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Weak representation theory in the calculus of relations

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by

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has met the dissertation requirements of Iowa State University

Signature was redacted for privacy.

Major Professor

Signature was redacted for privacy.

For the Major Program

To the one who knows me

All the best in the world

But all propositions of logic say the same thing. That is, nothing.

–Wittgenstein

The logician's net is too coarse-meshed to catch the fish that matter.

–David Stove

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1 Introduction

1.1 Introduction and definitions

We begin with an overview of the relevant history, open questions, and new results. Precise definitions will follow later.

A relation algebra $\mathfrak{A}$ is said to be *representable* if it is isomorphic to a proper relation algebra; it is said to be *weakly representable* if it has a representation respecting all the operations except $+$ and $-$. Let wRRA denote the class of weakly representable relation algebras. In his 1959 paper [6], Jónsson proved that wRRA is a quasi-variety, and hence is closed under the class operators **S** and **P**, by establishing an infinite set (Γ) of quasi-identities that define wRRA over RA. At the end of the paper he asked whether (Γ) could be replaced by a finite set of formulas, or by identities. The first question was answered negatively: wRRA is not finitely axiomatizable (see [5]), nor is RRA finitely axiomatizable over wRRA (see [2]).

The second question—whether (Γ) can be replaced by identities—turned out to be ambiguous. In the 1959 paper [6] Jónsson referred to algebras of the form $\mathfrak{A} = \langle A, \cap, |, ^{-1}, I\rangle$, where A is a set of binary relations and I acts as an identity for $|$, as *algebras of relations.* Let $\mathcal{R}$ be the class of all such algebras of relations. Jónsson proved that (Γ) along with a few identities axiomatized $\mathcal{R}$. But he also showed that (Γ) axiomatized wRRA over RA. So there are really two questions: Can $\mathcal{R}$ be axiomatized by equations, and can wRRA?

In their 1995 paper [3] Andréka and Bredhikin published an answer to Jónsson's question: No. They were referring to the first interpretation of the question, and showed that

$\mathcal{R}$ was not closed under $\mathbf{H}$. In chapter 3 we consider the question under the second interpretation.

The weakly representable but not representable algebras given in [2] all had weak representations over finite sets. In chapter 2 we will take up the question whether there is a weakly representable but not representable relation algebra with weak representation over a finite set. We will exhibit weak representations on finite set for some small algebras (too small, however, to be non-representable).

In order to motivate the definition of a relation algebra, we discuss those structures of which they are the abstract analogue. In what follows, we assume that the reader is familiar with boolean algebras and basic boolean arithmetic. For more on boolean arithmetic see [1] or [7].

1.1.1 Proper relation algebras

Let Sb(X) denote the power set of X, and let Re(X) denote the power set of $X \times X$.

Definition 1.1.1. A *proper relation algebra* is an algebra $\langle A, \cup, ^-, |, ^{-1}, \mathrm{Id}_E\rangle$, where $A \subseteq \mathrm{Sb}(E)$ for a nonempty equivalence relation E, and $\mathrm{Id}_E = \{\langle x, x\rangle : \langle x, y\rangle \in E \text{ for some y}\}$. The operation $|$, called *relative multiplication* or *composition*, is given by $R|S = \{\langle x, z\rangle : \exists y\ \langle x, y\rangle \in R \text{ and } \langle y, z\rangle \in S\}$, and the operation $^{-1}$ is given by $R^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in R\}$.

The set of all such algebras is denoted by PRA (for proper relation algebra).

Theorem 1.1.2. A set $E \subseteq U \times U$ (for some U) is an equivalence relation iff $E = E|E^{-1}$.

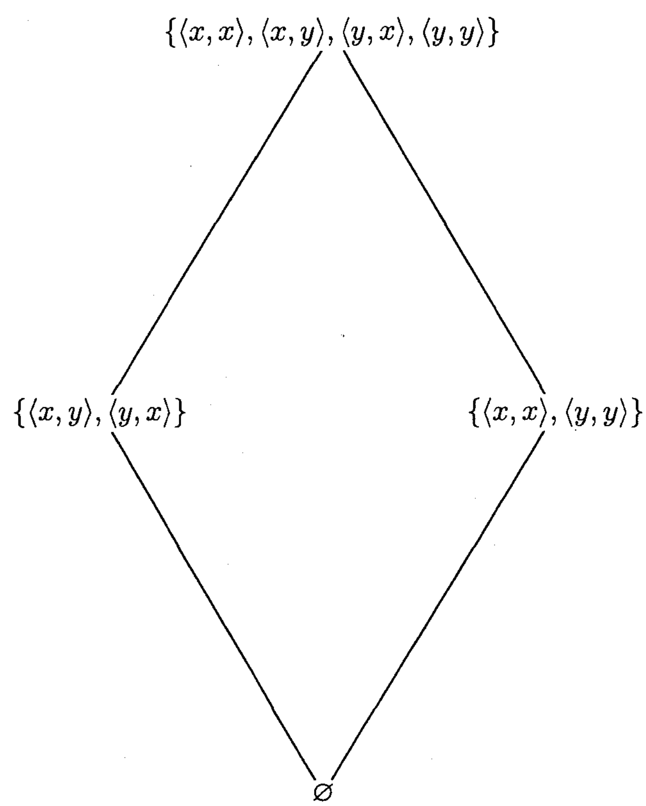
The set U above could be any set containing the *field of* E, which is $\{x : \exists y \langle x, y\rangle \in E \text{ or } \langle y, x\rangle \in E\}$. Informally, the field of a relation is the set of points which are related to something else by the relation in question.

Proof. By definition, E is an equivalence relation precisely when E satisfies $E^{-1} = E$ (symmetry) and $E|E \subseteq E$ (transitivity). So to prove the "only if" direction, let E be an equivalence relation. Then $E = E|E^{-1} \subseteq E$. Also, if xEy then $xExEy$, since E is reflexive over its field, and so we have $E \subseteq E|E = E|E^{-1}$. Thus $E = E|E^{-1}$ as desired. For the "if" direction, let $E = E|E^{-1}$. Then $E^{-1} = (E|E^{-1})^{-1} = E|E^{-1} = E$, so E is symmetric. Also, $E|E = E|E^{-1} = E$, so E is transitive. □

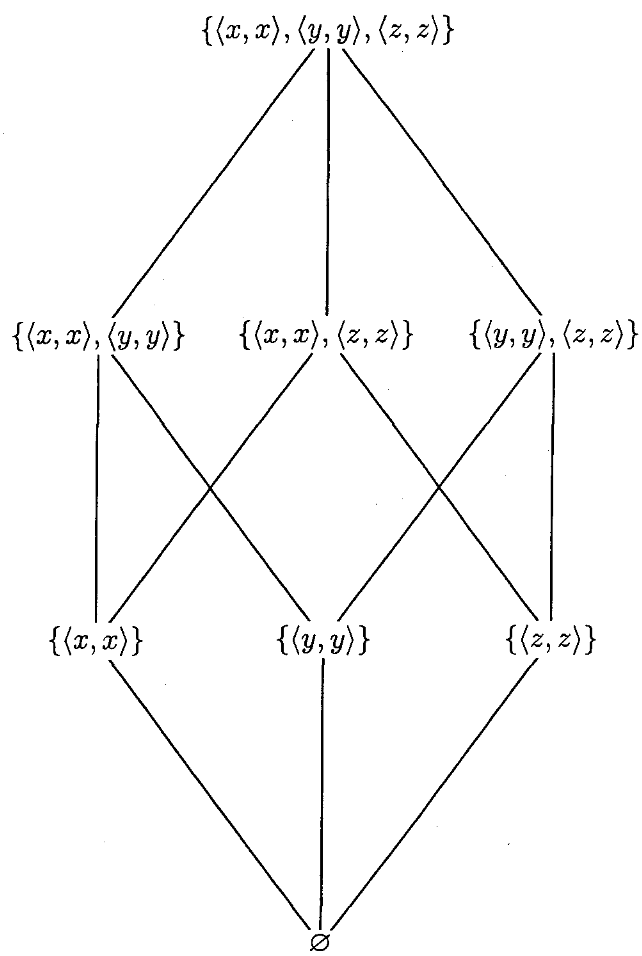
Algebras of the form $\langle \mathrm{Re}(U), \cup, ^{-}, |, ^{-1}, \mathrm{Id}_{U\times U}\rangle$ and their subalgebras are called *square* PRAs, since the boolean unit has the form $U \times U$.

1.1.2 Examples

The following Hasse diagram shows the boolean structure of a particular proper subalgebra of $\langle \mathrm{Re}(\{x, y\}), \cup, ^{-}, |, ^{-1}, \mathrm{Id}_{U\times U}\rangle$. This algebra is *square* (the boolean unit is of the form $U \times U$) but not *full* (it is not a power set algebra).



Another example is $\langle \mathrm{Sub}(\{\langle x, x\rangle, \langle y, y\rangle, \langle z, z\rangle\}), \cup, ^{-}, |, ^{-1}, \mathrm{Id}_{\{x,y,z\}}\rangle$. In this instance, the boolean unit is also the relational identity. This algebra is *full* but not *square*. See the following Hasse diagram:



1.1.3 Abstract relation algebras

Definition 1.1.3. A *relation algebra* is an algebra $\mathfrak{A} = \langle A, +, -, ;, \breve{}, 1' \rangle$ that satisfies the following equations.

$$(x + y) + z = x + (y + z) \tag{1.1}$$

$$x + y = y + x \tag{1.2}$$

$$x = \overline{\overline{x} + y} + \overline{\overline{x} + \overline{y}} \tag{1.3}$$

$$x \,;(y \,; z) = (x \,; y) \,; z \tag{1.4}$$

$$(x + y) \,; z = x \,; z + y \,; z \tag{1.5}$$

$$x\,;1' = x \tag{1.6}$$

$$\breve{\breve{x}} = x \tag{1.7}$$

$$(x+y)\breve{} = \breve{x} + \breve{y} \tag{1.8}$$

$$(x\,;y)\breve{} = \breve{y}\,;\breve{x} \tag{1.9}$$

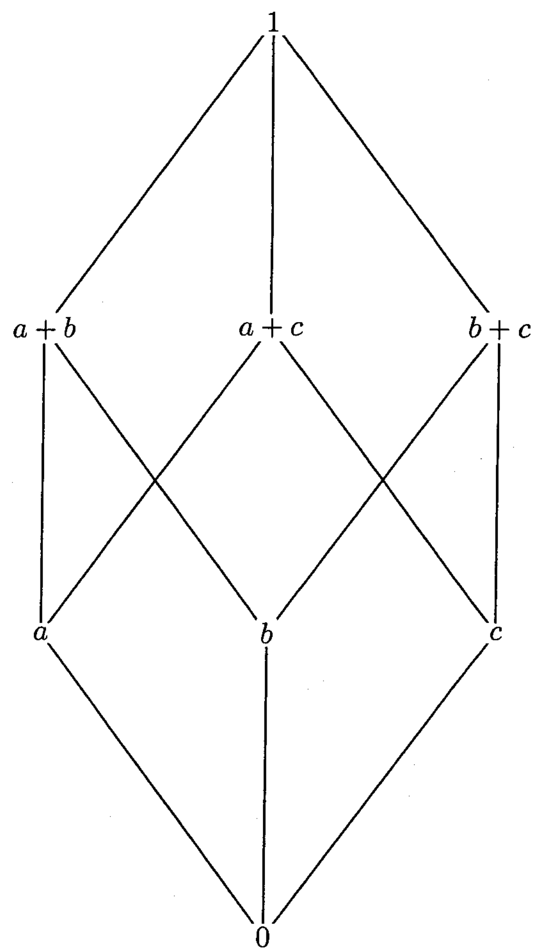
$$\bar{y} + \breve{x}\,;\overline{x\,;y} = \bar{y} \tag{1.10}$$

The first three axioms say that $\mathfrak{A}$ is a boolean algebra with additional operations. The remaining axioms are relational identities that hold in every PRA, translated into the abstract algebraic language. The class of all relation algebras is denoted by RA.

