LAMB WAVE MODES PROPAGATING ALONG ARBITRARY DIRECTIONS IN AN ORTHOTROPIC PLATE

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INTRODUCTION

The recent interests [1-5] on the Lamb wave study of orthotropic sheet materials have been motivated by nondestructive evaluation (NDE) and characterization of the fiber composite plate. The dispersive properties of Lamb waves in an isotropic plate have been well known since the original work of Lamb [6]. However, the problem of Lamb waves in anisotropic plates is, as yet, largely unexplored. For the propagation of elastic waves in composites reinforced with large diameter fibers, such as the SiC fibers employed to reinforce both ceramic and metallic matrix materials, the dynamic effects of the microstructuring must be considered if the fiber is large enough to equal a longitudinal acoustic quarter-wavelength in the range of 10 to 20 MHz. However, in most fiber composite systems, the fiber diameter is small enough to permit modelling of the material as a homogeneous, but anisotropic, medium which retains the symmetry of the composite, but ignores its microstructural nature.

Many researchers [1-5] have considered the composite plate as an equivalent homogeneous orthotropic material and studied the dispersion characteristics of Lamb waves using the effective stiffness matrix. However, in all these theories, they only considered Lamb waves propagating along the symmetric principal directions. This will simplify greatly the Lamb wave motion because the SH wave modes polarized parallel to the plate surface are not coupled to the dilatational and flexural modes in the principal directions. In this paper, we will develop a general theory of the coupled Lamb waves in the orthotropic fiber composite plate for propagation along an arbitrary direction. There will no longer be a family of SH modes independent the dilatational and flexural modes in the nonsymmetric directions, and Lamb waves can only be classified as antisymmetric and symmetric modes with respect to the median plane. The dispersion relations of coupled Lamb waves are derived, and dispersive characteristics of coupled Lamb waves are analyzed numerically for a transversely isotropic unidirectional fiber reinforced composite.

ELASTIC WAVE IN AN ORTHOTROPIC PLATE

As shown in Figure 1, we suppose that the fiber direction is parallel to the plate surface, and the (x_1, x_2, x_3) is the symmetric material coordinate system. In the material coordinate system, the stress-strain relation can be written as

$$\begin{pmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{pmatrix}=\begin{pmatrix}c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{pmatrix} \tag{1a}$$

where the strain ε_{ij} is related to the displacement U_i

$$\varepsilon_{ij}=\left(U_{i,j}+U_{j,i}\right)/2, \quad (i,j=1,2,3). \tag{1b}$$

The dynamic elastic wave equations is

$$\rho\ddot{U}_i=\sum_{j=1}^{3}\tau_{ij,j}, \quad (i=1,2,3), \tag{2a}$$

with the stress free boundary conditions at surfaces $x_3=\pm h$

$$\sum_{j=1}^{3}\tau_{ij}n_j=0, \quad (i=1,2,3), \tag{2b}$$

where $(n_1,n_2,n_3)=(0,0,\pm1)$ are the normal vectors at the surfaces. Now, we consider the Lamb wave propagating along an arbitrary direction ϑ in (x_1-x_2) plane with the wavenumber k, as shown in Figure 1,

$$U_i(x_1,x_2,x_3,t)=\tilde{U}_i(k,\vartheta,x_3,t)\exp ik(x_1\cos\vartheta+x_2\sin\vartheta), \tag{3}$$

then Eq.(2) can be expressed by

$$\hat{L}(\tilde{U})=\rho\ddot{\tilde{U}}, \quad -h<x_3<h, \tag{4a}$$

$$\hat{B}(\tilde{U})=0, \quad \text{at } x_3=\pm h, \tag{4b}$$

where $\tilde{U}=(\tilde{U}_1,\tilde{U}_2,\tilde{U}_3)^{\mathrm{T}}$ (the superscript T represents transpose), the operator $\hat{L}$ is a symmetric matrix defined by the elements

$$L_{11}=-a_{11}+c_{55}d^2/dx_3^2, \quad L_{12}=-a_{12}, \quad L_{13}=-a_{11}+ia_{13}d/dx,$$
$$L_{22}=-a_{22}+c_{44}d^2/dx_3^2, \quad L_{23}=ia_{23}d/dx,$$
$$L_{33}=-a_{33}+c_{33}d^2/dx_3^2,$$

