ANALYSIS OF MANY-DEFECT SYSTEMS

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ABSTRACT

In the general problem of quantitative NDE, the majority of past approaches are based upon the questionable assumption that the dominant defect or flaw can be identified before the beginning of the main body of the analysis. This concept is fundamental to most (but not all) treatments of probabilistic failure prediction and accept/reject optimization. This oversimplification and associated logical tangles are obviated by a more comprehensive approach to defect characterization and probabilistic failure prediction, in which it is assumed in the pertinent stochastic models that the various significant types of defects occur in all possible numbers. In this paper, we present a version of such an approach that involves a single type of defect and a promising approximation methodology.

1. INTRODUCTION

This paper is devoted to the development of an NDE formalism that circumvents many of the difficulties associated with the dominant defect approximation. This formalism deals with probabilistic failure prediction from a point of view in which the many-defect nature of the test specimen is emphasized.

An example of the dominant defect point of view can be found in several investigations of probabilistic failure prediction for ceramics (e.g., Richardson and Evans, 1980; Richardson and Fertig, 1981). In a more recent paper (Richardson and Fertig, 1982) dealing again with ceramics but with machining damage as the origin of surface defects, the inadequacy of the dominant defect approximation was confronted and certain aspects of a many-defect formulation were discussed, even though the actual computations employed the dominant defect assumption as a means for obtaining a lower bound to the probability of failure.

In the following sections, we discuss how a convenient formalism for treatment of many-defect systems can be set up. In Section 2, we

discuss the representation of many-defect states and several related matters. In Section 3, we discuss probabilistic failure prediction for such a system using a simple model of the failure process. In Section 4, we consider the problem in estimating the a posteriori density in single-defect state space based upon a set of nondestructive measurement. In Section 5, we present a practical approximation procedure for the calculation of the a priori and a posteriori densities in single-defect state space. Finally, in Section 6, we present a brief summary and discuss possible directions of future work.

2. PRELIMINARIES

We start the discussion of the representation of the state of a many-defect system with the consideration of the state of a single defect. We assume for the sake of simplicity that this state is represented by an N-dimensional vector z with continuous-valued components or, in other words, by a point in z-space. For example, if the defect were a spherical void, then z would be a four-dimensional vector whose components are composed of the sphere radius and three coordinates defining the position of its center. Many more realistic situations involving two or more different kinds of defects require modifications and extensions of the above representation, but for the sake of brevity we will not discuss them here.

It is appropriate to represent the state of a many-defect system by a set of representation points in single-defect state space. To avoid the somewhat awkward problems associated with the nonunique labeling of the representative points, it is convenient to adopt an occupation number formalism. To this end, we partition z-space into a large number of cells, D_i, $i = 1, \ldots, p$, and denote the volume of D_i by δz_i and its nominal center position by z_i. In Fig. 1, we illustrate this representation for the case of a two-dimensional single-

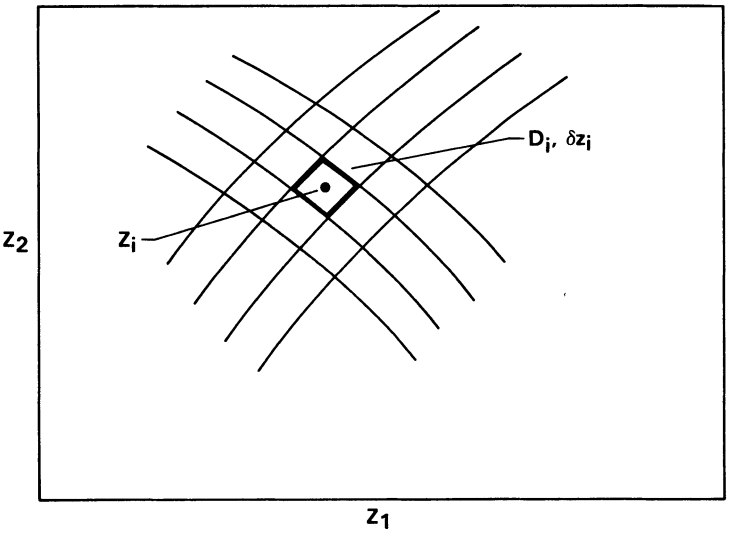

Fig. 1 Cells in single-defect state space.