We introduce a defined binary operation, $x \cdot y = \overline{\bar{x} + \bar{y}}$. We can define a partial order $\leq$ on an RA by $x \leq y$ iff $x + y = y$. Also, $x < y$ will mean $x \leq y$ and $x \neq y$. An element a is called an *atom* if $a > 0$ and $(\forall x)(x < a \Rightarrow x = 0)$. An RA is called *atomic* if every nonzero element x has an atom a with $a \leq x$. An atom a is called an *identity atom* if $a \leq 1'$ and is called a *diversity atom* if $a \leq -1' = 0'$. An RA is called *symmetric* if conversion is the identity function ($x = \breve{x}$). An RA is called *integral* if $x = 0$ or $y = 0$ whenever $x\,;y = 0$. This last condition is equivalent to the condition that 1' be an atom. (See Th. 1.2.9)

1.1.4 An example

Consider the following RA with boolean structure as follows:



All elements of this algebra are self-converse ($\breve{x} = x$). Relative multiplication is given by $x \,;y = x \cdot y$. If such an equational definition of ; is not available, composition for finite RAs can be specified by a multiplication table on the atoms. For this example we have

;	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

It is sufficient to specify ; on the atoms because in a finite relation algebra every non-zero element is the join of all of the (finitely many) atoms below it, and ; distributes over $+$. (Of course, each entry in the table should be the join of some atoms.) For example, if $x = a + b$ and $y = b + c$, then to compute $x \,; y$ we do the following: $x \,; y = (a + b) \,; (b + c) =$

$a\,;b{+}a\,;c{+}b\,;b{+}b\,;c$. Now each of these terms in the sum can be read off the multiplication table for atoms.

It is easy to see that this abstract relation algebra is isomorphic to the second algebra given in 1.1.2, which is a PRA.

1.2 Arithmetic in RA

We derive some useful results that hold in RA. First we show that left-distributivity of ; over + follows from the axioms. Note that only right-distributivity is assumed explicitly.

Theorem 1.2.1. $x\,;(y+z) = x\,;y + x\,;z$.

We use $(a\,;b)\breve{} = \breve{b}\,;\breve{a}$ to "turn around" the composition, and use right-distributivity.

Proof.

$$\begin{aligned} x\,;(y+z) &= \left([x\,;(y+z)]\breve{}\right)\breve{} && \text{by (1.7)} \\ &= \left((y+z)\breve{}\,;\breve{x}\right)\breve{} && \text{by (1.9)} \\ &= \left((\breve{y}+\breve{z})\,;\breve{x}\right)\breve{} && \text{by (1.8)} \\ &= \left(\breve{y}\,;\breve{x}+\breve{z}\,;\breve{x}\right)\breve{} && \text{by (1.5)} \\ &= \left((x\,;y)\breve{}+(x\,;z)\breve{}\right)\breve{} && \text{by (1.9)} \\ &= \left([x\,;y+x\,;z]\breve{}\right)\breve{} && \text{by (1.8)} \\ &= x\,;y+x\,;z && \text{by (1.7)} \end{aligned}$$

□

Theorem 1.2.2. The operation $\breve{}$ is an automorphism of the boolean reduct of any $\mathfrak{A} \in$ RA. In particular, $\breve{\bar{x}} = \bar{\breve{x}}$ (or $-\breve{x} = (\bar{x})\breve{}$).

Proof. The operation $\breve{}$ is bijective, since $\breve{\breve{x}} = x$. $(x+y)\breve{} = \breve{x}+\breve{y}$ is axiom (1.8).

$$x \le y \overset{\text{(def)}}{\iff} x+y=y \overset{(1.7,8)}{\iff} \breve{x}+\breve{y} = (x+y)\breve{} = \breve{y} \overset{\text{(def)}}{\iff} \breve{x} \le \breve{y}$$

so conversion is order-preserving (or "monotone").

Now since $\breve{}$ preserves order, we have $\breve{1} \le 1$ and $1 = (\breve{1})\breve{} \le \breve{1}$, so $\breve{1} = 1$. Also, we have $\breve{0} \ge 0$ and $0 = (\breve{0})\breve{} \ge \breve{0}$, so $\breve{0} = 0$. Hence $\breve{}$ preserves 0 and 1.

To prove $\breve{\bar{x}} = \bar{\breve{x}}$, first note that $x + \bar{x} = 1$, so $\breve{x} + \breve{\bar{x}} = (x + \bar{x})\breve{} = \breve{1} = 1$. Therefore $\bar{\breve{x}} \le \breve{\bar{x}}$. Similarly, $\breve{x} + \bar{\breve{x}} = 1$, so by a parallel argument $\bar{x} \le (\bar{\breve{x}})\breve{}$. By monotonicity of conversion, we have $\breve{\bar{x}} \le (\bar{\breve{x}})\breve{}\breve{} = \bar{\breve{x}}$. Therefore $\breve{\bar{x}} = \bar{\breve{x}}$.

Hence conversion is an automorphism of the boolean reduct. □

Theorem 1.2.3. If $x \le y$ then $x\,;z \le y\,;z$ and $z\,;x \le z\,;y$ (monotonicity of ;)

Proof. Assume $x \le y$. Then $x\,;z \le x\,;z + y\,;z \stackrel{(1.5)}{=} (x+y)\,;z = y\,;z$. Also, $z\,;x \le z\,;x + z\,;y \stackrel{(1.2.1)}{=} z\,;(x+y) = z\,;y$. □

The following theorem is known as the *cycle law* or *Peircean Law*, after C. S. Peirce (pronounced "purse").

Theorem 1.2.4. $x\,;y \cdot \breve{z} = 0 \iff y\,;z \cdot \breve{x} = 0$

Sometimes we will make use of this in its equivalent form, $x\,;y \cdot \breve{z} \ne 0 \iff y\,;z \cdot \breve{x} \ne 0$. We will cite this theorem when referring to either equivalent form.

Proof. Suppose that $x\,;y \cdot \breve{z} = 0$. This is equivalent to $\overline{x\,;y} \ge \breve{z}$. Then $(\overline{x\,;y})\breve{} \ge z$ by Th 1.2.2, $-(x\,;y)\breve{} \ge z$ by Th 1.2.2 again, and $-(\breve{y}\,;\breve{x}) \ge z$ by (1.9). Finally we have $y\,;\overline{\breve{y}\,;\breve{x}} \ge y\,;z$ by monotonicity of ;.

(1.10) tells us that $y\,;\overline{\breve{y}\,;\breve{x}} \le -\breve{x}$. Combining the last two inequalities, we get $y\,;z \le y\,;\overline{\breve{y}\,;\breve{x}} \le -\breve{x}$, and hence $y\,;z \cdot \breve{x} = 0$.

The converse follows by an alternate assignment of the roles of x, y, z. □

Note that neither of the facts in the following theorem were assumed explicitly. (1' was assumed to be a *right* identity.)

Theorem 1.2.5. $1'\breve{} = 1'$, $1'\,;x = x$, $0\,;x = 0 = x\,;0$

Proof. $1'^{\smile} = 1'^{\smile} ; 1' = 1'^{\smile} ; (1'^{\smile})^{\smile} = (1'^{\smile} ; 1')^{\smile} = 1'^{\smile\smile} = 1'$. Also, $1' ; x = (\breve{x} ; 1'^{\smile})^{\smile} = (\breve{x} ; 1')^{\smile} = \breve{\breve{x}} = x$, where the second equality is true by the just-derived $1'^{\smile} = 1'$.

To prove $0 ; x = 0$, note that by (1.2.4) we have $y ; x \cdot \breve{z} = 0 \iff x ; z \cdot \breve{y} = 0$. Now let x be arbitrary, let $y = 0 = \breve{0}$, and let $z = 1 = \breve{1}$. Then we get $0 ; x \cdot 1 = 0 \iff x ; 1 \cdot 0 = 0$. Now the right side of this biconditional is always true; therefore the left is also. Thus $0 ; x = 0 ; x \cdot 1 = 0$, as desired. The derivation of $x ; 0 = 0$ is similar.

□

Theorem 1.2.6. A symmetric algebra satisfies $x ; y = y ; x$.

Proof. Suppose $\breve{x} = x$ for all x. Then $x ; y = (x ; y)^{\smile\smile} = (x ; y)^{\smile} \overset{(1.9)}{=} \breve{y} ; \breve{x} = y ; x$. □

Theorem 1.2.7. $x \cdot y ; z = x \cdot y ; (z \cdot \breve{y} ; x)$

Proof.

$$\begin{aligned} x \cdot y ; z &= x \cdot y ; (z \cdot 1) \\ &= x \cdot y ; (z \cdot (\breve{y} ; x + \overline{\breve{y} ; x})) \\ &= x \cdot y ; (z \cdot \breve{y} ; x) + x \cdot y ; (\overline{\breve{y} ; x} \cdot z)) \\ &= x \cdot y ; (z \cdot \breve{y} ; x) + 0 \end{aligned}$$

To justify the last step, $x \cdot y ; (\overline{\breve{y} ; x} \cdot z)) = 0$, note that by (1.10) we have $y ; \overline{\breve{y} ; x} \leq \bar{x}$ and hence $x \cdot y ; (\overline{\breve{y} ; x} \cdot z) \leq x \cdot \bar{x} = 0$. □

Theorem 1.2.8. $x \leq x ; \breve{x} ; x$

Proof. We use (1.2.7), letting $y = x$ and $z = 1'$. Then

$$\begin{aligned} x &= x \cdot x ; 1' \\ &= x \cdot x ; (1' \cdot \breve{x} ; x) && \text{by the above} \\ &= x \cdot x ; \breve{x} ; x \end{aligned}$$

$\leq x\,;\breve{x}\,;x$

□

Theorem 1.2.9. If $x, y \leq 1'$, then $\breve{x} = x$ and $x\,;y = x \cdot y$.

Proof. By (1.2.8) and monotonicity, $x \leq x\,;\breve{x}\,;x \leq 1'\,;\breve{x}\,;1' = \breve{x}$. So $x \leq \breve{x}$. Then $\breve{x} \leq \breve{\breve{x}} = x$, and $x = \breve{x}$.

For the second result, $x\,;y \leq 1'\,;y = y$ and $x\,;y \leq x\,;1' = x$ by monotonicity, so $x\,;y \leq x \cdot y$. Also, by monotonicity and the (1.2.7) $x \cdot y \leq (x \cdot y)\,;(x \cdot y)^{\smallsmile}\,;(x \cdot y) \leq x\,;(x \cdot y)^{\smallsmile}\,;y \leq x\,;(1'\,;1')^{\smallsmile}\,;y = x\,;y$. Hence $x\,;y = x \cdot y$. □

Theorem 1.2.10. Let $\mathfrak{A} \in \mathsf{RA}$ have at least two elements. Then $\mathfrak{A}$ is integral iff $1' \in \operatorname{At}\mathfrak{A}$.

Proof. First, show $1' \notin \operatorname{At}\mathfrak{A} \implies \mathfrak{A}$ not integral.

By hypothesis, $\exists x\ 0 < x < 1'$. Let $y = \bar{x} \cdot 1'$. Note that $y \neq 0$. Then $x\,;y = x \cdot y = x \cdot \bar{x} \cdot 1' = 0$. So $\mathfrak{A}$ is not integral.

Conversely, suppose $1' \in \operatorname{At}\mathfrak{A}$. We want $x\,;y \neq 0$ for $x \neq 0 \neq y$.

First we have $0 \neq \breve{x} = 1'\,;\breve{x} \cdot \breve{x}$. By (1.2.4), $\breve{x}\,;x \cdot 1' \neq 0$. Since $1'$ is an atom, we have $\breve{x}\,;x \geq 1'$. Therefore by monotonicity we have $1'\,;y \leq (\breve{x}\,;x)\,;y \leq (\breve{x}\,;1)\,;1 = \breve{x}\,;1$. So $y = 1'\,;y \leq \breve{x}\,;1$, and $\breve{x}\,;1 \cdot y \neq 0$. Then by (1.2.4), $x\,;y \cdot 1 \neq 0$, and $x\,;y \neq 0$. □

Theorem 1.2.11. $(1\,;x\,;1)^{\smallsmile} = 1\,;x\,;1$

Proof.

$1\,;x\,;1 \leq 1\,;x\,;(1 \cdot (1\,;x)^{\smallsmile}\,;1)$	(1.2.7)
$\leq 1\,,1\,,(\breve{x}\,;\breve{1}\,;1)$	(1.9) and mono.
$= (1\,;1)\,;(\breve{x}\,;(\breve{1}\,;1))$	(1.4)
$\leq 1\,;\breve{x}\,;1$	mono.

Then, taking $\breve{x}$ in place of x, we get $1\,;\breve{x}\,;1 \leq 1\,;\breve{\breve{x}}\,;1 = 1\,;x\,;1$. □

It is useful to refer to $1\,;x\,;1$ as the *closure* of x, especially in a proper relation algebra, where $E|R|E = \bigcup\{U_\alpha \times U_\alpha : U_\alpha$ is an equivalence class of E, $(U_\alpha \times U_\alpha) \cap R \neq \emptyset\}$.