where the a_{ij} $(i,j=1,2,3)$ are

$$a_{11}=k^2(c_{11}\cos^2\vartheta+c_{66}\sin^2\vartheta), \quad a_{12}=\tfrac{1}{2}k^2(c_{12}+c_{66})\sin 2\vartheta,$$
$$a_{13}=k(c_{13}+c_{55})\cos\vartheta, \quad a_{22}=k^2(c_{66}\cos^2\vartheta+c_{22}\sin^2\vartheta),$$
$$a_{23}=k(c_{23}+c_{44})\sin\vartheta, \quad a_{33}=k^2(c_{55}\cos^2\vartheta+c_{44}\sin^2\vartheta),$$

and the boundary operator $\hat{B}$ is

$$\hat{B}=\begin{pmatrix}c_{55}\partial/\partial x_3 & 0 & ic_{55}k\cos\vartheta\\ 0 & c_{44}\partial/\partial x_3 & ic_{44}k\sin\vartheta\\ ic_{31}k\cos\vartheta & ic_{32}k\sin\vartheta & c_{33}\partial/\partial x_3\end{pmatrix}.$$

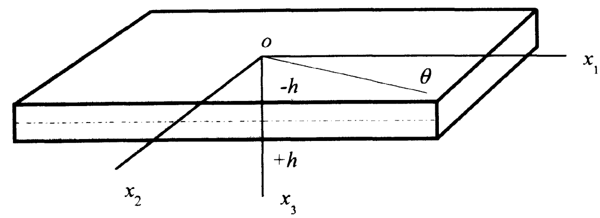

Figure 1. Geometry of an orthotropic fiber composite plate, showing the Lamb wave propagating direction ϑ.

COUPLED LAMB WAVE MODES

The problem of coupled Lamb wave modes is to solve the eigenvalue problem of the operator $\hat{L}$, as follows,

$$\hat{L}[\psi_n] = -\rho\omega_n^2\psi_n, \qquad -h < x_3 < h, \tag{5a}$$

$$\hat{B}[\psi_n] = 0, \qquad \text{at } x_3 = \pm h, \tag{5b}$$

where the ω_n are eigen-frequencies and the $\psi_n = (\psi_{1n}, \psi_{2n}, \psi_{3n})^T$ are corresponding eigen-modes. The relations between ω_n and k are the dispersion equation of the Lamb wave modes. The problem of wave propagation in an isotropic plate can be analyzed by two different methods, namely, the potential function analysis and the partial wave technique. The latter has two significant advantages that it leads more directly to wave solutions and also provides more insight into the physical nature of the waves. In anisotropic plate problems there is no real choice of analytical method and only the partial wave fields is suitable. The general solution of the Eq.(5a) can be constructed by the standing wave forms of the partial waves $\exp(i\lambda_l x_3)$ and $\exp(-i\lambda_l x_3)$ (l=p,q,r)

$$\psi_{jn} = i\sum_{l=p,q,r} \lambda_l \chi_{jl}(B_l \sin\lambda_l x_3 - A_l \cos\lambda_l x_3), \quad (j = 1 \text{ and } 2) \tag{6a}$$

$$\psi_{3n} = \sum_{l=p,q,r}(A_l \sin\lambda_l x_3 + B_l \cos\lambda_l x_3), \tag{6b}$$

where the partial wavenumbers λ_l^2 $(l = p,q,r)$ are three roots of the algebraic equation

$$\det(\Lambda_{ij}) = 0, \tag{7}$$

where Λ is the symmetric matrix with elements

$$\Lambda_{11} = (\rho\omega_n^2 - a_{11}) - c_{55}\lambda_l^2, \quad \Lambda_{12} = -a_{12} - c_{45}\lambda_l^2, \quad \Lambda_{13} = -a_{13}\lambda_l,$$
$$\Lambda_{22} = (\rho\omega_n^2 - a_{22}) - c_{44}\lambda_l^2, \quad \Lambda_{23} = -a_{23}\lambda_l,$$
$$\Lambda_{33} = (\rho\omega_n^2 - a_{33}) - c_{33}\lambda_l^2 .$$

Eq.(7) is derived by substituting the partial waves $\exp(i\lambda_l x_3)$ into Eq.(5a) and taking the determinant of a system of three homogeneous linear equations to vanish. The coefficients χ_{jl} $(j = 1 \text{ and } 2)$ are

$$\chi_{1l} = (a_{13}\Lambda_{22} + a_{12}a_{23}) / \Delta, \quad \chi_{2l} = (a_{23}\Lambda_{11} + a_{13}a_{21}) / \Delta,$$
$$\Delta = \Lambda_{11}\Lambda_{22} - a_{21}{}^2 .$$

