PHASE SHIFT DETERMINATION OF SCATTERED FAR-FIELDS
AND ITS APPLICATION TO AN INVERSE PROBLEM

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INTRODUCTION

The phase of the scattered far-field from flaws is focused and the phase shift analysis is carried out to quantify the scattered waveforms. The basic tool of the phase shift analysis is the integral representation of the scattered wave field. The scattering amplitude of the scattered far-field is first defined. The phase shift of the scattered far-field is then introduced by expanding the scattered far-field into partial waves with spherical wave components and taking into account the energy relation. The phase shift introduced represents the shift of phase in the scattered wave from the phase in an incident wave. The far-field integral representation for the scattered field is utilized to derive the explicit expression of the phase shift, and the integral representation of the phase shift is obtained as a surface integral over the flaw. This integral representation is true for arbitrary flaw shape and it relates to the flaw geometry and the boundary conditions on the flaw surface. The boundary element method is adopted for the determination of boundary quantities on the flaw surface. As an application of the phase shifts in the scattered far-field, a flaw shape is reconstructed from the information on the phase shifts.

The phase shift analysis is well known in the field of electromagnetics [1] and the atomic collision theory [2 ~ 5]. The attempt of this paper is to apply the partial wave decomposition to the characterization of scattered wave field from flaw and to relate the partial wave decomposition with the boundary element method. The present treatment is restricted to three dimensional analysis for the scalar wave equation of Helmholtz type.

INTEGRAL REPRESENTATION

The scattering problem for scalar quantity \( u \) is governed by the Helmholtz equation:

\[
\Delta u + k^2 u = 0 \quad \text{in} \, D
\]

where \( k = \omega/c \) is the wavenumber, \( \omega \) and \( c \) are the angular frequency and the wave velocity. The total field \( u \) is decomposed into

\[
u = u^I + u^S
\]
where $u^I$ is the plane incident wave and $u^S$ is the scattered wave. The integral representation for the total field reduces to

$$u(x) = u^I(x) + \int_S U(x,y) \frac{\partial u(y)}{\partial n_y} dy - \int_S \frac{\partial U(x,y)}{\partial n_y} u(y) dy, \quad x \in D$$

(3)

where $U$ is the fundamental solution for three dimensional Helmholtz equation (1) and has the form

$$U(x,y) = \frac{1}{4\pi} \frac{e^{ikr}}{r}$$

(4)

and $r = |x - y|$ is the distance of two points $x$ and $y$. This form of the fundamental solution represents the outgoing wave for the time factor $\exp(-i\omega t)$.

**SCATTERING AMPLITUDE**

The incident wave is assumed to be a plane wave traveling along the $x_3$ axis:

$$u^I(x) = e^{i k x_3}$$

(5)

and we consider the case that the scatterer has the revolutional symmetry with respect to the $x_3$ axis. Then, in the far $(x \to \infty)$ field, the total wave field can be expressed as

$$u(x) = u^I + u^S \sim e^{ikx_3} + \sum_{l=0}^{\infty} \frac{(2l + 1)i^l j_l(kx)}{2i} P_l(\cos \theta)$$

(6)

where, $x = (x, \theta, \varphi)$ and $x = |x| = \sqrt{x_1^2 + x_2^2}$, and the origin of the polar coordinate $(x, \theta, \varphi)$ is located in the center of the scatterer in this expression. In Eq.(6), $f(\theta)$ is the scattering amplitude and $\theta$ is the scattering angle measured from the positive $x_3$ axis. The scattering amplitude $f(\theta)$ is relating to the scattered far field $u^S$, and thus, to the scatterer (the shape and the boundary conditions on the surface of the scatterer).

