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Discrete Ornstein-Uhlenbeck process in a stationary dynamic enviroment

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Discrete Ornstein-Uhlenbeck process in a stationary dynamic environment

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DEDICATION

This thesis is dedicated to my parents Tonglin and Xuelan and my sister Nana. Without their patience, understanding, support, and most of all love, the completion of this work would not have been possible.

Also, this thesis is dedicated to my fiancé Darren who has been a great source of motivation and inspiration.

Finally, this thesis is dedicated to all those who believe in the richness of learning.

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ABSTRACT

The thesis is devoted to the study of solutions to the following linear recursion:

$$X_{n+1} = \gamma X_n + \xi_n,$$

where $\gamma \in (0,1)$ is a constant and $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of normal variables with *random* means and variances. More precisely, we assume that

$$\xi_n = \mu_n + \sigma_n \varepsilon_n,$$

where $(\varepsilon)_{n\in\mathbb{Z}}$ is an i.i.d. sequence of standard normal variables and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic process independent of $(\varepsilon_n)_{n\in\mathbb{Z}}$, which serves as an exogenous dynamic environment for the model. This is an example of a so called SV (stands for stochastic variance or stochastic volatility) time-series model. We refer to the stationary solution of this recursion as a discrete Ornstein-Uhlenbeck process in a stationary dynamic environment.

The solution to the above recursion is well understood in the classical case, when ξ_n form an i.i.d. sequence. When the pairs mean and variance form a two-component finite-state Markov process, the recursion can be thought as a discrete-time analogue of the Langevin equation with regime switches, a continuous-time model of a type which is widely used in econometrics to analyze financial time series.

In this thesis we mostly focus on the study of general features, common for all solutions to the recursion with the innovation/error term ξ_n modulated as above by a random environment (μ_n, σ_n), regardless the distribution of the environment. In particular, we study asymptotic behavior of the solution when γ approaches 1. In addition, we investigate the asymptotic behavior of the extreme values $M_n = \max_{1\leq k\leq n} X_k$ and the partial sums $S_n = \sum_{k=1}^n X_k$. The case of Markov-dependent environments will be studied in more detail elsewhere.

The existence of general patterns in the long-term behavior of X_n, independent of a particular choice of the environment, is a manifestation of the universality of the underlying mathematical framework. It turns out that the setup allows for a great flexibility in modeling yet maintaining tractability, even when is considered in its full generality. We thus believe that the model is of interest from both theoretical as well as practical points of views; in particular, for modeling financial time series.

CHAPTER 1. Introduction

In this chapter we introduce a random process which is studied in this thesis (Section 1.1) and briefly review some closely related models in order to motivate our interest in this random process (Sections 1.3.1-1.4). The structure of the thesis is presented in Section 1.2.

1.1 Discrete Ornstein-Uhlenbeck process

The thesis is devoted to the study of the following *linear recursion* (often also called *stochastic difference equation*):

$$X_{n+1} = \gamma X_n + \xi_n, \tag{1.1}$$

where $\gamma \in (0,1)$ is a constant multiplicative factor and $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of normal variables.

The stationary solution to Equation (1.1) is very well understood when the normal random variables ξ_n form an i.i.d. sequence, in which case the random series $(X_n)_{n\geq 0}$ defined by (1.2) is a *first-order autoregressive process* (usually abbreviated as AR(1)-process); see, for instance, [18, 26, 42, 68]. The AR(1)-process can be thought as a discrete-time version of the regular Ornstein-Uhlenbeck process; see, for instance, [108, p. 121] and also Section 1.3.2 below. The AR(1) process serves to model discrete-time dynamics of a stochastic volatility of financial assets and interest rates in, for instance, [28, 29, 71, 128, 131, 136, 139] (see also Section 1.3.3 below).

Equation (1.1) with i.i.d. but not necessarily Gaussian coefficients $(\xi_n)_{n\in\mathbb{Z}}$ has been considered, for example, in [50, 85, 93, 95, 96, 102, 111, 114, 121, 122, 135]. Among applied fields where Equation (1.1) with i.i.d. coefficients ξ_n serves as a model are condensed matter physics, image recognition, and investment theory.

In this thesis we focus on the case when ξ_n are dependent random variables. More precisely, we consider the coefficients ξ_n with the parameters (the mean and variance) of the normal distribution *modulated* by a stationary and ergodic process. By this we mean that ξ_n can be represented in the form

$$\xi_n = \mu_n + \sigma_n \varepsilon_n, \qquad n \in \mathbb{Z},$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a standard (i.i.d., zero mean, variance one) Gaussian sequence and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is independent of it stationary and ergodic process (exogenous *random environment*). For instance, the mean and variance form a two-dimensional Markov process. See Section 2.2 below for full details of the model definition.

In [95, 122], the authors refer to the stationary solution of (1.1) as with i.i.d. but not necessarily Gaussian coefficients as *discrete-time (generalized) Ornstein-Uhlenbeck process*. We adopt here a similar terminology, and call the model, which is formally introduced below in Section 2.2, *discrete-time Ornstein-Uhlenbeck process in a stationary dynamic environment*.

1.2 Structure of the thesis

The thesis is organized as follows. The rest of Chapter 1 is devoted to a discussion of some models closely related to the setting introduced above, in Section 1.1. A common feature of these models is the presence of an additive noise term (typically represented by a standard Gaussian or, more generally, Lévy process) *modulated* by an exogenous random (typically, Markovian) environment. The main goal of this chapter is to give a perspective on our setting within the applied context. Sections 1.3 and 1.4 contain a survey of the literature, including a discussion of general stochastic difference equations with i.i.d. and non-i.i.d. coefficients (Section 1.3.1 and Section 1.4, respectively), continuous-time Ornstein-Uhlembeck process, (Section 1.3.2), stochastic volatility models of financial mathematics (Section 1.3.3), and their particular case, where the coefficients of the model are randomized in a manner using "random environment", similar to the approach adopted in this thesis.

Chapter 2 includes the required mathematical background and preliminaries. Our main results are collected in Chapter 3, which also contains the proofs. Finally, Chapter 4 is devoted to the concluding remarks and discussion.

1.3 Related models: Literature survey

1.3.1 General stochastic difference equations

Linear recursion (1.1) is a particular case of a more general stochastic difference equation

$$X_{n+1} = \rho_n X_n + \xi_n, \tag{1.2}$$

with random coefficients $(\rho_n, \xi_n)_{n\in\mathbb{Z}}$. Both one-dimensional and multivariate versions of (1.2) have been extensively studied in the literature. In the latter case, for some integer $d \geq 1$, the underlying variables X_n and the additive coefficients ξ_n are random d-dimensional vectors while ρ_n are random $d \times d$ matrices. Although in this thesis we concentrate on the one-dimensional case only, most of our results (presented in Chapter 3 below) can be extended to the multivariate version in a fairly straightforward manner.

Equation (1.2) has a remarkable variety of real-world applications; see, for instance, [41, 120, 141] for an extensive account. In a basic example (cf. [97, 141]), X_n is the balance of a saving account at time n, ξ_n is the deposit made just before time n, and ρ_n is the interest rate. In insurance mathematics the time series generated by (1.2) is called a *perpetuity* or a *perpetual annuity* [41], and X_n is interpreted as the present value of an asset with a fluctuating annual premium and a random future discounting. Other applications of Equation (1.2) include, for example, evolution of the water density in Norwegian fjords, evolution of a stock of a radioactive material, study of fluctuations in brightness of the Milky Way, environmental pollution models, investment models, study of effect of environmental changes on crop production, models of cultural and genetical inheritance, atomic cascade models.

For applications in theoretical probability see, for instance, [59, 60, 62, 88, 106]. Examples of applications in economics are given, for instance, in [9, 25, 51, 52, 53, 79, 109, 110, 118]. See, for instance, [32] and [56] for a discussion of the model in comparison with similar linear and quasi-linear iterated recursions.

1.3.2 Langevin equation and the Ornstein-Uhlenbeck process

The classical and well-studied setup in Equation (1.1) occurs if one assumes that ξ_n are i.i.d. zero-mean normal variables. To motivate this setup, we start with the following equation describing a discrete-time motion in the line of a particle with mass m, in the presence of a random potential and a viscosity force proportional to velocity:

$$m(X_n - X_{n-1}) = -\kappa X_{n-1} + F_n, \qquad n = 1, 2, \ldots \tag{1.3}$$

Here random variable X_n represents the velocity of the particle at time n, κ is a constant damping coefficient, and F_n is a random force applied to the particle at time $n \in \mathbb{N}$. Setting $\gamma = 1 - m^{-1}\kappa$ and $\xi_n = m^{-1}F_n$, we obtain (1.1) as an equivalent form of (1.3).

Equation (1.3) is a discrete-time counterpart of the one-dimensional Langevin stochastic differential equation (SDE) [24, 140] which is often informally written as

$$m\dot{X}_t = -\kappa X_t + F_t, \qquad t \geq 0. \tag{1.4}$$

Usually, it is assumed that F_t is a (Gaussian) white noise process. Applications of (1.4) with a non-Gaussian term F_t are addressed for instance in [4, 7, 16, 94, 117].

We will now compare solutions of (1.3) and (1.4). Iterating (1.1) we obtain:

$$X_n = \xi_n + \gamma\xi_{n-1} + \gamma^2\xi_{n-2} + \cdots + \gamma^{n-1}\xi_1 + \gamma^n X_0. \tag{1.5}$$

If $\xi_n = \sigma(B_n - B_{n-1})$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion (cf. Section 2.1.4 below), (1.5) yields

$$X_n = \gamma^n X_0 + \sigma\gamma^{n-1}\sum_{k=1}^{n}\gamma^{-(k-1)}(B_k - B_{k-1}) = \gamma^n X_0 + \sigma\gamma^{n-1}\int_0^n \gamma^{-[s]}dB_s, \tag{1.6}$$

where $[s]$ stands for the integer part of s. This gives a representation of X_n as *Ito's stochastic integral* with respect to Brownian motion. This is a very special example of stochastic integral because for the simple deterministic process $f(s) = \gamma^{-[s]}$ the integral is defined as a Riemann sum, which is not the case in general [108].

On the other hand, using the embedding $\xi_n = \sigma(B_n - B_{n-1})$ and letting $\kappa = 1 - \gamma$, Equation (1.1) can be rewritten as (compare to (1.3))

$$X_{t+1} - X_t = -\kappa X_t + \sigma(B_{t+1} - B_t), \qquad t = 1, 2, \ldots.$$

The continuous-time analogue of this equation is the stochastic differential equation

$$dX_t = -\kappa X_t dt + \sigma dB_t, \qquad t \geq 0, \tag{1.7}$$

which is the formal version of (1.4). The (unique) solution of this equation is given by [108]

$$X_t = e^{-\kappa t} X_0 + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dB_s, \tag{1.8}$$

which is the celebrated *Ornstein-Uhlenbeck process.*

The continuous Ornstein-Uhlenbeck process can be recovered from the solution to Equation (1.1) as follows. Fix $\tau > 0$ and consider a sequence of processes $(X_{\tau,n})_{n\in\mathbb{N}}$ defined by $X_{\tau,0} = X_0$ and

$$X_{\tau,n+1} = \gamma_\tau X_{\tau,n} + \sqrt{\tau} \cdot \xi_n,$$

where $\gamma_\tau = 1 - \kappa\tau$. In particular, $X_{1,n} = X_n$. Notice that $\gamma_\tau \sim (1-\kappa)^\tau = \gamma^\tau$ as $\tau \to 0$ (think of γ_τ as an aggregate interest rate over a short time interval τ). Furthermore, according to (1.6), for any $t > 0$,

$$\begin{aligned} X_{\tau,[t/\tau]} &= \gamma_\tau^{[t/\tau]} X_0 + \sigma\sqrt{\tau} \cdot \gamma^{[t/\tau]-1} \int_0^{[t/\tau]} \gamma^{-[s]} dB_s, \\ &\sim e^{-\kappa t} X_0 + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dB_s, \end{aligned}$$

where in the last step we used the scaling property of the Brownian-motion $\sqrt{\tau}\, B_s \sim B_{\tau s}$ (see Section 2.1.4 below). The above argument can be made rigorous, and it can be shown that in fact the process $X_{\tau,[t/\tau]}$ converges weakly to the continuous Ornstein-Uhlenbeck process when τ approaches zero (see for instance [49, 112] and references therein). The conclusion in words is thus that if the interval between successive observations is taken to zero and the coefficients in Equation (1.1) are adjusted properly, then the obtained sequence of AR(1)-processes has the continuous Ornstein-Uhlenbeck process as its scaling limit.