The eigen-modes [Eq.(6)] can be classified as symmetric and antisymmetric with respect to the median plane (z=0),

$$\psi_j^{na} = i \sum_{l=p,q,r} \lambda_l \chi_{jl} B_l \sin\lambda_l x_3, \quad (j = 1 \text{ and } 2), \tag{8a}$$

$$\psi_3^{na} = \sum_{l=p,q,r} B_l \cos\lambda_l x_3, \tag{8b}$$

for antisymmetric modes and

$$\psi_j^{ns} = -i \sum_{l=p,q,r} \lambda_l \chi_{jl} A_l \cos\lambda_l x_3, \quad (j = 1 \text{ and } 2), \tag{9a}$$

$$\psi_3^{ns} = \sum_{l=p,q,r} A_l \sin\lambda_l x_3, \tag{9b}$$

for symmetric modes. Substituting Eq.(8) or (9) into the boundary condition [Eq.(5b)] yields the dispersion relations

$$\det(\mathrm{B}_{il}) = 0, \tag{10}$$

where B_{il} $(i = 1,2,3$ and $l = p,q,r)$ are

$$\mathrm{B}_{1l} = \lambda_l^2 \chi_{1l} + k\cos\vartheta, \quad \mathrm{B}_{2l} = \lambda_l^2 \chi_{2l} + k\sin\vartheta,$$

$$\mathrm{B}_{3l} = \left(c_{31}\chi_{1l} k\cos\vartheta + c_{32}\chi_{2l} k\sin\vartheta + c_{33}\right)\lambda_l\left(\tan\lambda_l h\right)^{(\pm 1)}.$$

Here the (+1) and (-1) in the elements B_{3l} are corresponding to the antisymmetric and symmetric modes, respectively.

DISPERSION CURVES

The dispersive characteristics of coupled Lamb waves are analyzed numerically for a unidirectional fiber reinforced composite. The fiber direction is along x_1 axis, then the

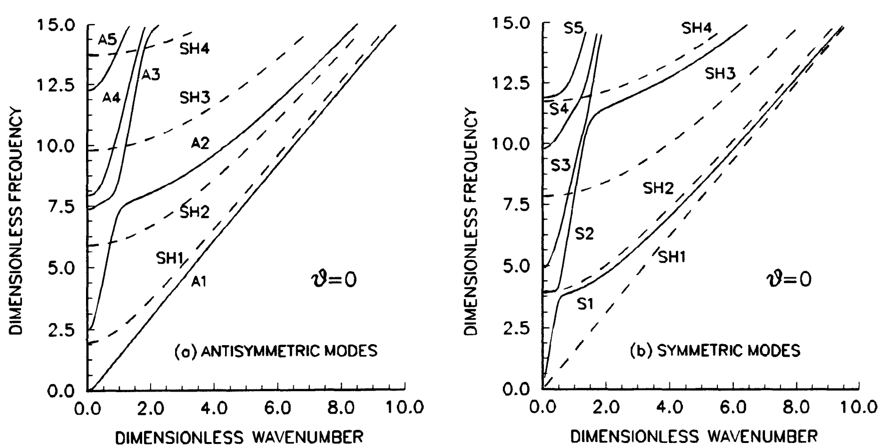

Figure 2. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes in a unidirectional fiber reinforced composite plate propagating along the fiber direction. The vertical axis is kh and the horizontal axis is $\omega h/\mathrm{C}$ $(\mathrm{C} = 10^3 m/\sec)$.

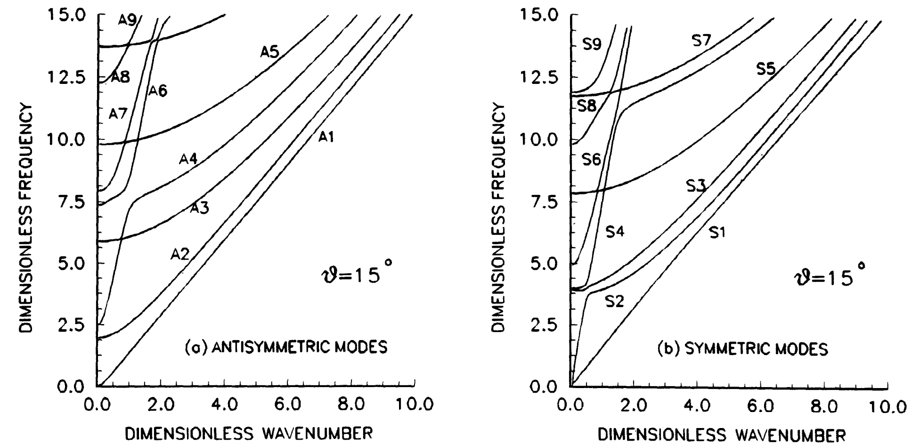

Figure 3. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes propagating along $\vartheta = 15°$ off the fiber direction.