Definition 1.2.12. Let $\mathfrak{A} \in \mathsf{RA}$, and $I \subseteq A$. I is said to be a *relational ideal* if for all x, y,

i. $y \in I,\ x \leq y\ \Rightarrow x \in I$

ii. $x, y \in I \Rightarrow x + y \in I$

iii. $x \in I \Rightarrow 1\,;x,\ x\,;1,\ \breve{x} \in I$

We will say that a relation ideal ideal I is *proper* if $1 \notin I$. It is a straightforward exercise to show that iii. above is equivalent to $x \in I \Rightarrow 1\,;x\,;1 \in I$.

Relational ideals are the kernels of homomorphisms. That is, if $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then $h^{-1}(0)$ is a relational ideal on $\mathfrak{A}$. Conversely, if I is a relational ideal on $\mathfrak{A}$, then $\Theta = \{\langle x, y\rangle : x \cdot \bar{y} + \bar{x} \cdot y \in I\}$ is a congruence; this induces a homomorphism $h : \mathfrak{A} \to \mathfrak{A}/\Theta$, $h(x) = x/\Theta$.

Definition 1.2.13. An algebra $\mathfrak{A}$ is *simple* if $\mathfrak{A}$ has at least two elements and the only homomorphisms from $\mathfrak{A}$ onto similar algebras are either injective or else are mappings from $\mathfrak{A}$ to the degenerate (1-element) algebra.

Note that if $\mathfrak{A}$ is simple and g is a homomorphism with domain $\mathfrak{A}$, then $\ker g = \{x \in A : g(x) = g(0)\}$ is either $\{0\}$ or A. Thus the only relational ideals on $\mathfrak{A}$ are $\{0\}$ and A. This next theorem provides a useful characterization of the simple relation algebras.

Theorem 1.2.14. Let $\mathfrak{A} \in \mathsf{RA}$. Then $\mathfrak{A}$ is simple iff for all $x \neq 0$, $1\,;x\,;1 = 1$.

Proof. We prove both directions by contrapositive. Suppose $\mathfrak{A}$ is not simple. Then there is a relational ideal I on A such that $\{0\} \subsetneq I \subsetneq A$. Thus there is some $x \in I$, $x \neq 0$ and $1\,;x\,;1 \in I$. But I is proper, so $1 \notin I$, and $1\,;x\,;1 < 1$.

Conversely, suppose that there is some $x \neq 0$ so that $1\,;x\,;1 < 1$. Then $I = \{z : z \leq 1\,;x\,;1\}$ is a relational ideal, and $\{0\} \subsetneq I \subsetneq A$. □

1.3 Representable relation algebras

RAs are algebraic generalizations of PRAs. It is natural to ask whether every RA is isomorphic to some PRA.

Definition 1.3.1. A relation algebra is said to be *representable* if it is isomorphic to some proper relation algebra. The class of all representable relation algebras is denoted by RRA.

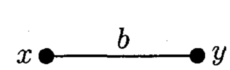
Theorem 1.3.2 (Lyndon, 1950). RRA$\neq$RA.

Proof. We exhibit a non-representable relation algebra. Lyndon found a large non-representable relation algebra. The following algebra, which is one of the smallest, is due to MacKenzie.

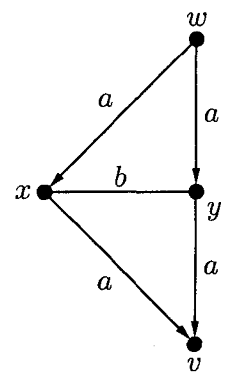
Let $\mathfrak{A}$ be an algebra with four atoms $1', a, \breve{a}, b$ ($b = \breve{b}$). The multiplication table for diversity atoms is as follows:

;	a	$\breve{a}$	b
a	a	1	$a+b$
$\breve{a}$	1	$\breve{a}$	$\breve{a}+b$
b	$a+b$	$\breve{a}+b$	$\bar{b}$

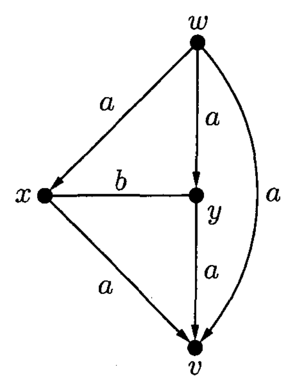
We will show that this cannot be the multiplication table for a proper relation algebra. Suppose that $1', a, \breve{a}, b$ are real relations, and that $1'$ is an identity relation. All these atoms are non-zero, so they all contain a pair. Let $\langle x, y\rangle \in b$.



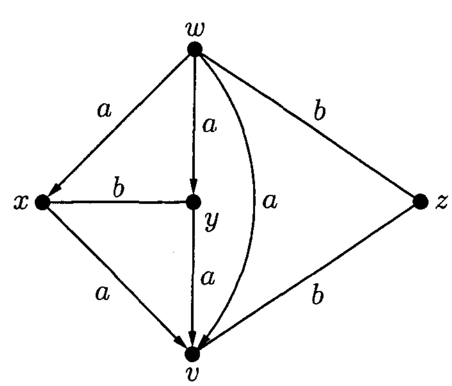
$b \leq a\,;\breve{a} = \breve{a}\,;a$, so there exist w, v so that



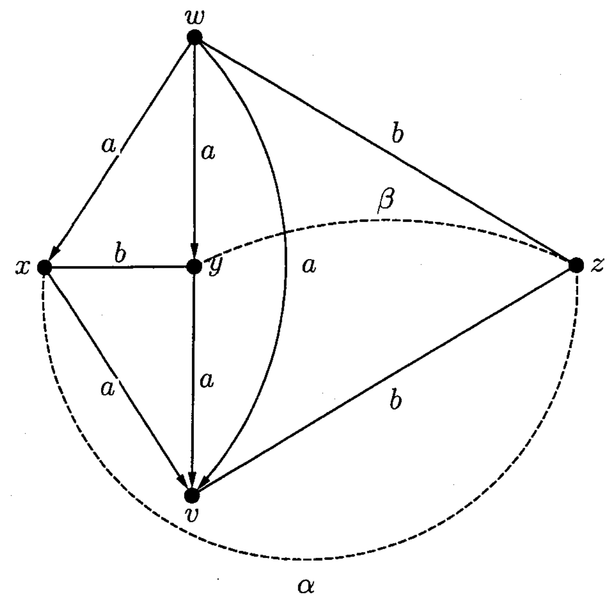
Then $\langle w, v\rangle \in a\,;a = a$, so we can draw the edge below and label it "a".



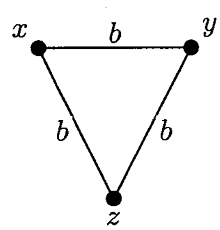
Now $a \leq b\,;b$, so there exists z which is distinct from v, w, x, y such that



Note that $z \neq v, w, x, y$ since $\langle z, x\rangle \in b\,;a \leq 0'$ and $\langle z, v\rangle, \langle z, w\rangle \in b \leq 0'$. Now since $\langle z, x\rangle \in b\,;a$ and $b\,;a$ is the join of finitely many atoms, there is an atom that contains $\langle z, x\rangle$; likewise for $\langle z, y\rangle$. Hence we have the following edges that need labels:



Now $\langle x, z\rangle$, $\langle y, z\rangle \in \breve{a}\,;b \cdot a\,;b = b$. Therefore the edges marked α, β can be labeled b. Then x, y, and z form a "monochromatic triangle":



But then $b \cdot b\,;b \neq 0$, which contradicts $b\,;b = \bar{b}$ from the multiplication table. Hence $\mathfrak{A}$ is not representable. □

Definition 1.3.3. An algebra $\mathfrak{A}$ is called *weakly representable* if there is an injective function h to a proper relation algebra such that $h(x \cdot y) = h(x) \cap h(y)$, $h(x\,;y) = h(x)|h(y)$, $h(\breve{x}) = h(x)^{-1}$, and $h(1') = \mathrm{Id}_E$.

Definition 1.3.4. If $\mathfrak{A}$ is a relation algebra and $r \in A$, then the *relativization of* $\mathfrak{A}$ *to* r is defined as follows:

$$\langle \{x \in A : x \leq r\}, +, {}^{-r}, ;^{r}, {}^{\smile r}, 1'_r \rangle$$

where $\bar{x}^r = \bar{x} \cdot r$, $x \,;^r y = (x\,;y) \cdot r$, $\breve{x}^r = \breve{x} \cdot r$, and $1'_r = 1' \cdot r$.

Definition 1.3.5. If K is a class of algebras, let $\mathbf{H}\mathsf{K}$ be the class of all homomorphic images of members of K; let $\mathbf{S}\mathsf{K}$ be the class of all subalgebras of members of K; let $\mathbf{P}\mathsf{K}$ be the class of all direct products of members of K; let $\mathbf{Up}\mathsf{K}$ denote the class of all ultraproducts of members of K.

A class K is called a *variety* if $\mathsf{K} = \mathbf{H}\mathsf{K} = \mathbf{S}\mathsf{K} = \mathbf{P}\mathsf{K}$. K is called a *quasivariety* if $\mathsf{K} = \mathbf{S}\mathsf{K} = \mathbf{P}\mathsf{K} = \mathbf{Up}\mathsf{K}$.

Definition 1.3.6. We say that an algebra $\mathfrak{A}$ has the *congruence extension property* if whenever $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and θ a congruence on $\mathfrak{B}$, then there is a congruence ψ on $\mathfrak{A}$ such that $\psi \cap (B \times B) = \theta$.

Theorem 1.3.7. If $\mathfrak{A}$ has the congruence extension property, then $\mathbf{HS}\{\mathfrak{A}\} = \mathbf{SH}\{\mathfrak{A}\}$.

We will use these definitions and facts in the two chapters to follow. In chapter 2, we will consider whether there exists a non-representable relation algebra with a weak representation on a finite set. We give examples of weak representations over finite sets, and find a method for searching for a non-representable relation algebra with a weak representation on a finite set. In chapter 3, we address the question whether the class of all weakly representable relation algebras is a variety, and get a partial result: we show that any homomorphic image of a subalgebra of a 'full' weakly representable algebra is again weakly representable. (See chapter 3 for a precise formulation of this idea.)

2 Finite weak representations

2.1 Definitions and background

The main focus of the study of relation algebras has been representability. Since not all relation algebras are representable, we might consider whether there is a weaker sense of representability which would encompass all relation algebras.

Theorem 2.1.1 (Jónsson-Tarski 1952). For every relation algebra $\mathfrak{A}$ there is some equivalence relation E and an injective function $h : \mathfrak{A} \to \langle \mathrm{Sb}(E), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_E \rangle$ that preserves $+$, $;$, $\breve{}$, 1', but not necessarily $\cdot$ or $-$. In other words, $h(x+y) = h(x) \cup h(y)$, $h(x\,;y) = h(x)|h(y)$, $h(\breve{x}) = h(x)^{-1}$, and $h(1') = \mathrm{Id}_E$.

Thus, in particular, the relational part (i.e. the operations $;$, $\breve{}$, 1') can be represented for every relation algebra. This is interesting enough, but no truly new questions arise; any questions about algebras representable in this fashion are just questions about all relation algebras. There is another weaker notion of representation, whose definition has provided more fertile ground. (We repeat here definition 1.3.3.)

Definition 2.1.2. A relation algebra $\mathfrak{A}$ is said to be *weakly representable* if there is some equivalence relation E and an injective function $h : \mathfrak{A} \longrightarrow \langle \mathrm{Sb}(E), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_E \rangle$ such that $h(x \cdot y) = h(x) \cap h(y)$, $h(x\,;y) = h(x)|h(y)$, $h(\breve{x}) = h(x)^{-1}$, $h(1') = \mathrm{Id}_E$. In this case h is called a weak representation of $\mathfrak{A}$. We will denote the class of all weakly representable relation algebras by *wRRA*. More generally, for algebras $\mathfrak{A}$ and $\mathfrak{B}$ we say that $h : A \to B$ is a *weak embedding* if it is an injective function such that $h(x \cdot y) = h(x) \cdot h(y)$, $h(x\,;y) = h(x)\,;h(y)$, $h(\breve{x}) = h(x)\breve{}$, $h(1') = 1'$, and $h(0) = 0$. In this case we write $\mathfrak{A} \xrightarrow{\mathrm{w}} \mathfrak{B}$.

Here is an easy theorem.

Theorem 2.1.3. $\mathfrak{A} \in$ *wRRA* if and only if $\mathfrak{A} \xrightarrow{\mathsf{w}} \langle Re(U), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_{U\times U}\rangle$.

