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Embedding subfields of quasi-local rings in residue fields

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EMBEDDING SUBFIELDS OF QUASI-LOCAL RINGS IN RESIDUE FIELDS

by

Henry Gilbert Bray

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I. INTRODUCTION

In this Introduction, we shall set down some conventions, state some definitions, and quote some well-known results; our purpose is to give background and motivation for the material considered in the body of this paper.

Unless explicit mention is made to the contrary, the terminology which we shall use will be that of [5] and [6]; we shall also have occasion to quote standard results which are either readily derived or are to be found in [5] and [6]. We shall denote the set of positive integers by "I^+". We call attention to the following facts:

1) Suppose that A is a commutative ring with identity, and that N is a maximal ideal in A. Then the quotient ring A/N is a commutative ring with identity which has no proper ideals; hence, A/N is a field.

2) Suppose that A is a commutative ring with identity, and that the set of non-units of A is an ideal, N, in A. Then N is a maximal ideal in A and A has precisely one maximal ideal, viz., N; furthermore, if B is any ideal in A and $B \neq A$, then $B \subseteq N$.

We pause to make a definition.

Definition 1: We say that A is a quasi-local ring iff A is a commutative ring with identity and the set of non-units of A is an ideal in A.

We shall always denote the maximal ideal of non-units by "N". We continue with another fact.

3) Suppose that A is a quasi-local ring which contains a field K, and that the identity of K is the identity of A; let N be the ideal of non-units in A, and let ϕ be the natural homomorphism of A onto A/N. Then, the restriction, $\phi|_K$, of ϕ to K is an isomorphism, and $\phi(K)$ is a subfield of A/N. ($\text{Ker } \phi|_K = K \cap N = \{0\}$).

Definition 2: Suppose that A is a quasi-local ring; let N be the ideal of non-units in A, and let ϕ be the natural homomorphism of A onto A/N. We say that _B is a set of representatives_ (or that _B is a complete residue set_) iff B satisfies the properties:

i) $B \subseteq A$.

ii) $\phi|_B$ is one-to-one on B.

iii) $\phi(B) = A/N$.

Suppose that A is a quasi-local ring, and that B is a set of representatives in A. In general, B need not be closed under either addition or multiplication; there are cases, however, where B is not only closed under both addition and multiplication, but is in fact a field.

Definition 3: Suppose that A is a quasi-local ring. We say that _B is a coefficient field_ (or that _B is a field of representatives_) iff B satisfies the two conditions:

i) B is a set of representatives in A; and

ii) B is a field.

It is of some interest, at this point, to raise the following questions:

Suppose that A is a quasi-local ring which contains a field K, and is such that the identity of K is the identity of A.

Question I: When does there exist a coefficient field?

Question II: When does there exist a coefficient field containing K?

(In connection with these questions, see the Remark on p. 46 of [1].)

Before discussing these two questions, we give two more definitions:

<u>Definition 4</u>: Let A be a quasi-local ring. We say that <u>A is equicharacteristic</u> iff A and A/N have the same characteristic.

<u>Definition 5</u>: We say that <u>A is a local ring</u> iff A satisfies the properties:

i) A is a quasi-local ring

ii) $\bigcap_{i=1}^{\infty} N^i = \{0\}$.

Remark 1: For a given $i \in I^+$, the symbol "N^i" denotes the ideal consisting of all sums of products, i in length, of elements of N.

Remark 2: We emphasize that the only distinction that we are making between "quasi-local" and "local" is the one given by Property ii of Definition 5.

Remark 3: In [5], Zariski and Samuel define a local ring as a Noetherian quasi-local ring; Corollary 2, p. 217 of [5] shows that Property ii of Definition 5 holds in a Noetherian quasi-local ring.

Property ii of Definition 5 is used rather extensively to construct a Hausdorff topology on a local ring (the so-called "Zariski topology" - see pp. 251-253 of [6]).

In case a local ring is complete in the Zariski topology (and is equicharacteristic), one can prove the following theorem, due to I. S. Cohen:

"An equicharacteristic complete local ring A admits a field of representatives."

(This is Theorem 27, p. 304, of [6].) Cohen's theorem, then, gives sufficient conditions for answering Question I.

In this paper, we shall give sufficient conditions for answering Questions I and II; we shall not make any use of Property ii of Definition 5, but shall replace said property by other conditions, to be specified below. We now define the basic structure in which we shall be working.

<u>Definition 6</u>: We say that <u>A is a quasi-local p-algebra</u> iff A satisfies the conditions:

1) A is a quasi-local ring

2) A contains a field (say, K).

3) The identity of K is the identity of A.

4) K has characteristic p, p a prime.

5) $F = A/N$ is a purely-inseparable extension of $\emptyset(K)$ with bounded exponent (i.e., $\exists$ i $\in I^+ \ni F^{p^i} \subseteq \emptyset(K)$).

We shall call K the <u>base-field</u> of the algebra A.

Remark 4: If one assumes that A is a commutative ring with identity which satisfies Conditions 2, 3, and 4 of Definition 6, then one can show that A has characteristic p, and that A is equicharacteristic.

Remark 5: In connection with Condition 5, Definition 6, we quote Corollary 2, p. 280, of [6]:

"Let A be a complete local ring having the same characteristic p as its residue field A/N. Then $\exists$ a subfield L of A $\ni$ A/N is purely-inseparable over $\emptyset(L)$."

<u>Definition 7</u>: Suppose that A is a quasi-local p-algebra, and let $F = A/N$; $\forall$ i $\in I^+$, we make the definitions

$$A^{p^i} = \{x \mid x \in A \text{ and } \exists\, a \in A \ni x = a^{p^i}\}$$
$$N^{p^i} = \{x \mid x \in N \text{ and } \exists\, n \in N \ni x = n^{p^i}\}$$
$$F^{p^i} = \{x \mid x \in F \text{ and } \exists\, f \in F \ni x = f^{p^i}\}.$$

We emphasize that the exponentiation (e.g., the "p^i" appearing in "N^{p^i}") is being used in a different sense from its use in Property ii of Definition 5.

We shall make extensive use of the existence and properties of p-bases. The term "p-basis" is defined on p. 129 of [5].

In Chapter II, we shall derive a necessary condition (our "intersection property") for a quasi-local p-algebra to possess a coefficient field containing the base field K.

To this necessary condition, we shall add another condition (our "joint-independence property") and demonstrate that in a quasi-local p-algebra, these two conditions are sufficient to answer Question II.

The intersection property is quite strong; we give some indication of its strength in Chapter III by showing that it is sufficient by itself to guarantee the existence of a coefficient field (but not necessarily one containing the base-field K) in a quasi-local p-algebra.

We bring the paper to a close by giving two examples in Chapter IV; one of these illustrates the joint-independence property, while the other gives a case in which the intersection property breaks down.

II. $G = G'$

Let A be a quasi-local p-algebra with base-field K, maximal ideal N, and let $F = A/N$; let ϕ be the natural homomorphism of A onto A/N.

Lemma 1: $\forall\ i \in I^+$, $\phi(A^{p^i} \cap K) \subseteq F^{p^i} \cap \phi(K)$.

Proof: Take $y \in \phi(A^{p^i} \cap K)$; then $\exists\ z \in A^{p^i} \cap K \ni y = \phi(z)$, and $\exists\ a \in A,\ k \in K \ni z = a^{p^i} = k$. Now, $\phi(z) = \phi(a^{p^i}) = [\phi(a)]^{p^i} = \phi(k)$. $\phi(a) = f \in F$, so that $y = \phi(z) = f^{p^i} = \phi(k)$; $f^{p^i} \in F^{p^i}$, $\phi(k) \in \phi(K)$, and it follows that

$$y \in (F^{p^i} \cap \phi(K)).$$

This shows that $\forall\ i \in I^+$, $\phi(A^{p^i} \cap K) \subseteq F^{p^i} \cap \phi(K)$, Q.E.D.

Since K is a field, we know from above (see Fact 3, Introduction) that $\phi|_K$ is one-to-one; it follows that $\forall\ i \in I^+$, $\phi|_{A^{p^i} \cap K}$ is one-to-one.

Lemma 2: If $\exists$ a coefficient field $\hat{F} \supseteq K$, then $\forall\ i \in I^+$, $\phi|_{A^{p^i} \cap K}$ maps $A^{p^i} \cap K$ isomorphically onto $F^{p^i} \cap \phi(K)$.

Proof: In view of Lemma 1 and the fact that $\forall\ i \in I^+$, $\phi|_{A^{p^i} \cap K}$ is one-to-one, we need show only that $\forall\ i \in I^+$, $\phi|_{A^{p^i} \cap K}$ is a homomorphism and that $F^{p^i} \cap \phi(K) \subseteq \phi(A^{p^i} \cap K)$.

Since $\forall\ i \in I^+$ $A^{p^i} \cap K$ is a subring of A, we have that $\forall\ i \in I^+$, $\phi|_{A^{p^i} \cap K}$ is a homomorphism.

Take $z \in F^{p^i} \cap \phi(K)$. Then $\exists\ f \in F,\ k \in K \ni z = f^{p^i} =$

$\emptyset(k)$; $\exists\ \hat{f} \in \hat{F} \ni f = \emptyset(\hat{f})$. Thus,

$$z = [\emptyset(\hat{f})]^{p^i} = \emptyset(\hat{f}^{p^i}) = \emptyset(k) \tag{1}$$

Now $\hat{f}^{p^i} \in \hat{F}$, $k \in \hat{F}$ (since $\hat{F} \supseteq K$) and $\emptyset|_{\hat{F}}$ is one-to-one (since $\hat{F}$ is a field in A). It follows from Equation 1 that $\hat{f}^{p^i} = k$, so that $\hat{f}^{p^i} \in K$. Now $\hat{f} \in A$, so that $\hat{f}^{p^i} \in A^{p^i}$, and hence $\hat{f}^{p^i} \in A^{p^i} \cap K$; it follows that $z = \emptyset(\hat{f}^{p^i}) \in \emptyset(A^{p^i} \cap K)$. We thus have that $\forall\ i \in I^+$, $F^{p^i} \cap \emptyset(K) \subseteq \emptyset(A^{p^i} \cap K)$, Q.E.D.

We have shown that ($\forall\ i \in I^+$, $A^{p^i} \cap K \cong F^{p^i} \cap \emptyset(K)$) is a necessary condition for the existence of a coefficient field $\hat{F} \supseteq K$. We emphasize that the isomorphism alluded to in this necessary condition is actually the mapping $\emptyset|_{A^{p^i} \cap K}$.