The incident wave in Eq.(6) can be expressed as

$$e^{ikx_3} = e^{ikx \cos \theta} = \sum_{l=0}^{\infty} (2l + 1)i^l j_l(kx) P_l(\cos \theta)$$

$$\sim \sum_{l=0}^{\infty} i^l \frac{(2l + 1)}{kx} \sin(kx - \frac{l\pi}{2}) P_l(\cos \theta)$$

$$= \sum_{l=0}^{\infty} \frac{(2l + 1)}{2ikx} \left\{ e^{ikx} - (-1)^l e^{-ikx} \right\} P_l(\cos \theta)$$

(7)

in the far-field, where Rayleigh's formula and the asymptotic expression of spherical Bessel function $j_l(\cdot)$ has been used. $P_l(\cdot)$ is the Legendre polynomials. The scattering amplitude $f(\theta)$ is expanded by the complete system $P_l(\cos \theta)$ as

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} \frac{2l + 1}{2i} a_l P_l(\cos \theta)$$

(8)

where $a_l$ is the unknown coefficients. Substitution of Eqs.(7) and (8) into Eq.(6) yields

$$u(x) \sim \sum_{l=0}^{\infty} \frac{2l + 1}{2ikx} \left\{ (1 + a_l)e^{ikx} - (-1)^l e^{-ikx} \right\} P_l(\cos \theta).$$

(9)
PHASE SHIFT

The energy conservation requires that

\[ |1 + a_i| = 1. \quad (10) \]

It suggests to set

\[ 1 + a_i = e^{2\pi \delta_i}. \quad (11) \]

Introducing Eq.(11) into Eq.(8), we have the following expressions

\[
\begin{align*}
 f(\theta) &= \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) [e^{2\pi \delta_l} - 1] P_l(\cos \theta) \\
\end{align*}
\]

(12)

for the scattering amplitude. In Eq.(12), \( f(\theta) \) is now expressed by the unknown quantity \( \delta_l \) \((l = 0, 1, 2 \cdots)\). The \( \delta_l \) is called phase shift.

Equations (9) and (11) lead to the following far field expression for the total field:

\[
\begin{align*}
 u(\mathbf{x}) &\sim \sum_{l=0}^{\infty} \frac{(2l + 1)}{kx} e^{i\pi \delta_l} e^{ikx/2} \sin \left( kx - \frac{l\pi}{2} + \delta_l \right) P_l(\cos \theta). \\
\end{align*}
\]

(13)

Comparing this Eq.(13) with the incident wave form in Eq.(7):

\[
\begin{align*}
 u^t(\mathbf{x}) &= e^{ikx_3} \sim \sum_{l=0}^{\infty} \frac{(2l + 1)}{kx} e^{il\pi/2} \sin(kx - \frac{l\pi}{2}) P_l(\cos \theta), \\
\end{align*}
\]

(14)

we know that the quantity \( \delta_l \) represents the phase shift of the total field measured from the incident field.

DETERMINATION OF PHASE SHIFT

In the expression of fundamental solution (4), the term \( \exp(ikr)/r \) can be expanded in the following partial wave form (Sunakawa[1], p.244):

\[
\begin{align*}
 \frac{e^{ikr}}{4\pi r} &= \frac{ik}{4\pi} \sum_{l=0}^{\infty} (2l + 1) j_l(ky) h_l^{(1)}(kx) P_l(\cos \chi), \quad (x > y) \\
 &= \frac{ik}{4\pi} \sum_{l=0}^{\infty} (2l + 1) j_l(kx) h_l^{(1)}(ky) P_l(\cos \chi), \quad (y > x) \\
\end{align*}
\]

(15a)

(15b)

where \( j_l(\cdot) \) is the spherical Bessel function and \( h_l^{(1)} \) is the spherical Hankel function defined as

\[
 j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+\frac{1}{2}}(z), \quad n_l(z) = \sqrt{\frac{\pi}{2z}} N_{l+\frac{1}{2}}(z)
\]

and

\[
h_l^{(1)}(z) = j_l(z) + in_l(z). \quad (16)
\]