defect state vector $z^T = (z_1, z_2)$. The state of the many-defect system is given by the set of occupation numbers $n_1, \ldots, n_p$ implying that there there are n_i defects with representative points in D_i. More explicitly, the many-defect state x is defined by

$$x^T = (n_i, \ldots, n_p) \quad . \tag{2.1}$$

Since we intend to take the limit* $\delta z_i \to 0$, i = 1, ..., p (with a compensating increase in the dimensionality p) at a later stage of the analysis, it is reasonable to assume that the probability of multiple occupancy (i.e., $n_i > 1$ for any i) is negligible and thus we will limit the possible values of each n_i to 0 and 1. For example, if only one defect were present with a single-defect state z_i, the many-defect state x would take the form

$$x^T = (0, \ldots, n_i = 1, \ldots, 0) \quad . \tag{2.2}$$

The generalization to larger numbers of defects is obvious.

Since the many-defect state x is discrete-valued, the a priori statistical ensemble of values of x is represented by a probability P(x) (as contrasted with a probability density). The associated first order mean density of representative points in z-space (henceforth called the first order density) is defined by

$$\rho(z_i)\delta z_i \triangleq En_i \equiv \sum_x n_i P(x) \quad . \tag{2.3a}$$

The corresponding second and third order densities are given by

$$\rho(z_i, z_j)\delta z_i \delta z_j \triangleq \sum_x n_i n_j P(x) \tag{2.3b}$$

$$\rho(z_i, z_j, z_k)\delta_z i \delta z_j \delta z_k \triangleq \sum_x n_i n_j n_k P(x) \quad . \tag{2.3c}$$

The a posteriori (conditional) version of the above results will be discussed later.

3. PROBABILISTIC FAILURE PREDICTION

We now turn to the consideration of the problem of determining the probability of failure of a specified many-defect system under an assumed stress environment (deterministic or stochastic), given the results of a set of nondestructive measurements. We will limit our discussion to the case of a single time situation, i.e., the case in-

*It is of course necessary to assume that each domain D_i must shrink down in in all directions.

volving a set of nondestructive measurements associated with single time, followed almost immediately by a single accept/reject decision. We will treat the failure process in an extrinsic "black box" manner and thus the possible time-dependent aspects of the failure process are not of explicit concern to us.

The performance of the system under the measured environment is represented by the binary variable c, which takes the value 0 if the system fails (according to a prescribed definition) and the value 1 if the system survives. The results of a set of nondestructive measurements are represented by the M-dimensional vector y. The a priori probability of c is denoted by P(c), where P(c=o) is, of course, the a priori probability of failure. The a posteriori probability of c, i.e., the probability of c conditioned on the results of nondestructive measurements, is denoted by $P(c|y)$, where obviously $P(c=o|y)$ is the a posteriori (or conditioned) probability of failure. The optimal accept/reject decision is given by comparing the latter probability with a threshold whose value is determined by the assumed loss function and the a priori probability of failure.

The various probabilities and probability densities of direct concern to us can be expressed in terms of the joint probability function* P(c,y) in accordance with the following equations:

$$P(c) = \int dy\, P(c,y) \tag{3.1}$$

$$P(y) = \sum_{c} P(c,y) \tag{3.2}$$

$$P(c|y) = P(c,y)/P(y) \tag{3.3}$$

where dy is a differential volume element in y-space and where the integration on y spans the domain of definition of y.

It is useful to introduce the many-defect state vector x into the present probabilistic failure analysis. We now make the assumption that x has the property

$$P(c,y|x) = P(c|x)P(y|x) \quad , \tag{3.4}$$

i.e., if x is known, c and y are statistically independent. The above relation is equivalent to either of the following alternative relations:

$$P(c|y,x) = P(c|x) \tag{3.5a}$$

$$P(y|c,x) = P(y|x) \quad . \tag{3.5b}$$

The first of the above alternative relations implies that the information contained in x is so comprehensive that the probability of c conditioned on x is unaffected by the further conditioning on y. The

* The term "probability function" denotes an entity that is a probability with respect to discrete-valued variables and a probability density with respect to continuous-valued variables.

second relation is described by a similar statement with c and y interchanged.

