EXACT SOLUTION OF PROBABILISTIC INVERSE PROBLEM PERTAINING TO THE SCATTERING OF ELASTIC WAVES FROM GENERAL INHOMOGENEITIES

John M. Richardson

Rockwell International Science Center

Thousand Oaks, CA 91360

ABSTRACT

Probabilistic inversion methodology (e.g., finding the most probable inhomogeneity given the measurements) has been applied to many inverse scattering problems with either Gaussian or non-Gaussian statistical models of possible scatterers. All of our past investigations have dealt either with weak inhomogeneities, for which the Born approximation is valid, or with highly reflective scatterers, for which the Kirchhoff approximation is presumed to be adequate. In this paper we consider an approach to the general probabilistic inversion problem involving strong inhomogeneities by the application of the Pontryagin maximum principle generalized to 3D space. The key feature of the present approach to inverse scattering is the treatment of the wave equation, relating each scattered field to each incident field and the state of the inhomogeneity, as a continuous set of side conditions. This set of side conditions is handled by the Lagrange multiplier method. Our procedure, in analogy with that of Pontryagin, is to first determine the most probable state of the inhomogeneity given both the scattered wave fields and the conjugate Lagrange multiplier fields. The final stationarization with respect to both fields yields finally a set of nonlinear coupled integral equations defined on the localization domain.

INTRODUCTION

An obvious approach to the solution of the probabilistic inverse problem associated with the scattering of elastic waves from a general, locally isotropic, inhomogeneity is the following: (1) solve the direct scattering problem by largely computational means for a large number of assumed inhomogeneities; (2) deduce the a posteriori probabilities (probabilities conditioned on actual scattering measurements) of the above assumed inhomogeneities; (3) attempt to deduce from these

---

*This work was supported by Rockwell Independent Research and Development Funds.

results the direction of the global maximum probability; (4) deduce a new set of assumed inhomogeneities; and (5) repeat the process. It is our conviction that this procedure can lead to computational problems involving combinatorial explosions of staggering proportions, particularly if the inhomogeneity is significantly strong (i.e., the scattering amplitude is decidedly nonlinear in the property deviations).

In this paper we describe another procedure that gives one hope of dealing with tractable computational problems. This procedure is suggested by Pontryagins (Gamkrelidze, 1978) treatment of a certain class of optimal deterministic control problems. In one simple version, this treatment has the salient feature of treating the dynamical equations as a continuous set of constraints that are handled by the Lagrange multiplier method in the minimization of a suitable cost function. We pursue an analogous procedure in the maximinization of the conditional probability with the p.d.e. for scattering regarded as a continuous set of constraints.

Although this paper was originally intended to deal with elastic waves scattering from inhomogeneities in solids, we have decided for the sake of brevity to limit the discussion to the much simpler problem of acoustical waves scattering from velocity inhomogeneities in fluids.

GENERALITIES

A frequently considered form of the scalar wave equation is

$$\nabla^2 p + k^2(1 + \phi)p = 0 \tag{2.1}$$

where $p = p(\vec{r},\omega)$ is the excess pressure at the position $\vec{r}$ and temporal frequency ω. The parameter k is the unperturbed wave number given by

$$k = \frac{\omega}{c_o} \tag{2.2}$$

where c_o is the propagation velocity of acoustical waves in the unperturbed medium. The above equation describes the propagation of waves in a fluid with a nonuniform acoustic velocity $c = c(\vec{r})$ but with a uniform density. The function $\phi = \phi(\vec{r})$ represents the deviation of the perturbed acoustic velocity relative to the unperturbed in accordance with the relation

$$\left(\frac{c_o}{c}\right)^2 = 1 + \phi \quad . \tag{2.3}$$

In the subsequent discussion we will make use of the causal and anticausal forms of Green's function for the unperturbed wave equation, namely

$$(\nabla^2 + k^2)G_{\pm}(\vec{r},\omega) = \delta(\vec{r}) \quad , \tag{2.4}$$

in which the + and - subscripts denote the causal and anticausal forms, respectively. As is well known these two forms of Green's function are given by

$$G_{\pm}(\vec{r},\omega) = -\frac{1}{4\pi}\frac{\exp(\pm ikr)}{r} \tag{2.5}$$

where $r = |\vec{r}|$. In this expression we have assumed the implicit time factor $\exp(-i\omega t)$, a frequently employed convention in scattering theory.