Since $K \cong \emptyset(K)$ (under $\emptyset|_K$), we shall from here on identify K and $\emptyset(K)$ elementwise; i.e., $\forall\ k \in K$ we set $\emptyset(k) = k$. We shall also identify P and $\emptyset(P)$ elementwise if P is a subset of Q, where Q is a subring of K and $\emptyset|_Q$ maps Q isomorphically onto $\emptyset(Q) \subseteq \emptyset(K)$. We shall also write $K = \emptyset(K)$, $P = \emptyset(P)$; with this understanding, we may write the above necessary condition as

$$(\forall\ i \in I^+,\ A^{p^i} \cap K = F^{p^i} \cap K) \tag{2}$$

Definition 1: Let A be a quasi-local p-algebra with base-field K, maximal ideal N, and let $F = A/N$. We say that <u>A has the intersection property</u> iff $\forall\ i \in I^+$, $A^{p^i} \cap K = F^{p^i} \cap K$.

We shall always be dealing with the case where F is a purely-inseparable extension of $\emptyset(K) = K$ with bounded exponent. This means that $\exists$ a smallest $n \in I^+ \ni F^{p^n} \subseteq K$.

Definition 2: Let A be a quasi-local p-algebra with base-field K, maximal ideal N, and let $F = A/N$. We say that A is of class n iff n is the smallest positive integer $\ni F^{p^n} \subseteq K$.

Lemma 3: Let A be a quasi-local p-algebra with base-field K, maximal ideal N, and let $F = A/N$; let $\emptyset$ be the natural homomorphism of A onto A/N. It is the case that $N \cap A^p = N^p$.

Proof: We have the inclusions $N \subseteq A$, $N^p \subseteq A^p$, $N^p \subseteq N$; hence

$$N^p \subseteq N \cap A^p \tag{3}$$

To show the opposite inclusion, take $x \in N \cap A^p$. Then $\exists$ $n \in N$, $a \in A \ni x = n = a^p$. It follows that

$$\emptyset(x) = \emptyset(n) = \emptyset(a^p) = [\emptyset(a)]^p = 0 \tag{4}$$

($n \in N = \text{Ker } \emptyset$). Now $\emptyset(a) \in F$, and F is a field; hence we have $\emptyset(a) = 0$, and $a \in N$. Thus, $x = a^p \in N^p$, and hence $x \in N^p$. It follows that

$$N \cap A^p \subseteq N^p \tag{5}$$

Equations 3 and 5 give us $N \cap A^p = N^p$, Q.E.D.

We pause to call attention to the following fact: If A is a quasi-local p-algebra which has the intersection property, then $\forall\, i \in I^+$, $A^{p^i} \cap K$ is a field. To show this, we note that $\forall\ i \in I^+$, $F^{p^i} \cap \emptyset(K)$ is a field (since $\forall\ i \in I^+$, F^{p^i} is a field, and $\emptyset(K)$ is a field). Since A has the intersection property, we have that $\forall\ i \in I^+$, $A^{p^i} \cap K \cong F^{p^i} \cap \emptyset(K)$, and hence, $\forall\ i \in I^+$, $A^{p^i} \cap K$ is a field.

<u>Lemma 4</u>: Let A be a quasi-local p-algebra with base-field K, maximal ideal N, and let $F = A/N$. Then $\forall\ i, j, m \in I^+ \cup \{0\}$, we have $(K^{p^i} \cap F^{p^j})^{p^m} = K^{p^{i+m}} \cap F^{p^{j+m}}$.

Proof: Take $x \in (K^{p^i} \cap F^{p^j})^{p^m}$. $\exists\ y \in K^{p^i} \cap F^{p^j} \ni x = y^{p^m}$. Now $\exists\ f \in F, k \in K \ni y = k^{p^i} = f^{p^j}$. Then, $x = y^{p^m} = k^{p^{i+m}} = f^{p^{j+m}}$, and hence, $x \in K^{p^{i+m}} \cap F^{p^{j+m}}$. It follows that

$$(K^{p^i} \cap F^{p^j})^{p^m} \subseteq K^{p^{i+m}} \cap F^{p^{j+m}} \tag{6}$$

Now take $z \in K^{p^{i+m}} \cap F^{p^{j+m}}$. $\exists\ k \in K,\ f \in F \ni z = k^{p^{i+m}} = f^{p^{j+m}}$, i.e., $(k^{p^i})^{p^m} = (f^{p^j})^{p^m}$, and so

$$0 = (k^{p^i})^{p^m} - (f^{p^j})^{p^m} = (k^{p^i} - f^{p^j})^{p^m} \tag{7}$$

Since k, f $\in$ F and F is a field, we have from Equation 7 that

$$k^{p^i} = f^{p^j} \tag{8}$$

Let $c_i = k^{p^i}$; it follows from Equation 8 that $c_i \in K^{p^i}$, $c_i \in F^{p^j}$, so that $c_i \in K^{p^i} \cap F^{p^j}$. Finally, we get

$$z = c_i^{p^m} \in (K^{p^i} \cap F^{p^j})^{p^m},$$

and hence

$$K^{p^{i+m}} \cap F^{p^{j+m}} \subseteq (K^{p^i} \cap F^{p^j})^{p^m} \tag{9}$$

Combining Equations 6 and 9 we get

$$(K^{p^i} \cap F^{p^j})^{p^m} = K^{p^{i+m}} \cap F^{p^{j+m}},$$

Q.E.D.

Definition 3: We say that A is a quasi-local p-V_n-algebra iff A is a quasi-local p-algebra of class n, and A has the intersection property.

Lemma 5: Suppose that A is a quasi-local p-V_{n+1}-algebra ($n \in I^+$). Then A^p is a quasi-local p-V_n-algebra.

Proof: We first note that A^p is a commutative ring with identity. Let $K_o = A^p \cap K = F^p \cap K$. Since K_o is a field and $K_o \subseteq A^p$, we may consider A^p as an algebra over K_o. We now wish to show that:

a) The set of non-units of A^p is an ideal in A^p; in fact, this set is just N^p.

b) $A^p/N^p \cong F^p$.

c) A^p is of class n.

d) A^p has the intersection property.

Proof of a): It is easy to check that N^p is an ideal in A^p. Since N is the set of non-units of A, we can say that the set of non-units of A^p is $N \cap A^p = N^p$. (See Lemma 3.) This proves a.

Proof of b): We first show that $\emptyset|_{A^p}$ maps A^p homomorphically onto F^p. First of all, A^p is a subring of A; it follows that

$\emptyset|_{A^p}$ is a homomorphism. Pick $z \in \emptyset(A^p)$; $\exists\ u \in A^p \ni z = \emptyset(u)$. Also, $\exists\ a \in A \ni u = a^p$; then $z = \emptyset(u) = \emptyset(a^p) = [\emptyset(a)]^p \in F^p$, since $\emptyset(a) \in F$. Hence, $z \in F^p$, and

$$\emptyset(A^p) \subseteq F^p \tag{10}$$

Now pick $x \in F^p$; $\exists\ f \in F \ni x = f^p$. Also, since $\emptyset$ maps A onto F, $\exists\ a \in A \ni f = \emptyset(a)$; thus, $x = f^p = [\emptyset(a)]^p = \emptyset(a^p) \in \emptyset(A^p)$, and $x \in \emptyset(A^p)$. It follows that

$$F^p \subseteq \emptyset(A^p) \tag{11}$$

Combining Equations 10 and 11, we see that $\emptyset(A^p) = F^p$, so that $\emptyset|_{A^p}$ is a homomorphism of A^p onto F^p.

Now $N = \text{Ker}\ \emptyset$; it follows that $\text{Ker}\ \emptyset|_{A^p} = N \cap A^p = N^p$ (see Lemma 3). We thus have $A^p/(\text{Ker}\ \emptyset|_{A^p}) \cong F^p$, i.e., $A^p/N^p \cong F^p$; this proves b.

To prove c, we reason as follows: Since A is a quasi-local p-algebra of class n+1, we have that n+1 is the smallest positive integer $\ni F^{p^{n+1}} \subseteq K$. Now $F^{p^{n+1}} \subseteq F^p$, and so $F^{p^{n+1}} \subseteq F^p \cap K = K_0$; i.e., n is the smallest positive integer $\ni (F^p)^{p^n} \subseteq K_0$. This proves c. (Recall that we are considering K_0 as the base-field of A^p, and that we are identifying K_0 and $\emptyset(K_0)$).

We now prove d. $\forall\ j \in I^+ - \{1\}$, we have $A^{p^j} \subseteq A^p$, $F^{p^j} \subseteq F^p$, so that $A^{p^j} = A^p \cap A^{p^j}$, $F^{p^j} = F^p \cap F^{p^j}$, and hence $\forall\ j \in I^+ - \{1\}$, we have $A^{p^j} \cap K = A^{p^j} \cap A^p \cap K = A^{p^j} \cap K_0$, and $F^{p^j} \cap K = F^{p^j} \cap F^p \cap K = F^{p^j} \cap K_o$. Using the fact that A has the intersection property, we may write, $\forall\ j \in I^+ - \{1\}$ $A^{p^j} \cap K_o = A^{p^j} \cap K = F^{p^j} \cap K = F^{p^j} \cap K_o$, i.e., $(A^p)^{p^{j-1}} \cap K_o = (F^p)^{p^{j-1}} \cap K_o$. Let $i = j - 1$; we then have $\forall\ i \in I^+$, $(A^p)^{p^i} \cap K_o = (F^p)^{p^i} \cap K_o$, which proves d.

a and b (and the fact that K_o is a field in A^p) tell us that A^p is a quasi-local p-algebra with base-field K_o, maximal ideal N^p, and that we may identify F^p and A^p/N^p. c and d tell us that A^p is a quasi-local p-V_n-algebra, Q.E.D.

We shall assume from here on that A is a quasi-local p-V_{n+1}-algebra ($n \geq 1$) with base-field K, maximal ideal N, and that $F = A/N$; furthermore, we will set $K_o = A^p \cap K = F^p \cap K$.