The stationary solution to Equation (1.7) has zero-mean. The general Ornstein-Uhlenbeck process solves

$$dX_t = \kappa(\mu - X_t)dt + \sigma dB_t, \qquad t \geq 0. \tag{1.9}$$

The solution to (1.9) has mean $\mu \in \mathbb{R}$ and is mean-reverting. That is, it tends to drift toward the mean. In fact, the drift is proportional to the value of the current deviation $|X_t - \mu|$ from the mean. In financial mathematics, the Ornstein-Uhlenbeck process is used to model the dynamics of interest rates and currency exchange rates. The parameter σ then represents the degree of volatility around the mean, which is caused by stochastic *shocks* (random fluctuations of the underlying process), while κ is the rate by which these shocks dissipate and the process reverts towards the mean. The corresponding model is called the *Vasicek model* if κ is a constant and is called the *Hull-White model* if $\kappa = \kappa(t)$ is assumed to be a non-constant deterministic function of time (see, for instance, [2, 81, 82] and [80, Chapter 21]).

1.3.3 Time series analysis: GARCH-type and SV models

In applied statistical and economical models, *time series* are described as an ordered sequence of associated together data which is observed at equally spaced time intervals. Time series are modeled by sequences of random variables or, more generally, random vectors. Usually, the dependence between members of the sequence is given in an explicit functional (recursive) form with an *error/noise term* as, for instance, in Equation (1.1). Times series models are used for predictive forecasting and monitoring in a variety of applications, including (see, for instance, [26, 68]) census analysis, inventory studies, quality control, and stock market analysis. Time series are often understood as discrete approximations for their scaling limits, continuous-time diffusions [28, 89, 112, 119].

Linear recursion (1.1) is a classical, and probably the most studied, time series model. A time series $(X_n)_{n \in \mathbb{N}}$ is a linear process if it admits representation

$$X_n = \sum_{k=0}^{\infty} c_k \xi_{n-k}, \qquad n \in \mathbb{N},$$

where ξ_n are standard (mean-zero with unit variance) normal variables and c_n are constants such that $\sum_{k=0}^{\infty} c_k^2 < \infty$. Linear processes are commonly modeled using moving average (MA) and autoregressive (AR) models. These models are often intertwined to generate new models. For example, the autoregressive moving average model (ARMA) combines the (AR) model and the (MA) model [43, 44]. An important framework for modeling financial time-series is represented by a widely used in econometrics *Generalized Autoregressive Conditional Heteroskedasticity* (GARCH) class of models [12, 13, 43, 44, 45, 139]. The term heteroskedasticity means "having random volatility". GARCH models assume the current variance to be a function of both the actual sizes of the previous time periods' error terms as well as the previous values of the underlying financial asset. GARCH models are employed commonly in modeling financial time series that exhibit time-varying volatility clustering (persistence of shocks), i.e. periods of swings followed by periods of relative calm. Two key ideas behind the GARCH model is that 1) the volatility of an asset changes over the time; 2) the price of the asset and its volatility (which, in particular, reflects the measure of risk involved in the investment in this asset) are correlated and partially determined by their past values. Therefore, the philosophy of GARCH is that volatility should not be considered as an exogenous factor to a price-dynamic model, but rather should be treated as one of the components of the underlying vector of variables. In fact, technically, GARCH-type models can typically be reduced to a multivariate version of (1.2) (see, for instance, [109]).

An alternative to GARCH-type models framework is formed by *stochastic volatility* (SV) models, which treat the volatility as an exogenous factor to a price-dynamic model. For a comparative study of two approaches and their relative advantages we refer the reader to [26, 36, 44, 67, 80, 139]; the topic is outside of the scope of this thesis. The model presented in Section 1.1 is belongs to the class of discrete-time SV models. One of the major motivation for our study is the model proposed in [128], where the logarithm of the stochastic volatility satisfies Equation 1.1 or its continuous-time counterpart Equation (1.7) (see, for instance, [28, 29, 71, 78, 128, 131, 136, 139]).

1.3.4 Geometric Brownian motion with regime switches

The celebrated Black-Scholes model of a financial market (see, for instance, [80]) assumes that the value of a stock S_t (observed at time $t > 0$) evolves with time as a *geometric Brownian motion*. That is,

$$dS_t = S_t \cdot \big[\mu dt + \sigma dB_t\big],$$

where the *average* μ and the *volatility* σ are constants, and B_t is a standard Brownian motion. Letting $X_t = \log S_t$ and applying Itô's formula [108], one obtains

$$dX_t = \Big(\mu - \frac{\sigma^2}{2}\Big)dt + \sigma dB_t.$$

Thus X_t is a Brownian motion with drift. One of the main criticisms of this model focuses on the underlying assumption that the volatility does not change with time. The recognition that many financial time-series, such as stock returns and exchange rates, exhibit changes in volatility over time goes back to at least [47, 48, 104]. These changes are due for example to seasonal effects, response to the news, and dynamics of the market.

The notion of *regime shifts* or *regime switches* were proposed by Hamilton in his seminal paper [67] in order to explain the cyclical feature of macroeconomic variables. The model for the financial market with regime switching was further investigated and developed by many authors, see for instance [20, 33, 40, 63, 64, 65, 66, 84, 137]. Its one-dimensional version can be formally stated as follows. Consider a standard Brownian motion B_t and a continuous-time finite-state Markov chain $y = (y_t)_{t\geq 0}$. Denote by $\mathcal{D}$ the state-space of this Markov chain. That is, for every $t \geq 0$ we have $y_t \in \mathcal{D} = \{1, 2, \cdots, d\}$ for some $d \in \mathbb{N}$. The random variable y_t is observable and represents the regime of the economy at time $t \geq 0$. The integer $d = |\mathcal{D}|$ is the number of feasible regimes and $\mathcal{D}$ is the index set for enumeration of the regimes. Assume that the stochastic processes B_t and y_t are independent, and let S_t represent the price of a risky asset. Then the process $(S_t)_{t\geq 0}$ satisfies the *Markov-modulated* (cf. Definition 1.1 below) stochastic differential equation

$$dS_t = S_t \cdot \big[\mu_{y_t} dt + \sigma_{y_t} dB_t\big],$$

where, $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ represent, respectively, the long-term average and the volatility when the economy is at state $i \in \mathcal{D}$. Applying Itô's formula [108], yields the following stochastic differential equation for $X_t = \log S_t$

$$dX_t = \Big(\mu_{y_t} - \frac{\sigma_{y_t}^2}{2}\Big)dt + \sigma_{y_t} dB_t. \tag{1.10}$$

Remarkably, in [40] the authors studied a more general stochastic differential equation

$$dX_t = \big(b_{y_t} + \nu_{y_t} X_t\big)dt + \sigma_{y_t} dB_t, \tag{1.11}$$

which is reduced to (1.10) if one sets $\nu_i \equiv 0$ and $b_i = \mu_i - \frac{\sigma_i^2}{2}$ for $i \in \mathcal{D}$. The solution to (1.11) can be thought as a *Markov-modulated Ornstein-Uhlenbeck process.*

1.4 Linear recursions with non-i.i.d. coefficients

In a series of papers [25, 31, 54, 70, 106, 126], the results of [56, 86] and [58, 61] have been extended to the case where the coefficients $(\rho_n, \xi_n)_{n\in\mathbb{Z}}$ in Equation (1.2) are modulated by a Markov chain. The extension is desirable in many, especially financial, applications, see for instance [9, 10, 25, 67, 118].

By the coefficients induced by a stochastic process we mean the following setup.

Definition 1.1. The coefficients $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ in Equation (1.2) are said to be *modulated* by a sequence of random variables $(\omega_n)_{n\in\mathbb{Z}}$, each valued in a set $\mathcal{D}$, if there exist independent random variables $(\xi_{n,i}, \rho_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}} \in \mathbb{R}^2$ such that for a fixed $i \in \mathcal{D}$, $(\xi_{n,i}, \rho_{n,i})_{n\in\mathbb{Z}}$ are i.i.d,

$$\xi_n = \xi_{n,\omega_n} \qquad \text{and} \qquad \rho_n = \rho_{n,\omega_n}, \tag{1.12}$$

and $(\xi_{n,i}, \rho_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}}$ is independent of $(\omega_n)_{n\in\mathbb{Z}}$.

Notice that the randomness of the coefficients induced by a sequence $(\omega_n)_{n\in\mathbb{Z}}$ (which serves here as a random environment for the underlying process X_n) is due to two factors:

1) to the randomness of the underlying auxiliary process $(\omega_n)_{n\in\mathbb{Z}}$, which can be thought as representative of the "state of the external world" or "regime",

   and, given the value of ω_n,

2) to the "intrinsic" randomness of characteristics of the system which is captured by the random pairs $(\xi_{n,\omega_n}, \rho_{n,\omega_n})$.

Note that when $(\omega_n)_{n\in\mathbb{Z}}$ is a finite Markov chain, (1.12) defines a *Hidden Markov Model* (HMM). See for instance [46] for a survey of HMM and their applications in various areas.

In [121], it is considered the asymptotic behavior of the solution to a multivariate version of the Langevin equation (1.1), driven by a force ξ_n induced by certain Gibbs' states. In a "regular variation in, regular variation out" setup it is shown that many results known for regular random walks with i.i.d. heavy-tailed increments can be carried over to the random motion of the Langevin's particle (with dependent and non-stationary increments). Similar models in dimension one with i.i.d. coefficients $(\xi_n)_{n\in\mathbb{Z}}$ were considered with applications to the physics of a moving particle in mind for instance in [95, 96, 114, 122]. For a general multivariate version of Equation (1.2) *with i.i.d. coefficients*, convergence of X_n to stable laws was recently shown in [22].

The key technical tool used in [54, 70, 121] is an extension to the Markovian setup of a lemma of [61] (cf. [58, Lemma 2]), which states a "persistent additive propagation" of regularly varied distribution tails during the iteration from X_{n-1} to X_n in (1.2). This "additive propagation" of the tail structure is related to the corresponding property of the so called *stable* distributions (see, for instance, Definitions 1.1.3 and 1.1.4 in [127, Section 1.1]), which are the only possible limits for a weak convergence of (suitably scaled) partial sums of i.i.d. random variables. In fact, a necessary and sufficient condition for such convergence in a non-Gaussian case is a regular variation of the distribution tails of the underlying random variables [39, 127]. The Gaussian random variables is an extreme case of 2-stable distributions (because of the "propagation of the tails structure" property stated in part (b) of Proposition 2.15 below; see, for instance, Example 1.1.3 in [127, Section 1.1]). Thus, from the mathematical point of view, this thesis can be viewed is a natural continuation of/complementary study to [121].

CHAPTER 2. MATHEMATICAL BACKGROUND

In this chapter we overview some general mathematical topics closely related to our model. The goal is to provide both conceptual and technical backgrounds to our results, which are presented below in Chapter 3. In Section 2.1 we recall general basic concepts of probability theory, which are necessary for the rigorous definition of the model and statement of our main results. The underlying model (namely, a particular setting of coefficients in Equation (1.1)) is formally introduced in Section 2.2. In Section 2.3 we recall some well-known results regarding the existence of the limiting distribution for the model and its properties. Finally, in Section 2.4 we discuss certain paradigms in the asymptotic behavior of the partial sums of random variables (in particular, classical random walks), which are shown to be shared by our model in Section 3.4 below.

2.1 Review of probability concepts

The aim of this section is threefold. First, in Subsection 2.1.1 we define a general formal framework for stochastic models, namely the *probability space*. Next, in Subsections 2.1.2 and 2.1.3 we introduce random variables and discuss various modes of convergence for sequences of random variables. Finally, Subsection 2.1.4 presents a review of Gaussian processes.

All the definitions and the results, that for the reader's convenience are collected in Subsections 2.1.1–2.1.3, can be found in Chapters 1 and 2 of [39]. The discussion in Subsection 2.1.4 is based mostly on the material adapted from [37, Chapter 12] and [72, Chapter 2].

2.1.1 Probability space

Probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set called *sample space*, $\mathcal{F}$ is a collection of subsets of Ω called *events*, and $\mathbb{P} : \mathcal{F} \to [0, \infty)$ is a function that assigns *probabilities* to events. Elements of Ω are denoted by ω and are called *outcomes*.

A possible interpretation of the probability space is as follows. The outcome ω includes full information about certain system but is not accessible to the observer. Full information means that all characteristics of the system (i.e. *random variables*) are deterministic functions of ω. Since ω is not known, the observer can only measure "chances that the outcome belongs to a certain set", i.e. probabilities of events.

To develop a well-founded theory capable for realistic modeling of "random phenomena", we need to impose some structural assumptions on $\mathcal{F}$ and $\mathbb{P}$. In fact, it is assumed that $\mathcal{F}$ is a *σ-algebra* (*σ-field* is a synonymous) and $\mathbb{P}$ is a *probability measure*.

Definition 2.1. Let ω be a set and $\mathcal{F}$ be a collection of its subsets. Then $\mathcal{F}$ is called a *σ-algebra* (or *σ-field*) if

1. $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$, where $\emptyset$ is an empty set.

2. $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$, where $A^c := \{x \in \Omega : x \notin A\}$ is the complement of A in Ω.

3. $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ whenever $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$.