Proof. The "if" direction follows from the definition of wRRA. For the "only if" direction, suppose $\mathfrak{A} \in$ wRRA. Then $\mathfrak{A} \xrightarrow{\mathsf{w}} \langle \mathrm{Sb}(E), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_E\rangle$ for some equivalence relation E. Thus it will suffice to note that

$$\langle \mathrm{Sb}(E), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_E\rangle \xrightarrow{\mathsf{w}} \langle \mathrm{Re}(U), \cup, \bar{}, \cap, |, ^{-1}, \mathrm{Id}_{U\times U}\rangle$$

for any U such that $U \times U \supseteq E$. Let $h : \mathrm{Sb}(E) \to \mathrm{Re}(U)$ be the inclusion function ($h(R) = R$). It is easy to see that h preserves all operations except complementation, which is relative to the largest relation. □

It was noted by Jónsson [6] that McKenzie's 4-atom algebra, originally offered as an example of a small non-representable algebra (indeed, there is no smaller example), is not weakly representable. (See [4] or [1] for more about this algebra.) Thus we conclude that wRRA≠RA.

The question remains whether wRRA = RRA, i.e., whether all weakly representable relation algebras are in fact representable (in the strong sense). This question was answered negatively by Andréka in [2], where she exhibited a countable collection of algebras in wRRA \ RRA. All these algebras were finite but weakly representable over infinite sets in the following sense: for every nonzero element x, $h(x)$ is a relation that contains infinitely many pairs (if h is the weak representation). Thus the underlying set is infinite. This raises the interesting question whether weak representations are possible over finite sets. We show below that they are. But first let us dispense with a terminological issue.

By a *strictly weak representation* we simply mean a weak representation that is not a representation (note that all representations are weak representations, trivially.) Such a weak representation is one that definitely fails to preserve either $+$ or $-$. Note that if $+$ fails then $-$ must fail also, since $x + y = -(\bar{x} \cdot \bar{y})$. More precisely, if $-$ were preserved in addition to $\cdot$, then $+$ would be preserved also by the previous equation.

So, to reformulate the question, is there an algebra with a strictly weak representation over a finite set? We consider some examples in the next section.

2.2 Examples of strictly weak representations

2.2.1 Boolean relation algebras

Recall that a relation algebra is called boolean if it satisfies 1' = 1. In this case we have $x\,;y = x \cdot y$ and $\breve{x} = x$, so the relational operations are uninteresting. It is easy to see that any such algebra can have a strictly weak representation.

Suppose $\mathfrak{A}$ is a boolean relation algebra. Let h be any representation of $\mathfrak{A}$. Let e be an element not appearing in any pair in $h(1)$. Define $\hat{h}(x) = h(x)$ for all $x < 1$, and let $\hat{h}(1) = h(1) \cup \{\langle e, e\rangle\}$. It is not hard to see that $\hat{h}$ preserves $\cdot$, ;, $\breve{}$, and 1', but that $\hat{h}(x + \bar{x}) \neq \hat{h}(x) \cup \hat{h}(\bar{x})$ in general.

Composition plays no real role here, since for boolean relation algebras composition is just intersection ($x\,;y = x \cdot y$). For non-boolean relation algebras it is not so trivial to build a strictly weak representation, since we cannot just "throw in" an extra pair as we did above. For instance, suppose that $x, y < 1$, and $x\,;y = 1$. If $\langle e, e\rangle \in h(1)$, then it must be the case that for some e', e'', $\langle e, e'\rangle \in h(x)$ and $\langle e'', e\rangle \in h(y)$, since composition in the target algebra is real relational composition and is preserved by h. Any pair in $h(1)$ must be comprised of individuals already present in pairs in $h(x)$ and $h(y)$. It is thus via composition that the extra pairs emerge that cause union not to be preserved.

2.2.2 Non-boolean relation algebras

For the following discussion of non-boolean relation algebras we will restrict ourselves to the finite integral symmetric case.

When one says that one has found a representation of an algebra $\mathfrak{A}$, what one has often done is to have drawn a complete graph G on n vertices for some $n \leq \omega$ whose edges are colored by the diversity atoms of $\mathfrak{A}$. As is customary, we let a graph be a set of vertices G

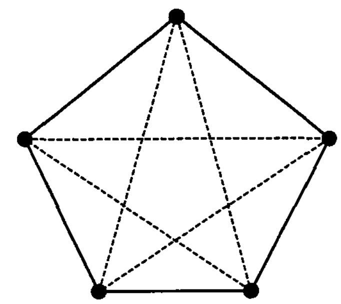
and a set of edges $E = \{\{x, y\} \subseteq G : x \neq y\}$. We color the edges of G in such a way that (1.) if $\{x, y\}$ is colored by a diversity atom a and $a \leq b\,;c$ for diversity atoms b and c, then there is some $z \in G$ such that $\{x, z\}$ is colored by b and $\{z, y\}$ is colored by c, (2.) if $\{x, y\}$ is colored by a diversity atom a and $a \not\leq b\,;c$ for diversity atoms b and c, then there is no $z \in G$ such that $\{x, z\}$ is colored by b and $\{z, y\}$ is colored by c, and (3.) for all $x \in G$ and for all diversity atoms a there is a $y_a \in G$ with $\{x, y_a\}$ colored by a.

I want to call any such colored graph an *atom graph.* An atom graph G for an algebra $\mathfrak{A}$ yields a representation in the following way. Let $R_a = \{\langle v, w\rangle \in G \times G : \{v, w\}$ is colored by $a\}$ for all diversity atoms a, and let $R_{1'} = \{\langle v, v\rangle : v \in G\}$. Then the function defined by $h(x) = \bigcup\{R_a : a \in \text{At}\,\mathfrak{A}, a \leq x\}$ is a representation of $\mathfrak{A}$. Loosely speaking, the colored graph shows how to represent the atoms of $\mathfrak{A}$; for non-atoms, take all pairs in all atoms underneath. Thus the graph "blows up" to a representation of the entire algebra.

As an example consider the algebra with atoms 1', a, b, all symmetric, whose multiplication table is

;	1'	a	b
1'	1'	a	b
a	a	$1' + b$	$a + b$
b	b	$a + b$	$1' + a$

The algebra is representable, and the following edge-colored graph is an atom graph for the algebra.



The solid edges represent the atom a, while the dotted edges represent b.

2.2.3 Strictly weak representations for small RAs

Let's examine the relation algebras with two or fewer atoms. First, we notice that the one-element algebra cannot have a *strictly* weak representation, since its image would contain only one relation (the empty relation). Neither can the two-element algebra, since its image would contain only one non-empty relation (supposing the the image of 0 is $\emptyset$).

There are three algebras with two atoms (and hence four elements). These three satisfy, respectively, 0';0'= 0, 0';0'= 1', and 0';0'= 1. We consider each in turn.

The algebra satisfying 0';0'= 0 is boolean. Thus 1' $= 1$. It has two sub-identity atoms, x and y. The multiplication table on atoms is

;	x	y
x	x	0
y	0	y

Define

$$R_0 = \emptyset$$

$$R_x = \{\langle x, x\rangle\}$$

$$R_y = \{\langle y, y\rangle\}$$

$$R_1 = \{\langle x, x\rangle, \langle y, y\rangle, \langle z, z\rangle\}$$

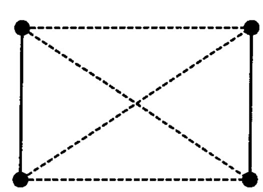
Then the map $h(a) = R_a$ is a weak representation, and not a representation. ($R_x \cup R_y \subsetneq R_1$.)

Consider the algebra satisfying 0';0'= 1'. The multiplication table on atoms is

;	1'	0'
1'	1'	0'
0'	0'	1'

A representation on a two-point set is given by

where the single edge denotes the diversity relation. Notice that the fact that 0';0' = 1' means that in the atom graph, the edges denoting the diversity relation will be a perfect matching. We can have a weak representation on a four-point set by coloring an incomplete graph with the one diversity atom:



Here again, the solid edges represent the diversity relation, while the dotted edges represent the additional edges to be included in the unit relation, besides the edges in the identity and diversity relations. In symbols, falling back on the notation of the previous example and naming the points in the above graph 0, 1, 2, and 3, clockwise from the upper left,

$$R_0 = \emptyset$$

$$R_1\text{'} = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$$

$$R_0\text{'} = \{\langle 0,3\rangle, \langle 3,0\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}$$

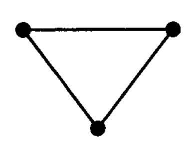
$$R_1 = 4 \times 4$$

where $4 = \{0, 1, 2, 3\}$.

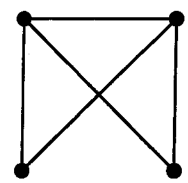
Finally, we consider the algebra satisfying 0'; 0'= 1. Its multiplication table is

;	1'	0'
1'	1'	0'
0'	0'	1' + 0'

The following graph gives a representation:



Again, the solid edges represent the diversity atom. The following graph gives a weak representation:



Notice that this is an incomplete graph; the graph is "completed" via composition. If $R_{0'}$ denotes the set of pairs connected by edges in this graph, then $R_{0'} ; R_{0'}$ is equal to 4×4, where 4 denotes the four-point set. Number the vertices zero through four, clockwise from the upper left; then it can easily be checked that for

$$R_0 = \emptyset$$

$$R_{1'} = \{\langle x, x \rangle : x \in 4\}$$

$$R_{0'} = \{\langle x, y \rangle : x, y \in 4,\ x \neq y,\ \{x, y\} \neq \{1, 2\}\}$$

$$R_1 = 4 \times 4$$

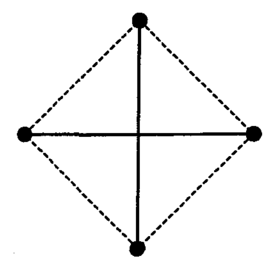
the function $h(a) = R_a$ preserves $\cdot$, $;$, $\breve{}$, and $1'$, but $h(1') \cup h(0') \subsetneq h(1)$.

2.3 A suggestive example

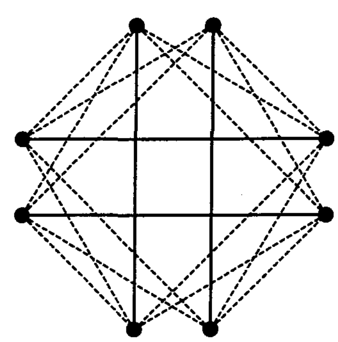
Let's consider an algebra $\mathfrak{A}$ on three atoms, $1'$, r, b ("r" for "red", "b" for "blue"). Again, all atoms are symmetric. The multiplication table is below.

;	1'	r	b
1'	1'	r	b
r	r	1'	b
b	b	b	$1' + r$

$\mathfrak{A}$ is representable. Letting the solid edges stand for red, and dashed for blue, the following yields a representation:



Now if we split each vertex, preserving red as a matching, we get

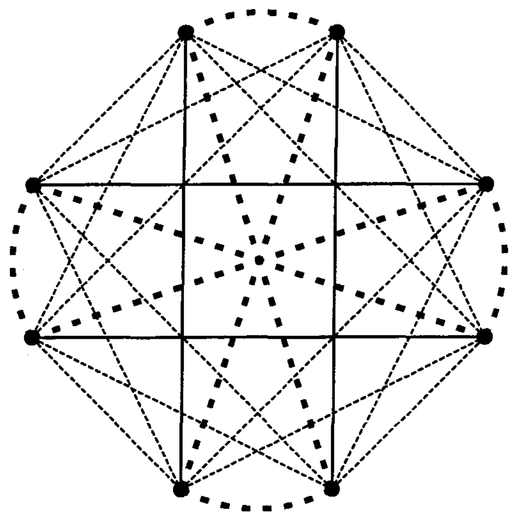


which yields a weak representation. As usual, let $R_0 = \emptyset$, let $R_{1'}$ be the identity, and let R_r and R_b be the pairs of vertices connected by red, respectively blue, edges. Let R_E be the set of missing or uncolored edges. These edges will "emerge" (hence the "E") in the composition $R_b|R_b$. While the multiplication table gives $b\,;b = 1' + r$, the graph yields $R_b|R_b = R_{1'} \cup R_r \cup R_E$. Note that the edges R_E do not appear in any atom. We define a weak representation h. Let

$$h(x) = \begin{cases} \bigcup\{R_a : a \in \operatorname{At}\mathfrak{A}, a \leq x\}, & x \not\geq b\,;b \\ \bigcup\{R_a : a \in \operatorname{At}\mathfrak{A}, a \leq x\} \cup R_E, & x \geq b\,;b \end{cases}$$

It is a somewhat tedious but straightforward matter to check that h is a weak representation. Clearly, h fails to be a representation.