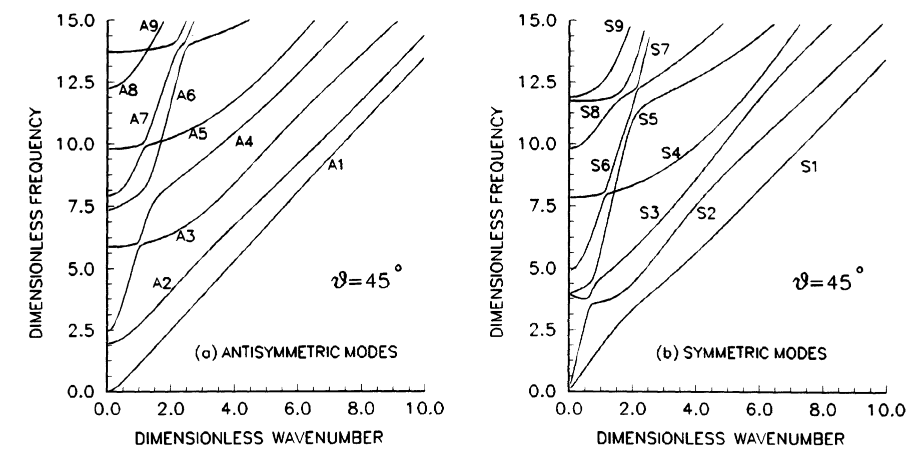

Figure 4. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes propagating along $\vartheta = 45°$ off the fiber direction.

$(x_2 - x_3)$ plane is an isotropic plane. There are only five independent stiffness constants for a transversely isotropic composite: $c_{33} = c_{22}$, $c_{44} = (c_{33} - c_{23})/2$, $c_{13} = c_{12}$, $c_{66} = c_{55}$. In simulations, five independent elastic constants are $c_{11} = 155.44$ GPa, $c_{22} = 15.90$GPa, $c_{12} = 9.10$ GPa, $c_{23} = 8.14$ GPa, and $c_{55} = 6.08$ GPa. Figures 2-7 depict the frequency spectrum of the Lamb wave modes along different propagating directions.

(1) Along the fiber direction ($\vartheta = 0$): the three types of free plate modes (the pure SH, dilatational, and flexural) can be obtained, as shown in Figure 2. The SH modes are not coupled to the dilatational and flexural modes, as evidenced by their crossings. The dispersion curves are somewhat more complicated than the isotropic plate curves, and most striking difference between the anisotropic and isotropic flexural and dilatational curves is the oscillatory manner in which they approach their asymptotic limits.

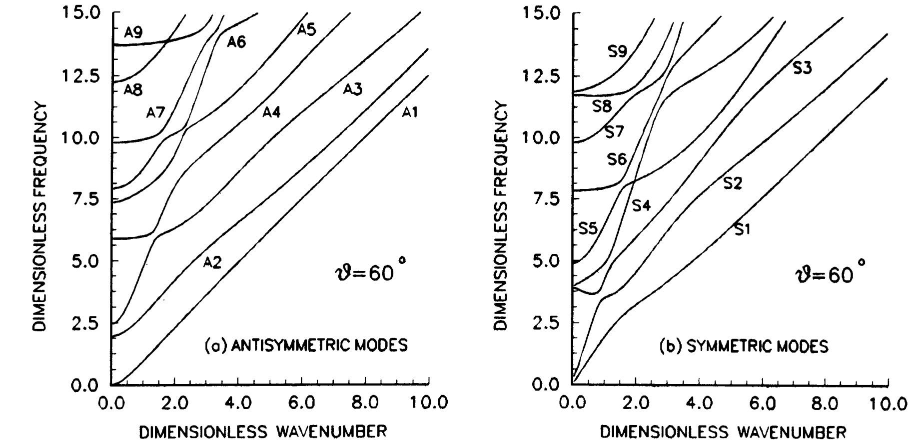

Figure 5. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes propagating along $\vartheta = 60°$ off the fiber direction.