If $\mathcal{F}$ is a σ-algebra, any its element is called an event or, alternatively, a measurable set (with respect to $\mathcal{F}$). In particular, if A and B are events, then "not A", "A or B", "A and B", "not A and B", etc. are events as well. Extension of these conventions to a countable number of sets is essential for most of the applications.

If $\mathcal{F}$ is a σ-algebra of subsets of a set Ω, the pair $(\Omega, \mathcal{F})$ is called a *measurable space*. Note that different probability measures can be introduced in the same probability space.

Definition 2.2. Let $(\Omega, \mathcal{F})$ be a measurable set. A function $\mathbb{P} : \mathcal{F} \to [0, \infty)$ that assigns non-negative values to the events is called a *measure* if

1. For all $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$.

2. $\mathbb{P}(\emptyset) = 0$.

3. If $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, are disjoint, then $\mathbb{P}\big(\cup_{n\in\mathbb{N}} A_n\big) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

If in addition $\mathbb{P}(\Omega) = 1$, then $\mathbb{P}$ is called a *probability measure*.

2.1.2 Random variables

Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{T})$ be given measurable spaces.

Definition 2.3. A function $X : \Omega \to S$ is called *measurable* or, alternatively, a $(S, \mathcal{T})$*-valued random variable* if $X^{-1}(A) \in \mathcal{F}$ whenever $A \in \mathcal{T}$.

Intuitively, X is measurable if $X^{-1}(A) = \{\omega \in \Omega : X(\omega)\}$ is an event that can be considered in the model whenever A is an event. Note that the probability measure $\mathbb{P}$ is irrelevant to this definition, which allows to consider the same random variables in different stochastic models.

If $S = \mathbb{R}^d$ and $\mathcal{T}$ are Borel sets, a measurable map X is called *random vector*, and if $d = 1$ it is usually called random variable, omitting the words "real-valued".

If $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ are Borel sets, is a random variable, then the probability measure $\mu_X(A) := \mathbb{P}(X \in A)$ is called the *distribution* of X or, alternatively, the *law* of the random variable X. The function $F_X(x) = \mathbb{P}(X \leq x) : \mathbb{R} \to [0,1]$ is called the *distribution function* of X. It can be shown that μ_X is the unique probability measure on $(\mathbb{R}, \mathcal{B})$ that has the property $\mu_X\big((a,b]\big) = F(b) - F(a)$ for all real numbers a, b. This means that the distribution function completely determines the law of a random variable. If $F_X(x) = \int_{-\infty}^{x} f_X(y)dy$ for some $f_X : \mathbb{R} \to \mathbb{R}_+$, we say that X has *density function* f_X. Equivalently, a random variable X has density $f_X(x)$ if $\mathbb{P}(X \in A) = \int_A f_X(x)dx$ for any Borel subset A of $\mathbb{R}$.

The Lebesgue integral $\int_{\mathbb{R}} x\mu_X(dx) = \int_{\Omega} X(\omega)\mathbb{P}(d\omega)$ is called the *expectation* of X and is denoted by either $\int X d\mathbb{P}$ or $\mathbb{E}[X]$. For non-negative random variables, we have the following useful formula

$$\mathbb{E}[Y] = \int_0^{\infty} \mathbb{P}(Y > x)dx \geq \sum_{n=1}^{\infty} \mathbb{P}(Y > n) \tag{2.1}$$

Indeed, using Fubini's theorem to interchange the integrals, we obtain

$$\begin{aligned} \mathbb{E}[Y] &= \int_{\mathbb{R}} y\mu_Y(dy) = \int_0^{\infty} \Big(\int_0^{y} dx\Big)\mu_Y(dy) = \int_0^{\infty} \Big(\int_x^{\infty} \mu_Y(dy)\Big)dx \\ &= \int_0^{\infty} \mathbb{P}(Y > x)dx \geq \sum_{n=1}^{\infty} \mathbb{P}(Y > n), \end{aligned}$$

where the last inequality follows from the monotonicity of the function $G_Y(x) := \mathbb{P}(Y > x)$.

If Y is a random variable defined in the same probability space with X, we denote by $\mathbb{E}[X|Y]$ the *conditional expectation* of X given Y. We denote by $\mathrm{COV}(X,Y)$ or by $\mathrm{COV}_{\mathbb{P}}(X,Y)$ (*covariance* of X and Y) the expectation $\mathbb{E}\big[\overline{X}\cdot\overline{Y}\big]$, where $\overline{X}=X-\mathbb{E}[X]$ and $\overline{Y}=Y-\mathbb{E}[Y]$. We denote $\mathrm{COV}(X,X)$ by either $\mathrm{VAR}(X)$ or $\mathrm{VAR}_{\mathbb{P}}(X)$ (*variance* of X).

2.1.3 Modes of convergence

Here we collect some definitions and basic facts about various modes of convergence for random sequences. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and X be a random variable defined in the same probability space $(\Omega,\mathcal{F},\mathbb{P})$. Denote the distribution function of X_n by F_n and the distribution function of X by F_X.

Definition 2.4.

(i) We say that X_n converges to X *almost surely* (or *with probability one*) if there exists $E\in\mathcal{F}$ such that $\mathbb{P}(E)=1$ and

$$\lim_{n\to\infty} X_n(\omega)=X(\omega), \qquad \forall\ \omega\in E.$$

We write $\lim_{n\to\infty}X_n=X$, a.s. to denote the almost sure convergence.

(ii) We say that X_n converges to X *in probability* if $\lim_{n\to\infty}\mathbb{P}(|X_n-X|>\varepsilon)=0$ for all $\varepsilon>0$. We write $X_n\xrightarrow{\mathbb{P}}X$ to denote convergence in probability.

Proposition 2.5. *Almost sure convergence implies convergence in probability, the reverse statement is not true in general.*

Theorem 2.6.

(i) X_n converges to X in probability if and only if from every subsequence X_{n_m} can be extracted a further subsequence that converges to X a.s.

(ii) If X_n converges to X in probability and $f:\mathbb{R}\to\mathbb{R}$ is continuous, then $f(X_n)$ converges to $f(X)$ in probability.

Definition 2.7. We say that X_n converges to X *in distribution* (also *in law*, also *weakly*) if $\lim_{n\to\infty}F_n(x)=F_X(x)$ for all $x\in\mathbb{R}$ for which the limiting distribution function F_X is continuous. We write $X_n \Rightarrow X$ or $X_n \stackrel{\mathbb{P}}{\Rightarrow} X$ (to emphasize the underlying probability measure) to denote the weak convergence.

Theorem 2.8. *Each of the following statements is equivalent to the weak convergence:*

(i) $\lim_{n\to\infty} \mathbb{E}\big[f(X_n)\big] = \mathbb{E}\big[f(X)\big]$ *for all bounded and continuous functions* $f : \mathbb{R} \to \mathbb{R}$.

(ii) $\lim_{n\to\infty} \mathbb{E}\big[e^{itX_n}\big] = \mathbb{E}\big[e^{itX}\big]$ *for all* $t \in \mathbb{R}$.

The important example of random variables with density is addressed in the next statement.

Proposition 2.9.

1. *If all* X_n *are integer valued, then* X_n *converges to* X *weakly if and only if*

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)$$

   for all $k \in \mathbb{Z}$.

2. *(Scheffe's theorem) If each* X_n *has a density* f_n *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in \mathbb{R}$, *then* f *is a density of a random variable* X *and* X_n *converges to* X *weakly.*

The weak convergence is the one which is involved in the statement of the central limit theorem. In general, it is weaker than the almost sure convergence.

Proposition 2.10. *Let* $c \in \mathbb{R}$ *be a constant.*

1. *If* $X_n \to_{n\to\infty} X$ *in probability, then* $X_n \to_{n\to\infty} X$ *weakly.*

2. *If* $X_n \to_{n\to\infty} c$*, weakly then* $X_n \to_{n\to\infty} c$ *in probability.*

3. *If* $X_n \to_{n\to\infty} X$ *weakly and* $Y_n \to_{n\to\infty} c$ *weakly, then* $X_n + Y_n \to_{n\to\infty} X + c$ *weakly.*

The following serves as a counterexample to some "extensions" of the above proposition.

Example 2.11. *If* $-X$ *has the same distribution as* X *(i.e., the law of* X *is symmetric) and* $X_n = -X$ *for all* $n \in \mathbb{N}$*, then*

(i) X_n *converges to* X *weakly, but not in probability.*

(ii) $X_n + X = 0$ *and does not converge weakly to* $X + X = 2X$.

We conclude this subsection with two results, useful in checking whether or not $\mathbb{E}(X_n)$ converges as $n \to \infty$ to $\mathbb{E}(X)$.

Theorem 2.12 (dominated convergence)**.** *If* $X_n \geq 0$, X_n *converges to* X *in probability, and there exists a random variable* Y *with* $\mathbb{E}(Y) < \infty$ *such that* $\mathbb{P}\big(|X_n| \leq Y\big) = 1$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

In the special case when Y is an a.s. constant, the above result is called the *bounded convergence theorem.*

Theorem 2.13 (monotone convergence)**.** *If* $0 \leq X_n \leq X_{n+1}$ *for all* $n \in \mathbb{N}$ *and* X_n *converges to* X *a.s., then* $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

2.1.4 Gaussian processes

The model that we consider in this thesis is formally introduced in Section 2.2 below. The model belongs to the class of "processes in random environment" having two levels of randomness, explicitly separated in the construction of the model. The first level models an exogenous environment while the second captures intrinsic random factors influencing operation of the underlying system in a fixed environment. Once the environment is fixed, the model is a Gaussian stochastic process. The goal of this section is to present a background on Gaussian processes, which is needed for definition and analysis of our model in a fixed environment.

We start with a definition of a single Gaussian random variable.

Definition 2.14. For any $\mu \in \mathbb{R}$ and $\sigma > 0$, a random variable X with density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is called *Gaussian* or *normally distributed.* We denote this by $X \sim \mathcal{N}(\mu, \sigma^2)$. To extend this definition to $\sigma = 0$, we convene that $X \sim \mathcal{N}(\mu, 0)$ means $\mathbb{P}(X = \mu) = 1$.

For any constants $\mu \in \mathbb{R}$ and $\sigma > 0$, we denote by Φ_{μ,σ^2} the distribution function of a normal random variable with mean μ and variance σ^2. That is,

$$\Phi_{\mu,\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{t} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

We set $G_{\mu,\sigma^2}(t) := 1 - \Phi_{\mu,\sigma^2}(t)$.

In the next proposition we summarize some basic properties of Gaussian random variables.

Proposition 2.15.

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Then

$$\mathbb{E}[X] = \mu, \qquad \mathbb{E}\big[|X|\big] = \mathbb{E}\Big[|\mu| + \sqrt{\frac{2\sigma^2}{\pi}}\Big], \qquad \mathbb{E}[X^2] = \sigma^2 + \mu^2.$$

(b) Let $(Y_k)_{k\geq 0}$ be a sequence of independent random variables with $Y_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$ for some constants $\mu_k \in \mathbb{R}$ and $\sigma_k \geq 0$. Then for any sequence of reals $(\lambda_k)_{k\geq 0}$ we have

$$\sum_{k=1}^{n} \lambda_k X_k \sim \mathcal{N}(a_n, b_n^2)$$

with $a_n = \sum_{k=1}^n \lambda_k \mu_k$ and $b_n^2 = \sum_{k=1}^n \lambda_k^2 \sigma_k^2$.

(c) If $X \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

(d) If $X \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$\mathbb{E}\big[e^{itX}\big] = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

for any $t \in \mathbb{R}$.

It will be convenient to extend Definition 2.14 to random parameters μ and σ.

Definition 2.16. Let (μ, σ) be a random $\mathbb{R}^2$-valued vector with $\mathbb{P}(\sigma > 0) = 1$. We say that a random variable X has $\mathcal{N}(\mu, \sigma^2)$-distribution (in words, *normal-*(μ, σ^2) *distribution*) and write $X \sim \mathcal{N}(\mu, \sigma^2)$ if

$$\mathbb{P}(X \leq t) = \mathbb{E}\big[\Phi_{\mu,\sigma^2}(t)\big], \qquad t \in \mathbb{R}.$$

That is, conditional on (μ, σ^2), distribution of X is a normal distribution with mean μ and variance σ^2.

We have the following definition.

Definition 2.17. Let Λ be any set. A collection of random variables $X = \{X_\lambda : \lambda \in \Lambda\}$ defined in the same probability space is called a *Gaussian process*, if any finite linear combination $\sum_{k=1}^{n} a_k X_{\lambda_k}$, where $a_k \in \mathbb{R}$, $\lambda_k \in \Lambda$, $n \in \mathbb{N}$, is a Gaussian random variable.

Gaussian processes are used in regression and classification tasks in a wide variety of applications including, for instance, finance, computer vision, rational learning, and control theory.