What is suggestive about this example is that the edges R_E that are missing actually form an edge set that behaves like an atom. Indeed, let us add these edges to the graph, assigning them the color green (given by the bold dashed edges).



This new graph is the atom graph of a relation algebra on four atoms, representable on a finite set. Let's call this algebra $\mathfrak{A}'$. Its table is below:

;	1'	r	b	g
1'	1'	r	b	g
r	r	1'	b	g
b	b	b	1'rg	b
g	g	g	b	1'r

Here we have used juxtaposition of atoms to denote their join, for aesthetic purposes. Consider the "principal minor"

;	1'	r	b	g
1'	1'	r	b	g
r	r	1'	b	g
b	b	b	1'rg	b
g	g	g	b	1'r

If we delete the "g" row and column from this table, and delete all remaining appearances of g, we get the table for $\mathfrak{A}$. Thus $\mathfrak{A}$ is equal to $\mathfrak{A}'$ relativized to $-g$. This suggests a method of searching for a non-representable algebra with a strictly weak representation on a finite set: look for non-representable relativizations of algebras representable on a finite set.

Now, not all such relativizations are strictly weakly representable algebras. So now we must look for necessary and sufficient conditions for the relativization to yield a strictly weakly representable algebra.

2.4 Weak embeddings

In this section we take advantage of the fact that for $\mathfrak{B} \in \mathsf{RRA}$, it is sufficient for $\mathfrak{A}$ to be in wRRA that $\mathfrak{A} \xrightarrow{\mathsf{w}} \mathfrak{B}$. We weakly embed some 3-atom algebras into 4-atom representable algebras, and give representations of the 4-atom algebras, which in turn yield weak representations of the 3-atom algebras.

First, we note that the algebra with atoms 1', r, and b from the previous section weakly embeds into the larger algebra with atoms 1', r, b, g via

$$1' \longmapsto 1'$$

$$r \longmapsto r$$

$$b \longmapsto b$$

$$1'r \longmapsto 1'rg$$

$$1\text{'}b \longmapsto 1\text{'}b$$

$$rb \longmapsto rb$$

$$1\text{'}rb \longmapsto 1\text{'}rbg$$

Example 1

Now let $\mathfrak{A}_1$ have 3 atoms 1', a, b, all symmetric, with composition given by

;	1'	a	b
1'	1'	a	b
a	a	1'a	b
b	b	b	1'a

Let $\mathfrak{B}_1$ have 4 atoms 1', a, b, c, all symmetric, with composition given by

;	1'	a	b	c
1'	1'	a	b	c
a	a	1'ac	b	a
b	b	b	1'ac	b
c	c	a	b	1'

Notice that $\mathfrak{A}_1$ is a relativization of $\mathfrak{B}_1$ to $-c$: by deleting the row and column for c in the table for $\mathfrak{B}_1$ along with all other appearances of c, we get the table for $\mathfrak{A}_1$.

$\mathfrak{B}_1$ is representable on a 10-point set. Let $10 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We define a function ρ_1 on the atoms of $\mathfrak{B}_1$ as follows:

$$1\text{'} \longmapsto \{\langle n, n\rangle : n \in 10\}$$

$$a \longmapsto \{\langle n, m\rangle : |n - m| \in \{2, 4, 6, 8\}\}$$

$$b \longmapsto \{\langle n, m\rangle : |n - m| \in \{1, 3, 7, 9\}\}$$

$$c \longmapsto \{\langle n, m\rangle : |n - m| \in \{5\}\}$$

Then ρ_1 extends to a representation of $\mathfrak{B}_1$. Also $\mathfrak{A}_1$ weakly embeds into $\mathfrak{B}_1$ via φ_1 given by

$$1' \longmapsto 1'$$

$$a \longmapsto a$$

$$b \longmapsto b$$

$$1'a \longmapsto 1'ac$$

$$1'b \longmapsto 1'b$$

$$ab \longmapsto ab$$

$$1'ab \longmapsto 1'abc$$

Then $\rho_1 \circ \varphi_1$ is a weak representation of $\mathfrak{A}_1$.

Example 2

Now let $\mathfrak{A}_2$ have 3 atoms 1', a, b, all symmetric, with composition given by

;	1'	a	b
1'	1'	a	b
a	a	$1'a$	b
b	b	b	$1'ab$

Let $\mathfrak{B}_2$ have 4 atoms 1', a, b, c, all symmetric, with composition given by

;	1'	a	b	c
1'	1'	a	b	c
a	a	$1'a$	b	ac
b	b	b	$1'abc$	b
c	c	ac	b	$1'a$

Again, notice that $\mathfrak{A}_1$ is a relativization of $\mathfrak{B}_1$ to $-c$.

$\mathfrak{B}_2$ is representable on a 12-point set. Let $12 = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. We define a function ρ_2 on the atoms of $\mathfrak{B}_2$ as follows:

$$1' \longmapsto \{\langle n,n\rangle : n \in 12\}$$

$$a \longmapsto \{\langle n,m\rangle : |n-m| \in \{6\}\}$$

$$b \longmapsto \{\langle n,m\rangle : |n-m| \in \{1,4,5,8,9,11\}\}$$

$$c \longmapsto \{\langle n,m\rangle : |n-m| \in \{2,3,7,10\}\}$$

Then ρ_2 extends to a representation of $\mathfrak{B}_2$. Also $\mathfrak{A}$ weakly embeds into $\mathfrak{B}_2$ via φ_2 given by

$$1' \longmapsto 1'$$

$$a \longmapsto a$$

$$b \longmapsto b$$

$$1'a \longmapsto 1'ac$$

$$1'b \longmapsto 1'b$$

$$ab \longmapsto ab$$

$$1'ab \longmapsto 1'abc$$

Then $\rho_2 \circ \varphi_2$ is a weak representation of $\mathfrak{A}_2$.

Example 3

Now let $\mathfrak{A}_3$ have 3 atoms 1', a, b, all symmetric, with composition given by

;	1'	a	b
1'	1'	a	b
a	a	1'ab	ab
b	b	ab	1'ab

Let $\mathfrak{B}_3$ have 4 atoms 1', a, b, c, all symmetric, with composition given by

;	1'	a	b	c
1'	1'	a	b	c
a	a	1'abc	abc	abc
b	b	abc	1'abc	abc
c	c	abc	abc	1'abc

Again, notice that $\mathfrak{A}_1$ is a relativization of $\mathfrak{B}_1$ to $-c$.

$\mathfrak{B}_3$ is representable on a 19-point set. Let $19 = \{0, 1, 2, 3, \ldots, 18\}$. We define a function ρ_3 on the atoms of $\mathfrak{B}_3$ as follows:

$$1' \longmapsto \{\langle n, n\rangle : n \in 19\}$$

$$a \longmapsto \{\langle n, m\rangle : |n - m| \in \{1, 7, 8, 11, 12, 18\}\}$$

$$b \longmapsto \{\langle n, m\rangle : |n - m| \in \{2, 3, 5, 14, 16, 17\}\}$$

$$c \longmapsto \{\langle n, m\rangle : |n - m| \in \{4, 6, 9, 10, 13, 15\}\}$$

Then ρ_3 extends to a representation of $\mathfrak{B}_3$. Also $\mathfrak{A}_3$ weakly embeds into $\mathfrak{B}_3$ via φ_3 given by

$$1' \longmapsto 1'$$

$$a \longmapsto a$$

$$b \longmapsto b$$

$$1'a \longmapsto 1'a$$

$$1'b \longmapsto 1'b$$

$$ab \longmapsto abc$$

$$1'ab \longmapsto 1'abc$$

Then $\rho_3 \circ \varphi_3$ is a weak representation of $\mathfrak{A}_3$.

3 On the variety question for wRRA

3.1 Introduction

Recall that a relation algebra $\mathfrak{A}$ is said to be *representable* if it is isomorphic to a proper relation algebra; it is said to be *weakly representable* if it has a representation respecting all the operations except $+$ and $-$. Let wRRA denote the class of weakly representable relation algebras. In his 1959 paper [6], Jónsson proved that wRRA is a quasi-variety, and hence is closed under the class operators **S** and **P**, by establishing an infinite set (Γ) of quasi-identities that define wRRA over RA. At the end of the paper he asked whether (Γ) could be replaced by a finite set of formulas, or by identities. The first question was answered negatively: wRRA is not finitely axiomatizable (see [5]), nor is RRA finitely axiomatizable over wRRA (see [2]).

The second question—whether (Γ) can be replaced by identities—turned out to be ambiguous. In the 1959 paper [6] Jónsson referred to algebras of the form $\mathfrak{A} = \langle A, \cap, |, ^{-1}, I\rangle$, where A is a set of binary relations and I acts as an identity for $|$, as *algebras of relations.* Let $\mathcal{R}$ be the class of all such algebras of relations. Jónsson proved that (Γ) along with a few identities axiomatized $\mathcal{R}$. But he also showed that (Γ) axiomatized wRRA over RA. So there are really two questions: Can $\mathcal{R}$ be axiomatized by equations, and can wRRA?

In their 1995 paper [3] Andréka and Bredhikin published an answer to Jónsson's question: No. They were referring to the first interpretation of the question, and showed that $\mathcal{R}$ was not closed under **H**. Here we consider the question under the second interpretation.

3.2 Point relations

Definition 3.2.1. Let E be a non-empty equivalence relation. We define the *points of* E,

$$\mathrm{Pt}_E := \{p \subseteq E : E|p|E = E,\ p|E|p \subseteq \mathrm{Id}_E\}$$

where $\mathrm{Id}_E = \{\langle x, x\rangle \in E\}$.

The collection of all ordered pairs of points for an equivalence relation E will be a useful tool for proving the main result. We first need to establish the following facts about points. Item i. gives a characterization of the point relations.

Lemma 3.2.2. The following hold for any non-empty equivalence relation E.

(i.) $p \in \mathrm{Pt}_E$ iff for all equivalence classes U of E, $\exists u \in U$ so that $p \cap U^2 = \{\langle u, u\rangle\}$.

(ii.) $p \in \mathrm{Pt}_E \Rightarrow p = p^{-1} \subseteq \mathrm{Id}_E$.

(iii.) $R, S \subseteq E$, $p, q \in \mathrm{Pt}_E$; then

- (a) $E|p|R|q|E \cap E|p|S|q|E = E|p|(R \cap S)|q|E$
- (b) $E|R|p|E \cap E|p|S|E = E|R|p|S|E$
- (c) $E|p|R|q|E = E|q|R^{-1}|p|E$

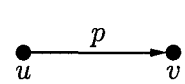
(iv.) $\forall R \subseteq E\ \exists p, q \in \mathrm{Pt}_E\ E|R|E = E|p|R|q|E$

(v.) $\forall R, S \subseteq E\ \exists p \in \mathrm{Pt}_E\ E|R|S|E = E|R|p|S|E$

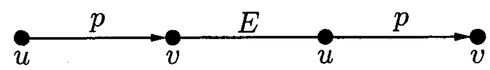
Proof. For this proof we will abbreviate "$\langle u, v\rangle \in p$" by "$upv$". So the string "$upvEupv$" indicates that

$$\langle u, v\rangle \in p,\ \langle v, u\rangle \in E,\ \langle u, v\rangle \in p$$

. We will also use pictures: "$\langle u, v\rangle \in p$" will be represented by



Similarly, "$upvEupv$" will be represented by



This will reduce somewhat the number of graphs that need to be drawn.

(i.) ($\Rightarrow$): Assume $p \in \mathrm{Pt}_E$. Let U be a (nonempty) equivalence class of E. $E|p|E = E$ implies that $p \cap U^2$ is nonempty. So choose $\langle u, v\rangle \in p \cap U^2$. Then $upvEupv$ since $\langle u, v\rangle \in p$ and uEv. But $p|E|p \subseteq \mathrm{Id}_E$ by hypothesis, so $\langle u, v\rangle \in \mathrm{Id}$ and $u = v$. Therefore $p \cap U^2 \subseteq \mathrm{Id}_E$.

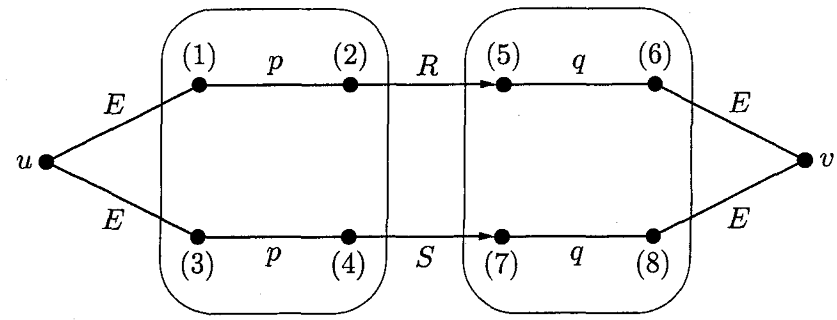
Now suppose that $\langle u, u\rangle$, $\langle v, v\rangle \in p \cap U^2$. Then $upuEvpv$, so $u(p|E|p)v$, but $p|E|p \subseteq \mathrm{Id}$, so $u = v$.