The addition theorem of the spherical function is

\[
 P_l(\cos \chi) = P_l(\cos \theta) P_l(\cos \tilde{\theta})
 + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta) P_l^m(\cos \tilde{\theta}) \cos m(\varphi - \bar{\varphi}) \quad (17)
\]
where \( P^m_l(\cdot) \) is the associated Legendre function, furthermore \( \theta, \varphi, \varphi \), and \( \varphi \) are defined as \( x = (x, \theta, \varphi) \) and \( y = (y, \theta, \varphi) \) as shown in Fig.1. For \( x > y \), Eq.(15) is now written as

\[
\frac{e^{ikr}}{4\pi r} = \frac{ik}{4\pi} \sum_{l=0}^{\infty} (2l + 1)j_l(ky)h_l^{(1)}(kx)P_l(\cos \theta)P_l(\cos \bar{\theta})
\]

\[
+ [\text{terms containing } \cos m(\varphi - \varphi)]
\]

with the help of Eq.(17). In the far field, the function \( h_l^{(1)}(\cdot) \) behaves

\[
h_l^{(1)}(kx) \sim (-i)^{l+1}e^{ikr}/kx, \quad (x \to \infty).
\]

From Eqs.(18) and (19), the fundamental solution (4) has the following expression in the far field

\[
U(x, y) = \frac{1}{4\pi} \frac{e^{ikr}}{r} \sim \frac{-1}{4\pi} \frac{e^{ikz}}{z} \sum_{l=0}^{\infty} (2l + 1)(-i)^{l+2}j_l(ky)P_l(\cos \theta)P_l(\cos \bar{\theta})
\]

\[
+ [\text{terms containing } \cos m(\varphi - \varphi)], \quad (x \to \infty).
\]

Substitution of Eq.(20) into the integral representation (3) leads to the following far field representation:

\[
u(x) \sim e^{ikx_3} + \frac{e^{ikz}}{x} \left[ \frac{-1}{4\pi} \int_S \frac{\partial u(y)}{\partial n_y} \sum_{l=0}^{\infty} (2l + 1)(-i)^{l+2}j_l(ky)P_l(\cos \theta)P_l(\cos \bar{\theta})dS_y
\]

\[
- \frac{1}{4\pi} \int_S u(y)ikK(x, y) \sum_{l=0}^{\infty} (2l + 1)(-i)^{l+2}j_l(ky)P_l(\cos \theta)P_l(\cos \bar{\theta})dS_y \right] , (x \to \infty)
\]

where the relation

\[
\frac{\partial U(x, y)}{\partial n_y} = n_i(y) \frac{\partial U(x, y)}{\partial y_i} = -ik \left( 1 + \frac{i}{kx} \right) n \cdot \hat{r}U(x, y)
\]

\[
\sim -ikK(x, y)U_{\text{far}}(x, y)
\]

has been used and the expression of \( U_{\text{far}}(x, y) \) has been defined in Eq.(20), and \( K(x, y) \) has the form

\[
K(x, y) = \left\{ 1 + \frac{i}{kx} \left( 1 + \frac{y\cos \chi}{x} \right) \right\} u(y) \cdot \hat{r}(y)
\]

and the terms containing \( \cos m(\varphi - \varphi) \) in Eq.(20) reduce to zero in the integration process over the surface \( S \).

![Fig.1. Geometrical relation.](image-url)
The comparison of Eqs.(21) and (6) leads to the integral representation for the scattering amplitude

\[
 f(\theta) = \frac{-1}{4\pi} \int_S \left\{ \frac{\partial u(y)}{\partial n_y} + ikK(x, y)u(y) \right\}
 \times \sum_{l=0}^{\infty} (2l + 1)(-i)^{l+2} j_l(ky)P_l(\cos \theta)P_l(\cos \vartheta)dS_y
\]

\[
 = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)(-i)^{l+2} \left[ \frac{-2ik}{4\pi} \int_S \left\{ \frac{\partial u(y)}{\partial n_y} + ikK(x, y)u(y) \right\} \right]
 \times j_l(ky)P_l(\cos \vartheta)dS_y \right\}P_l(\cos \theta).
\]