The distribution of a Gaussian process has a convenient parametrization. Namely, the mean vector $\mu = (\mu_\lambda : \lambda \in \Lambda)$ and the covariance matrix $\Sigma = (\rho_{\lambda,\kappa} : \lambda, \kappa \in \Lambda)$, where $\rho_{\lambda,\kappa} = \mathrm{COV}(X_\lambda, X_\kappa)$, completely determine the distribution of a Gaussian process. In particular, two Gaussian random variables are independent if and only if they are uncorrelated, that is their covariance is zero.

A real-valued Gaussian process $X = \{X_\lambda : \lambda \in \Lambda\}$ is said to be

(i) a discrete parameter Gaussian process (or a *Gaussian sequence*) if Λ is a subset of $\mathbb{Z}$,

(ii) a continuous parameter Gaussian processess if Λ is an interval in $\mathbb{R}$.

The basic example of a continuous-time Gaussian process is Brownian motion.

Definition 2.18. A one-dimensional *Brownian motion* is a real-valued stochastic process B_t, $t \geq 0$, with the following properties:

(i) With probability one, $t \to B_t$ is a continuous function.

(ii) For any $0 \leq s < t$, the increment $B_{t+s} - B_s$ is a Gaussian variable with mean zero and variance t.

(iii) For any $0 < t_0 < t_1 < \cdots < t_n$, B_{t_0}, $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent random variables.

The special role of Brownian motion can be explained in many (sometimes related) ways. For instance,

(i) Brownian motion is a *scaling limit* of partial sums of independent random variables, and hence it nicely describes an accumulative effect of a big number of roughly independent small contributions;

(ii) Brownian motion is the only process with *continuous paths* and stationary *independent increments*;

(iii) some explicit computations are possible with Brownian motion;

(iv) Brownian motion is an example, each time typical in a sense, of a few important classes of stochastic processes. Namely it is a *Markov process* with continuous paths, it is a *square-integrable martingale*, it is a *Gaussian process*, and it is a *Lévy process* with finite variance.

The *central limit theorem* and its *functional form* (*invariance principle*) suggest that Brownian motion inherits many properties of the *simple symmetric random walk*. To produce stochastic processes with different properties one might consider solutions of the following equation

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \tag{2.2}$$

where μ and σ are either random or deterministic "nice" functions. While the first integral in the above equation is the usual Riemann integral, the second one is a *stochastic integral* that in general cannot be defined path-wise because, with probability one, a path (i.e., a random realization) of Brownian motion is a very irregular, non-smooth function of time. In particular, with probability one, the derivative $\frac{dB_t}{dt}$ does not exists for any $t \geq 0$. If functions μ and σ are smooth enough, there is a unique solution (*diffusion* or *Itô's process*) to (2.2) (see, for instance, [108]). The Ornstein-Uhlenbeck process, which solves (1.7), is a classical example of an Itô's process. In contrast to Brownian motion, Ornstein-Uhlenbeck process admits a stationary distribution given by $X_t = \sigma e^{-\kappa t} \int_{-\infty}^t e^{\kappa s} dB_s$ (cf. Equation (1.8)). Up to linear transformations of the time and space variables, this is the only nontrivial stationary, Gaussian, and Markov process [34], [19, Section 16.1]. Similarly to the Brownian motion, the Ornstein-Uhlenbeck process is often considered as a scaling limit of discrete models. For a classical application to the queueing theory see, for instance, [83].

We conclude this section with an *extreme value theorem* for Gaussian processes. The notion of extreme value theorem usually refers to a weak limit theorem for the running maxima $M_n = \max_{1 \leq k \leq n} X_k$ of a sequence of random variables $(X_n)_{n \in \mathbb{N}}$. For random process with continuous paths (in a suitable for the underlying state space topology) the extreme value theory is closely related to the study of *first passage times* $T_a = \inf\{t > 0 : X_t > a\}$ (through the identity of the events $\{T_a > s\}$ and $\{\max_{t\le s} X_t < a\}$). For the extreme value theory for i.i.d. and weakly dependent (mixing) processes see, for instance, [123, 99, 100]. For extensions to linear time series models in econometrics, which are driven by i.i.d. coefficients, see [30] and the monograph [42]. There is an extensive literature discussing asymptotic behavior of maxima of Gaussian processes (see, for instance, [23, 27, 76, 77, 107]. The following result suffices for our purposes (see the original article [142] or Theorem A in [23]). For $n \in \mathbb{N}$, let

$$a_n = \sqrt{2\log n} \qquad \text{and} \qquad b_n = a_n - \frac{\log a_n + \log\sqrt{2\pi}}{a_n}. \tag{2.3}$$

Theorem 2.19. *[142] Let $(X_n)_{n\in\mathbb{Z}}$ be a Gaussian sequence with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$. Let $\rho_{ij} = \mathbb{E}[X_iX_j]$ and $M_n = \max_{1\le k\le n} X_k$. If*

(i) $\delta := \sup_{i<j} |\rho_{ij}| < 1$.

(ii) For some $\lambda > \frac{2(1+\delta)}{1-\delta}$,

$$\frac{1}{n^2}\sum_{1\le i<j\le n} |\rho_{ij}|\cdot\log(j-i)\cdot\exp\big\{\lambda|\rho_{ij}|\cdot\log(j-i)\big\} \to 0, \qquad \text{as } n\to\infty, \tag{2.4}$$

then, for any $y \in \mathbb{R}$, $\mathbb{P}\big(M_n \le b_n + a_n^{-1}y\big) \to \exp\big\{-e^{-y}\big\}$ *as* $n \to \infty$.

The conditions of the theorem hold for instance for a standard i.i.d. Gaussian sequence, but in general do not assume even stationarity of the underlying process. The result states that the random sequence $a_n(M_n - b_n)$ converges weakly to a non-degenerate limit. In particular, M_n converges to a_n in probability. The latter result implies a sharp concentration of the running maximum around its long-term asymptotic average. The limiting distribution in Theorem 2.19 is called the *standard Gumbel distribution.* Gumbel distribution functions $\mathbb{P}(Y \le y) = \exp\big\{-e^{-(y-\mu)/\beta}\big\}$ constitute one of three parametric classes, which describe all possible limits in the extreme value theory of i.i.d. sequences (see, for instance, [99, 100, 123]).

2.2 The model

In Chapter 3 we study the random sequence of $(X_n)_{n\in\mathbb{Z}}$ defined by Equation (1.1) with $\gamma \in (0, 1)$ and a particular choice for the underlying class of the coefficients $(\xi_n)_{n\in\mathbb{Z}}$. The goal of this section is to introduce our assumptions about the sequence ξ_n.

Let $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ be a *stationary* and *ergodic* sequence of random pairs with $\mu_n \in \mathbb{R}$ and $\sigma_n > 0$. Denote

$$\omega_n = (\mu_n, \sigma_n) \in \mathbb{R}^2, \qquad n \in \mathbb{Z}, \tag{2.5}$$

and $\omega = (\omega_n)_{n\in\mathbb{Z}}$. We refer to the sequence ω as a *random dynamic environment* or simply *dynamic environment*.

The stationarity means that [39, p. 335] the law of ω is invariant under time shifts, that is the random sequence $(\omega_{k+n})_{n\in\mathbb{Z}}$ has the same distribution as ω for any $k \in \mathbb{Z}$. For a rigorous definition and treatment of ergodic processes we refer the reader to [39, Chapter 6]. However, the following remark is in order:

Remark 2.20.

(a) Intuitively, ergodicity means irreducibility of a certain type for general stochastic processes.

(b) In particular, i.i.d. sequences, irreducible countable Markov chains, m-dependent, and strongly mixing sequences are ergodic (cf. [39, Chapter 6]).

(c) Since, generally speaking, time shifts of a functional of a stationary ergodic sequence produce a stationary and ergodic sequence, usual linear models of econometrics (such as AR, MA, ARMA, ARCH, GARCH, etc.) yield stationary and ergodic sequences (see, for instance, [113]).

(d) A number of equivalent definitions of ergodicity can be given. In particular, Birkhoff's ergodic theorem [39, p. 341] states that a stationary random sequence X_n is ergodic if and only if the strong law of large numbers holds for partial sums of $Y_n = f(X_n)$ for any bounded and measurable function $f : \mathbb{R} \to \mathbb{R}$.

Let $\mathcal{B}$ and $\mathcal{B}_+$ be the σ-algebras of the Borel sets of $\mathbb{R}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, respectively. We denote the probability law of the random environment on the measurable space $\big((\mathbb{R}\times\mathbb{R}_+)^{\mathbb{Z}}, (\mathcal{B}\otimes\mathcal{B}_+)^{\otimes\mathbb{Z}}\big)$ by P and denote the corresponding expectation operator by E_P.

We assume that the underlying probability space is enlarged to include a sequence $(\xi_n)_{n\in\mathbb{Z}}$, defined as follows. Let $(\xi_n)_{n\in\mathbb{Z}}$ be a sequence of random variables, independent in a fixed environment ω, and such that (in the sense of Definition 2.16)

$$\xi_n \sim \mathcal{N}(\mu_n, \sigma_n^2), \qquad n \in \mathbb{Z}.$$

There is a useful alternative way to write Equation (1.1). Let

$$\epsilon_n := \frac{\xi_n - \mu_n}{\sigma_n}, \qquad n \in \mathbb{Z}.$$

Then, according to part (c) of Proposition 2.15, $(\epsilon)_{n\in\mathbb{Z}}$ is an i.i.d. sequence with $\epsilon_n \sim \mathcal{N}(0,1)$. Therefore, one can write Equation (1.1) as

$$X_{n+1} = \gamma X_n + \mu_n + \sigma_n \epsilon_n, \qquad n \in \mathbb{Z}. \tag{2.6}$$

Definition 2.21. Let $\omega = (\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequences of pairs of real-valued random variables. We say that a sequence $(\xi_n)_{n\in\mathbb{Z}}$ is a *Gaussian process modulated by* ω if

$$\xi_n = \mu_n + \sigma_n \epsilon_n, \qquad n \in \mathbb{Z},$$

for a standard (i.e., i.i.d. with mean zero and variance one) Gaussian sequence $(\epsilon_n)_{n\in\mathbb{Z}}$.

We denote the conditional law of $(\xi_n)_{n\in\mathbb{Z}}$, given an environment ω, by P_ω and the corresponding expectation by E_ω. To emphasize the existence of two levels of randomness in the model, the first one due to the random environment and the second due to the randomness of $(\varepsilon_n)_{n\in\mathbb{N}}$, we will use the notations $\mathbb{P}$ and $\mathbb{E}$ for, respectively, the unconditional distribution of $(\xi_n)_{n\in\mathbb{Z}}$ (and $(X_n)_{n\in\mathbb{Z}}$) and the corresponding expectation operator.

We are now in a position to state our assumptions on the multiplicative coefficient γ and the random environment ω.

Assumption 2.22. *Assume that:*

(A1) The sequence of pairs $(\omega_n)_{n\in\mathbb{Z}}$ is stationary and ergodic.

(A2) $E_P\big(\log^+|\mu_0| + \log^+|\sigma_0|\big) < +\infty$, where $x^+ := \max\{x, 0\}$ for $x \in \mathbb{R}$.

(A3) $\gamma \in (0,1)$ is a constant.

The conditions on the coefficients of the model, which are stated in Assumption 2.22, insure in particular the existence of the limiting distribution for X_n (see Theorem 2.24 below). In fact, assuming (A1) and (A2) above, we can distinct between the following qualitatively different regimes in the range of the parameter γ :

1. If $|\gamma| < 1$, Equation (1.1) admits a unique stationary solution given by

$$X_n = \sum_{k=-\infty}^{n} \gamma^{n-k} \xi_k. \tag{2.7}$$

2. If $\gamma = 1$, $X_n = X_0 + \sum_{k=1}^{n} \xi_k$ is a random walk (cf. Section 2.4 below).

3. If $|\gamma| > 1$, dividing both the sides of the identity (1.5) by γ^n, it can be shown that X_n/γ^n converges with probability one to a proper random variable $X_0 + \sum_{k=1}^{\infty} \gamma^{-k} \xi_k$. That is, in this case X_n grows exponentially.

4. If $\gamma = -1$ and $\mu_n \equiv 0$, then $X_n = X_0 + \sum_{k=1}^{n} (-1)^k \xi_k$ is again a random walk since the distribution of ξ_k is symmetric in this case.

With possible applications in mind and taking in account the above characterization of the process for different values of the parameter γ, we focus in this thesis in the case $\gamma \in (0, 1)$. The asymptotic behavior of the limiting distribution when $\gamma \to 1^-$ is considered in Section 3.2. Notice that the limiting distribution, as a function of the parameter γ, is discontinuous at one. In fact, when $\gamma = 1$, the limiting distribution is not well-defined and a scaling is required to obtain a limit of $X_n = X_0 + \sum_{k=1}^{n} \xi_k$.