($\Leftarrow$): Suppose for all equivalence classes U, $\exists u\ p \cap U^2 = \{\langle u, u\rangle\}$. Show $E|p|E = E$: The inclusion $\subsetneq$ always holds. To show $\supseteq$, let $\langle x, y\rangle \in E$. Then there is some equivalence class U so that $x, y \in U$. We also have $p \cap U^2 = \{\langle u, u\rangle\}$. Then $xEupuEy$, and so $\langle x, y\rangle \in E|p|E$.

Show $p|E|p \subseteq \mathrm{Id}_E$: Let $\langle x, y\rangle \in p|E|p$. Then $\langle x, x\rangle, \langle y, y\rangle \in p$ and $xpxEypy$. But since xEy, x and y are in the same equivalence class U. So then $\langle x, y\rangle \in p \cap U^2$, which implies $x = y$. Therefore $p|E|p \subseteq \mathrm{Id}_E$.

(ii.) follows from i.

(iii.) (a) We want $E|p|R|q|E \cap E|p|S|q|E = E|p|(R \cap S)|q|E$. So let $\langle u, v\rangle \in E|p|R|q|E \cap E|p|S|q|E$. Then there exists points (1)–(8) such that



(Edges labeled p are now drawn undirected in light of ii.)

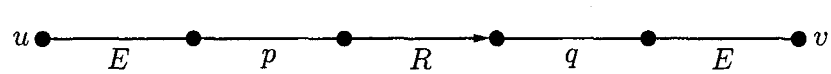
By i., (1),(2),(3),(4) are all the same point. Likewise, (5),(6),(7),(8) are all the same point. So then $\langle u, v\rangle \in E|p|(R \cap S)|q|E$.

The proof of $\supseteq$ is trivial by $|$-monotonicity.

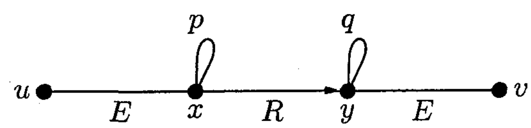
(b) The proof of $E|R|p|E \cap E|p|S|E = E|R|p|S|E$ is similar to the previous.

(c) $E|p|R|q|E = E|q|R^{-1}|p|E$:

Let $\langle u, v\rangle \in E|p|R|q|E$:



This gives



Now uEy and vEx, so we can write $uEyqyR^{-1}xpxEv$, and $\langle u,v\rangle \in E|q|R^{-1}|p|E$. The inclusion $\supseteq$ is similar.

So then $E|p|R|q|E = E|q|R^{-1}|p|E$.

(iv.) Let $\{U_\alpha\}_{\alpha\in I}$ be the equivalence classes of E. Let $R \subseteq E$. Let $R_\alpha = R \cap U_\alpha^2$. For all non-empty R_α, pick $\langle u_\alpha, v_\alpha\rangle \in R_\alpha$. For R_α empty, let $\langle u_\alpha, v_\alpha\rangle \in U_\alpha$. Let $p := \{\langle u_\alpha, u_\alpha\rangle : \alpha \in I\}$ and $q := \{\langle v_\alpha, v_\alpha\rangle : \alpha \in I\}$. Then $E|R|E = E|p|R|q|E$.

(v.) Let $R, S \subseteq E$. Let $\{U_\alpha\}_{\alpha\in I}$ be as above. When $(R|S) \cap U_\alpha^2 \neq \emptyset$, pick $\langle x,y\rangle \in (R|S) \cap U_\alpha^2$. For every such alpha, $\exists u_\alpha \in U_\alpha$, $xRu_\alpha Sy$. Let $p := \{\langle u_\alpha, u_\alpha\rangle : \alpha \in I\}$. Then $E|R|S|E = E|R|p|S|E$.

□

3.3 The main result

We will prove in this section that any homomorphic image of a 'full' weakly representable algebra (all subrelations of an equivalence relation E) has a subdirect representation into simple algebras, each of which is weakly representable. The proof given here is adapted from a proof that RRA is a variety, a proof which appears in Maddux's book [7].

We begin with some notational matters. As is usual, we let $\mathrm{Sb}(E)$ denote the collection of all subsets of E, and $\mathrm{Re}(U)$ denote $\mathrm{Sb}(U \times U)$, the set of all binary relations over U.

Definition 3.3.1. Let $\mathfrak{Sb}_{\oplus,\sim}(E) = \langle \mathrm{Sb}(E), \oplus, \sim, \cap, \emptyset, |, ^{-1}, \mathrm{Id}_E \rangle$ be a relation algebra where E is an equivalence relation; $\cap$, $|$, and $^{-1}$ are the usual set-theoretic operations; and $\oplus$ and $\sim$ are boolean plus and minus operations, but not necessarily set union and complement. For aesthetic purposes we will hereafter write $\mathfrak{Sb}(E)$ instead of $\mathfrak{Sb}_{\oplus,\sim}(E)$, recalling that $\mathfrak{Sb}(E)$ depends not only on E but on $\oplus$ and $\sim$.

Let $\mathfrak{Re}(U) = \langle \mathrm{Re}(U), \oplus, \sim, \cap, \emptyset, |, ^{-1}, \mathrm{Id}_U \rangle$, where again $\cap$, $|$, and $^{-1}$ are the usual set-theoretic operations; and $\oplus$ and $\sim$ are boolean plus and minus operations. Note that we have written Id_U rather than $\mathrm{Id}_{U\times U}$ for brevity. We trust that this will not cause confusion.

Let $\mathfrak{Re}^*(U) = \langle \mathrm{Re}(U), \cap, \emptyset, |, ^{-1}, \mathrm{Id}_U \rangle$. Here, all the operations are the usual set-theoretic ones. This algebra is not similar to relation algebras; the star is meant to suggest this.

For relation algebras $\mathfrak{A}, \mathfrak{B}$, etc, we will denote their respective domains by A, B, etc.

All of these different algebras will play a role in the proof. In order to be able to discuss interactions between algebras that are not similar, we need the following definition.

Definition 3.3.2. Let U be a set. A *heteromorphism* h is a function from a relation algebra $\mathfrak{A}$ to an algebra of the form $\mathfrak{Re}^*(U)$ that preserves the common operations and 'forgets' the others, i.e., for all $x, y \in A$, h satisfies

$$h(x \cdot y) = h(x) \cap h(y) \tag{3.1}$$

$$h(0) = \emptyset \tag{3.2}$$

$$h(x\,;y) = h(x)|h(y) \tag{3.3}$$

$$h(\breve{x}) = h(x)^{-1} \tag{3.4}$$

$$h(1') = \mathrm{Id}_U \tag{3.5}$$

If, instead of (3.5), h satisfies

$$h(1') \supseteq \mathrm{Id}_U \tag{3.6}$$

then we will call h a *near-heteromorphism.*

The following lemma establishes a sufficient condition for weak representability.

Lemma 3.3.3. Let $\mathfrak{A} \in \mathsf{RA}$, and let U be a set. Let $g : \mathfrak{A} \longrightarrow \mathfrak{Re}^*(U)$ be an injective near-heteromorphism, i.e., g satisfies (1)-(4) and (6) of the previous definition. Then $g(1')$ is an equivalence relation and $h : \mathfrak{A} \longrightarrow \mathfrak{Re}(U/g(1'))$ given by

$$h(a) = \{\langle r/g(1'), s/g(1')\rangle : \langle r, s\rangle \in g(a)\}$$

is an injective homomorphism.

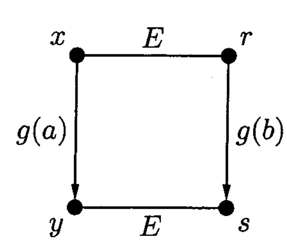
Proof. Let us abbreviate the equivalence relation $g(1')$ by E.

We define the function $h : A \longrightarrow \mathrm{Re}(U/E)$ by

$$h(a) = \{\langle r/E, s/E\rangle : \langle r, s\rangle \in g(a)\}$$

First we will show that h is 1-1 if g is.

Let $a, b \in A$, $a \neq b$. Then $g(a) \neq g(b)$. Suppose without loss of generality that $g(a) \setminus g(b) \neq \emptyset$. We want to show that $h(a) \neq h(b)$. It will suffice to show that if $\langle x, y\rangle \in g(a) \setminus g(b)$, then $\langle x/E, y/E\rangle \in h(a) \setminus h(b)$. So let $\langle x, y\rangle \in g(a) \setminus g(b)$. We want $\langle x/E, y/E\rangle$ to be distinct from all $\langle r/E, s/E\rangle$, where $\langle r, s\rangle \in g(b)$. Suppose by way of contradiction that there is some $\langle r, s\rangle \in g(b)$ so that $\langle x/E, y/E\rangle \doteq \langle r/E, s/E\rangle$. Then xEr and yEs. So we have



So then

$$\begin{aligned} \langle x, y\rangle \in E|g(b)|E &= g(1')|g(b)|g(1') && E = g(1') \\ &= g(1' ; b ; 1') && g \text{ a hom.} \end{aligned}$$

$$= g(b)$$

This stands in contradiction to the assumption that $\langle x, y\rangle \in g(a) \setminus g(b)$. Therefore $\langle x/E, y/E\rangle \in h(a) \setminus h(b)$, and $h(a) \neq h(b)$, and so h is 1-1 also.

We define the binary operation $\oplus$ on the set $h(A) \subseteq \mathrm{Re}(U/E)$ by

$$x \oplus y = h(h^{-1}(x) + h^{-1}(y))$$

where $+$ is the operation in $\mathfrak{A}$. Also define the unary operation $\sim$ by

$$\sim x = h(-h^{-1}(x))$$

where $-$ is the operation in $\mathfrak{A}$. Letting the set $h(A) \subseteq \mathrm{Re}(U/E)$ also have the operations of intersection, relative multiplication, and conversion, we form the algebra $\mathfrak{Re}(U/E) = \langle \mathrm{Re}(U/E), \oplus, \sim, \cap, |, ^{-1}, \mathrm{Id}_{U/E}\rangle$. This algebra is similar to relation algebras.

We now want to show that h is a homomorphism. The function h respects $+$ and $\sim$ by design. We check $h(a \cdot b) = h(a) \cap h(b)$:

$$\begin{aligned}
\langle r/E, s/E\rangle \in h(a \cdot b) &\iff \langle r, s\rangle \in g(a \cdot b) = g(a) \cap g(b) \\
&\iff \langle r, s\rangle \in g(a) \text{ AND } \langle r, s\rangle \in g(b) \\
&\iff \langle r/E, s/E\rangle \in h(a) \text{ AND } \langle r/E, s/E\rangle \in h(b) \\
&\iff \langle r/E, s/E\rangle \in h(a) \cap h(b)
\end{aligned}$$

We check $h(a\,;b) = h(a)|h(b)$:

$$\begin{aligned}
\langle r/E, s/E\rangle \in h(a\,;b) &\iff \langle r, s\rangle \in g(a\,;b) = g(a)|g(b) \\
&\iff \exists t\ \langle r, t\rangle \in g(a) \text{ AND } \langle t, s\rangle \in g(b)
\end{aligned}$$

$$\iff \exists t\ \langle r/E, t/E\rangle \in h(a) \text{ AND } \langle t/E, s/E\rangle \in h(b)$$
$$\iff \langle r/E, s/E\rangle \in h(a)|h(b)$$

We check $h(\breve{a}) = h(a)^{-1}$:

$$\begin{aligned}\langle r/E, s/E\rangle \in h(\breve{a}) &\iff \langle r, s\rangle \in g(\breve{a}) = g(a)^{-1}\\ &\iff \langle s, r\rangle \in g(a)\\ &\iff \langle s/E, r/E\rangle \in h(a)\\ &\iff \langle r/E, s/E\rangle \in h(a)^{-1}\end{aligned}$$

We check $h(1') = \mathrm{Id}_{U/E}$:

$$\begin{aligned}\langle r/E, s/E\rangle \in h(1') &\iff \langle r, s\rangle \in g(1') = E, \text{ (an equivalence relation)}\\ &\iff rEs\\ &\iff r/E = s/E\\ &\iff \langle r/E, s/E\rangle \in \mathsf{Id} \cap (U/E)^2\end{aligned}$$

Thus $h : \mathfrak{A} \longrightarrow \mathfrak{Re}(U)$ is a homomorphism. Since RA is equational, $h(\mathfrak{A}) \in$ RA. Since the operations $\cap$, $|$, $^{-1}$ on $h(A)$ are the usual set-theoretic operations, $h(\mathfrak{A}) \in$ wRRA. □

Lemma 3.3.4. Let $h : \mathfrak{Sb}(E) \twoheadrightarrow \mathfrak{B}$ be a surjective homomorphism; $\mathfrak{B}$ is a non-degenerate algebra of the type of RAs. Define $\sigma : \mathfrak{Sb}(E) \longrightarrow \mathfrak{Re}^*(\mathrm{Pt}_E)$ by

$$\sigma(R) = \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : h(E) = h(E|p|R|q|E)\}$$

Then σ is a near-hetermorphism, i.e.