We can now write

$$P(c,y) = \sum_x P(C,y|x)P(x)$$
$$= \sum_x P(c|x)P(y|x)P(x) \quad , \tag{3.6}$$

where P(x) is the a priori probability of x. The above expression clearly decomposes the total calculation of the conditional probability of performance (i.e., failure when c=o) into three separate parts corresponding to the three factors on the last line. The factors are $P(c|x)$ representing the failure model, $P(y|x)$ representing the measurement model, and finally P(x) embracing the a priori statistics. The abstract structure of this calculation is represented by the block diagram in Fig. 2. From Eq. (3.6), we can readily derive two additional relations of direct interest, i.e.,

$$P(c) = \sum_x P(c|x)P(x) \tag{3.7}$$

$$P(c|y) = \sum_x P(c|x)P(x|y) \quad , \tag{3.8}$$

which form the basis for some of our later results.

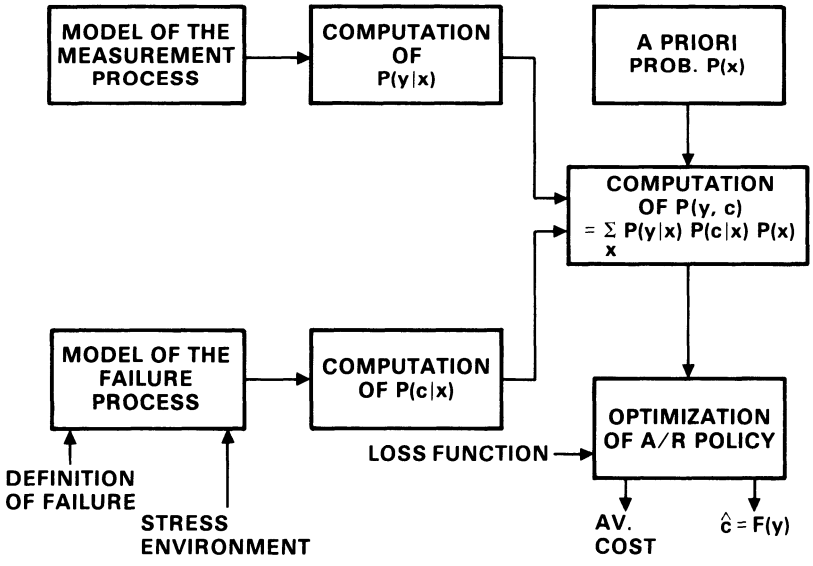

Fig. 2 Structure of probabilistic failure analysis.

We assume a relatively simple form of the a priori probability given by

$$P(x) = \exp(\lambda + \alpha N)\, P^{o}(x) \tag{3.9}$$

where N is the total number of defects given by

$$N = \sum_i n_i \quad , \tag{3.10}$$

α is a real constant and λ is a normalization constant defined by the expression

$$\exp(-\lambda) = \sum_x \exp(\alpha N)\, P^{o}(x) \quad . \tag{3.11}$$

The summation on x involves all possible values of the occupation numbers n_i. The quantity $P^{o}(x)$ is the standard probability given by the expression

$$P^{o}(x) = \prod_i P^{o}(n_i) \tag{3.12}$$

where

$$P^{o}(n_i = 0) = 1 = \rho^{o}(z_i)\, \delta z_i \tag{3.13a}$$

$$P^{o}(n_i = 1) = \rho^{o}(z_i)\, \delta z_i \quad . \tag{3.13b}$$

In the above expressions $\rho^{o}(z_i)$ is the standard density in single-defect state space, i.e., z-space, evaluated at $z = z_i$. The form of the a priori probability P(x), defined by the above equation, has two salient properties: 1) the occupation numbers of different cells are statistically independent, and 2) the a priori density in z-space is given by

$$\rho(z) = \exp(\alpha)\rho^{o}(z) \tag{3.14}$$

in the limit of infinitesimal cell size, i.e., the a priori density has a fixed shape, but a variable magnitude depending on α.