It was pointed out in [139, Section 3.5] (see also [138, pp. 74-75] and the paragraph following Equation 4 in [69]) that the case when γ is close to one is of a special interest in the context of stochastic volatility models. On the purely mathematical side, our study of the asymptotic behavior of the limiting distribution when $\gamma \to 1^-$ is to a large extent inspired by [11, 132], which investigate a *weak disorder* asymptotic expansion for a Lyapunov exponent associated with a random Fibonacci sequence. We remark that the asymptotic behavior of the first passage time for certain special cases of discrete Ornstein-Uhlenbeck process was investigated in [95, 96, 102].

In the manuscript in preparation [55] we intend to explore a specific variant of the above model, where coefficients ξ_n are modulated by a Markov-dependent process. More precisely, in [55] we study Equation (2.6) under the following assumption on the environment ω.

Assumption 2.23. *Let $(y_n)_{n\in\mathbb{Z}}$ be an irreducible Markov chain defined on a finite state space $\mathcal{D}$, and suppose that the sequence $(\xi_n)_{n\in\mathbb{Z}}$ is induced (modulated) by $(y_n)_{n\in\mathbb{Z}}$ as follows.*

Assume that for each $i \in \mathcal{D}$ there exists an i.i.d. sequence of pairs of reals $\omega_i = (\mu_{i,n}, \sigma_{i,n})_{n\in\mathbb{Z}}$ and these sequence are independent each of other. Further, suppose that (A2) of Assumption 2.22 hold for each $i \in \mathcal{D}$, with (μ_0, σ_0) replaced by $(\mu_{i,0}, \sigma_{i,0})$.

Finally, define

$$\mu_n = \mu_{y_n,n} \qquad \text{and} \qquad \sigma_n = \sigma_{y_n,n}.$$

In what follows, in fact without loss of generality, we assume that $\mathcal{D} = \{1, \ldots, d\}$ for some $d \in \mathbb{N}$. For future reference, we denote

$$\bar{\mu}_i = E_P[\mu_{i,0}], \qquad i \in \mathcal{D}.$$

We denote by H transition matrix of the underlying Markov chain, that is

$$H(i,j) = P(y_1 = j | y_0 = i), \qquad i,j \in \mathcal{D}.$$

Since H is assumed to be irreducible, there exists a unique stationary distribution of $(y_n)_{n\in\mathbb{Z}}$ [39, Chapter 5]. The stationary distribution is a d-vector

$$\pi = \big(\pi_1, \ldots, \pi_d\big),$$

such that [39, p. 300] $\pi = \pi H$ and

$$P(y_n = j) = \sum_{i\in\mathcal{D}} \pi_i H^n(i,j), \qquad n \geq 0,\, i \in \mathcal{D}.$$

The case of Markovian environments is of a special interest both because of its relative tractability (cf. Section 1.4 above) as well as because in this case the model becomes a discrete counterpart of the continuous-time Markov-modulated Ornstein-Uhlenbeck process, which is discussed at the end of Section 1.3.4 above.

2.3 Limiting distribution of X_n

The distribution of the underlying random variables is the key component in modeling time series. Conditions under which X_n defined by (1.2) converges in distribution to the stationary solution have been stated by Vervaat [141] and Brandt [17]. This convergence is related to the existence to the solutions of the distribution equation $X \overset{d}{=} \rho X + \xi$, where the pair (ρ, ξ) is independent of X [56, 103]. For integers $n \in \mathbb{Z}$ and $t \leq n-1$ let $\Pi_{n,t} = \prod_{k=t}^{n-1} \rho_k$. It follows from (1.2) that

$$X_n = \Pi_{n,0} X_0 + \sum_{t=1}^{n-1} \Pi_{n,n-t} \xi_t. \tag{2.8}$$

The following result can be deduced from (2.8) (see for instance [17]):

Theorem 2.24. *[17] Assume that*

(i) The sequence of pairs $(\xi_n, \rho_n)_{n \in \mathbb{Z}}$ is stationary and ergodic.

(ii) $\mathbb{E}\big[\log^+ |\xi_0|\big] < +\infty$, where $x^+ := \max\{x, 0\}$ for $x \in \mathbb{R}$.

(iii) $\mathbb{E}\big[\log^+ |\rho_0|\big] < 0$.

Then, for any initial value X_0, the series X_n defined by (1.2) converges in distribution, as $n \to \infty$, to the random variable

$$X = \sum_{k=0}^{\infty} \Pi_{0,-k} \xi_{-k}, \tag{2.9}$$

which is the unique initial value making $(X_n)_{n \geq 0}$ into a stationary sequence.

Remark 2.25. *Notice that the above theorem implicitly states the almost sure absolute convergence of the series in the right-hand side of (2.9).*

When ρ_n is not a constant random variable (i.e., (1.2) is not reduced to (1.1)), the explicit form of the distribution function of X is not known in general, with practically a single exception [141, 56]. Notice that while (1.5) gives a moving-average representation for X_n in terms of the coefficients appearing in Equation (1.1), it does not yield in general an explicit form of its distribution function.

The *distribution tails* $\mathbb{P}(X > t)$ and $\mathbb{P}(X < -t)$ of X were shown to be *regularly varying* as $t \to \infty$ in [86] and [56, 58, 61], provided that the pairs $(\xi_n, \rho_n)_{n\in\mathbb{Z}}$ form an i.i.d. sequence. These results attracted much attention over the last three decades and have been extended by many authors. Recall that $f : \mathbb{R} \to \mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha \in \mathbb{R}$ where $L(t)$ is a slowly varying function, that is $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. Here and henceforth $f(t) \sim g(t)$ for deterministic real-valued functions f and g means $\lim_{t\to\infty} f(t)/g(t) = 1$.

The questions of the singularity and the existence of a density with respect to the Lebesgue measure were addressed (under different assumptions), for instance, in [3]. In fact, [5] and [22] studied invariant (infinite) measures in the appealing critical case when $\mathbb{E}\big[\log^+ |\rho_0|\big] = 0$.

2.4 Theory of random walks

The asymptotic behavior of partial sums $\sum_{k=1}^n X_k$ defined by Equation (1.2) is the topic of, for instance, [8, 22, 75, 92, 120, 121]. For most of the results obtained in these papers, the strategy is to compare relevant properties of the above partial sums to those of a suitably chosen random walk. In this section we discuss some basic properties of classical random walks, which are extended to our setup in Section 3.4 below.

We start with the following definition:

Definition 2.26. Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables. Then a sequence of random variables $(X_n)_{n\geq 0}$ such that $\mathbb{P}(X_0 = x) = 1$ for some $x \in \mathbb{R}$ and

$$X_n = X_{n-1} + \xi_n = x + \sum_{k=1}^{n} \xi_k,$$

is called a *random walk* on $\mathbb{R}$ starting at point x. The random variables ξ_n are called the *increments* of the random walk.

Notice that when $\gamma = 1$ and $(\xi_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence, Equation (1.1) defines a one-dimensional random walk with increments ξ_n. We remark that often, the term "random walk" refers to the partial sums of random variables ξ_n even if the latter are not independent. While in the remainder of this section the term "random walk" always refers to the partial sum of an i.i.d. sequence, in Chapter 3 we will adopt the wide-sense definition.

The random walks represent arguably the most popular and the most studied type of random processes. For various aspects of the theory of random walks see, for instance, [15, 98, 125, 134]. Many of the properties of random walks can be carried over to a wide range of stochastic processes for which, in contrast to the random walks, explicit computations are not directly available. The following theorem (see, for instance, [39, Chapter 3]) describes the basic asymptotic behavior of random walks under mild general conditions on their increments.

Theorem 2.27. *Let* $X = (X_n)_{n\in\mathbb{N}}$ *be a random walk on* $\mathbb{R}$ *with i.i.d. increments* $(\xi_n)_{n\in\mathbb{N}}$ *and drift* $\mu := \mathbb{E}[\xi_n] \in [-\infty, +\infty]$. *We have:*

(a) Recurrent behavior: if $\mu = 0$ *and (the trivial case is excluded)* $\mathbb{P}(\xi_k = 0) < 1$,

$$\limsup_{n\to\infty} X_n = +\infty \qquad \text{and} \qquad \liminf_{n\to\infty} X_n = -\infty, \qquad a.s.$$

(b) Law of large numbers:

$$\lim_{n\to\infty} X_n/n = \mu, \qquad a.s.$$

(c) Central limit theorem: if $\sigma^2 = \mathbb{E}[\xi_k^2] \in (0,\infty)$, *then*

$$\frac{X_n - n\mu}{\sqrt{n\sigma^2}} \Rightarrow \mathcal{N}(0,1).$$

Note that according to the law of large numbers, if $\mu \neq 0$ then

$$\lim_{n\to\infty} X_n = +\infty \cdot \operatorname{sign}(\mu), \qquad \text{a.s.}$$

that is the random walk is *transient*. We remark that condition $\sigma^2 < \infty$ of the central limit theorem implies (for instance, by Jensen's inequality; see [39, p. 464]) that $|\mu| < \infty$.

Since the increments of Brownian motion are independent Gaussian random variables, the CLT reveals a tight connection between random walks and Brownian motion. In fact, the CLT can be strengthened to a claim (*Donsker's invariance principle*, also known as a *functional form* of the CLT) that the whole path of the random walk, i. e. the sequence $(X_n)_{n\in\mathbb{Z}_+}$, being properly normalized, converges in certain sense to the Brownian motion; see [39, pp. 406, 410]. One important consequence of this result [39, p. 406] is that continuous functions of a properly normalized path of balanced random walk (i.e. with $\mu = 0$) converge weakly, as $n \to \infty$, to the corresponding characteristics of Brownian motion. For instance, $\frac{1}{\sqrt{n\sigma^2}} \max_{1\le k\le n} X_k$, converges in distribution to $\sup_{0\le t\le 1} B_t$.

Theorem 2.27 shows that, under very mild conditions, all random walks exhibit the same pattern of asymptotic behavior. The asymptotic behavior described in Theorem 2.27 is universal for several important classes of stochastic processes (for instance, the law of large numbers is a particular case of Birkhoff's ergodic theorem). One of the main goals in time series analysis is construction of random processes which, while preserve some of the properties possessed by the partial sums of i.i.d. sequences, differ from the latter in significant specific aspects. We next list a few recipes which have been used in the literature (sometimes, in combination) in order to accomplish this goal.

1. Consider models either driven by or having as scaling limits general Lévy (self-similar, with independent increments and scaling factors different from the usual $\sqrt{t}$) rather than Gaussian processes. See, for instance, [4, 7, 15, 105, 117, 121, 124]. To mention one example, remove the second moment assumption in the conditions of Theorem 2.27 to obtain (so called *stable*) limit theorems of a type different from the classical CLT (cf. [39, Section 2.7]).

2. Remove the i.i.d. assumption and introduce instead a long-term dependence between the elements of the sequence $(X_n)_{n\in\mathbb{N}}$. See [6, 14, 35, 94, 110] for examples in applied fields. A popular and practically important implementation of this idea is the replacement of the ordinary Brownian motion by a fractional Brownian motion as the underlying stochastic process in the definition of the model (see, for instance, [115, 116, 130, 133]).

3. Randomize parameters of the model or introduce the dependence structure through additional (non-observable) variables serving as hidden parameters of the model. Define a joint (interactive) dynamics of the underlying variables and the non-observable parameters together. See, for instance, [12, 43, 46, 124]. The classical ARCH and GARCH models can serve as an example.

4. Assume that the distribution of X_n is changing according to the current state of an (observable) random environment, which is exogenous to the model. See, for instance,

[40, 49, 63, 80, 129] for the SV and Markov-modulated models in financial mathematics, and [125, 143] for an implementation in the context of the abstract theory of random walks. References [79] and [126] give two different examples in the context of the stochastic difference equations.

The approach to the AR(1) model taken in this work falls into the forth category above.

CHAPTER 3. MAIN RESULTS

Main results of this thesis, regarding the asymptotic behavior of the sequence $(X_n)_{n\in\mathbb{Z}}$ defined by Equation (1.1), are presented in this chapter. Section 3.1 establishes the distribution of the random limit X introduced in (2.7). In Section 3.2 we study the asymptotic behavior of this distribution when $\gamma \to 1$. In Sections 3.3 and 3.4 we investigate the joint distribution of the first n members of the sequence $(X_k)_{k\in\mathbb{N}}$. The main result of Section 3.3 is a limit theorem for the extreme values of the process $(X_k)_{k\in\mathbb{Z}}$. In Section 3.4 we focus on the asymptotic behavior of the random walk $S_n = \sum_{k=1}^n X_k$.