(i.) $\sigma(\emptyset) = \emptyset$

(ii.) $\sigma(E) = \mathrm{Pt}_E \times \mathrm{Pt}_E$

(iii.) $\sigma(R \cap S) = \sigma(R) \cap \sigma(S)$

(iv.) $\sigma(R|S) = \sigma(R)|\sigma(S)$

(v.) $\sigma(R^{-1}) = \sigma(R)^{-1}$

(vi.) $\sigma(\mathrm{Id}_E) \supseteq \mathrm{Id}_{\mathrm{Pt}_E}$

Proof. (i.) $\sigma(\emptyset) = \{\langle p, q\rangle : h(E) = h(E|p|\emptyset|q|E\} = \emptyset$, since $h(E) \neq h(\emptyset)$.

(ii.) $E = E|p|E|q|E$ for all p, q, so $h(E) = h(E|p|E|q|E)$, and ii. holds.

(iii.) $\sigma(R \cap S) = \sigma(R) \cap \sigma(S)$:

Let $p, q \in \mathrm{Pt}_E$. Then

$$\begin{aligned}
\langle p, q\rangle \in \sigma(R \cap S) &\iff h(E) = h(E|p|(R \cap S)|q|E) && \text{def.}\\
&\iff h(E) = h(E|p|R|q|E) \cdot h(E|p|S|q|E) && \text{lemma 3.2.2, } h \text{ a hom}\\
&\iff h(E) = h(E|p|R|q|E)\\
&\qquad \text{AND } h(E) = h(E|p|S|q|E) && h(E) \text{ is the bool. } 1\\
&\iff \langle p, q\rangle \in \sigma(R) \cap \sigma(S)
\end{aligned}$$

Therefore $\sigma(R \cap S) = \sigma(R) \cap \sigma(S)$.

(iv.) $\sigma(R|S) = \sigma(R)|\sigma(S)$:

Let $\langle p, q\rangle \in \sigma(R|S)$. So $h(E) = h(E|p|R|S|q|E)$. By lemma 3.2.2, $\exists\, r \in \mathrm{Pt}_E\ E|p|R|S|q|E = E|p|R|r|S|q|E$. Therefore $E|p|R|S|q|E = E|p|R|r|S|q|E \subseteq E|p|R|r|E \subseteq E$ (since $S|q|E \subseteq E$), and $E|p|R|S|q|E = E|p|R|r|S|q|E \subseteq E|r|S|q|E \subseteq E$ (since $E|p|R \subseteq E$). So then

$$\begin{aligned} h(E) &= h(E|p|R|S|q|E) && \\ &= h(E|p|R|S|q|E \cap E|p|R|r|E) && \text{lemma 3.2.2} \\ &= h(E|p|R|S|q|E) \cdot h(E|p|R|r|E) && h \text{ a hom.} \\ &= h(E) \cap h(E|p|R|r|E) && \text{hyp.} \\ &= h(E|p|R|r|E) && h \text{ a hom.} \end{aligned}$$

And so $\langle p, r\rangle \in \sigma(R)$. Similarly, $\langle r, q\rangle \in \sigma(S)$. So $\langle p, q\rangle \in \sigma(R)|\sigma(S)$.

Conversely, let $\langle p, q\rangle \in \sigma(R)|\sigma(S)$. Then $\exists\, r \in \mathrm{Pt}_E\ \langle p, r\rangle \in \sigma(R),\ \langle r, q\rangle \in \sigma(S)$. So $h(E|p|R|r|E) = h(E) = h(E|r|S|q|E)$. Then

$$\begin{aligned} h(E) &= h(E|E) && \\ &= h(E)\,;h(E) && h \text{ a hom.} \\ &= h(E|p|R|r|E)\,;h(E|r|S|q|E) && \text{hyp.} \\ &= h(E|p|R|r|E|E|r|S|q|E) && h \text{ a hom.} \\ &= h(E|p|R|r|S|q|E) && (r|E|E|r = r|E|r = r,\ r \in \mathrm{Pt}_E) \\ &= h(E|p|R|S|q|E) && \text{hyp.} \end{aligned}$$

Hence $\langle p, q\rangle \in \sigma(R|S)$.

(v.) $\sigma(R^{-1}) = \sigma(R)^{-1}$:

$$\begin{aligned}\langle q, p\rangle \in \sigma(R^{-1}) &\iff h(E) = h(E|q|R^{-1}|p|E) = h(E|p|R|q|E)\\ &\iff \langle p, q\rangle \in \sigma(R)\\ &\iff \langle q, p\rangle \in \sigma(R)^{-1}\end{aligned}$$

(vi.) $\sigma(\mathrm{Id}_E) \supseteq \mathrm{Id}_{\mathrm{Pt}_E}$:

If $\langle p, p\rangle \in \mathrm{Id}_{\mathrm{Pt}_E}$, then $h(E) = h(E|p|\mathrm{Id}_E|p|E)$ since $E = E|p|E = E|p|\mathrm{Id}_E|E = E|p|\mathrm{Id}_E|(E|p|E) = E|p|\mathrm{Id}_E|p|E$. So then $\langle p, p\rangle \in \sigma(\mathrm{Id}_E)$.

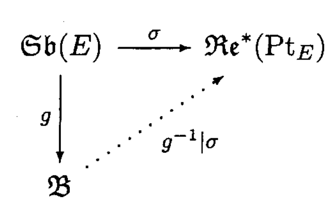
Thus σ is a near-heteromorphism. □

Lemma 3.3.5. Let $g : \mathfrak{Sb}(E) \twoheadrightarrow \mathfrak{B}$ be a surjective homomorphism with maximal kernel; $\mathfrak{B}$ is a non-degenerate algebra of the type of RAs. Then $\mathfrak{B} \in$ wRRA.

Proof. Let $\sigma : \mathfrak{Sb}(E) \longrightarrow \mathfrak{Re}^*(\mathrm{Pt}_E)$ be given by

$$\sigma(R) = \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|R|q|E)\}$$

Consider $g^{-1}|\sigma \subseteq B \times \mathrm{Sb}(\mathrm{Pt}_E \times \mathrm{Pt}_E)$.



It is easy to see that $g^{-1}|\sigma$ is functional, since $\sigma(R) = \sigma(S)$ whenever $g(R) = g(S)$. We want to show that $g^{-1}|\sigma$ is a near-heteromorphism.

$g^{-1}|\sigma(a \cdot b) = g^{-1}|\sigma(a) \cap g^{-1}|\sigma(b)$:

We need to show that for $R \in g^{-1}(a \cdot b),\ S \in g^{-1}(a),\ T \in g^{-1}(b)$, $\sigma(R) = \sigma(S) \cap \sigma(T)$.

$g(E\|p\|R\|q\|E) = g(E\|p)\,;g(R)\,;g(q\|E)$	g a hom.
$= g(E\|p)\,;(a \cdot b)\,;g(q\|E)$	$g(R) = a \cdot b$
$= g(E\|p)\,;[g(S) \cdot g(T)]\,;g(q\|E)$	
$= g(E\|p)\,;g(S \cap T)\,;g(q\|E)$	g a hom.
$= g(E\|p\|(S \cap T)\|q\|E)$	g a hom.
$= g(E\|p\|S\|q\|E \cap E\|p\|T\|q\|E)$	lemma 3.2.2
$= g(E\|p\|S\|q\|E) \cdot g(E\|p\|T\|q\|E)$	g a hom.

Therefore

$$
\begin{aligned}
\sigma(R) &= \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|R|q|E)\} \\
&= \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|S|q|E) \cdot g(E|p|T|q|E)\} \\
&= \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|S|q|E) \text{ AND } g(E) = g(E|p|T|q|E)\} \\
&= \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|S|q|E)\} \\
&\quad \cap \{\langle p, q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|T|q|E)\} \\
&= \sigma(S) \cap \sigma(T)
\end{aligned}
$$

Now we show $g^{-1}|\sigma(a\,;b) = (g^{-1}|\sigma(a))\big|(g^{-1}|\sigma(b))$:

Let $R \in g^{-1}(a\,;b),\ S \in g^{-1}(a),\ T \in g^{-1}(b)$. We want to show that $\sigma(R) = \sigma(S)|\sigma(T)$.

$$
\begin{aligned}
g(E|p|R|q|E) &= g(E|p)\,;g(R)\,;g(q|E) && g \text{ a hom.}\\
&= g(E|p)\,;(a\,;b)\,;g(q|E) &&\\
&= g(E|p)\,;[g(S)\,;g(T)]\,;g(q|E) &&\\
&= g(E|p)\,;g(S|T)\,;g(q|E) && g \text{ a hom.}\\
&= g(E|p|(S|T)|q|E) && g \text{ a hom.}\\
&= g(E|p|S|r|T|q|E) && \text{for some } r \in \mathrm{Pt}_E, \text{ by lem. 3.2.2}\\
&= g(E|p|S|r|E \cap E|r|T|q|E) && \text{lemma 3.2.2}\\
&= g(E|p|S|r|E) \cdot g(E|r|T|q|E) && g \text{ a hom.}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sigma(R) &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|R|q|E)\}\\
&= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : \exists r \in \mathrm{Pt}_E,\ g(E) = g(E|p|S|r|E) \cdot g(E|r|T|q|E)\}\\
&= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : \exists r \in \mathrm{Pt}_E,\ g(E) = g(E|p|S|r|E) \text{ AND } g(E) = g(E|r|T|q|E)\}\\
&= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : \exists r \in \mathrm{Pt}_E,\ \langle p,r\rangle \in \sigma(S) \text{ AND } \langle r,q\rangle \in \sigma(T)\}\\
&= \sigma(S)|\sigma(T)
\end{aligned}
$$

Now we show $g^{-1}|\sigma(\breve{a}) = [g^{-1}|\sigma(a)]^{-1}$:

Let $R \in g^{-1}(\breve{a}),\ S \in g^{-1}(a)$. We want to show $\sigma(R) = \sigma(S)^{-1}$.

Then

$$
\begin{aligned}
g(E|p|R|q|E) &= g(E|p)\,;g(R)\,;g(q|E) && g \text{ a hom.}\\
&= g(E|p)\,;\breve{a}\,;g(q|E)\\
&= g(E|p)\,;g(S)^{\smile}\,;g(q|E)\\
&= g(E|p)\,;g(S^{-1})\,;g(q|E) && g \text{ a hom.}\\
&= g(E|p|S^{-1}|q|E) && g \text{ a hom.}\\
&= g(E|q|S|p|E) && \text{by lemma 3.2.2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sigma(R) &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|R|q|E)\}\\
&= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|q|S|p|E)\}\\
&= \{\langle q,p\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|S|q|E)\}\\
&= \sigma(S)^{-1}
\end{aligned}
$$

Now we show $g^{-1}|\sigma(0) = \emptyset$:

Let $R \in g^{-1}(0)$. Then $g(R) = 0$, so there is no pair of point relations satisfying $g(E) = g(E|p|R|q|E)$. Hence $\sigma(R) = \emptyset$.

Now we show $g^{-1}|\sigma(1') \supseteq \mathrm{Id}_{\mathrm{Pt}_E}$:

Let $\langle p,p\rangle \in \mathrm{Id}_{\mathrm{Pt}_E}$. Let $S \in g^{-1}(1')$.

Then

$$\begin{aligned}\sigma(S) &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|S|q|E)\}\\ &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p)\,;g(S)\,;g(q|E)\}\\ &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p)\,;1'\,;g(q|E)\}\\ &= \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|q|E)\}\end{aligned}$$

Now since $p \subseteq \mathrm{Id}_E$, we have $p|p = p \cap p = p$, and thus $E|p|p|E = E|p|E = E$ (by def. of point relations). Therefore $g(E|p|p|E) = g(E)$, and $\langle p,p\rangle \in \sigma(S)$.