(24)

Furthermore, the comparison of Eqs.(24) and (12) leads to the integral representation for the phase shift

\[
 e^{2i\delta_l} - 1 = -(-i)^{l+1} \frac{k}{2\pi} \int_S \left\{ \frac{\partial u(y)}{\partial n_y} + ikK(x, y)u(y) \right\} j_l(ky)P_l(\cos \vartheta)dS_y,
\]

\[
 (l = 0, 1, 2 \cdots)
\]

(25)

where \( K(x, y) \) has been defined in Eq.(23).

Knowing the boundary quantities \( u(y) \) and \( \partial u(y)/\partial n \), we can determine the phase shifts \( \delta_l \) \( (l = 0, 1, 2 \cdots) \) from Eq.(25). From the asymptotic expansion of Bessel functions around the origin, \( j_l(ky) \) in Eq.(25) behaves

\[
 j_l(ky) \sim \frac{(ky)^l}{(2l + 1)!! \left( 1 - \frac{(ky)^2}{2(2l + 3)} + \cdots \right)}, \quad \text{when} \quad ky \to 0
\]

(26)

where

\[
 (2l + 1)!! = (2l + 1)(2l - 1)(2l - 3) \cdots 5 \cdot 3 \cdot 1.
\]

(27)

When the wavelength \( \lambda (= 2\pi/k) \) of the incident wave is large compared with the typical dimension of the scatterer, the convergence of the algebraic system in Eq.(25) to determine \( \delta_l \) \( (l = 0, 1, 2 \cdots) \) is expected to be fairly good.

Boundary quantities on the right-hand side in Eq.(25) are determined by solving the boundary integral equations for the given boundary condition on the surface of the scatterer.

Tables 1 and 2 show the phase shifts \( \delta_l \) \( (l = 0, 1, 2 \cdots) \) calculated from Eq.(25). For both tables the boundary condition is \( u = 0 \) on the scatterer. In Table 1, the shape of the scatterer is sphere. The phase shifts are designated for nondimensional wavenumbers \( ak = 0.1 \) and \( ak = 0.5 \), where \( a \) is the radius of sphere and \( k \) is the wave number. For the case of sphere, the integration in Eq.(25) can be performed analytically, and the analytical values of \( \delta_l \) have been shown in the column of “ANAL” in the table. The values in the column of “BEM” were obtained from Eq.(25) and the unknown \( \partial u/\partial n \) was evaluated with the help of the boundary element method. For both cases of \( ak = 0.1 \) and 0.5, the convergence for \( l \) \( (l = 0, 1, 2 \cdots) \) is quite good. In Table 2, the phase shifts for several shapes of scatterer are shown for \( ak = 0.5 \).

AN INVERSE PROBLEM

When we know the phase shifts \( \delta_l \) \( (l = 0, 1, 2 \cdots) \) in the far field, we now want to determine the shape of the scatterer \( S \).
Table 1. Convergence check of the phase shifts \( \delta_l (l = 0, 1, 2, \cdots) \) for sphere.

<table>
<thead>
<tr>
<th>ak=0.1</th>
<th>ak=0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_0 )</td>
<td>( \delta_0 )</td>
</tr>
<tr>
<td>0</td>
<td>-5.64E+0</td>
</tr>
<tr>
<td>1</td>
<td>-1.85E-2</td>
</tr>
<tr>
<td>2</td>
<td>E-5</td>
</tr>
<tr>
<td>3</td>
<td>E-7</td>
</tr>
</tbody>
</table>

Table 2. Phase shifts \( \delta_l (l = 0, 1, 2, \cdots) \) for \( ak = 0.5 \).