We now begin a discussion of a property which will eventually lead to a set of sufficient conditions for the existence of a coefficient field in A containing K. Let B_K be a p-basis for K chosen in the following way: $B_K = G \cup H$, where $G \subseteq K - K_o$, $H \subseteq K_o - K^p$; let $H = H^* \cup H_o$, where $H^* \subseteq K_o - (F^{p^2} \cap K)$, $H_o \subseteq F^{p^2} \cap K$. Now define for each $i \in I^+$ a field K_i by the relation $K_i = K \cap F^{p^{i+1}}$. Note that the fact that A is of class n+1 allows us to write $K_n = K \cap F^{p^{n+1}} = F^{p^{n+1}}$, and $K_{n+1} = K \cap F^{p^{n+2}} = F^{p^{n+2}} \subseteq K^p$. ($F^{p^{n+1}} \subseteq K$, and hence $F^{p^{n+2}} \subseteq K^p$.) The p-independence of H_o in K tells us that $H_o \cap K^p = \Lambda$ (Λ = the empty set), so that $H_o \cap K_{n+1} =$

$H_0 \cap F^{p^{n+2}} = \Lambda$. In general, for $\forall$ i = 2,3,$\cdots$,n+1, we have $H_0 \cap F^{p^i} \neq \Lambda$. We may then reason as follows.

Pick $h \in H = H^* \cup H_0$. Then there is exactly one i, ($i \in I^+$), $1 \leq i \leq n + 1$, $\ni$ $h \in (K_{i-1} - K_i)$. It follows that $\exists\, f \in (F - F^p) \ni h = f^{p^i}$. ($h \in [K \cap F^{p^i} - K \cap F^{p^{i+1}}]$.) In fact, there is exactly one such f; for suppose f, $f_1 \in F$ and $f^{p^i} = h = f_1^{p^i}$. Then $f^{p^i} - f_1^{p^i} = (f - f_1)^{p^i} = 0$, and $f - f_1 = 0$. (f, f_1, and $f - f_1$ are all in F, and F is a field with characteristic p.) If we let h range over H, we thus pick out in $F - F^p$ a set of elements which are p^i-th roots of elements in H for some i, $1 \leq i \leq n + 1$. We denote this set by "$\bar{\bar{H}}$", and give it a special name.

<u>Definition 4</u>: Let $G \cup H$ be a p-basis for K chosen as indicated above. We say that <u>$\bar{\bar{H}}$ is an origin set for H</u> iff

$$\bar{\bar{H}} = \{f \mid f \in F - F^p \text{ and } \exists\, i \in I^+,$$

$$1 \leq i \leq n+1, \ni f^{p^i} \in H \cap (K_{i-1} - K_i)\}.$$

We emphasize that $\bar{\bar{H}}$ is a uniquely determined set, once H has been selected.

<u>Definition 5</u>: Let $G \cup H$ be a p-basis for K chosen as indicated above, and let $\bar{\bar{H}}$ be an origin set (in $F - F^p$) for H. We say that <u>A has the joint-independence property</u> iff $G \cup \bar{\bar{H}}$ is p-independent in F.

<u>Lemma 6</u>: Suppose A is a quasi-local p-V_{n+1}-algebra ($n \geq 1$) which has the joint-independence property. Then A^p is a quasi-local p-V_n-algebra which has the joint-independence property.

Proof: In view of Lemma 5, we need only prove that A^p has the joint-independence property.

Consider the set $\overline{H}^p$. We note that $\overline{H}^p \subseteq (F^p - F^{p^2})$. Furthermore, $H^* \subseteq \overline{H}^p$; for, given $h^* \in H^*$, $\exists$ a unique $\overline{h} \in \overline{H} \ni \overline{h}^p = h^*$. ($h^* \in H^* \subseteq (K_o - K_1)$, and $\overline{H}$ is an origin set for $H = H^* \cup H_o$.) In fact, the inclusion $H^* \subseteq \overline{H}^p$ is proper; for, suppose it were not. Then we would have $\overline{H}^p = H^*$, and $\overline{H}^{p^2} = (H^*)^p \subseteq K_o^p \subseteq K^p$, giving us

$$\overline{H}^{p^2} \subseteq K^p \tag{12}$$

Now take $h_o \in H_o \cap (F^{p^2} - F^{p^3})$; then, using Equation 12, we may say that $\exists$ $\overline{h} \in \overline{H} \ni h_o = \overline{h}^{p^2}$, i.e., $h_o = \overline{h}^{p^2} \in \overline{H}^{p^2} \subseteq K^p$, so that $h_o \in K^p$, which contradicts the p-independence of H_o in K. Hence, the inclusion $H^* \subseteq \overline{H}^p$ is proper. It follows that $\exists$ a set $H' \neq \Lambda$ with the following properties:

1) $\overline{H}^p = H^* \cup H'$.

2) $H' \subseteq F^p - F^{p^2}$.

3) $H' \cap K_o = \Lambda$.

Let $G_o = G^p \cup H^*$. We then claim that

i) $G_o \cup H_o$ is a p-basis for K_o.

ii) H' is an origin set (in $F^p - F^{p^2}$) for H_o.

iii) $G_o \cup H'$ is p-independent in F^p.

We prove ii first. Take $h_0 \in H_0$; $\exists$ a unique $\bar{h} \in \bar{H}$ and a unique i, $2 \leq i \leq n+1$, $\ni$ $\bar{h}^{p^i} = h_0$, i.e.,

$$(\bar{h}^p)^{p^{i-1}} = h_0 \tag{13}$$

We show now that $\bar{h}^p \in H'$. ($\bar{h}^p \in \bar{H}^p = H^* \cup H'$.) Suppose that $\bar{h}^p \in H^*$. Then, we would have $h_0 = \bar{h}^{p^i} \in (H^*)^{p^{i-1}} \subseteq (H^*)^p \subseteq K_0^p \subseteq K^p$, giving us that $h_0 \in K^p$, which is a contradiction. It follows that $\bar{h}^p \in H'$. Let $h_1 = \bar{h}^p$, and $j = i-1$. From our comments above and Equation 13, we may state that to each $h_0 \in H_0$ there correspond a unique $h_1 \in H'$ and a unique $j \in I^+$, $1 \leq j \leq n$, $\ni$ $h_0 = h_1^{p^j}$. This completes the proof of ii.

We next prove iii. We note that

$$(G \cup \bar{H})^p = G^p \cup \bar{H}^p = G^p \cup H^* \cup H'$$
$$= (G^p \cup H^*) \cup H' = G_0 \cup H'.$$

Since $G \cup \bar{H}$ is p-independent in F, $(G \cup \bar{H})^p = G_0 \cup H'$ is p-independent in F^p. This proves iii.

To prove i, we proceed as follows: We first show that $K^p(H) = K_0$. Clearly, $K^p(H) \subseteq K_0$. Suppose that $K_0 - K^p(H) \neq \Lambda$, and pick $a \in (K_0 - K^p(H))$. Now $a \in K_0 \subseteq K = K^p(G \quad H)$, and so $\exists$ a subset $\{g_1, \cdots, g_s\}$ of G $\ni$ $a \in K^p(H, g_1, \cdots, g_s)$, but $a \notin K^p(H, g_1, \cdots, g_{s-1})$. From the exchange property (see p. 129 of [5]), it follows that

$$g_s \in K^p(H, g_1, \cdots, g_{s-1}, a) \tag{14}$$

Now, $K^p(H,a) \subseteq K_o \subseteq F^p$, and hence

$$K^p(H,g_1,\cdots,g_{s-1},a) \subseteq F^p(g_1,\cdots,g_{s-1}) \tag{15}$$

Combining Equations 14 and 15, we find that

$$g_s \in F^p(g_1,\cdots,g_{s-1}),$$

which contradicts the p-independence of $\{g_1,\cdots,g_s\}$ in F. It follows that

$$K^p(H) = K_o \tag{16}$$

We now show that H is a minimal set of generators for K_o over K^p. Suppose that H is not minimal (in this sense). Then $\exists$ a set $S \neq \Lambda$ satisfying the conditions $S \subseteq H$,

$$K_o = K^p(H - S) \tag{17}$$

We now note that $K_o(G) = K$; for it is clear that $K_o(G) \subseteq K$, and $K_o(G) \supseteq K^p$, $K_o(G) \supseteq H$, and $K_o(G) \supseteq K^p(G \cup H) = K$. Using Equation 17, we may then write

$$S \subseteq K \subseteq K_o(G) = K^p(H - S)(G) \tag{18}$$

Pick $s \in S$; from Equation 18, we have $s \in K^p((H - S) \cup G)$, which means that the set $\{s\} \cup (H - S) \cup G$ is not p-independent in K. Since $\{s\} \cup (H - S) \cup G \subseteq G \cup H$, we conclude that $G \cup H$ is not p-independent in K, which is a contradiction. It follows that H is a minimal set of generators for K_o over

K^p. We note that

$$K^p = K_0^p(G^p), \tag{19}$$

(since $K = K_0(G)$). With some change in the wording, we now quote Lemma 5, p. 15 of [4]: "If B is a p-independent subset of the field Q and $C \subseteq Q \supseteq C$ is a minimal set of generators for $Q^p(B \cup C)$ over $Q^p(B)$, then $B \cup C$ is p-independent in $Q^p(B \cup C)$". Since G is p-independent in F, G^p is p-independent in F^p; it follows that G^p is p-independent in K_0. We have shown above that H is a minimal set of generators for $K_0 = K_0^p(G^p \cup H)$ over $K^p = K_0^p(G^p)$. We may now apply the above-quoted lemma (setting $B = G^p$, $Q = K_0$, and $C = H$) to conclude that $G^p \cup H = (G^p \cup H^*) \cup H_0 = G_0 \cup H_0$ is p-independent in K_0. We proceed to show that $K_0 = K_0^p(G_0 \cup H_0)$. Since K_0^p, G_0, and H_0 are all subsets of K_0, we have $K_0^p(G_0 \cup H_0) \subseteq K_0$. It remains to show that $K_0 \subseteq K_0^p(G_0 \cup H_0)$. We have from above that $K_0(G) = K$, $K_0^p(G^p) = K^p$. Since $K^p(H) = K_0$, we may write $K_0^p(G_0 \cup H_0) \supseteq K_0^p(G \cup H) = K_0^p(G^p)(H) = K^p(H) = K_0$.

We have thus shown that $G_0 \cup H_0$ is p-independent in K_0, and that $K_0 = K_0^p(G_0 \cup H_0)$. It follows that $G_0 \cup H_0$ is a p-basis of K_0, and we have proved i. We have thus shown that i, ii, and iii all hold. It follows that A^p has the joint-independence property, Q.E.D.