As usual in scattering problems, we write p in the form

$$p = p^i + p^s \tag{2.6}$$

where p^i is the incident wave satisfying the unperturbed wave equation

$$(\nabla^2 + k^2)p^i = 0 \tag{2.7}$$

(with appropriate boundary conditions) and where p^s is the socalled scattered wave satisfying the usual radiation condition at a large distance from the localization domain. Substituting (2.6) into (2.1) and subtracting (2.7) we obtain

$$[\nabla^2 + k^2(1 + \phi)]p^s + k^2\phi p^i = 0 \tag{2.8}$$

With p^i and ϕ given, the above equation can be solved for the scattered wave p^s subject to the radiation condition on a sphere of very large radius R, i.e.,

$$\nabla p^s = ik\vec{n}p^s \quad , \quad |\vec{r}| = R \tag{2.9}$$

where $\vec{n}$ is the outward-pointing normal vector.

THE MEASUREMENT PROCESS

The experimental set-up is schematically represented in Fig. 1 where a transmitting transducer T_m produces an incident wave p^i_m propagating in the direction of the localization domain D_L and the receiving transducer T_n observes the scattered wave p^s_m in its neighborhood thereby producing the output waveform f_{mn}. The transmitting transducer T_m (henceforth called the m-th transmitter) is one of many transmitters (m = 1,...,M) or, equivalently, is the same transducer in many positions and orientations. Similarly, the receiving transducer (henceforth called the n-th receiver) is one of many receivers (n = 1,...,N) or, as before, the same transducer in many configurations. In fact, in some cases a transmitter and receiver could be the same transducer operating in transmitting and receiving modes. We assume that a large spherical surface (not shown) exists with every surface element in the far field of the scatterer and all of the transducers. The inhomogeneity of acoustic velocity is assumed a priori to be confined to the localization domain D_L.

We can express the measurement process in the form of a stochastic measurement model represented by the relation

$$f_{mn} = \int d\vec{r} H_n p^s_m + \nu_{mn} \tag{3.1}$$

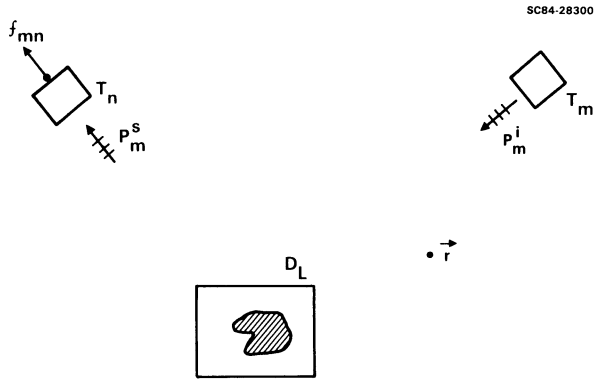

Fig. 1 Experimental setup.

where

$f_{mn} = f_{mn}(\omega)$ is a possible waveform obtained in the mn-th measurement, i.e., the output of the n-th receiver responding to the scattered wave p_m^s associated with the m-th transmitter.

$H_n = H_n(\vec{r},\omega)$ is the transfer function relating the output voltage of the n-th receiver to the scattered wave amplitude at the position $\vec{r}$ and frequency ω.

$p_m^s = p_m^s(\vec{r},\omega)$ is the scattered wave amplitude associated with the incident wave $p_m^i = p_m^i(\vec{r},\omega)$ produced by the m-th transmitter.

$\nu_{mn} = \nu_{mn}(\omega)$ is an additive noise representing the error in the mn-th measurement.

Clearly, the receiver responds to the total amplitude $p = p^i + p^s$ in its neighborhood and thus the response to p^s must be separated out by some standard procedure (e.g., time windowing, subtraction of the response to p^i, etc.).

The production of the incident wave amplitude p_m^i by the m-th transmitter should be described by a source function $s = s(\vec{r},t)$ related to the input voltage $V^i(\omega)$ by a relation of the form

$$s(\vec{r},\omega) = H_m(\vec{r},\omega)V^i(\omega) \tag{3.2}$$

where H_m is a transfer function describing the action of the m-th transmitter. The source function should appear on the r.h. side of the unperturbed version of (2.1). Although a more explicit formulation, achieved by this line of development, would bring out the reciprocal relation between transmitters and receivers in a clear manner, we will for the sake of brevity not pursue this matter further in the present paper and consequently regard p_m^i as produced by the m-th transmitter in the same undefined manner.

The position vector $\vec{r}$ takes continuous values in all of 3D space, the frequency ω takes values that are integral multiples of $2\pi/T$ where T is the duration of the observation interval. As stated earlier, the indices m and n take the integral values $1,\ldots M$ and $1,\ldots N$ respectively.