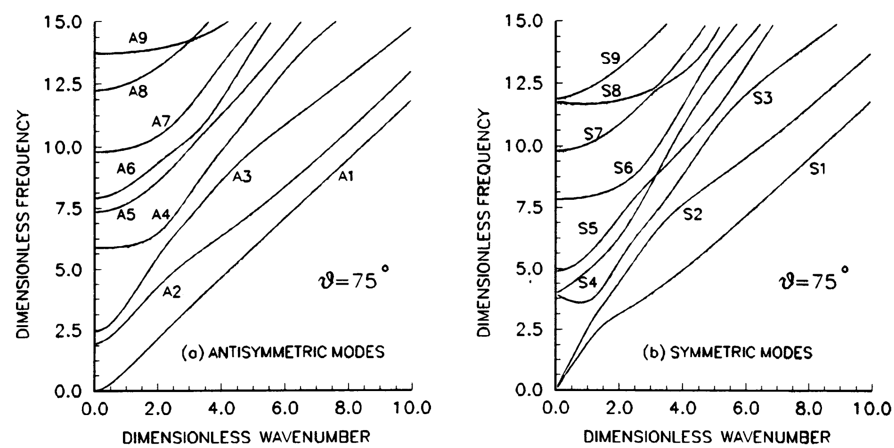

Figure 6. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes propagating along $\vartheta = 75°$ off the fiber direction.

(2) Near the fiber direction: as shown in Figure 3 for $\vartheta = 15°$, Superficially the dispersion curves in Figure 3 appear much like the curves of Figure 2 for $\vartheta = 0°$, but a closer examination reveals a significant difference, namely, there will no longer be a family of SH modes independent of the flexural and dilatational modes and every crossing in Figure 2 corresponds to a splitting in Figure 3. All partial waves are coupled and the free plate modes can only be classified as symmetric and antisymmetric with respect to the median plane.

(3) As the propagating direction departs greatly from the fiber direction, as shown in Figures 4, 5 and 6 for $\vartheta = 45°$, 60° and 75°. The mode splitting is much more apparent than that near the fiber direction [compared Figures 4, 5 and 6 with Figure 3].

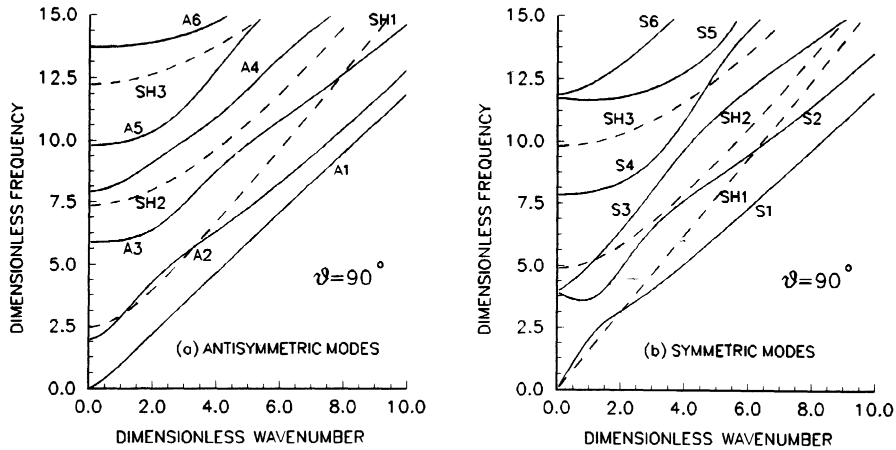

Figure 7. Dispersion curves of the antisymmetric (a) and symmetric (b) Lamb wave modes propagating along $\vartheta = 90°$ perpendicular to the fiber direction.

(4) Along the x_2-axis perpendicular to the fiber direction ($\vartheta = \pi/2$), as shown in Figure 7, we can also obtain the decoupled SH wave independent of the flexural and dilatational motions. In addition, the dispersion curves are same as the isotropic plates because the $(x_2 - x_3)$ plane is isotropic.

CONCLUSION

In summary, the dispersion relation of coupled Lamb wave modes in an orthotropic fiber composite plate along an arbitrary propagating directions has been presented and the dispersion curves are analyzed numerically for a unidirectional fiber reinforced composite, in which the dispersion displays an interesting transition from strongly anisotropic along the fiber direction to transversely isotropic perpendicular to the fiber direction. The formalism provides a quantitative tool for analyzing excitation of Lamb waves and determining material parameters in an orthotropic fiber composite plate.

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