It is to be noted that we have not allowed A to be of class 1 in the statement and proof of Lemma 6; we pause to make some comments about this situation. Suppose that A is a quasi-local p-V_1-algebra. Then, we have $F^p \subseteq K$, $F^{p^2} \subseteq K^p$. We may still select a p-basis, $G \cup H$, of K as indicated on p. 13; in this case, however, we must have $H = H^*$, $H_0 = \Lambda$. To see this, we reason as follows: H_0 is a p-independent subset of K, and $H_0 \subseteq F^{p^2} \subseteq K^p$, which implies that $H_0 = \Lambda$. We may still find an origin set $\bar{H}$ (in $F - F^p$) for H; however, in this case, we have

$$\bar{H}^p = H^*,\ H' = \Lambda \text{ ---- for } \bar{H}^{p^2} \subseteq F^{p^2} \subseteq K^p.$$

(See pp. 15-16 for a discussion of H'.) We thus see that we may still speak about the joint-independence property for a quasi-local p-V_1-algebra; however, we must be careful to take into account the above-mentioned degenerations in doing so.

For convenience, we shall assume from here on that G is a denumerable (at most) set.

<u>Lemma 7</u>: Suppose that

1) A is a quasi-local p-V_n-algebra which has the joint-independence property.
2) $\exists$ a coefficient field in A^p for F^p. (Let "$\hat{E}$" denote this field.)
3) $\hat{E} \supseteq K_0$.

Then $\hat{E}[G]$ is a field, and $\hat{E}[G] \supseteq K$.

Proof: Let $E = F^p$. Suppose $G = \{g_1, g_2, \cdots\}$. For each

$i \in I^+$, define a ring $\hat{E}_i$ by the relations

$$\hat{E}_1 = \hat{E}[g_1], \quad \hat{E}_{i+1} = \hat{E}_i[g_{i+1}] \tag{20}$$

We shall show that $\forall$ $i \in I^+$, $\hat{E}_i$ is a field. The proof is by induction. Let $I = \{i | i \in I^+$ and $\hat{E}_i$ is a field$\}$. We shall show that $I = I^+$.

a) We show that $1 \in I$:

We have $g_1^p \in K^p \subseteq K_o \subseteq \hat{E}$; set $g_1^p = k_1 \in \hat{E}$. g_1 satisfies the polynomial equation $x^p - k_1 = 0$. We wish to show that $x^p - k_1$ is irreducible over $\hat{E}$. Suppose, then that $x^p - k_1$ is reducible over $\hat{E}$. Then $\exists$ $\hat{e}_1 \in \hat{E} \ni \hat{e}_1^p = k_1 = g_1^p$; thus, we may write

$$\emptyset(\hat{e}_1^p) = \emptyset(g_1^p), \text{ i.e., } [\emptyset(\hat{e}_1)]^p = [\emptyset(g_1)]^p \tag{21}$$

Set $\emptyset(\hat{e}_1) = e_1 \in E$; since $g_1 \in K$, $\emptyset(g_1) = g_1$. (Recall that we are identifying G and $\emptyset^{-1}(G)$ elementwise.) From Equation 21, we may write $e_1^p = g_1^p$, and hence $e_1 = g_1$. It follows that $g_1 \in E$, i.e., $g_1 \in F^p$, which is a contradiction. Hence, $x^p - k_1$ is irreducible over $\hat{E}$, and $\hat{E}_1$ is a field. We have shown that $1 \in I$.

b) Suppose that $i \in I$; we shall show that $i+1 \in I$.

We are assuming that $\hat{E}_i$ is a field. We have that $g_{i+1}^p \in K^p \subseteq K_o \subseteq \hat{E}$; set $g_{i+1}^p = k_{i+1} \in \hat{E} \subseteq \hat{E}_i$. Suppose that $x^p - k_{i+1}$ is reducible over $\hat{E}_i$. Then $\exists$ $\hat{e}_{i+1} \in \hat{E}_i \ni \hat{e}_{i+1}^p = k_{i+1} = g_{i+1}^p$; thus we may write

$$\phi(\hat{e}^p_{i+1}) = \phi(g^p_{i+1}),$$

i.e.,
$$[\phi(\hat{e}_{i+1})]^p = [\phi(g_{i+1})]^p \tag{22}$$

Now $\hat{e}_{i+1} \in \hat{E}_i = \hat{E}(g_1, g_2, \cdots, g_i)$. Hence,

$$\phi(\hat{e}_{i+1}) \in \phi(\hat{E})(\phi(g_1), \phi(g_2), \cdots, \phi(g_i)),$$

i.e.,
$$\phi(\hat{e}_{i+1}) \in E(g_1, g_2, \cdots, g_i) \subseteq F \tag{23}$$

(Note that $\forall$ $i \in I^+$, $\phi(g_i) = g_i$). Then $\phi(g_{i+1}) = g_{i+1}$; if we set $\phi(\hat{e}_{i+1}) = e_{i+1} \in E(g_1, g_2, \cdots, g_i)$, we have from Equation 22 that $\phi(\hat{e}_{i+1}) = \phi(g_{i+1})$, i.e.,

$$e_{i+1} = g_{i+1} \tag{24}$$

Now, since $e_{i+1} \in E(g_1, g_2, \cdots, g_i)$, $\exists$ a polynomial f_{i+1} in $\{g_1, g_2, \cdots, g_i\}$ with coefficients in

$$E = F^p \ni e_{i+1} = f_{i+1}(g_1, g_2, \cdots, g_i) \tag{25}$$

Combining Equations 24 and 25, we have

$$g_{i+1} = f_{i+1}(g_1, g_2, \cdots, g_i),$$

which contradicts the p-independence of $\{g_1, g_2, \cdots, g_i, g_{i+1}\}$ in F. (Note that f_{i+1} has coefficients in $E = F^p$.) It follows that $x^p - k_{i+1}$ is irreducible over $\hat{E}_i$; hence $\hat{E}_{i+1}$ is a field, and $i+1 \in I$. By the principle of induction, we conclude that $I = I^+$; it follows that $\hat{E}[G]$ is a field, which we may denote by "$\hat{E}(G)$".

Now $\hat{E} \supseteq K_o$, and $K_o \supseteq K^p$, $K_o \supseteq H$; hence $\hat{E}(G) \supseteq K^p$, $\hat{E}(G) \supseteq H$, $\hat{E}(G) \supseteq G$. Thus, we have $\hat{E}(G) \supseteq K^p(G \cup H)$, i.e., $\hat{E}(G) \supseteq K$, Q.E.D.

Suppose that A is a quasi-local p-V_n-algebra which has the joint-independence property. Since G is p-independent in F, we may augment G to a p-basis of F; i.e., we can find a set M having the properties: $M \neq \Lambda$, $M \subseteq F - F^p$, $G \cup M$ is a p-basis of F. We assume in what follows that M is denumerable (at most).

<u>Definition 6</u>: Let A be a quasi-local p-V_n-algebra which has the joint-independence property, and let $G \cup M$ be a p-basis of F selected as indicated above. We say that <u>M' is a set of counter-images of M</u> (<u>under</u> $\emptyset$) iff M' is a set having the properties: $M' \subseteq A$, $M' \neq \Lambda$, $\emptyset|_{M'}$ is one-to-one, and $\emptyset(M') = M$.

<u>Lemma 8</u>: Suppose that the hypotheses of Lemma 7 hold, and that $G \cup M$ is a p-basis for F selected as indicated above. Then we can find a set $\overline{M}$ satisfying the conditions:

a) $\overline{M} \cap \hat{E}(G) = \Lambda$.

b) $\overline{M}$ is a set of counter-images of M under $\emptyset$.

c) $\overline{M}^p \subseteq \hat{E}$.

Proof: We first point out that $E[G]$ is a field in F, and that

$$E[G] \cap M = \Lambda \qquad (26)$$

To show this, we reason as follows: $\emptyset|_{\hat{E}}$ maps $\hat{E}$ isomorphically onto $E = F^p$; since $G \subseteq K$, we have that $\emptyset|_{\hat{E}[G]}$ maps $\hat{E}[G]$ isomorphically onto $E[\emptyset(G)] = E[G]$. Since $\hat{E}[G]$ is a field, it follows that $E[G]$ is a field, which we denote by "$E(G)$". Now suppose that $E(G) \cap M \neq \Lambda$, and pick $m \in E(G) \cap M$. Then $m \in E(G) = F^p(G)$, so that $\{m\} \cup G$ is not p-independent in F; this contradicts the p-independence of $G \cup M$ in F. Hence, we have $E(G) \cap M = \Lambda$. Now, let M' be any set of counter-images of M under $\emptyset$. We claim that $\hat{E}(G) \cap M' = \Lambda$; for, suppose not. Take $c \in \hat{E}(G) \cap M'$; then $c \in \hat{E}(G)$, $c \in M'$ so that $\emptyset(c) \in E(G)$, and $\emptyset(c) \in M$, which implies that $\emptyset(c) \in (E(G) \cap M)$, giving us a contradiction. Hence, if M' is any set of counter-images of M under $\emptyset$, then $\hat{E}(G) \cap M' = \Lambda$. We now wish to show that (as a direct sum of groups)

$$A^p = \hat{E} \oplus N^p \qquad (27)$$

First of all, we have that

$$\hat{E} \cap N^p = \{0\} \qquad (28)$$

Equation 28 follows from the fact that $\hat{E}$ is a field and from the fact that N^p is the set of non-units in A^p. We now show that

$$A^p = \hat{E} + N^p \qquad (29)$$

It is clear that

$$\hat{E} + N^p \subseteq A^p \qquad (30)$$

To show the opposite inclusion, we reason as follows: Take $z \in A^p$; since $\hat{E}$ is a coefficient field in A^p for F^p, it follows that $\exists\ \hat{e} \in \hat{E} \ni \emptyset(z) = \emptyset(\hat{e})$. We must then have

$$z - \hat{e} \in \text{Ker } \emptyset|_{A^p}, \text{ i.e., } z - \hat{e} \in N^p;$$

hence, $\exists\ b \in N^p \ni z - \hat{e} = b$, i.e., $z = \hat{e} + b \in \hat{E} + N^p$. We thus have $z \in \hat{E} + N^p$; it follows that

$$A^p \subseteq \hat{E} + N^p \qquad (31)$$

It follows from Equations 30 and 31 that

$$A^p = \hat{E} + N^p \qquad (32)$$

Equations 28 and 32 imply that (as a direct sum of groups)

$$A^p = \hat{E} \oplus N^p \qquad (33)$$

We now proceed to construct the set $\overline{M}$. Let M' be any set of counter-images of M under $\emptyset$, and pick $m' \in M'$. Now $(m')^p \in A^p$; from Equation 33, we may assert that corresponding to m' $\exists$ unique elements $\hat{e} \in \hat{E}$,

$$b \in N^p \ni (m')^p = \hat{e} + b \qquad (34)$$

Now, corresponding to b $\exists$ a set N_b having the properties:

$$N_b \subseteq N, \text{ and } \forall\ n \in N_b,\ n^p = b \qquad (35)$$

Now pick $n \in N_b$; from Equations 34 and 35, we may write

$(m')^p = \hat{e} + b = \hat{e} + n^p$, and hence,

$$(m')^p - n^p = (m' - n)^p = \hat{e} \in \hat{E} \qquad (36)$$

Let $\bar{m} = m' - n$; we note that $\phi(\bar{m}) = \phi(m') \in M$, and that $\bar{m}^p \in \hat{E}$. $(n \in N, \phi(n) = 0.)$