| \hline \( \delta_0 \) | \( \delta_1 \) | \( \delta_2 \) | \( \delta_3 \) |
| sphere | semi-sphere | semi-sphere | strip-sphere |
| 0 | -2.82E+1 | -2.28E+1 | -2.70E+1 | -2.10E+1 |
| 1 | -2.03E+0 | -1.53E+0 | -6.10E-1 | -1.76E-1 |
| 2 | -2.93E-2 | -6.78E-2 | -7.11E-2 | -1.12E-1 |
| 3 | E-4 | E-4 | E-4 | E-4 |

For example, we consider the case of the boundary condition \( u = 0 \) on \( S \). Then, the integral representation (25) to determine the phase shifts \( \delta_l \) reduces to

\[
e^{2i\delta_l} - 1 = -(-i)^{l+1} \frac{k}{2\pi} \int_S \frac{\partial u(y)}{\partial n_y} j_l(ky) P_l(\cos \theta) dS_y, \quad (l = 0, 1, 2, \cdots).
\]  

On the surface of the scatterer \( S \), the following integral equation holds

\[
\int_S \frac{\partial u(y)}{\partial n_y} U(x, y) dS_y = -u^I(x), \quad x \in S
\]  

(28)

From Eq.(3). For the given incident field \( u^I \), we now suppose that the phase shifts \( \delta_l \) (\( l = 0, 1, 2, \cdots \)) have been measured at the far field. In this case, Eqs.(28) and (29) can be considered as the coupled integral equations to determine the boundary shape \( S \) of the scatterer. Equation (29) is the integral equation to determine the boundary quantity \( \partial u/\partial n \) and Eq.(28) is the equation to determine the boundary shape \( S \).

In general, the surface \( S \) has infinite number of degree of freedom. Equation (28) also has, in principle, infinite number of equations for each \( \delta_l \) (\( l = 0, 1, 2, \cdots \)) to determine the surface parameters. In practice the boundary surface \( S \) may be represented as a function of \( n \) parameters of \( \beta_i \) (\( i = 1, 2, \cdots, n \)). In this case, Eq.(28) with \( n \) phase shifts \( \delta_l \) (\( l = 0, 1, 2, \cdots, n-1 \)) can be considered as \( n \) equations to determine the \( n \) boundary shape parameters \( \beta_i \) (\( i = 1, 2, \cdots, n \)). The other choice of \( n \) equations is to use Eq.(28) for the zeroth order phase shifts \( \delta_0(k_i) \) for \( n \) different wavenumbers \( k_i \) (\( i = 1, 2, \cdots, n \)). In this case, it is necessary to calculate \( \partial u/\partial n \) for \( n \) different wavenumbers \( k_i \) from Eq.(29).
Now we restrict our attention to the ellipsoidal scatterer which has revolutional symmetry with respect to the $x_3$ axis in the following form:

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{a}\right)^2 + \left(\frac{x_3}{b}\right)^2 = 1 \quad (30)$$

where

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta \quad (31)$$
in the polar coordinate. Substitution of Eq.(31) into (30) leads to the expression

$$r^2 = \frac{1}{\left\{ \left(\frac{\sin \theta \cos \varphi}{a}\right)^2 + \left(\frac{\sin \theta \sin \varphi}{a}\right)^2 + \left(\frac{\cos \theta}{b}\right)^2 \right\} } . \quad (32)$$

Equation (31) with $r$ in Eq.(32) yields the expression of the ellipsoidal surface and it contains two parameters $a$ and $b$.

To determine two parameters $a$ and $b$, Newton's method is used. First we consider the following two equations

$$F_1(a, b) = 0$$
$$F_2(a, b) = 0 \quad (33)$$

We assume that

$$F_1(a + \Delta a, b + \Delta b) = 0$$
$$F_2(a + \Delta a, b + \Delta b) = 0 \quad (34)$$