We assume a priori that $\nu_{mn}(\omega)$ is a Gaussian random process with the properties

$$E\nu_{mn}(\omega) = 0 \tag{3.3a}$$

$$E\nu_{mn}(\omega)\nu_{m'n'}(\omega')^* = T\delta_{mm'}\delta_{nn'}\delta_{\omega\omega'}C_\nu(\omega) \tag{3.3b}$$

where $E(\cdot)$ is the a priori averaging operator. Implicit in (3.3b) is the assumption that $C_\nu(t-t')$, the time-domain representation of $C_\nu(\omega)$, is periodic with period T.

The a priori statistical properties of the scatterer state represented by the function $\phi(\vec{r})$ must also be defined. Since this function does not appear directly in the measurement model some additional discussion is required. In the above measurement model only p_m^s in the neighborhood of the n-th receiver appears explicitly. Clearly p_m^s is related to p_m^i by the partial differential equation (2.8) describing the scattering process in the localization domain D_L. This process leads to a dependence of each scattered wave amplitude $p_m^s(\vec{r},\omega)$ upon the function $\phi(\vec{r})$ assuming, of course, that each incident wave amplitude is given.

We are ultimately interested in determining the most probable form of the state of the scatterer represented by the function $\phi(\vec{r})$, given the results of measurements on a particular scattering system, i.e., given the actual measured values of the waveforms $f_{mn}(\omega)$ for all m, n, and ω. This process involves the consideration of the conditional probability density given by

$$\log P(\phi|f) = \log P(f|\phi) + \log P(\phi) - \log P(f) \quad . \tag{3.4}$$

In the above equation we used a rather abbreviated notation in which ϕ represents the complete function $\phi(\vec{r})$ defined in the localization volume D_L, i.e., the set of values of $\phi(\vec{r})$ for all $\vec{r}\varepsilon D_L$, and in which f represents the set of values of $f_{mn}(\omega)$ for all m, n and ω. Actually, each $f_{mn}(\omega)$ should be accompanied by values of the parameters defining the receiving and transmitting transducers (i.e., the receiver transfer function $H_n(\vec{r},\omega)$ and the incident wave $p_m^i(\vec{r},\omega)$ associated with the m-th transmitter), for the sake of brevity we will regard these as known without explicit indication of them.

In any case, the function $\phi(\vec{r})$ does not appear explicitly in the measurement model (3.1). Instead, only the scattered amplitudes $p_m^s(\vec{r},\omega)$ appear in (3.1) and thus the conditioning on ϕ in $P(f|\phi)$ is equivalent to conditioning on the set $p_m^s(\vec{r},\omega)$ for all m, $\vec{r}$, and ω, a set represented compact symbol p^s. Thus we can replace $P(f|\phi)$ by $P(f|p^s)$ and can rewrite (3.4) in the form

$$\log P(\phi|f) = \log P(f|p^s) + \log P(\phi) - \log P(f) \quad . \tag{3.5}$$

Now, the relation between p^s and ϕ is determined by (2.8) and associated boundary conditions, all regarded as a continuous set of constraints. The term log P(f) is a given quantity in the process of finding the most probably $\phi(\vec{r})$ and thus can be ignored. The relative *a priori* probability density of $\phi(\vec{r})$ is assumed to be given by

$$\log P(\phi) = -\int_{D_L} d\vec{r}\; g(\vec{r},\phi(\vec{r}))$$

aside from an ignorable additive constant.

From the Gaussianity of $p_{mn}(\omega)$ we readily deduce that $P(f|p^s)$ is given by

$$\log P(f|p^s) = -\frac{1}{2}\sum_{m,n}\sum_{\omega}(TC_\nu)^{-1}\left|f_{mn} - \int d\vec{r}H_n p^s{}_m\right|^2 \quad ,$$

again ignoring additive constants.

THE LAGRANGE MULTIPLIER METHOD

As is well known, a set of constraints in a maximization problem can be replaced by another problem involving an additive linear combination of the quanitities formerly constrained. In the present case, the quantity to be maximized is (3.5) and the set (now continuous) of constraints is given by (2.8).