We now define our set $\bar{M}$ as follows: Let M' be any set of counter-images of M under ϕ; to each $m' \in M'$ we correspond the element $b \in N^p$ appearing above in Equations 34 and 35. We then pick <u>exactly one element $n \in N_b$</u> and set $\bar{m} = m' - n$; we denote the set of all $\bar{m}$'s thus obtained by "$\bar{M}$". It is to be emphasized that there may well be many sets that will be obtained in this way from a given M'; this is due to our freedom in picking an element n from each N_b.

Let us then construct one such set $\bar{M}$; we then claim that a, b, and c hold. To show this, we proceed as follows: Let $n_{m'}$ be that unique element of $N_b \ni \bar{m} = m' - n_{m'}$, and consider the mapping ψ: $M' \longrightarrow \bar{M}$ defined by $\psi(m') = m' - n_{m'}$; ψ is certainly onto. We wish to show that ψ is one-to-one. Suppose that $\psi(m') = \psi(m_1')$, for $m', m_1' \in M'$; then, we have

$$m' - n_{m'} = m_1' - n_{m_1'},$$

so that

$$\phi(m') = \phi(m_1') \qquad (37)$$

$(n_{m'}, n_{m_1'} \in N$, and $\phi(N) = \{0\}.)$ Since ϕ is one-to-one on M', it follows from Equation 37 that $m' = m_1'$; this establishes that ψ is one-to-one on M'. It is now an easy matter to check that $\emptyset(\bar{M}) = M$, and that $\emptyset$ is one-to-one on $\bar{M}$. ($\forall\ \bar{m} \in \bar{M}\ \exists$ a unique $m' \in M' \ni \emptyset(\bar{m}) = \emptyset(\psi(m'))$.) It follows that $\bar{M}$ is a set of counter-images of M under $\emptyset$; it is immediate from our previous comments that $\bar{M} \cap \hat{E}(G) = \Lambda$. Also, it follows from Equation 36 and the comments following it that $\bar{M}^p \subseteq \hat{E}$. We have thus established a, b, and c of Lemma 8, Q.E.D.

<u>Lemma 9</u>: Suppose that the hypotheses of Lemma 7 hold, and that $G \cup M$ is a p-basis for F selected as indicated above. Suppose further that $\bar{M}$ is a set satisfying a, b, and c of Lemma 8. Then $\hat{E}(G)[\bar{M}]$ is a field, and in fact, $\hat{E}(G)[\bar{M}]$ is a coefficient field in A for $F \ni \hat{E}(G)[\bar{M}] \supseteq K$.

Proof: The denumerability of M implies that M' and that of $\bar{M}$. Suppose $\bar{M} = \{\bar{m}_1, \bar{m}_2, \cdots\}$. For each $i \in I^+$, define a ring P_i by the relations

$$P_1 = \hat{E}(G)[\bar{m}_1],\ P_{i+1} = P_i[\bar{m}_{i+1}] \qquad (38)$$

We shall show that $\forall\ i \in I^+$, P_i is a field. The proof is by induction. We note that $\bar{m}_1^p \in \hat{E}$; set $\bar{m}_1^p = r_1 \in \hat{E} \subseteq \hat{E}(G)$. $\bar{m}_1$ satisfies the equation $x^p - r_1 = 0$. Suppose that $x^p - r_1$ is reducible over $\hat{E}(G)$. Then $\exists\ \hat{e}_1 \in \hat{E}(G) \ni \hat{e}_1^p = r_1 = \bar{m}_1^p$; we then have $\emptyset(\hat{e}_1^p) = \emptyset(\bar{m}_1^p)$, i.e.,

$$[\emptyset(\hat{e}_1)]^p = [\emptyset(\bar{m}_1)]^p \qquad (39)$$

Set $\emptyset(\bar{m}_1) = m_1 \in M$, and $\emptyset(\hat{e}_1) = e_1 \in \emptyset(\hat{E})(\emptyset(G)) = E(G) \subseteq F$.

We have, from Equation 39, that $e_1^p = m_1^p$; since $e_1, m_1 \in F$, it follows that $e_1 = m_1$, so that $m_1 \in E(G) = F^p(G)$, which contradicts the p-independence of $G \cup \{m_1\}$ in F. It follows that $x^p - r_1$ is irreducible over $\hat{E}(G)$, and hence, P_1 is a field.

Now suppose that P_i is a field; we wish to show that P_{i+1} is a field. We have that $\bar{m}_{i+1}^p \in \hat{E}$; set

$$\bar{m}_{i+1}^p = r_{i+1} \in \hat{E} \subseteq P_i.$$

Suppose that $x^p - r_{i+1}$ is reducible over P_i. Then

$$\exists \hat{e}_{i+1} \in P_i \ni \hat{e}_{i+1}^p = r_{i+1} = \bar{m}_{i+1}^p;$$

thus, we may write

$$[\phi(\hat{e}_{i+1})]^p = [\phi(\bar{m}_{i+1})]^p \qquad (40)$$

Set $\phi(\bar{m}_j) = m_j \in M$ for $j = 1, 2, \cdots, i+1$. Now $\hat{e}_{i+1} \in P_i = \hat{E}(G)(\bar{m}_1, \bar{m}_2, \cdots, \bar{m}_i)$; hence,

$$\phi(\hat{e}_{i+1}) \in \phi(\hat{E})(\phi(G))(\phi(\bar{m}_1), \cdots, \phi(\bar{m}_i)),$$

i.e., $\phi(\hat{e}_{i+1}) \in E(G)(m_1, m_2, \cdots, m_i) \subseteq F.$

Set $\phi(\hat{e}_{i+1}) = e_{i+1} \in E(G)(m_1, \cdots, m_i).$

From Equation 40, we thus have $e_{i+1}^p = m_{i+1}^p$, so that $e_{i+1} = m_{i+1}$, giving us $m_{i+1} \in E(G)(m_1, m_2, \cdots, m_i)$; hence, we have that $m_{i+1} \in F^p(G \cup \{m_1, m_2, \cdots, m_i\})$, which contradicts the p-independence of $G \cup \{m_1, \cdots, m_i, m_{i+1}\}$ in F. It follows that $x^p - r_{i+1}$ is irreducible over P_i, and hence, P_{i+1} is a field. By induction, we have thus established that $\forall\ i \in I^+$, P_i is a field. It follows that $\hat{E}(G)[\overline{M}]$ is a field, which we denote by "$\hat{E}(G)(\overline{M})$". Now $\hat{E}(G)(\overline{M}) \supseteq K$, since $\hat{E}(G) \supseteq K$. We also have

$$\emptyset(\hat{E}(G)(\overline{M})) = \emptyset(\hat{E})(\emptyset(G))(\emptyset(\overline{M}))$$
$$= E(G)(M) = F^p(G \cup M) = F,$$

since $\overline{M}$ is a set of counter-images of M under $\emptyset$ and $G \cup M$ is a p-basis of F. Set $\hat{F} = \hat{E}(G)(\overline{M})$; it follows that $\hat{F}$ is a coefficient field in A for $F \ni \hat{F} \supseteq K$, Q.E.D.

<u>Lemma 10</u>: Suppose that

1) A is a quasi-local p-V_n-algebra which has the joint-independence property.
2) $\exists$ a coefficient field (say, $\hat{E}$) in A^p for $F^p \ni \hat{E} \supseteq K_0$.

Then $\exists$ a coefficient field (say, $\hat{F}$) in A for $F \ni \hat{F} \supseteq K$.

Proof: We augment G to a p-basis, $G \cup M$, of F as indicated above; the hypotheses of Lemmas 7 and 8 then hold, and we apply both of these lemmas. Lemma 8 guarantees the existence of a special set $\overline{M}$, and the hypotheses of Lemma 9 hold; we then apply Lemma 9. The constructions involved in Lemmas 7, 8, and 9 then yield a field ($\hat{E}(G)(\overline{M})$ of Lemma 9) which satisfies all of the requirements on our field $\hat{F}$, Q.E.D.

<u>Theorem 1</u>: $\forall\ n \in I^+$, we have: Every quasi-local p-V_n-algebra which has the joint-independence property possesses a

coefficient field containing the base field.

Proof: The proof is by induction. Let

$C = \{i \mid i \in I^+$ and every quasi-local p-V_i-algebra which has the joint-independence property possesses a coefficient field containing the base field$\}$.

We shall prove that $C = I^+$.

a) We first show that $1 \in C$. Suppose that A is a quasi-local p-V_1-algebra which has the joint-independence property. In this case, we have $F^p \subseteq K$, so that $K_0 = F^p \cap K = F^p$. Since $\emptyset$ maps $A^p \cap K$ isomorphically onto $F^p \cap \emptyset(K) = F^p$, it follows that $A^p \cap K = K_0$ is a coefficient field in A^p for F^p; furthermore $A^p \cap K \supseteq K_0$. We may apply Lemma 10 (with n = 1) to obtain a coefficient field (say, $\hat{F}$) in A for $F \ni \hat{F} \supseteq K$. This shows that $1 \in C$.

b) We now show that $(\forall m)\ (m \in C \longrightarrow m+1 \in C)$.