The conditional probability of performance $P(c|x)$ is based upon the following arguments. We make the rather simplistic assumption that survival of the many-defect system is achieved if, and only if, no cell causes failure, i.e.

$$c = \prod_i c_i \quad , \tag{3.15}$$

where $c_i = o$ if the cell i involves a process causing failure and $c_i = 1$ otherwise. We readily infer that

$$P(c=1|x) = E(c|x)$$

$$= \prod_i E(c_i|n_i) \quad , \tag{3.16}$$

where the last line follows from the a priori statistical independence of the n_i assumed in the last paragraph. We now make the further assumption that $E(c_i|n_i)$ is given by

$$E(c_i|n_i) = P(c_i=1|n_i)$$
$$= 1-n_i\ P_f(z_i) \quad , \tag{3.17}$$

where $P_f(z_i)$ is the probability that a single defect with the state z_i causes failure. From Eq. (3.7), we readily obtain the result

$$P(c=1) = \prod_i [1- \rho(z_i)P_f(z_i)\delta z_i]$$
$$\to \exp[-\int dz \rho(z)P_f(z)] \quad , \tag{3.18}$$

where the last line involves the limit of vanishing cell size.

The conditional probability density $P(y|x)$, representing the measurement process, is significantly more complex than either $P(c|x)$ or $P(x)$ and, for this reason, the next several sections will be devoted to it and related matters.

The discussion of the performance probability conditioned on nondestructive measurements, i.e., $P(c|y)$, should logically follow the ensuing discussions of $P(y|x)$. Nevertheless, we will present a brief discussion of $P(c|y)$ from an anticipatory point of view. First of all, we note that the calculation of $P(c|y)$ is significantly more complicated than that of $P(c)$ because the statistical independence of the n_i assumed a priori does not hold a posteriori, i.e., this assumption does not survive the process of conditioning on nondestructive measurements. However, under certain conditions (e.g., the system is sufficiently dilute a posteriori), the a posteriori statistical independence assumption is an acceptable approximation. In this case, we can write

$$P(c=1|y) = \exp[-\int dz \rho(z|y)P_f(z)] \quad , \tag{3.19}$$

a simple extension of Eq. (3.18). In this approximation, the problem of determining the probability of failure (or survival) conditioned on nondestructive measurements reduces to the determination of the conditional first order density defined by

$$\rho(z_i|y)\delta z_i = \sum_x n_i P(x|y) \quad , \tag{3.20}$$

followed by the process of taking the limit of zero cell size.

4. MEASUREMENT MODEL AND STATE ESTIMATION

The treatment of $P(x|y)$, and ultimately the determination of $\rho(z|y)$, involves the formulation of a measurement model. A satisfactory model, including random aspects, is given by the expression

$$y = Fx + \nu \quad , \tag{4.1}$$

where y and x have already been defined earlier in Section 2. The M-dimensional vector ν represents the measurement error and the Mxp

dimensional matrix F gives the characteristics of the measurement system in the absence of error.

The term Fx requires further discussion. In more explicit terms, it can be written in the form

$$Fx = \sum_i f(z_i)\, n_i \tag{4.2}$$

where $f(z_i)$ is the noiseless measurement vector that would be obtained if the total many-defect system contained only one defect and this were located at $z = z_i$ in single-defect state space. It is to be stressed that the measurement vector y represents all of the measurements made on the system being inspected. Clearly, the subvector of y associated with a measurement at one location on the specimen will be weakly dependent upon z_i's associated with distant locations beyond the range of this measurement.

The description of the measurement model must include a discussion of the a priori statistical properties of x and ν. We assume that x and ν are statistically independent. The measurement error ν is assumed to be Gaussian random vector with the properties

$$E\nu = 0 \tag{4.3a}$$

$$E\nu\nu^T = C_\nu, \tag{4.3b}$$

where C_ν is the covariance matrix of ν. The a priori statistical properties of x were discussed in the last section.

In the subsequent analysis, we will, for the sake of generality, assume the complex quantities. This assumption simply means in the writing of formulas that the transpose $(\ )^T$ must be replaced by the Hermitian conjugate $(\ )^\dagger$. The situation in which quantities are real may now, of course, be regarded as a special case.