3.1 Limiting distribution of X_n

In this section we consider random variable X introduced in (2.7). Our first result is a characterization of X as a mixture of Gaussian random variables under general Assumption 2.22. Let

$$\theta = \sum_{k=0}^{\infty} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau = \Big(\sum_{k=0}^{\infty} \gamma^{2k} \sigma_{-k}^2\Big)^{1/2}, \tag{3.1}$$

where μ_n and σ_n are introduced in Section 2.2. Notice that by Theorem 2.24, the random variables θ and τ are well-defined functions of the environment (see Remark 2.25). Recall Definition 2.16. We have

Theorem 3.1. *Let Assumption 2.22 hold. Then $X \sim \mathcal{N}(\theta, \tau^2)$, where θ and τ are random variables defined in (3.1).*

Proof. Recall the form of the characteristic function (Fourier transform) of a normal random variable from part (d) of Proposition 2.15. Thus, by Theorem 2.8, it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\big[e^{itX_n}\big] = E_P\Big[e^{i\theta t - \frac{\tau^2 t^2}{2}}\Big], \qquad t \in \mathbb{R}.$$

Since environment ω is a stationary sequence, it follows from (1.5) that $\mathbb{E}\big[e^{itX_n}\big] = \mathbb{E}\big[e^{itY_n}\big]$ where $Y_n \sim \mathcal{N}(\theta_n, \tau_n^2)$ with τ_n and θ_n given by

$$\theta_n = \sum_{k=0}^{n} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau_n = \Big(\sum_{k=0}^{n} \gamma^{2k} \sigma_{-k}^2\Big)^{1/2}.$$

It follows from (3.1) and Theorem 2.24 (see also Remark 2.25) that

$$\lim_{n\to\infty} \theta_n = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \tau_n = \tau, \quad P-\text{a.s.}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\big[e^{itX_n}\big] &= \lim_{n\to\infty} \mathbb{E}\big[e^{itY_n}\big] = \lim_{n\to\infty} E_P\Big[E_\omega\big[e^{itY_n}\big]\Big] \\
&= \lim_{n\to\infty} E_P\Big[e^{i\theta_n t - \frac{\tau_n^2 t^2}{2}}\Big] = E_P\Big[\lim_{n\to\infty} e^{i\theta_n t - \frac{\tau_n^2 t^2}{2}}\Big] = E_P\Big[e^{i\theta t - \frac{\tau^2 t^2}{2}}\Big].
\end{aligned}$$

To justify interchanging of the limit and expectation operator in the last but one step, observe that $\left|e^{i\theta_n t - \tau_n^2 t^2}\right| \le 1$, and therefore the bounded convergence theorem [39, p. 466] can be applied. □

Corollary 3.2. *Let Assumption 2.22 hold. Then the distribution of X is absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ algebra of the Borel sets of $\mathbb{R}$.*

Proof. By Fubini's theorem [39, p. 470], for any Borel set $A \subset \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}(X \in A) = \mathbb{E}\big[\mathcal{N}(\theta, \tau^2) \in A\big] &= \int \left(\frac{1}{\sqrt{2\pi\tau^2}} \int_A e^{-\frac{(x-\theta)^2}{2\tau^2}} dx\right) dP(\omega) \\
&= \int_A \left(\int \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{(x-\theta)^2}{2\tau^2}} dP(\omega)\right) dx,
\end{aligned}$$

and, furthermore, the integral $\int \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{(x-\theta)^2}{2\tau^2}} P(d\omega)$ exists for m-a.e. x, where m denotes the Lebesgue measure of the Borel subsets of $\mathbb{R}$. □

Corollary 3.3. *Let Assumption 2.22 hold. Then, provided that the moments in the right-hand side exist, we have the following identities:*

(i) $\mathbb{E}[X] = \dfrac{E_P[\mu_0]}{1-\gamma}$.

(ii) $\mathrm{VAR}_{\mathbb{P}}(X) = \dfrac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta)$.

Proof. It follows from Theorem 3.1 that

$$m_{_X} := \mathbb{E}[X] = E_P\big[E_\omega\big[\mathcal{N}(\theta,\tau^2)\big]\big] = E_P[\theta] = \frac{E_P[\mu_0]}{1-\gamma}$$

and

$$\begin{aligned}\mathrm{VAR}_{\mathbb{P}}(X) &= E_P\big[E_\omega\big[X^2 - m_{_X}^2\big]\big] = E_P[\tau^2+\theta^2] - m_{_X}^2 \\ &= E_P[\tau^2] + \mathrm{VAR}_P(\theta) = \frac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta),\end{aligned}$$

where we used the fact $m_{_X} = E_P[\theta]$, and therefore $\mathrm{VAR}_P(\theta) = E[\theta^2] - m_{_X}^2$. □

In the case of Markovian environment, $\mathrm{VAR}_P(\theta)$ can be expressed in terms of certain explicit transformations of the transition kernel of the underlying Markov chain. In the following lemma we compute $\mathrm{VAR}_P(\theta)$ under Assumption 2.23. To state the result we first need to introduce some notation. Let $a = E_P[\mu_0]$. Further, recall π_i and $\bar{\mu}_i$ from the last paragraph of Section model. Let $\mathfrak{m}_2$ denote the d-dimensional vector whose i-th component is $\pi_i\bar{\mu}_i^2$ and introduce a $d\times d$ matrix K_γ by setting

$$K_\gamma(i,j) = \frac{\gamma}{\bar{\mu}_i}\cdot H(i,j)\cdot\bar{\mu}_j, \qquad i,j=1,\ldots,d,$$

where H is transition kernel of the underlying Markov chain $(y_n)_{n\in\mathbb{Z}}$.

We have:

Lemma 3.4. *Let Assumption 2.23 hold. Then*

$$\mathrm{VAR}_P(\theta) = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2}\cdot\langle\mathfrak{m}_2,(I-K_\gamma)^{-1}\mathbf{1}\rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)},$$

where $\langle\mathbf{x},\mathbf{y}\rangle$ stands for the usual scalar product of two d-vectors $\mathbf{x}$ and $\mathbf{y}$.

Proof. For $n\in\mathbb{Z}$, let $\nu_n = \mu_{-n} - a$ and

$$\rho_n := E_P[\nu_i\nu_{n+i}] = \mathrm{COV}_P(\mu_{-i},\mu_{-i-n}).$$

Then, according to (3.1),

$$\begin{aligned}\mathrm{VAR}_P(\theta) &= E_P\Big[\Big(\sum_{n=0}^\infty\gamma^n\nu_n\Big)^2\Big] = E_P\Big[\sum_{n=0}^\infty\gamma^{2n}\nu_n^2\Big] + 2\sum_{n=0}^\infty\gamma^n\sum_{k=n+1}^\infty\gamma^k\rho_{k-n} \\ &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + 2\sum_{n=0}^\infty\gamma^n\sum_{m=1}^\infty\gamma^{n+m}\rho_m = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2}\cdot\sum_{m=1}^\infty\gamma^m\rho_m.\end{aligned}$$

It remains to compute ρ_n for $n \geq 1$. We have

$$\begin{aligned}\rho_n &= \sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i H^{n-1}(i,j) E\Big[(\bar{\mu}_i - a)(\bar{\mu}_j - a)\Big] \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i H^{n-1}(i,j) E_P\Big[(\bar{\mu}_i\bar{\mu}_j - a\bar{\mu}_i - a\bar{\mu}_j + a^2)\Big] \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i H^{n-1}(i,j) E_P\big[\bar{\mu}_i\bar{\mu}_j - a^2\big] = E\Big[\sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i\bar{\mu}_i H^{n-1}(i,j)\bar{\mu}_j\Big] - a^2.\end{aligned}$$

Define the following *Doob transform* of matrix H :

$$K(i,j) = \frac{1}{\bar{\mu}_i} H(i,j)\bar{\mu}_j, \qquad i,j = 1,\ldots,d.$$

Then, a routine induction argument shows that for any $n \in \mathbb{N}$,

$$K^n(i,j) = \frac{1}{\bar{\mu}_i} H^n(i,j)\bar{\mu}_j.$$

Using this formula, we obtain

$$\rho_n = E_P\Big[\sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i\bar{\mu}_i^2 K^{n-1}(i,j)\Big] - a^2 = \langle \mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2,$$

and hence

$$\begin{aligned}\mathrm{VAR}_P(\theta) &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2}\cdot\sum_{n=1}^{\infty}\gamma^n\big(\langle \mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2\big) \\ &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2}\cdot\langle \mathfrak{m}_2, (I-K_\gamma)^{-1}\mathbf{1}\rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)},\end{aligned}$$

The proof of the lemma is completed. □

Remark 3.5. *It is not hard to verify that with an appropriate modification of the definition of the Doob transform K_γ (as a positive integral kernel rather than a d-matrix), the statement of Lemma 3.4 remains true for a general, non-necessarily restricted to a finite-state, Markovian setup (for instance, it works for the setting described in Definition 1.1 of [126]).*

For an arbitrary moment of order $p > -1$ (including non-integer values of p) we have

Corollary 3.6. *Let Assumption 2.22 hold. Then* $\mathbb{E}\big[|X-\theta|^p\big] = \frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\cdot E_P[\tau^p]$ *for any* $p > -1$.

Formulas for $\mathbb{E}\big[|X|^p\big]$, explicit in terms of expectations of *confluent hypergeometric functions* of the quotient θ/τ, can also be given (cf. [1, 101]).

The next theorem shows that under a mild extra assumption, the tails of the distribution of X have asymptotically Gaussian structure. Recall the L_p norms $\|Y\|_p$ defined as follows for a constant $p \in [1,\infty]$ and a random variable Y :

$$\|Y\|_p = \Big(\mathbb{E}\big[|Y|^p\big]\Big)^{1/p}, \qquad p \in [1,\infty),$$

and

$$\|Y\|_\infty = \text{ess sup}\,|Y| := \sup\big\{y \in \mathbb{R} : \mathbb{P}\big(|Y| < y\big) < 1\big\}.$$

In the proof of the following theorem we will use the fact that for any random variable Y (see, for instance, [39, p. 466]),

$$\|Y\|_\infty = \lim_{p\to\infty} \|Y\|_p$$

Recall τ from (3.1) and notice that (3.1) implies:

$$\|\tau\|_\infty^2 \le \frac{\|\sigma_0\|_\infty^2}{1-\gamma^2}.$$

We have:

Theorem 3.7. *Let Assumption 2.22 and assume in addition that* $P\big(|\mu_0| + \sigma_0 < \lambda\big) = 1$ *for some constant* $\lambda > 0$. *Then*

$$\lim_{t\to\infty} \frac{1}{t^2}\log \mathbb{P}(X > t) = \lim_{t\to\infty} \frac{1}{t^2}\log \mathbb{P}(X < -t) = -\frac{1-\gamma^2}{2\Lambda^2},$$

where $\Lambda \in (0,\infty)$ *is the* L_∞*-norm of* τ.

Proof. We will only consider the upper tails $\mathbb{P}(X > t)$. The lower tails $\mathbb{P}(X < -t)$ can be treated in the same manner exactly, and therefore the proof for lower tails is omitted.

First, we will recall some well-known bounds for the tails of normal distributions. For the reader's convenience we will give a short derivation of these bounds here. Recall $G_{\mu,\sigma^2}(t)$ from Section 2.1.4. We have:

$$\begin{aligned} G_{0,\sigma^2}(t) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_t^\infty e^{-\frac{x^2}{2\sigma^2}}dx \le \frac{1}{\sqrt{2\pi\sigma^2}}\int_t^\infty \frac{x}{t}\, e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{2\sqrt{2\pi\sigma^2 t^2}}\int_{t^2}^\infty e^{-\frac{y}{2\sigma^2}}dy \\ &= \frac{\sigma^2}{\sqrt{2\pi\sigma^2 t^2}}\int_{t^2}^\infty \frac{1}{2\sigma^2}\, e^{-\frac{y}{2\sigma^2}}dy = \sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}. \end{aligned}$$

On the other hand, denoting $t_\sigma = t/\sigma$ and using l'Hôpital's rule,

$$\lim_{t\to\infty} \frac{G_{0,\sigma^2}(t)}{\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}} \quad = \quad \lim_{t_\sigma\to\infty} \frac{\int_{t_\sigma}^{\infty} e^{-\frac{x^2}{2}}\,dx}{t_\sigma^{-1}e^{-\frac{t_\sigma^2}{2}}} = 1.$$

Therefore, there exists $t_0 > 0$ such that if $t > \lambda t_0$, we have

$$G_{\theta,\tau^2}(t) \geq \frac{1}{2}\sqrt{\frac{\tau^2}{2\pi t^2}}e^{-\frac{t^2}{2\tau^2}} = \sqrt{\frac{\tau^2}{8\pi t^2}}e^{-\frac{t^2}{2\tau^2}}.$$

By Theorem 3.1,

$$\mathbb{P}(X > t) = E_P\big[P_\omega(X > t)\big] = E_P\big[G_{\theta,\tau^2}(t)\big].$$

To get the upper bond observe that

$$E_P\big[G_{\theta,\tau^2}(t)\big] \quad\leq\quad E_P\Big[\sqrt{\frac{\lambda^2}{2\pi t^2}}e^{-\frac{t^2}{2\tau^2}}\Big] \leq \sqrt{\frac{\lambda^2}{2\pi t^2}}E_P\big[e^{-\frac{t^2}{2\tau^2}}\big].$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X>t) \quad &\leq \quad \lim_{t\to\infty}\frac{1}{t^2}\log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_{\infty}\Big)\\ &= \quad \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}.\end{aligned}$$

For the lower bound, we first observe that for $t > \lambda t_0$,

$$E_P\big[G_{\theta,\tau^2}(t)\big] \quad\geq\quad E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}}e^{-\frac{t^2}{2\tau^2}}\Big].$$

Now, let $\varepsilon > 0$ any positive real number such $P(\tau > \varepsilon) > 0$. Then

$$E_P\big[G_{\theta,\tau^2}(t)\big] \quad\geq\quad E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}}e^{-\frac{t^2}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\Big] \geq \sqrt{\frac{\varepsilon^2}{8\pi t^2}}E_P\Big[e^{-\frac{t^2}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\Big],$$

which implies

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X>t) \quad &\geq \quad \lim_{t\to\infty}\frac{1}{t^2}\log\Big(\big\|e^{-\frac{1}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\big\|_{\infty}\Big)\\ &= \quad \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}.\end{aligned}$$

This completes the proof of the theorem. □

Next, we give a (somewhat degenerate) example of the situation when the distribution of τ can be explicitly computed, and the tails of X do not have the Gaussian asymptotic structure.