Suppose then that $m \in C$. This means that every quasi-local p-V_m-algebra which has the joint-independence property possesses a coefficient field containing the base field. We shall show that $m+1 \in C$. Let A be a quasi-local p-V_{m+1}-algebra which has the joint-independence property. Applying Lemma 6, we find that A^p is a quasi-local p-V_m-algebra which has the joint-independence property. By the hypothesis of induction, A^p possesses a coefficient field containing the base field of A^p. If we label said coefficient field by "$\hat{E}$", and note that the base field of A^p is K_0, we may say that

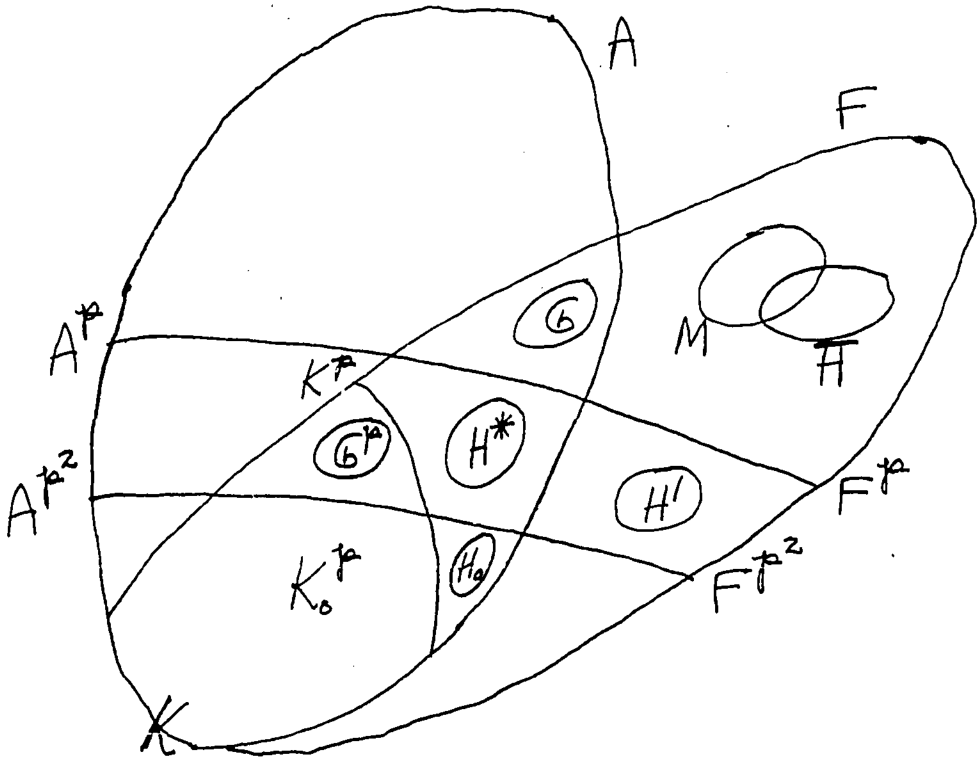
$\hat{E} \supseteq K_0$. We may now apply Lemma 10 (with $n = m+1$) to obtain the fact that A contains a coefficient field (say, $\hat{F}$) for $F \ni \hat{F} \supseteq K$. It follows that $m+1 \in C$.

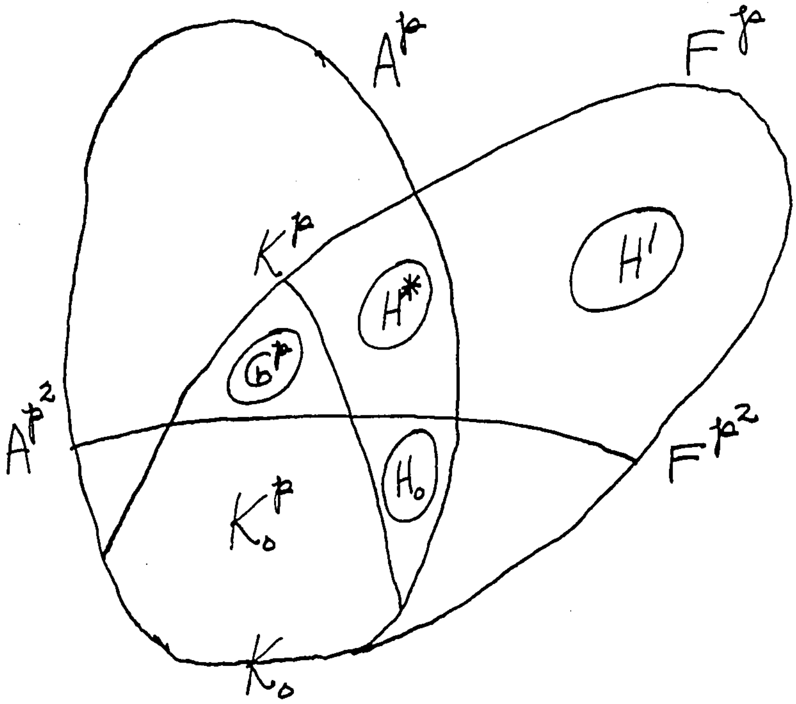
By the principle of induction, we get that $C = I^+$; this completes our proof.

We bring this chapter to a close with two drawings showing the structure of A and F, and A^p and F^p (Figures 1 and 2).

Figure 1. A and F

Figure 2. A^p and F^p


A
F
A^p
K^p
G
M
H
A^{p^2}
G^p
H^*
H'
F^p
K_0^p
H_0
F^{p^2}
K


A^p
F^p
K^p
H'
H^*
G^p
A^{p^2}
F^{p^2}
K_0^p
H_0
K_0

III. G POSSIBLY NOT EQUAL TO G'

Lemma 1: Suppose that

1) A is a quasi-local p-V_n-algebra.

2) $\exists$ a coefficient field (say, $\hat{E}$) in A^p for $E = F^p$.

Then $\exists$ a field, say $\hat{K}_o$, with the properties: $\hat{K}_o \subseteq \hat{E}$, $\phi|_{\hat{K}_o}$ maps $\hat{K}_o$ isomorphically onto $K_o = F^p \cap K$.

Proof: $\phi|_{\hat{E}}$ maps $\hat{E}$ isomorphically onto $E = F^p$; let $\phi|_{\hat{E}}^{-1} = \psi$. Then ψ maps E isomorphically onto $\hat{E}$. K_o is a subfield of $F^p = E$; hence $\psi|_{K_o}$ is an isomorphism of K_o onto $\psi(K_o)$. Set $\hat{K}_o = \psi(K_o)$; then, we may write, $\psi(K_o) = \phi|_{\hat{E}}^{-1}(K_o)$, so that $\hat{K}_o = \psi(K_o) \subseteq \hat{E}$. Also, $\phi(\hat{K}_o) = \phi(\psi(K_o)) = K_o$. Since $\hat{K}_o$ is a field, it follows that $\phi|_{\hat{K}_o}$ maps $\hat{K}_o$ isomorphically onto K_o, Q.E.D.

Let $G \cup H$ be a p-basis of K chosen as indicated above. We shall not assume that G is p-independent in F; however, we can say that $\exists$ a maximal set G' with the properties: $G' \subseteq G$, G' is p-independent in F. We now construct a special set $\bar{G}$ related to G' in much the same way that $\bar{M}$ was related to M' (see above).

Lemma 2: Suppose that

1) A is a quasi-local p-V_n-algebra.

2) $\exists$ a coefficient field (say, $\hat{E}$) in A^p for F^p.

3) $G \cup H$ is a p-basis of K chosen as indicated above.

4) $G' \subseteq G$, G' is p-independent in F.

Then $\exists$ a set $\overline{G}$ having the properties:

a) $\overline{G} \cap \hat{E} = \Lambda$

b) $\emptyset(\overline{G}) = \emptyset(G') = G'$

c) $\overline{G}^p \subseteq \hat{E}$.

Proof: Pick $g' \in G'$; we have $(g')^p \in (G')^p \subseteq K^p \subseteq K_o$. Set

$$(g')^p = k_o \in K_o \tag{1}$$

Corresponding to k_o $\exists$ exactly one element $\hat{k}_o \in \hat{K}_o \ni$

$$\emptyset(\hat{k}_o) = k_o = \emptyset(k_o) \tag{2}$$

($\hat{K}_o$ is mapped isomorphically onto K_o by $\emptyset|_{\hat{K}_o}$.) Now,

$$\hat{k}_o \in \hat{E} \subseteq A^p, \text{ and } k_o \in K_o \subseteq A^p.$$

If we recall that Ker $\emptyset|_{A^p} = N^p$, we may write, from Equation 2,

$$\exists\, z \in N^p \ni \hat{k}_o = k_o + z \tag{3}$$

Note that z is uniquely determined once $k_o = (g')^p$ has been given; we may associate with z a set N_z having the properties: $N_z \subseteq N$, and

$$\forall\, n \in N_z,\ n^p = z \tag{4}$$

Choose any $n \in N_z$; we may then write, in view of Equations 1, 3, and 4, that $\hat{k}_o = k_o + z = (g')^p + n^p = (g' + n)^p$, giving us that

$$(g' + n)^p \in \hat{K}_o \tag{5}$$

We now construct our set $\overline{G}$. Pick $g' \in G'$; then find $k_0 = (g')^p$, then $\hat{k}_0$, then z. Pick <u>exactly one</u> $n \in N_z$, and let $\overline{g} = g' + n$. We denote the set of $\overline{g}$'s thus obtained by "$\overline{G}$". We now verify that a, b, and c hold. It is immediate from Equation 5 that $\overline{G}^p \subseteq \hat{K}_0 \subseteq \hat{\hat{E}}$, which proves c. Now consider the mapping ψ: $G' \longrightarrow \overline{G}$, given by $\psi(g') = \overline{g} = g' + n$. ψ is certainly onto; we show that ψ is one-to-one. Suppose that

$$\psi(g') = \psi(g''), \tag{6}$$

for $g', g'' \in G'$. Then $\exists\, n', n'' \in N \ni \psi(g') = g' + n' = g'' + n'' = \psi(g'')$; hence, we may write

$$\phi(g') + \phi(n') = \phi(g'') + \phi(n'') \tag{7}$$

Since $\phi(N) = \{0\}$, it follows from Equation 7 that

$$\phi(g') = \phi(g'') \tag{8}$$

Now $G' \subseteq K$, so that ϕ is one-to-one on G'; it follows that $g' = g''$ (see Equation 8), which shows that ψ is one-to-one. Now pick $\overline{g} \in \overline{G}$; $\exists\, g' \in G', n \in N, \ni$

$$\phi(\overline{g}) = \phi(g' + n) = \phi(g') \tag{9}$$

It follows from Equation 9 and the one-to-one-ness of ψ that $\phi(\overline{G}) = \phi(G') = G'$, which proves b. To prove a, suppose that $\overline{G} \cap \hat{E} \neq \Lambda$, and pick $x \in \overline{G} \cap \hat{E}$. Then, we have

$$\phi(x) \in \phi(\overline{G}) = G', \phi(x) \in \phi(\hat{E}) = E,$$

so that $\phi(x) \in E \cap G'$. This gives us a contradiction, since $E \cap G' = F^p \cap G' = \Lambda$. It follows that a holds; we have proved a, b, c, Q.E.D.