$P(x|y)$, the a posteriori probability of x conditioned on the measurement vector y, is given by

$$P(x|y) = P(y|x)\, P(x)/P(y)$$

$$= P(y|x)\, P(x)/\sum_x P(y|x)\, P(x) \quad , \tag{4.4}$$

where $P(y|x)$ is, in turn, given by

$$P(y|x) = A \exp\left(-\frac{1}{2}\,(y - Fx)^\dagger\, C_\nu^{-1}\, (y - Fx)\right) \tag{4.5}$$

in which A is an ignorable normalization constant. Substituting Eq. (3.9) and (4.5) into Eq. (4.4), we obtain

$$P(x|y) = \exp\left(\mu - \frac{1}{2}\,(y - Fx)^\dagger\, C_\nu^{-1}\, (y - Fx) + \alpha N\right) P^o(x) \quad , \tag{4.6}$$

in which μ is a normalization constant given by

$$\exp(-\mu) = \sum_{x} \exp(-\frac{1}{2}(y - Fx)^{\dagger} C_{\nu}^{-1} (y - Fx) + \alpha N) P^{o}(x) \quad .(4.7)$$

The quantities α and $P^{o}(x)$ were defined in the previous section.

5. DENSIFICATION HIERARCHY

Although the exact solution is given in a formal sense by Eqs. (3.20) and (4.6), it is impossible to carry it out by any conceivable combination of analytical and computational processes. The reader is reminded that the components of the state vector x are not continuous variables. If they were and if log P(x) were quadratic in x, then the problem would be readily solvable. However, this is not the case and thus an approximate procedure must be devised. Among the many possible approximation procedures, we have selected one such procedure because of its relative simplicity and computational convenience.

We observe that as $\alpha \to -\infty$, i.e., the system of many-defects becomes very dilute a priori, the conditional mean density $\rho(z_i|y)$ has a limiting behavior given by

$$\exp(-\alpha)\ \rho(z|y) \to \exp(y^{\dagger} C_{\nu}^{-1} f(z) - \frac{1}{2} f(z)^{\dagger} C_{\nu}^{-1} f(z))\ \rho^{o}(z) \quad (5.1)$$

after taking the limit zero cell size. This relatively simple result suggests that it might be worthwhile to devise a method that is it based upon the consideration of the behavior of $P(z|y)$ as α increases from $-\infty$ to some desired value.

Such a method can be obtained by consideration of the differential equation

$$\frac{\partial}{\partial\alpha} P(x|y) = (N - E(N|y))\ P(x|y)$$

$$= \sum_{i} (n_i - E(n_i|y))\ P(x|y) \quad (5.2)$$

where, as previously stated, $N = \sum_{i} n_i$ is the total number of defects present.

The above result is simply derived from Eq. (4.6) by direct differentiation with respect to α and by noting from Eq. (4.7) that

$$\frac{\partial\mu}{\partial\alpha} = -E(N|y) \quad . \quad (5.3)$$

A similar equation can be derived for P(x), namely

$$\frac{\partial}{\partial\alpha} P(x) = (N - EN)\ P(x), \quad (5.4)$$

by differentiating Eq. (3.9) with respect to α and using Eq. (3.11) to deduce that

$$\frac{\partial \lambda}{\partial \alpha} = - EN \quad . \tag{5.5}$$

Due to the statistical independence of occupation numbers for different cells, the unconditional densities of various orders can be simply obtained in closed form and thus there is no need for an approximate procedure in this part of the problem.

Starting with the differential equation (5.2), we can derive an infinite hierarchy of equations for the various orders of conditional densities by multiplying both sides of Eq. (5.2) by n_j, $n_j n_k$, $n_j n_k n_l$, etc. and then summing on x. For the sake of brevity, we will not discuss here the detailed steps in the derivation. A more detailed discussion is given in a recent paper by Richardson and Salvado' (1984). In order to present the final results in a useful form, it is necessary to define some new functions and variables.