Example 3.8. *Let Assumption 2.23 hold and suppose that* $P(\mu_0 = 0) = 1$, $|\mathcal{D}| = 2$, *and*

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Further, assume that $\sigma_{1,n}^2$ *and* $\sigma_{2,n}^2$ *have strictly asymmetric* α*-stable distributions with index* $\alpha \in (0,1)$ *and "Laplace transform" given by*

$$E_P\big[e^{-\lambda \sigma_{i,n}^2}\big] = e^{-\theta_i \lambda^\alpha}, \qquad \lambda > 0,\ i = 1,2,$$

for some positive constants θ_i, $\theta_1 \neq \theta_2$. *In notation of [127], these distributions belong to the class* $S_\alpha(\theta,1,0)$ *(see Section 1.1 and also Propositions 1.2.11 and 1.2.12 in [127]). The stationary distribution of the underlying Markov chain is uniform on* $\mathcal{D}$, *and therefore for the Laplace transform of the limiting variance* τ^2 *introduced in* (3.1) *we have for any* $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}\big[e^{-\lambda\tau^2}\big] &= \frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{1,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{2,0}^2}\Big] \\
&\quad+\frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{2,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{1,0}^2}\Big] \\
&= \frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_1\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_2\lambda^\alpha\gamma^{(4k+2)\alpha}} + \frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_2\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_1\lambda^\alpha\gamma^{(4k+2)\alpha}} \\
&= \frac{1}{2}e^{-\frac{\theta_1\lambda^\alpha}{1-\gamma^{4\alpha}}}\cdot e^{-\frac{\theta_2\lambda^\alpha\gamma^{2\alpha}}{1-\gamma^{4\alpha}}} + \frac{1}{2}e^{-\frac{\theta_2\lambda^\alpha}{1-\gamma^{4\alpha}}}\cdot e^{-\frac{\theta_1\lambda^\alpha\gamma^{2\alpha}}{1-\gamma^{4\alpha}}} \\
&= \frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_1+\theta_2\gamma^{2\alpha})}{1-\gamma^{4\alpha}}} + \frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_2+\theta_1\gamma^{2\alpha})}{1-\gamma^{4\alpha}}}.
\end{aligned}$$

Therefore, Theorem 3.1 yields for $t \in \mathbb{R}$,

$$\mathbb{E}\big[e^{itX}\big] = \mathbb{E}\Big[e^{-\frac{\tau^2t^2}{2}}\Big] = \frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_1+\theta_2\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}} + \frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_2+\theta_1\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}}.$$

Thus X *is a mixture of two symmetric* (2α)*-stable distributions (see Definition 1.1.6 on p. 5 and Property 1.2.5 on p. 11 of [127]). In particular,* X *has power tails. Namely (see Property 1.2.15 on p. 16 of [127]) the following limits exist, are equivalent, and are both finite and strictly positive:*

$$\lim_{t\to\infty} t^{2\alpha}\cdot\mathbb{P}(X > t) = \lim_{t\to\infty} t^{2\alpha}\cdot\mathbb{P}(X < -t) \in (0,\infty).$$

Clearly, in this setting

$$\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X > t) = \lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X < -t) = 0,$$

in strike contrast to the result obtained under the conditions of Theorem 3.7.

We note that even though the above example is somewhat artificial and is cooked-up to enable explicit computations, whereas the setup of Theorem 3.7 can be considered as "generic" in applications, the mathematical construction on which the example relies is quite instructive. In fact, this construction illustrates one of the major driving forces behind "Lévy process-driven" models of the type considered, for instance, in [7].

We conclude this section with the following remark.

Remark 3.9. *In Section 2.2 we introduced a model without putting any essential restriction on the environment. One of the main goals of this thesis is to demonstrate that the mathematical analysis of this model is of interest even at this level of generality. However, additional assumptions are required in order to establish refined properties of the model, robust to transformation of the environment within a reasonably large class of environments.*

In particular, (1.5) *and* (2.6) *show that* X_n *can be represented as the sum*

$$X_n = X_{1,n} + X_{2,n},$$

where the summands in the right-hand side satisfy the following recursion equations for any $n \in \mathbb{Z}:$

$$X_{1,n+1} = \gamma X_{1,n} + \mu_n \qquad \text{and} \qquad X_{2,n+1} = \gamma X_{2,n} + \sigma_n \varepsilon_n.$$

Therefore, if we allow $P(\mu_n \neq 0) > 0$ *without any further restrictions, the Gaussian structure of the sequence* $X_{2,n}$ *can be, in principle, completely annihilated by the term* $X_{1,n}$*. This phenomenon is illustrated in Example 3.8 above. This example also suggests that this conceptual problem can be avoided by assuming that the distribution of* μ_n *has, generally speaking, sufficiently light tails (cf. [55]).*

In this work, for simplicity, we often assume in the sequel that $\mu_n = 0$ *with probability one. Such an assumption is standard for most of econometric models, where it is supposed that the structural evolution of the underlying system is captured by the multiplicative factor* γX_n *on the right-hand side of* (1.1)*, while the term* ξ_n *represents an error which fluctuates around zero mean.*

3.2 Asymptotic behavior of X when $\gamma \to 1^-$.

In this section we investigate the asymptotic behavior of the distribution of X when the parameter $\gamma \in (0,1)$ converges to 1. To emphasize the dependence of the stationary solution to Equation (1.1) on γ, we throughout this section use the notation X_γ for X.

To illustrate the main result of this section, consider first the case when the coefficients ξ_n in Equation (1.1) are independent and distributed according to $\mathcal{N}(0,\sigma^2)$ for some constant $\sigma > 0$. Then $X_\gamma \sim \frac{1}{\sqrt{1-\gamma^2}}\mathcal{N}(0,\sigma^2)$, and hence,

$$\sqrt{1-\gamma} \cdot X_\gamma \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{\sqrt{2}}\mathcal{N}(0,\sigma^2), \qquad \text{as } \gamma \to 1^-. \tag{3.2}$$

It is plausible to conjecture that (3.2) remains true under Assumption 2.23 (consider for instance, even though a degenerate in the obvious sense, Example 3.8). We are planning to explore this direction in [55]. In this thesis we prove the following general result, which shows that in a certain sense $(1-\gamma)^{-1/2}$ is always the proper scaling factor for the distribution of X_γ when $\gamma \to 1^-$.

Theorem 3.10. *Let Assumption 2.22 hold. Then* $\frac{\log|X_\gamma|}{\log(1-\gamma)} \overset{\mathbb{P}}{\longrightarrow} -\frac{1}{2}$ *as* $\gamma \to 1^-$, *where* $\overset{\mathbb{P}}{\longrightarrow}$ *means convergence in probability under the law* $\mathbb{P}$.

Proof. It suffices to prove that for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\Big|\frac{\log|X_\gamma|}{\log(1-\gamma)} + \frac{1}{2}\Big| > \varepsilon\Big) \to_{\gamma\to 1^-} 0 \tag{3.3}$$

This is equivalent to the following two claims:

$$\mathbb{P}\Big(\frac{\log|X_\gamma|}{\log(1-\gamma)} > -\frac{1}{2} + \varepsilon\Big) \to_{\gamma\to 1^-} 0$$

and

$$\mathbb{P}\Big(\frac{\log|X_\gamma|}{\log(1-\gamma)} < -\frac{1}{2} - \varepsilon\Big) \to_{\gamma\to 1^-} 0.$$

Since $\log(1-\gamma) < 0$, it is enough to show that, first,

$$\mathbb{P}\Big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\Big) \to_{\gamma\to 1^-} 0,$$

and, secondly,

$$\mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \to_{\gamma\to 1^-} 0.$$

Toward this end, observe that due to Assumption 2.22 the distribution of τ does not have an atom at zero, i.e. $P(\tau > 0) = 1$. Therefore, using the continuity property of probability measures [39, p. 2]

$$\lim_{\delta\to 0} P\big(\tau^2 \notin (\delta, \delta^{-1})\big) = 0, \tag{3.4}$$

Fix now arbitrary $\varepsilon > 0$ and $\delta \in (0,1)$, and let $I_\delta := (\delta, \delta^{-1})$. We have

$$\begin{aligned}\mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\big) &= \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau \in I_\delta\big) + \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau \notin I_\delta\big)\\ &\le \mathbb{P}\big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}; \tau^2 \in I_\delta\big) + P\big(\tau^2 \notin I_\delta\big).\end{aligned}$$

Similarly,

$$\mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \to_{\gamma\to 1^-} \le \mathbb{P}\big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}; \tau^2 \in I_\delta\big) + P\big(\tau^2 \notin I_\delta\big)$$

Using the "δ-truncated version" of τ^2, we obtain

$$\begin{aligned}\mathbb{P}\big(|X_\gamma|\sqrt{1-\gamma} < (1-\gamma)^{\varepsilon}; \tau^2 \in I_\delta\big) &= E_P\big[P_\omega\big(|X_\gamma|\sqrt{1-\gamma} < (1-\gamma)^{\varepsilon}\big); \tau^2 \in I_\delta\big]\\ &\ge P\big(|\mathcal{N}(0,\delta)| < (1-\gamma)^{\varepsilon}\big) \to_{\gamma\to 1^-} 0,\end{aligned} \tag{3.5}$$

and, similarly,

$$\mathbb{P}\big(|X_\gamma|\sqrt{1-\gamma} > (1-\gamma)^{-\varepsilon}; \tau^2 \in I_\delta\big) \le P\big(|\mathcal{N}(0,\delta^{-1})| < (1-\gamma)^{\varepsilon}\big) \to_{\gamma\to 1^-} 0. \tag{3.6}$$

Taking in account that $\delta \in (0,1)$ is arbitrary, and combining (3.5) and (3.6) together with (3.4), we obtain (3.3). The proof of the theorem is completed. □

3.3 Joint distribution of $(X_n)_{n\ge 0}$

In this section we consider a stationary sequence X_n that solves the linear recursion (1.1) in the case $E_P[\xi_n] = 0$. It follows from (2.9) that

$$X_n = \sum_{k=0}^{\infty} \gamma^k \xi_{n-k} = \gamma^n \sum_{j=-\infty}^{n} \gamma^{-j}\xi_j. \tag{3.7}$$

For any sequence of real constants $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$ we have

$$\sum_{k=0}^{n} c_k X_k \;=\; \sum_{k=0}^{n} c_k \sum_{j=-\infty}^{k} \gamma^{k-j}\xi_j = \sum_{j=-\infty}^{0} \xi_j \sum_{k=0}^{n} c_k\gamma^{k-j} + \sum_{j=1}^{n} \xi_j \sum_{k=j}^{n} c_k\gamma^{k-j},$$

where we used the absolute convergence of the series to interchange the summation signs. Therefore, under the measure P_ω, that is in a given environment ω,

$$\sum_{k=0}^{n} c_k X_k \sim \mathcal{N}(0, \eta^2_{\mathbf{c},n}),$$

where

$$\eta^2_{\mathbf{c},n} = \sum_{j=-\infty}^{0} \sigma_j^2\Big(\sum_{k=0}^{n} c_k\gamma^{k-j}\Big)^2 + \sum_{j=1}^{n} \sigma_j^2\Big(\sum_{k=j}^{n} c_k\gamma^{k-j}\Big)^2.$$

This shows that under P_ω, the process $(X_n)_{n\geq 0}$ is Gaussian (recall Definition 2.17).