So $g^{-1}|\sigma$ is a near-hetermorphism.

Now we wish to prove that $g^{-1}|\sigma$ is 1-1.

Note that since the kernel of g is maximal, $\mathfrak{B}$ is a simple relation algebra.

Let $b \in B,\ b \neq 0$. We will show $g^{-1}|\sigma(b) \neq 0$. Let $R \in g^{-1}(b)$. Show $\sigma(R) \neq \emptyset$. Now $\sigma(R) = \{\langle p,q\rangle \in \mathrm{Pt}_E \times \mathrm{Pt}_E : g(E) = g(E|p|R|q|E)\}$. By lemma 3.2.2, $\forall R \subseteq E\ \exists p,q \in \mathrm{Pt}_E$ so that $E|R|E = E|p|R|q|E$. Since $R \neq \emptyset,\ E|R|E \neq \emptyset$. Thus

$$\begin{aligned}g(E|p|R|q|E) &= g(E|R|E)\\ &= g(E)\,;g(R)\,;g(E) && g \text{ a hom.}\\ &= 1\,;b\,;1\\ &= 1 && \mathfrak{B} \text{ is simple and } b \neq 0\\ &= g(E)\end{aligned}$$

Therefore $\langle p,q\rangle \in \sigma(R)$, and hence $\sigma(R) \neq \emptyset$.

So we have shown that $g^{-1}|\sigma$ is an injective near-hetermorphism from $\mathfrak{B}$ into $\mathfrak{Re}^*(\mathrm{Pt}_E)$. Therefore by lemma 3.3.3, $\mathfrak{B} \in \mathsf{wRRA}$. □

We now want to consider the collection of all algebras of the form $\mathfrak{Sb}(E)$, where E ranges over equivalence relations and $\oplus$ and $\sim$ range over the possible valid boolean operations. We show that any homomorphic image of such an algebra is representable in the weak sense.

Theorem 3.3.6. $\mathsf{H}\{\mathfrak{Sb}(E) : E|E^{-1} = E, \oplus$ and $\sim$ are valid boolean operations$\} \subseteq \mathsf{wRRA}$.

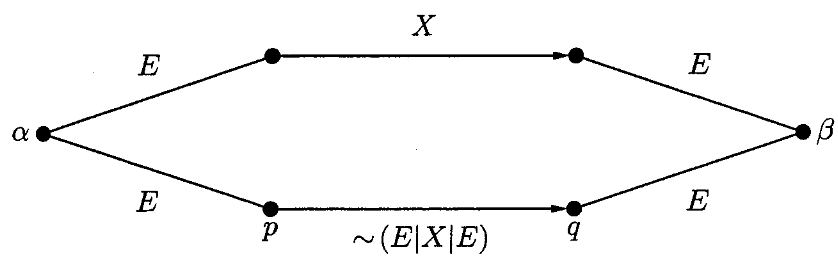
Proof. Let $\mathfrak{B} \in \mathsf{H}\{\mathfrak{Sb}(E) : E|E^{-1} = E, \oplus$ and $\sim$ are valid boolean operations$\}$ be non-degenerate. There is some homomorphism $g : \mathfrak{Sb}(E) \twoheadrightarrow \mathfrak{B}$. Let $I = g^{-1}(0)$. I is a relational ideal. Let $b, c \in B$, $b \neq c$. Then there are distinct relations $R, S \subseteq E$ with $g(R) = b$ and $g(S) = c$. Let $T :=\sim(E|(R \vartriangle S)|E)$, where $\vartriangle$ denotes symmetric difference, $R \vartriangle S = R \cap \sim S \oplus S \cap \sim R$.

Now we wish to find a maximal ideal containing T. If $T \in I$, extend I to a maximal ideal. (Use Zorn's Lemma.) If $T \notin I$, define $I' = \{X \subseteq E : \exists X_1 \in I,\ X \subseteq X_1 \oplus (E|T|E)\}$. I' is a relational ideal that includes I. We would also like to know that I' is proper; to see this, suppose the contrary, so that $E \in I'$. Then there is some $X_1 \in I$ so that $E = X_1 \oplus E|T|E$. Then we have

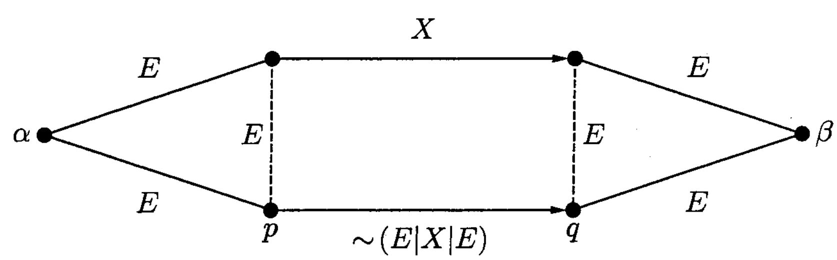
$$\begin{aligned} X_1 &\supseteq \sim(E|T|E) && \text{by boolean arithmetic} \\ &= \sim[E|\sim(E|(R \vartriangle S)|E)|E] && \text{def. of } T \\ &\supseteq E|(R \vartriangle S)|E && (\star) \end{aligned}$$

So $X_1 \supseteq E|(R \vartriangle S)|E$. But $X_1 \in I$, and since $E|(R \vartriangle S)|E$ is below X_1, $E|(R \vartriangle S)|E \in I$ also. But then $R \vartriangle S \in I$ too, which means that $g(R) = g(S)$, contrary to assumption.

To justify $(\star)$, we show that for $X \subseteq E$, $\sim[E|\sim(E|X|E)|E] \supseteq E|X|E$. Suppose $\langle \alpha, \beta\rangle \in E|\sim(E|X|E)|E$. Suppose by way of contradiction that $\langle \alpha, \beta\rangle \in E|X|E$ also. Then we have what is depicted in the following diagram:



Then the following edges can be added:



Then $\langle p, q\rangle \in E|X|E \cap \sim(E|X|E) = \emptyset$, a contradiction. Therefore $E|\sim(E|X|E)|E \cap E|X|E = \emptyset$, and hence $\sim[E|\sim(E|X|E)|E] \supseteq E|X|E$.

Therefore I' is proper, hence is included in a maximal ideal J. Now $J \supseteq I \cup \{T\}$. Then there is a homomorphism $h_J : \mathfrak{Sb}(E) \twoheadrightarrow \mathfrak{Sb}(E)/J$ with maximal kernel J. By lemma 3.3.5, $\mathfrak{Sb}(E)/J$ is isomorphic to some $\mathfrak{C}_{b,c} \in \mathsf{wRRA}$.

We show $g^{-1}|h_J : \mathfrak{B} \twoheadrightarrow \mathfrak{C}_{b,c}$ is a homomorphism that separates b and c. To see that $g^{-1}|h_J$ is a function, let $R, S \subseteq E$, with $g(R) = g(S)$. Thus $g(R \triangle S) = 0$, so $R \triangle S \in I \subseteq J$.

Therefore $h_J(R) = h_J(S)$.

To see that $g^{-1}|h_J$ is a homomorphism, consider a binary operation $*$. Let $R \in g^{-1}(a * b)$, $S \in g^{-1}(a)$, $T \in g^{-1}(b)$. We want $h_J(R) = h_J(S) * h_J(T)$, which is equivalent to $R \vartriangle (S * T) \in J$.

Note that $g(R) = a * b = g(S) * g(T) = g(S * T)$. Therefore $R \vartriangle (S * T) \in I \subseteq J$.

To show that $g^{-1}|h_J$ separates b and c, recall that $g(R) = b,\ g(S) = c$, and $J \supseteq I \cup \{T\}$. Thus

$$\begin{aligned}
T = \sim [E|(R \vartriangle S)|E] \in J &\Longrightarrow \sim [E|(R \vartriangle S)|E] \xmapsto{h_J} 0 \\
&\Longrightarrow E|(R \vartriangle S)|E \xmapsto{h_J} 1 \\
&\Longrightarrow \text{ it is not the case that } (R \vartriangle S \xmapsto{h_J} 0) \\
&\Longrightarrow R \vartriangle S \notin J \\
&\Longrightarrow h_J(R) \neq h_J(S)
\end{aligned}$$

So for each $b \neq c$ we get a separating homomorphism $g^{-1}|h_J$ to a weakly representable relation algebra. Thus we have a homomorphism

$$h : \mathfrak{B} \longrightarrow \prod_{\substack{b,c \in B \\ b \neq c}} \mathfrak{C}_{b,c}$$

given by $h(x) = \langle g^{-1}|h_J(x) : b, c \in \mathfrak{B},\ b \neq c\rangle$. h is an embedding into a product of weakly representable relations algebras. Hence $\mathfrak{B} \in \mathsf{SP}\mathsf{wRRA} = \mathsf{wRRA}$. □

Corollary 3.3.7. $\mathsf{HS}\{\mathfrak{Sb}(E) : E|E^{-1} = E, \oplus$ and $\sim$ are valid boolean operations$\} \subseteq \mathsf{wRRA}$.

Proof. Let $\mathfrak{B} \in \mathsf{HS}\{\mathfrak{Sb}(E) : E|E^{-1} = E, \oplus$ and $\sim$ are valid ops.$\}$. Since every RA has the congruence extension property, we can commute the operators H and S, giving us

$$\mathfrak{B} \in \mathsf{HS}\{\mathfrak{Sb}(E) : E|E^{-1} = E\} = \mathsf{SH}\{\mathfrak{Sb}(E) : E|E^{-1} = E\} \subseteq \mathsf{SwRRA} = \mathsf{wRRA}$$

as desired. □

Corollary 3.3.8. If every $\mathfrak{A} \in \mathsf{wRRA}$ embeds isomorphically into an $\mathfrak{Sb}(E)$ for some equivalence relation E and operations $\oplus$, $\sim$, then wRRA is a variety.

Proof. It remains to show only closure under H. Let $\mathfrak{B} \in \mathsf{HwRRA}$. There is some $\mathfrak{A} \in \mathsf{wRRA}$ with $\mathfrak{B} \in \mathsf{H}\{\mathfrak{A}\}$. Now $\mathfrak{A} \cong | \subseteq \mathfrak{Sb}(E)$ for some equivalence relation E, so $\mathfrak{B} \in \mathsf{HS}\{\mathfrak{Sb}(E) : E|E^{-1} = E\}$. Then

$$\mathfrak{B} \in \mathsf{HS}\{\mathfrak{Sb}(E) : E|E^{-1} = E\} = \mathsf{SH}\{\mathfrak{Sb}(E) : E|E^{-1} = E\} \subseteq \mathsf{SwRRA} = \mathsf{wRRA}$$

as desired. □

It remains to be shown that the hypothesis of the last corollary holds—if, in fact, it is true. Perhaps it can be shown that if the hypothesis fails, then wRRA is not a variety.

4 Conclusion

In the previous two chapters we studied weakly representable relation algebras. In chapter 2 we saw that weak representations that are not representations are possible on finite sets. We connected weak representability for finite algebras to relativizations of and weak embeddings into representable relation algebras. The search for a non-representable relation algebra with a weak representation over a finite set can proceed as follows: starting with a non-representable algebra (that is not known to be not weakly representable), search for a larger representable algebra into which the non-representable algebra can be weakly embedded.

In chapter 3 we considered the question whether wRRA was a variety, and we showed that it is if every weakly representable algebra embeds into a "full" weakly representable relation algebra. Whether this last condition holds is yet to be seen; if it does not, however, perhaps that will give a way to answer the question negatively. Given an algebra $\mathfrak{A} \in \mathsf{wRRA}$, if $\mathfrak{A}$ does not so embed, can we use this to construct a congruence θ so that $\mathfrak{A}/\theta \notin \mathsf{wRRA}$?

Let's collect in a list those questions that remain to be answered:

(i.) Does every $\mathfrak{A} \in \mathsf{wRRA}$ embed into an $\mathfrak{Sb}(E)$ for some E and $\oplus$, $\sim$?

(ii.) Whether wRRA is a variety or not, what can we say about its equational theory? Does a basis for the equational theory require infinitely many variables?

(iii.) Is there an $\mathfrak{A} \in \mathsf{wRRA} \setminus \mathsf{RRA}$ with weak representation over a finite set?

(iv.) Suppose $\mathfrak{A} \xrightarrow{\mathsf{w}} \mathfrak{B}$. Is $\mathfrak{A}$ always a relativization of $\mathfrak{B}$?

(v.) Is $\mathsf{RRA} \subset \mathsf{strwRRA}$?

(vi.) Is strwRRA elementary? A variety? Is it even closed under **S**?

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