Then it follows that

$$\frac{\partial F_1}{\partial a} \Delta a + \frac{\partial F_1}{\partial b} \Delta b = -F_1(a, b)$$
$$\frac{\partial F_2}{\partial a} \Delta a + \frac{\partial F_2}{\partial b} \Delta b = -F_2(a, b)$$

for the small increments of $\Delta a$ and $\Delta b$. These $\Delta a$ and $\Delta b$ can be obtained as

$$\left\{ \begin{array}{c} \Delta a \\ \Delta b \end{array} \right\} = - \left[ \begin{array}{cc} \frac{\partial F_1}{\partial a} & \frac{\partial F_1}{\partial b} \\ \frac{\partial F_2}{\partial a} & \frac{\partial F_2}{\partial b} \end{array} \right]^{-1} \left\{ \begin{array}{c} F_1(a, b) \\ F_2(a, b) \end{array} \right\} \quad (35)$$

from Eq.(35).

For the determination of two parameters $a$ and $b$, we adopt two zeroth order phase shifts $\delta_0(k_1)$ and $\delta_0(k_2)$ for two different wave numbers $k_1$ and $k_2$. Then the integral equations (28) and (29) to determine the boundary shape parameters and the boundary quantity $\partial u/\partial n$ can be written as

$$F_1(a, b) = \frac{-(-i) k_1}{2\pi} \int_{S(a, b)} \frac{\partial u(y)}{\partial n_y} j_0(k_1 y) P_0(\cos \varphi) dS_y - \left[ e^{2i\delta_0(k_1)} - 1 \right]$$

$$F_2(a, b) = \frac{-(-i) k_2}{2\pi} \int_{S(a, b)} \frac{\partial u(y)}{\partial n_y} j_0(k_2 y) P_0(\cos \varphi) dS_y - \left[ e^{2i\delta_0(k_2)} - 1 \right]$$

$$\int_{S(a, b)} \frac{\partial u(y)}{\partial n_y} U(x, y) dS_y = -u^f(x) \quad \text{for } k = k_1 \text{ and } k_2 \quad (37)$$

It is necessary to evaluate the $\partial F_1/\partial a$, $\partial F_1/\partial b$, $\partial F_2/\partial a$, $\partial F_2/\partial b$ to determine the increments $\Delta a$ and $\Delta b$ in Eq.(36). It can be done numerically. For example, $\partial F_1/\partial a$ can be evaluated at $(a_0, b_0)$ as
Fig. 2. Convergence process for the sphere of $a = b = 1$
(Initial values for $a$ and $b$ are $a_0 = 2$, $b_0 = 1.5$).

\[
\frac{\partial F_1(a_0, b_0)}{\partial a_0} = \frac{(-i)k_1}{2\pi} \left[ \int_{S(a_0 + \Delta \alpha, b_0)} \frac{\partial u}{\partial n} j_0(k_1y) P_0(\cos \bar{\theta}) dS_y - \int_{S(a_0, b_0)} \frac{\partial u}{\partial n} j_0(k_1y) P_0(\cos \bar{\theta}) dS_y \right] / \Delta \alpha
\]

(38)

In this expression, $\partial u / \partial n$ on the surfaces $S(a_0 + \Delta \alpha, b_0)$ and $S(a_0, b_0)$ can be calculated from integral equations

\[
\int_{S(a_0 + \Delta \alpha, b_0)} \frac{\partial u}{\partial n} U(x, y) dS_y = -u'(x)
\]

(39)

\[
\int_{S(a_0, b_0)} \frac{\partial u}{\partial n} U(x, y) dS_y = -u'(x)
\]

for the given wave number $k_1$.

Fig. 2 shows the convergence process for the sphere of $a = b = 1$. The initial values for $a$ and $b$ are chosen to be $a_0 = 2$ and $b_0 = 1.5$. The phase shifts used in Eq.(37) have been shown in Table 1 for $ak_1 = 0.1$ and $ak_2 = 0.5$. At the 3rd iteration, Newton’s method based on Eqs.(36) and (37) converges to the final values of $a = b = 1.0$ within an acceptable error bound.

REFERENCES

2. D. Bohm, Quantum Theory (Prentice-Hall, 1951) Part 5.