<u>Lemma 3</u>: Suppose that the hypotheses of Lemma 2, Chapter III hold, and that $\overline{G}$ is a set constructed as in said lemma.

Then $\hat{E}[\overline{G}]$ is a field.

Proof: The denumerability of G implies that of $\overline{G}$. Suppose $\overline{G} = \{\overline{g}_1, \overline{g}_2, \cdots\}$. For each $i \in I^+$, define a ring $\hat{E}_i$ by

$$\hat{E}_1 = \hat{E}[\overline{g}_1], \quad \hat{E}_{i+1} = \hat{E}_i[\overline{g}_{i+1}].$$

We shall show that $\forall\, i \in I^+$, $\hat{E}_i$ is a field. The proof is by induction. We have $\overline{g}_1^p \in \hat{E}$; set $\overline{g}_1^p = k_1 \in \hat{E}$. Suppose that $x^p - k_1$ is reducible over $\hat{E}$. Then $\exists\, \hat{e}_1 \in \hat{E} \ni \hat{e}_1^p = k_1 = \overline{g}_1^p$; thus, we have

$$[\phi(\hat{e}_1)]^p = [\phi(\overline{g}_1)]^p \tag{10}$$

Set $\phi(\hat{e}_1) = e_1 \in E$, $\phi(\overline{g}_1) = g_1' \in G'$. From Equation 10, we have $e_1^p = (g_1')^p$, so that $e_1 = g_1'$; we thus have $g_1' \in E = F^p$, which is a contradiction. Hence, $x^p - k_1$ is irreducible over $\hat{E}$, and $\hat{E}_1$ is a field. Now suppose that $\hat{E}_i$ is a field; we wish to show that $\hat{E}_{i+1}$ is a field. We have that $\overline{g}_{i+1}^p \in \hat{E}$; set $\overline{g}_{i+1}^p = k_{i+1} \in \hat{E} \subseteq \hat{E}_i$. Suppose that $x^p - k_{i+1}$ is reducible over $\hat{E}_i$. Then $\exists\, \hat{e}_{i+1} \in \hat{E}_i \ni \hat{e}_{i+1}^p = k_{i+1} = \overline{g}_{i+1}^p$; then, we have,

$$[\phi(\hat{e}_{i+1})]^p = [\phi(\overline{g}_{i+1})]^p \tag{11}$$

Set $\phi(\overline{g}_j) = g_j' \in G'$, for $j = 1, 2, \cdots, i+1$. Now $\hat{e}_{i+1} \in \hat{E}_i = \hat{E}(\overline{g}_1, \overline{g}_2, \cdots, \overline{g}_i)$, so that $\phi(\hat{e}_{i+1}) \in \phi(\hat{E})(\phi(\overline{g}_1), \cdots, \phi(\overline{g}_i))$, i.e.,

$$\phi(\hat{e}_{i+1}) = e_{i+1} \in E(g_1', g_2', \cdots, g_i').$$

From Equation 11, we then have $e_{i+1}^p = (g_{i+1}')^p$, so that $e_{i+1} = g_{i+1}'$; it follows that $g_{i+1}' \in E(g_1', g_2', \cdots, g_i')$, i.e., $g_{i+1}' \in F^p(g_1', g_2', \cdots, g_i')$, which contradicts the p-independence of $\{g_1', g_2', \cdots, g_i', g_{i+1}'\}$ in F. It follows that $x^p - k_{i+1}$ is irreducible over $\hat{E}_i$. Hence, $\hat{E}_{i+1}$ is a field. By induction, we have that $\hat{E}[\overline{G}]$ is a field, Q.E.D. We shall denote $\hat{E}[\overline{G}]$ by "$\hat{E}(\overline{G})$".

Since G' is p-independent in F, we may augment G' to a p-basis, say $G' \cup M$, of F. ($M \neq \Lambda$, $M \subseteq F - F^p$.) We now proceed essentially as in Lemma 8, Chapter II. We construct a set $\overline{M}$ in A having the properties

i) $\overline{M} \cap \hat{E}(\overline{G}) = \Lambda$.

ii) $\overline{M}^p \subseteq \hat{E}$.

iii) $\overline{M}$ is a set of counter-images of M under ϕ.

We point out that the proofs of ii and iii go through exactly as they did in Lemma 8, Chapter I. (Neither of these proofs depends in any way upon whether $\hat{E} \supseteq K_0$ or not; ii depends upon the fact that (as a direct sum of groups) $A^p = \hat{E} \oplus N^p$, while iii depends upon the fact that $\overline{m} = m' + n$ for $\overline{m} \in \overline{M}$, $m' \in M'$, $n \in N$.)

We prove i: Suppose that $\overline{M} \cap \hat{E}(\overline{G}) \neq \Lambda$, and pick $a \in \overline{M} \cap \hat{E}(\overline{G})$. Then, $\phi(a) \in \phi(\overline{M}) = M$, $\phi(a) \in \phi(\hat{E})(\phi(\overline{G}))$, i.e., $\phi(a) \in M$, $\phi(a) \in E(G')$; this contradicts the fact that

$M \cap E(G') = \Lambda$. (If $M \cap E(G') \neq \Lambda$, then

$\exists\ m \in M \ni m \in E(G')$,

contradicting the p-independence of $\{m\} \cup G'$ in F.) It follows that $\overline{M} \cap \hat{E}(\overline{G}) = \Lambda$, which proves i, Q.E.D.

Lemma 4: Suppose that the hypotheses of Lemma 2, Chapter III hold, and that $G' \cup M$ is a p-basis of F selected as indicated above; suppose also that $\overline{M}$ is a set in A having the properties i, ii, and iii listed in the comments following the proof of Lemma 3, Chapter III.

Then $\hat{E}(\overline{G})[\overline{M}]$ is a field; in fact, this field is a coefficient field in A for F.

Proof: We again assume that M is denumerable, which implies that $\overline{M}$ is denumerable. Let $\overline{M} = \{\overline{m}_1, \overline{m}_2, \cdots\}$. For each $i \in I^+$, define a ring P_i by the relations $P_1 = \hat{E}(\overline{G})[\overline{m}_1]$, $P_{i+1} = P_i[\overline{m}_{i+1}]$. We shall show that $\forall\ i \in I^+$, P_i is a field. The proof is by induction. $\overline{m}_1^p \in \hat{E}$; set $\overline{m}_1^p = r_1 \in \hat{E}$. Suppose that $x^p - r_1$ is reducible over $\hat{E}(\overline{G})$. Then $\exists$ $\hat{e}_1 \in \hat{E}(\overline{G}) \ni \hat{e}_1^p = r_1 = \overline{m}_1^p$; we then get

$$[\phi(\hat{e}_1)]^p = [\phi(\overline{m}_1)]^p \qquad (12)$$

Set $\phi(\overline{m}_1) = m_1 \in M$, and $\phi(\hat{e}_1) = e_1 \in \phi(\hat{E})(\phi(\overline{G})) = E(G') \subseteq F$. From Equation 12, we have $e_1^p = m_1^p$, so that $e_1 = m_1$; it follows that $m_1 \in E(G') = F^p(G')$, which contradicts the p-independence of $\{m_1\} \cup G'$ in F. It follows that $x^p - r_1$ is irreducible over $\hat{E}(\overline{G})$, and hence, P_1 is a field. Now suppose that P_i is a field; we wish to show that P_{i+1} is a field. We have that $\overline{m}_{i+1}^p \in \hat{E}$; set $\overline{m}_{i+1}^p = r_{i+1} \in \hat{E} \subseteq P_i$. Suppose that $x^p - r_{i+1}$ is reducible over P_i. Then

$$\exists\ \hat{e}_{i+1} \in P_i \ni \hat{e}_{i+1}^p = r_{i+1} = \overline{m}_{i+1}^p;$$

then, we have

$$[\phi(\hat{e}_{i+1})]^p = [\phi(\overline{m}_{i+1})]^p \qquad (13)$$

Set $\phi(\overline{m}_j) = m_j \in M$, for $j = 1, 2, \cdots, i+1$. Now, $\hat{e}_{i+1} \in P_i = \hat{E}(\overline{G})(\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_i)$, so that we may set

$$\phi(\hat{e}_{i+1}) = e_{i+1} \in \phi(\hat{E})(\phi(\overline{G}))(\phi(\overline{m}_1), \cdots, \phi(\overline{m}_i)),$$

i.e., $\qquad \phi(\hat{e}_{i+1}) = e_{i+1} \in E(G')(m_1, \cdots, m_i).$

From Equation 13, we then have $e_{i+1}^p = m_{i+1}^p$, so that $e_{i+1} = m_{i+1}$; it follows that $m_{i+1} \in E(G')(m_1, \cdots, m_i)$, i.e.,

$$m_{i+1} \in F^p(G')(m_1, \cdots, m_i),$$

which contradicts the p-independence of

$$\{m_1, m_2, \cdots, m_i, m_{i+1}\} \cup G'$$

in F. It follows that $x^p - r_{i+1}$ is irreducible over P_i; hence, P_{i+1} is a field. By induction, we have that $\hat{E}(\overline{G})[\overline{M}]$ is a field, which we denote by $\hat{E}(\overline{G})(\overline{M})$. Now we may write

$$\begin{aligned}\phi(\hat{E}(\overline{G})(\overline{M})) &= \phi(\hat{E})(\phi(\overline{G}))(\phi(\overline{M})) \\ &= E(G')(M) = F^p(G' \cup M) = F,\end{aligned}$$

since $G' \cup M$ is a p-basis of F. It follows that $\hat{E}(\overline{G})(\overline{M})$ is a coefficient field in A for F, Q.E.D.

We set $\hat{E}(\overline{G})(\overline{M}) = \hat{F}$.