Thus, we are lead to consider the variational function

$$\psi = \log P(f|p^s) + \log P(\phi)$$
$$+ \int_D d\vec{r} \sum_{\omega}\sum_{m} \lambda^*_m[(\nabla^2 + k^2(1+\phi))p^s_m + k^2\phi p^i_m] \qquad (4.1)$$

where the functions $\lambda_m = \lambda_m(\vec{r},\omega)$ constitute a set of Lagrange multipliers, i.e., for $m = 1,\ldots,M$ and for all ω and $\vec{r}$. The function ψ is to be maximized with respect to ϕ and to be at least stationarized with respect to λ_m and p^s_m. The variations with respect to these functions are now of course unconstrained.

Setting the variation of ψ with respect to λ_m equal to zero yields the original p.d.e. for the scattering process (i.e., (2.8) with the subscript m added to p^s and p^i), namely

$$(\nabla^2 + k^2(1+\phi))p^s_m + k^2\phi p^i_m = 0 \qquad (4.2)$$

However, setting the variation of ψ with respect to p^s_m equal to zero yields the more complex relation

$$\sum_{\omega}\sum_{m,n}(TC_\nu)^{-1}\left(f_{mn}-\int d\vec{r}H_n p_m^s\right)\int d\vec{r}H_n^*\delta p_m^{s*}$$

$$+\int_D d\vec{r}\sum_{\omega}\sum_{m}[(\nabla^2+k^2(1+\phi))\lambda_m]\delta p_m^{s*}$$

$$+\int_S d\vec{S}\cdot\sum_{\omega}\sum_{m}[\lambda\nabla\delta p_m^{s*}-(\nabla\lambda_m)\delta p_m^{s*}]=0 \qquad (4.3)$$

If we impose the radiation conditions on δp_m^s for $\vec{r}\varepsilon S$, i.e.,

$$\nabla\delta p_m^s = ik\vec{n}\delta p_m^s \quad , \qquad (4.4)$$

and the complex conjugate (i.e., anti-radiation) conditions on λ for $\vec{r}\varepsilon S$, i.e.,

$$\nabla\lambda_m = -ik\vec{n}\lambda_m \quad , \qquad (4.5)$$

where $\vec{n}$ is the outward pointing normal vector (i.e., in the direction $\vec{r}$) associated with the point $\vec{r}$ on the surface S. These conditions imply that the surface integral term in (4.3) vanishes. Since the variations δp_m^s in the remaining volume integral are linearly independent of each other at different points in $\vec{r}$-space. We obtain the result

$$(\nabla^2+k^2(1+\phi))\lambda_m$$

$$+\sum_n (TC)_\nu^{-1} H_n^*\left(f_{mn}-\int d\vec{r}H_n p_m^s\right)=0 \qquad (4.6)$$

Finally, the vanishing of the variation of ϕ yields

$$\int d\vec{r}\,\frac{\partial g}{\partial\phi}(\vec{r},\phi)\delta\phi$$

$$+\int d\vec{r}\sum_{\omega}\sum_{m}\lambda_m^* k^2(p_m^i+p_m^s)\delta\phi=0 \quad . \qquad (4.7)$$

Since the $\delta\phi$ for different values of $\vec{r}$ are linearly independent, it follows that

$$\frac{\partial f}{\partial\phi}(\vec{r},\phi)+\sum_{\omega}\sum_{m}\lambda_m^* k^2(p_m^i+p_m^s)=0 \qquad (4.8)$$

Since ϕ is assumed to vanish identically outside of the localization domain D_L, it follows that (4.8) can be confined to D_L. If $g(\vec{r},\phi)$ has a sufficiently simple form with respect to its dependence on ϕ than (4.8) can be solved analytically in closed form. For example, if

$$g(\vec{r},\phi) = \frac{1}{2}\alpha\phi^2 \tag{4.9}$$

then we obtain

$$\phi = -\alpha^{-1}\sum_{\omega}\sum_{m}\lambda_m^* k^2(p_m^i + p_m^s) \quad , \tag{4.10}$$

$$\vec{r}\varepsilon D_L$$

In the above expression α may be assumed to depend on $\vec{r}$, if one wishes.

Several remarks are in order concerning the nature of p_n^s and λ_m. First of all, the $p_m^s = p_m^s(\vec{r},\omega)$ constitute a set of physical fields that satisfy the radiation condition on the surface S. It follows that p_m^s is causal, i.e., its "response" to all sources will involve the usual causal Green function G_+. On the other hand the conjugate fields $\lambda_m = \lambda_m(\vec{r},\omega)$ are informational in nature and they satisfy the anti-radiation condition on the surface S. It follows that λ_m is anti-causal, i.e., its influence can propagate backwards in time and the informational sources affecting this field will involve the anti-causal Green function G_-. To be sure, λ_n is causal in the sense that it cannot depend upon measurements not yet made, but it is anti-causal in the sense that measurements recently made can influence inferences concerning earlier situations.