Following the common practice in classical statistical machanics, we will introduce correlation factors $g(z,z'|y)$, $g(z,z',z''|y)$, etc. defined by the relations

$$\rho(z,z'|y) = \rho(z|y)\ \rho(z'|y)\ g(z,z'|y) \tag{5.6}$$

$$\rho(z,z',z''|y) = \rho(z|y)\ \rho(z'|y)\ \rho(z''|y)\ g(z,z'|y)$$

$$\cdot\ g(z',z''|y)\ g(z'',z|y)\ g(z,z',z''|y) \tag{5.7}$$

etc. It is understood that all of the g-functions are unchanged by permuting the order of the various z-arguments, i.e., $g(z,z'|y) = g(z',z|y)$, etc. We will also introduce a new variable γ replacing α in accordance with the relation

$$\gamma = \exp(\alpha) \quad . \tag{5.8}$$

It can be shown that γ is proportional to $\rho(z|y)$ when γ is very small. Finally, we introduce a reduced density $\sigma(z|y)$ defined by the relation

$$\rho(z|y) = \gamma\sigma(z|y) \quad . \tag{5.9}$$

The reduced density approaches a function independent of γ as $\gamma \to 0$.

We now obtain the first member of the hierarchy in the form

$$\frac{\partial}{\partial \gamma} \log\ \sigma(z|y) = \int dz'\ \sigma(z'|y)(g(z,z'|y)-1) \tag{5.10}$$

and the second number in the more complex form

$$\frac{\partial}{\partial\gamma} \log g(z,z'|y) = \int dz'' \, \sigma(z''|y)[(g(z,z''|y)-1)(g(z',z''|y)-1)$$

$$+ g(z,z''|y)\, g(z',z''|y)(g(z,z',z''|y)-1)] \quad (5.11)$$

By analyzing the behavior of $E(n_i|y)$ and $E(n_i n_j|y)$ in the limit $\gamma \to 0$ (or $\alpha \to -\infty$), we obtain the initial conditions

$$\sigma(z|y)\Big|_{\gamma=0} = \exp(y^\dagger C_\nu^{-1} f(z) - \tfrac{1}{2} f(z)^\dagger C_\nu^{-1} f(z)) \, \rho^o(z) \quad (5.12)$$

$$g(z,z'|y)\Big|_{\gamma=0} = \exp(-f(z)^\dagger C_\nu^{-1} f(z')) \quad . \quad (5.13)$$

It is easy to show that all of the higher order g-functions have an initial value of 1.

To obtain a practical computational scheme for the solution of the above hierarchy, we must introduce a closure approximation that yields a finite hierarchy of tractable size. In the present treatment, we will consider the crudest meaningful closure approximation; namely, we will retain only the first member of the hierarchy, i.e., Eq. (5.10) with $g(z,z'|y)$ approximated by its initial form (infinite dilution form), i.e., given by Eq. (5.13). It should be noted that the initial form of $g(z,z'|y)$ is independent of the measurement vector y, but it depends (somewhat paradoxically) upon the fact that measurements are made.

With the above approximations, the total hierarchy reduces down to the single equation

$$\frac{\partial}{\partial\gamma} \log \sigma(z|y) = \int dz' \, \sigma(z'|y) \, h(z,z') \quad (5.14)$$

where

$$h(z,z') = \exp(-f(z)^\dagger C_\nu^{-1} f(z')) - 1 \quad . \quad (5.15)$$

and where the initial condition is given by (4.15).

With a suitable discretization in z-space, the computational solution of the above equation is equivalent to the computational solution of a large number of coupled nonlinear ordinary differential equations, subject to initial conditions given by a correspondingly discretized form of Eq. (5.12). The integration should proceed to a desired terminal value of γ.

6. DISCUSSION

We have considered a simple formulation of a many-defect system in which (a) only one type of defect is assumed, (b) there is no a priori correlation between defects, and (c) the failure process (at

least the crucial phase of it) involves no interaction between defects. It is a direct consequence of this formulation that the a priori probability of failure (of the entire many-defect system) depends only upon the probability $P_f(z)$ that a single defect with a state z causes failure and upon the a priori (mean) density $\rho(z)$ of representative points in single defect state space. We have derived an approximate relation of similar structure connecting the a posteriori probability of failure (i.e., conditioned on measurements) and the a posteriori density $\rho(z|y)$. A substantial effort was devoted to the approximate determination of $\rho(z|y)$ using a suitably truncated hierarchy of integro-differential equations.

It is quite clear how to modify the above formulation to achieve a higher degree of reality. One should consider (a) a more complex representation of many-defect states capable of handling several defect types, (b) a model of the failure process involving physical interactions between defects, and (c) an a priori probability entailing correlations between defects. Obviously, effort must also be expended on approximation methodologies involved in the solution of problems arising from these more realistic formulations.

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