Lemma 5: Suppose that

1) A is a quasi-local p-V_n-algebra

2) $\exists$ a coefficient field (say, $\hat{E}$) in A^p for $F^p = E$.

Then $\exists$ a coefficient field (say, $\hat{F}$) in A for F.

Proof: We may apply Lemmas 1 and 2 of Chapter III; we construct the sets $\overline{G}$ and $\overline{M}$, and then apply the constructions of Lemmas 3 and 4 of Chapter III. The field $\hat{E}(\overline{G})(\overline{M})$ will then serve as our required $\hat{F}$; this completes our proof.

Theorem 1: $\forall$ n $\in I^+$, we have: Every quasi-local p-V_n-algebra possesses a coefficient field.

Proof: The proof is by induction. Let

$C = \{i \mid i \in I^+$ and every quasi-local p-V_i-algebra possesses a coefficient field$\}$.

We shall show that $C = I^+$.

a) We show that $1 \in C$. Suppose that A is a quasi-local p-V_1-algebra. In this case, $F^p \subseteq K$, so that $F^p \cap K = K_o = F^p$. Now, $\emptyset|_{A^p \cap K}$ maps $A^p \cap K = K_o$ isomorphically onto $F^p \cap K = F^p$; it follows that $A^p \cap K$ is a coefficient field in A^p for F^p. We may then apply Lemma 5, Chapter III, with n = 1, $\hat{E} = K_o$, and conclude that $\exists$ a coefficient field, say $\hat{F}$, in A for F. This shows that $1 \in C$.

b) We now show that $(\forall m)(m \in C \longrightarrow m+1 \in C)$. Suppose then that $m \in C$. This means that every quasi-local p-V_m-algebra possesses a coefficient field. We wish to show that $m+1 \in C$. Suppose that A is a quasi-local p-V_{m+1}-algebra. We may apply Lemma 5, Chapter II, to conclude that A^p is a quasi-local p-V_m-algebra. By the hypothesis of induction, $\exists$ a coefficient field in A^p for F^p; we now apply Lemma 5, Chapter III (with $n = m+1$) to conclude that $\exists$ a coefficient field in A for F. It follows that $m+1 \in C$. By the principle of induction, we have that $C = I^+$. This completes the proof of our theorem.

IV. TWO EXAMPLES

Example 1: We give an example illustrating the joint-independence property.

Let Q be a perfect field with characteristic p, p a prime; let $K = Q(u,v,w,x,y)$, where u, v, w, x, and y are algebraically independent over Q. Let $F = K(a_1,a_2,a_3)$, where

$$a_1^{p^2} = u, \quad a_2^{p^2} = u + v^p \qquad\qquad (1)$$
$$a_3^{p^2} = u + v^p + w^p$$

Set $H_0 = \{u\}$, $H^* = \{v,w\}$, $G = \{x,y\}$, $M = \{a_1,a_2,a_3\}$, $\bar{H} = \{a_1,a_2-a_1,a_3-a_2\}$, $H = H_0 \cup H^*$. We note that

$$(a_2 - a_1)^{p^2} = a_2^{p^2} - a_1^{p^2} = v^p,$$

and

$$(a_3 - a_2)^{p^2} = a_3^{p^2} - a_2^{p^2} = w^p,$$

so that

$$(a_2 - a_1)^p = v, \quad (a_3 - a_2)^p = w \qquad\qquad (2)$$

Now $H = \{u,v,w\}$; it follows from Equations 1 and 2 that $\bar{H}$ is an origin set for H. We note that

$$F^p = Q(u^p,v^p,w^p,x^p,y^p)(a_1^p,a_2^p,a_3^p),$$

and

$$F^{p^2} = Q(u^{p^2},v^{p^2},w^{p^2},x^{p^2},y^{p^2})(u,u+v^p,u+v^p+w^p),$$

so that $F^{p^2} \subseteq K$, $F^p \nsubseteq K$. The fact that $G \cup H = \{u,v,w,x,y\}$

is a set of algebraically independent elements over Q tells us that $G \cup H$ is p-independent in K. Rygg shows on pp. 34-35 of [4] that G is p-independent in F, and that M is a minimal set of generators for $F^p(G \cup M)$ over $F^p(G)$; it follows from Lemma 5, p. 15 of [4] that $G \cup M$ is p-independent in $F^p(G \cup M)$. Similarly, we have that $\bar{H}$ is a minimal set of generators for $F^p(G \cup \bar{H})$ over $F^p(G)$; it follows that $G \cup \bar{H}$ is p-independent in $F^p(G \cup \bar{H})$. It is readily shown that $K = K^p(G \cup H)$, and that $F = F^p(G \cup M) = F^p(G \cup \bar{H})$. It follows that $G \cup H$ is a p-basis of K, $G \cup M$ is a p-basis of F, and that $G \cup \bar{H}$ is p-independent in F. (In fact, in this case, $G \cup \bar{H}$ is a p-basis of F.)

We draw a picture of our structure (Figure 3).

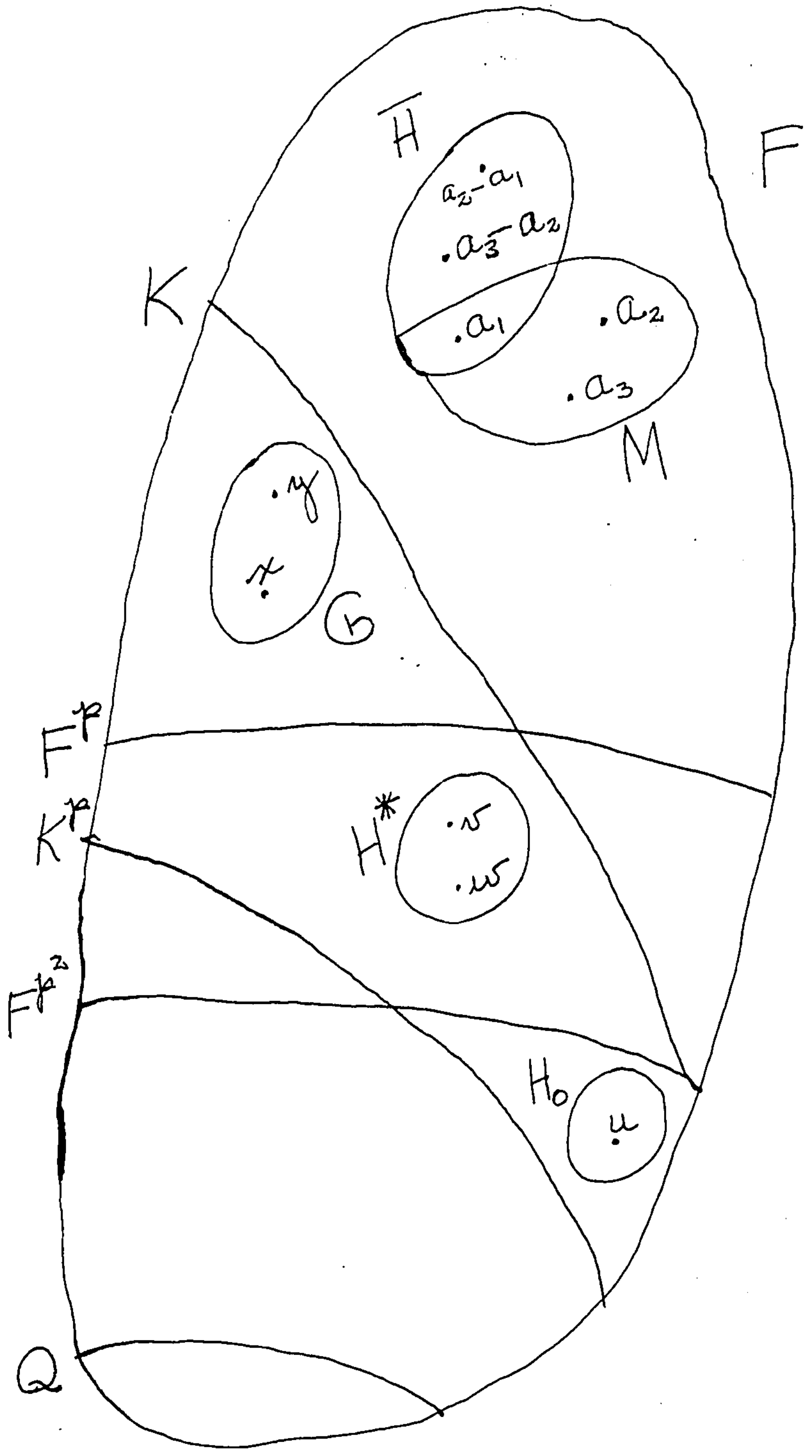
We now construct a quasi-local p-algebra of class 2 (say, A) having F as its residue field. To do this, we quote (with some change in the wording and terminology) a translation of Satz 22, p. 96 of [3]:

"Let L be a finite-degree purely-inseparable field extension of a field B, and let M be any field extension of B. Then $L \times_B M$ is a quasi-local algebra over B having as its residue field the unique composite, [L,M], of L and M".

We now set B = K, L = M = F, $A = F \times_K F$. In this case, we have $[F,F] \cong F$; it follows that A has F as its residue field.

We point out that, in this case, either of $\{1\} \times_K F$ or

Figure 3. $G \cup \bar{\bar{H}}$ p-independent in F



$\bar{H}$
$a_2 - a_1$
$a_3 - a_2$
a_1
a_2
a_3
F
K
M
y
x
G
F^p
K^p
H^*
v
w
F^{p^2}
H_0
u
Q

$F \underset{K}{x} \{1\}$ will serve as a coefficient field in A for F.

Example 2: We give an example of a quasi-local p-algebra which does not have the intersection property.

Let Q be a perfect field with characteristic p, p a prime; let $K = Q(s,t)$, where s and t are algebraically independent over Q. Set $L = K(a)$, $M = K(c)$, where

$$a^{p^2} = s, \quad c^{p^2} = st^p \tag{3}$$

Take $A = L \underset{K}{x} M$; we see that we may apply Satz 22, p. 96 of [3] (see above). We may then state that A is a quasi-local p-algebra having as its residue field

$$F = [L,M] = [K(a),K(c)] = K(a,c).$$

On pp. 18-19 of [2], Montgomery shows that $A^p \cap K \neq F^p \cap K$; it follows that A does not have the intersection property.

We note that in this case A cannot possess a coefficient field containing K.

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