of a P wave at which a shock first forms. These factors are illustrated in Fig. 5, which shows the dependence of the first three harmonics of the reflected P wave on distance \( r \) measured along the dilatational ray emanating from the origin according to linear theory, nondimensionalized by the shock formation distance \( \tilde{r} \) for the incident wave. If the linear Snell's law were applied, phase reversal would result in disappearance of the higher harmonics at \( r \approx 3\tilde{r} \). The absence of such cancellation when the nonlinear Snell's law is used is primarily due to the fact that signals arrive at a field point from a distributed set of points on the boundary. This behavior is contrasted by Fig. 6, which shows the position dependence of the harmonic components of the reflected SV waveform. The distance parameter for this figure is the same as that for Fig. 5, specifically distance along the linear dilatational ray through the origin. Since the reflected SV wave propagates without amplitude dispersion, the curves obtained by both linear and finite amplitude forms of Snell's law are almost identical. Note that because of the lack of propagative distortion in shear, an evaluation of the dependence on distance along a ray of the reflected shear wave would show that the harmonic amplitudes remain nearly constant.
The second situation we address is the case of two elastic half spaces with a bonded interface. The medium for the incident dilatational wave is steel, and the second medium is taken as a fictitious material whose linear dilatational and shear speeds are 5/3 and 10/9, respectively, of the dilatational speed in steel. The angle of incidence in this case is the first critical value according to linear theory, at which the transmitted shear wave begins to evanesce, and the distance from the source to the origin is maintained at 0.5r. The parameter p for the various curves is the ratio of the coefficient of nonlinearity in the two media, with p = 0 corresponding to the receiving medium being linear. The field point associated with Fig. 7 is at 0.2r along a line extending from the origin at an angle of elevation of 10° above the interface. The interesting feature of critical incidence is that due to the variability of the angle of transmission resulting from the nonlinear Snell's law, the shear wave is evanescent for part of its phase (maximum particle velocity) and propagative for the remainder. This substantially strengthens the nonlinear effect, as can be seen from the extreme exaggeration in the waveform distortion in Fig. 7.

CONCLUSIONS

We have developed a general theory governing the dominant nonlinear effects associated with reflection and transmission of planar stress waves obliquely incident on the boundary of an elastic half-space. The nonlinear version of Snell's law, which incorporates the velocity dependence of the phase speed, leads to some new phenomena. Regular incidence introduces asymmetry into the distortion of dilatational waves, and shear waves also are shown to be distorted, even though the nonlinear effect in the propagation of such waves is of a negligible order. For regular incidence, the nonlinear effect is quite weak, and probably immeasurable with current technology. Critical incidence, which in the nonlinear theory leads to a wave that shifts between evanescent to propagating features within one period, strengthens the effect to the

![Fig. 7. Waveforms at a typical field point at the critical angle of incidence for the transmitted dilatational wave.](1835)
point where certain features of the waveform might be observable. NDE techniques attuned to measuring the generation of higher harmonics, combined with improvements in instrumentation sensitivity and in the ability to generate high amplitude stress waves, might lead to improved flaw detection procedures.

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