DOMAIN REDUCTION

In the previous section we showed how the vanishing of the variations of ψ with respect to p_m^s and λ_m lead to p.d.e.'s to be solved in the large domain D. Here we will show that these equations can be transformed (by a procedure similar to that employed in the direct theory of scattering) into integral equations to be solved in the much smaller localization domain D_L.

By operating on (4.2) by G_+ we obtain

$$p_m^s = -G_+k^2\phi(p_m^s + p_m^i) \quad . \tag{5.1}$$

Here we must use G_+ (instead of G_-) because of the radiation condition (4.4). Similarly, by operating on (4.6) by G_- we obtain

$$\lambda_m = -G_-k^2\phi\lambda_m$$

$$-\sum_n (TC_\nu)^{-1}(G_-H_n^*)(f_{mn} - \int d\vec{r}(H_n p^s)^1) \quad . \tag{5.2}$$

In contrast with the previous case, we must use G_- because of the anti-radiation condition. In each equation we assume that ϕ is given by its optimal form that must satisfy (4.8).

We turn now to a consideration of how it is possible to confine $\vec{r}$ to the localization D_L in the solution of (5.1) and (5.2). We first observe that the r.h. side of (5.1) depends upon p_m^s and p_m^I only in D_L because ϕ vanishes outside of D_L. Furthermore, the optimal form of ϕ depends only on p_m^s and λ_m within D_L. In the r.h. side of (5.2) the same remark applies to λ_m. The remaining terms on the r.h. side of (5.2) require a more complex discussion. The term $\int d\vec{r}'(H_n p_m^s)$ depends upon p_m^s in a region far removed from D_L. However (5.1) can be used to relate p_m^s at any point outside of D_L to p_m^s at all points in D_L. Thus the r.h. sides of (5.1) and (5.2) can be regarded as depending only on p_m^s and λ_m in D_L. Therefore in the solution of these equations the l.h. sides can be similarly confirmed and therefore the entire solution process can be carried out with $\vec{r}\,\varepsilon D_L$.

The reduction of the solution domain from D to D_L is of course a great help in reducing the labor involved in a computational treatment. One can, for example, represent both p_m^s and λ_m in terms of a truncated set of basic functions of $\vec{r}$ defined on the domain D_L. For a given resolution requirement this set would involve far fewer members than would a basic set defined on the much larger domain D.

POSSIBLE EXTENSIONS AND MODIFICATIONS

We have shown how the solution of a relatively simple probabilistic inverse problem involving the scattering of scalar waves can be solved "exactly" in the sense that it can be reduced to a tractable computational procedure for modest accuracy requirements, but can be extended to satisfy more stringent accuracy requirements in an obvious manner.

The methodology can be extended in a conceptually straight forward manner to the full elastodynamic case involving vector waves represented in terms of the displacement field. We could consider a general, locally isotropic, inhomogeneity (defined by unknown spatial distributions of the density and the two Lame' constants in a specified localization domain D_L). We know that the deterministic version of this problem is soluble for the case of weak inhomogeneities if a sufficient diversity of very accurate pitch-catch measurements are made, even if these are limited to longitudinal modes. This result gives us the confidence to assume that the general probabilistic problem, in which the locally isotropic inhomogeneity is not necessarily weak, will not entail serious blind spots.

The electromagnetic case is somewhat simpler than the elastodynamic and hence the extension of our present inversion methodology to this case should be straightforward.

There are several other directions of extensions and modifications that should be considered independently of whether we are dealing with the acoustic, elastodynamic, or electromagnetic cases. Some of them are:

1. Treatment of the integral equations for scattering as constraints instead of the p.d.c.'s.

2. A perhaps clearer elucidation of the causal and anti-causal nature of the physical and conjugate fields by formulating the problem in the time domain instead of the frequency domains.

3. Reformulate the problem in such a manner that the reciprocal relations between transmitters and receivers is clearly manifested.

It is no doubt worthwhile to extend the present methodology toward cases involving less complexity of detail but still with significant conceptual challenge. An example of such an extension is the one-dimensional inverse scattering problem of the type treated in the deterministic version by Corones et al. (Corones, Davison and Krueger, 1983).

REFERENCES

J.P. Corones, M.E. Davison and R.J. Kroeger (1983), "Inverse Optics," Proceedings of the SPIE, Vol. 413.

R.V. Gamkrelidze (1978), "Principles of Optimal Control Theory," Plenum, New York.