Recall from Section 2.1.4 that the distribution of a mean-zero Gaussian sequence is entirely determined by its covariance structure. It follows from (2.8) that

$$X_{k+n} = \gamma^n X_k + \sum_{t=0}^{n-1} \gamma^t \xi_{n+k-t-1}, \qquad k\in\mathbb{Z},\ n\in\mathbb{N}.$$

Therefore, for any $k\in\mathbb{Z}$ and $n\in\mathbb{N}$, we have

$$\mathrm{COV}_\omega(X_k X_{k+n}) = E_\omega[X_k X_{k+n}] = \gamma^n E_\omega[X_k^2]. \tag{3.8}$$

In particular, random variables X_n and X_m are positively correlated for any $n, m\in\mathbb{Z}$. Let

$$\lambda_k^2 := E_\omega[X_k^2] = \sum_{j=0}^{\infty} \gamma^{2j}\sigma^2_{k-j}, \qquad k\in\mathbb{Z}. \tag{3.9}$$

We next study the asymptotic distribution of the random variables

$$L_n = \max_{0\leq k\leq n} \frac{X_k}{\lambda_k} \qquad \text{and} \qquad M_n = \max_{0\leq k\leq n} X_k, n\in\mathbb{N}.$$

We have:

Theorem 3.11. *Let Assumption 2.22 hold. Suppose in addition that $E_P[\mu_0] = 0$ and*

$$P\big(\sigma_0 \in (\delta, \delta^{-1})\big) = 1 \tag{3.10}$$

for some constant $\delta\in(0,1)$. Then *(a) For any constant $y \in \mathbb{R}$,*

$$\lim_{n\to\infty} P_\omega\big(a_n(L_n - b_n) \le y\big) = \exp\{-e^{-y}\}, \qquad P - a.s., \tag{3.11}$$

where a_n and b_n are defined in (2.3).

(b) Further,

$$\frac{\log M_n}{\log\log n} \xrightarrow{P_\omega} \frac{1}{2}, \qquad P - a.s.$$

Proof.

(a) Let

$$U_k = \frac{X_k}{\lambda_k}, \qquad k \in \mathbb{Z}.$$

Then $E_\omega[U_k] = 0$ and $E_\omega[U_k^2] = 1$. Furthermore, (3.8) implies for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\rho_{n,k+n} := \mathrm{COV}_\omega(U_k U_{k+n}) = E_\omega[U_k U_{k+n}] = \gamma^n \frac{\lambda_k}{\lambda_{k+n}}. \tag{3.12}$$

It suffices to verify that the conditions of Theorem 2.19 are satisfied for random variables U_n. Toward this end, observe that (3.9) implies

$$\lambda_{k+n}^2 = \gamma^{2n}\lambda_k^2 + \sum_{t=0}^{n-1} \gamma^{2t}\sigma_{k+n-t-1}^2,$$

and hence, in virtue of (3.9) and (3.10),

$$\frac{\lambda_{k+n}}{\gamma^n \lambda_k} = \sqrt{1 + \gamma^{-2n}\lambda_k^{-2}\sum_{t=0}^{n-1}\gamma^{2t}\sigma_{k+n-t-1}^2} > \sqrt{1+\gamma^{-2n}\lambda_k^{-2}\gamma^{2n-2}\sigma_k^2} > \sqrt{1+\gamma^{-2}\delta^4}.$$

Thus

$$\mathfrak{r} := \sup_{k\in\mathbb{Z}, n\in\mathbb{N}} \rho_{k,k+n} = \sup_{k\in\mathbb{Z}, n\in\mathbb{N}} \left\{\gamma^n \frac{\lambda_k}{\lambda_{k+n}}\right\} < 1.$$

Furthermore, it follows from (3.12) and (3.9) that, under condition (3.10), we have for any constant $\mathfrak{s} > 0$:

$$\begin{aligned}
&\frac{1}{n^2}\sum_{1\le i<j\le n}|\rho_{ij}|\cdot\log(j-i)\cdot\exp\bigl\{\mathfrak{s}|\rho_{ij}|\cdot\log(j-i)\bigr\}\\
&\quad\le\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{1}{\delta^4}\gamma^{(j-i)}\cdot\log(j-i)\cdot\exp\bigl\{\mathfrak{s}\delta^{-4}\cdot\log(j-i)\bigr\}\\
&\quad=\frac{1}{n^2\delta^4}\sum_{1\le i<j\le n}\gamma^{(j-i)}\log(j-i)\cdot(j-i)^{\mathfrak{s}\delta^{-4}}=\frac{1}{n^2\delta^4}\sum_{k=1}^{n-1}(n-k)\cdot\gamma^k\log k\cdot k^{\mathfrak{s}\delta^{-4}}\\
&\quad\le\frac{1}{n\delta^4}\sum_{k=1}^{\infty}\gamma^k\log k\cdot k^{\mathfrak{s}\delta^{-4}}\to 0,\qquad\text{as } n\to\infty.
\end{aligned}$$

Therefore, (3.11) holds for any $y\in\mathbb{R}$ by Theorem 2.19. The proof of part (a) of the theorem is completed.

(b) It follows from the conditions of the theorem that there exists $c_0 > 0$ such that for all $n\in\mathbb{Z}$,

$$c_0^{-1} < \frac{M_n}{L_n} < c_0,\qquad P-\text{a.s.}$$

Therefore, $P-$a.s., for any $\varepsilon > 0$, we have

$$P_\omega\Bigl(\frac{\log M_n}{\log\log n} > \frac{1}{2}+\varepsilon\Bigr)\quad=\quad P_\omega\bigl(M_n > (\log n)^{\frac{1}{2}+\varepsilon}\bigr)\le P_\omega\bigl(L_n > c_0(\log n)^{\frac{1}{2}+\varepsilon}\bigr).$$

Part (a) of the theorem implies that, for any $y\in\mathbb{R}$,

$$\lim_{n\to\infty}P_\omega\bigl(L_n\le ya_n^{-1}+b_n\bigr)=\exp\{-e^{-y}\}\qquad P-\text{a.s.}\tag{3.13}$$

Since for any fixed $y > 0$ and $\varepsilon > 0$, eventually (for all n, large enough) we have

$$ya_n^{-1}+b_n < c_0(\log n)^{\frac{1}{2}+\varepsilon}.$$

It follows from (3.13) (because we can use arbitrarily small y while $\lim_{y\to-\infty}\exp\{-e^{-y}\}=0$) that

$$\lim_{n\to\infty}P_\omega\Bigl(\frac{\log M_n}{\log\log n} > \frac{1}{2}+\varepsilon\Bigr)=0\qquad P-\text{a.s.}$$

Similarly, since $P-$a.s., for any $\varepsilon > 0$,

$$P_\omega\Bigl(\frac{\log M_n}{\log\log n} < 1-\varepsilon\Bigr)\quad=\quad P_\omega\bigl(M_n < (\log n)^{\frac{1}{2}-\varepsilon}\bigr)\le P_\omega\bigl(L_n < c_0^{-1}\cdot(\log n)^{\frac{1}{2}-\varepsilon}\bigr),$$

while for any $y \in \mathbb{R}$, eventually,

$$c_0^{-1} \cdot (\log n)^{1/2-\varepsilon} < y a_n^{-1} + b_n,$$

It follows from (3.13), using this time arbitrarily large values of y, that

$$\lim_{n\to\infty} P_\omega\Big(\frac{\log M_n}{\log\log n} < \frac{1}{2} - \varepsilon\Big) = 0 \qquad P - \text{a.s.}$$

The proof of the theorem is completed. □

3.4 Random walk $S_n = \sum_{k=1}^n X_k$

Random walk $S_n = \sum_{k=1}^n X_k$ associated with Equation (1.1) has been studied in [121] and [122]. The following decomposition of S_n, which is implied by (2.8), is useful:

$$\begin{aligned} S_n &= \sum_{k=1}^n \gamma^k X_0 + \sum_{k=1}^n \sum_{t=1}^k \gamma^{k-t}\xi_t = \sum_{k=1}^n \gamma^k X_0 + \sum_{t=1}^n \sum_{k=t}^n \gamma^{k-t}\xi_t \\ &= \sum_{k=1}^n \gamma^k X_0 + \sum_{t=1}^n \Big(\sum_{k=t}^\infty \gamma^{k-t} - \sum_{k=n+1}^\infty \gamma^{k-t}\Big)\xi_t \\ &= \sum_{k=1}^n \gamma^k X_0 + (1-\gamma)^{-1} \sum_{t=1}^n \xi_t - (1-\gamma)^{-1} \sum_{t=1}^n \gamma^{n+1-t}\xi_t. \end{aligned} \tag{3.14}$$

Similar decomposition was used for instance in [92] in the context of Equation (1.2) (with i.i.d. coefficients). Notice that, due to Assumption 2.22, the following inequalities hold with probability one (the right-most inequality in (3.16) is implied by Theorem 2.24):

$$\Big|\sum_{k=1}^n \gamma^k X_0\Big| \le |X_0| \cdot \sum_{k=0}^\infty \gamma^k < \infty, \tag{3.15}$$

and

$$\Big|\sum_{t=1}^n \gamma^{n+1-t}\xi_t\Big| \overset{D}{=} \Big|\sum_{t=-n+1}^0 \gamma^{1-t}\xi_t\Big| \le \sum_{k=0}^\infty \gamma^{k+1} \cdot |\xi_{-k}| < \infty, \tag{3.16}$$

where $\overset{D}{=}$ means equivalence of distributions. This shows that only the second term in the right-most expression of (3.14) contributes to the asymptotic behavior of S_n. More precisely, we have:

Lemma 3.12. *Let Assumption 2.22 hold. Then* *(a) For any sequence of reals $(a_n)_{n\in\mathbb{N}}$ increasing to infinity, we have*

$$\frac{1}{a_n}\sum_{k=1}^{n}\gamma^k X_0 \to_{n\to\infty} 0, \qquad \mathbb{P}-a.s.$$

and

$$\frac{1}{a_n}\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t \to_{n\to\infty} 0, \qquad \text{in probability.}$$

(b) If in addition $E_P\big[|\mu_0|\big] < \infty$ and $E_P\big[\sigma_0\big] < \infty$, then

$$\frac{1}{n}\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t \to_{n\to\infty} 0, \qquad \mathbb{P}-a.s.$$

Proof.

(a) The first claim of part (a) is a direct consequence of (3.15). The second claim follows from (3.16) as follows. For any $\varepsilon > 0$, we have in virtue of (3.16),

$$\mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big| > \varepsilon\Big) =$$
$$= \mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=-n+1}^{0}\gamma^{1-t}\xi_t\Big| > \varepsilon\Big) \le \mathbb{P}\Big(\frac{1}{a_n}\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}| > \varepsilon\Big) \to_{n\to\infty} 0,$$

which implies the result.

(b) We must show that for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big| > \varepsilon \text{ i.o.}\Big) = 0,$$

where the abbreviation "i.o." stands for *infinitely often*. By the Borel-Cantelli lemma [39, p. 47], it suffices to show that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big| > \varepsilon\Big) < \infty. \tag{3.17}$$

Using (3.16) and inequality (2.1), we obtain

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=1}^{n}\gamma^{n+1-t}\xi_t\Big| > \varepsilon\Big) \le \mathbb{P}\Big(\frac{1}{n}\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}| > \varepsilon\Big) \le \frac{1}{\varepsilon}\mathbb{E}\Big[\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|\Big]$$
$$= \frac{\gamma}{\varepsilon(1-\gamma)}\mathbb{E}\big[|\xi_0|\big].$$

Since ξ_k are Gaussian random variables under P_ω, part (a) of Proposition 2.15 implies

$$\mathbb{E}\big[|\xi_0|\big] = E_P\Big[|\mu_0| + \sqrt{\frac{2\sigma_0^2}{\pi}}\Big]. \tag{3.18}$$

It hence follows from the conditions of the lemma that $\mathbb{E}\big[|\xi_k|\big] < \infty$. This establishes (3.17) and therefore completes the proof of part (b) of the lemma. □

In particular, one can obtain the following strong law of large numbers.

Theorem 3.13. *Let Assumption 2.22 hold and in addition* $E_P\big[|\mu_0|\big] < \infty$ *and* $E_P\big[\sigma_0\big] < \infty$. *Then,*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X] = (1-\gamma)^{-1}\, E_P[\mu_0], \qquad \mathbb{P} - a.s. \tag{3.19}$$

Proof. Recall that under Assumption 2.22, $(\xi_n)_{n\in\mathbb{Z}}$ is stationary and ergodic sequence. Furthermore, (3.18) implies that $\mathbb{E}\big[|\xi_0|\big] < \infty$. Therefore, by the Birkhoff ergodic theorem [39, p. 341],

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi_k = \mathbb{E}[\xi_0] = E_P[\mu_0], \qquad \mathbb{P} - \text{a.s.} \tag{3.20}$$

It follows now from (3.14) and Lemma 3.17 that

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{1-\gamma}\,\frac{1}{n}\sum_{k=1}^{n} \xi_k = \frac{1}{1-\gamma}\, E_P[\mu_0], \qquad \mathbb{P} - \text{a.s.}$$

The proof of the theorem is completed. □

It follows from (3.14) that if $E_P[\mu_0] = 0$ and $b_n^{-1}\sum_{k=1}^{n} \sigma_k^2$ converges in distribution to a random variable G for a suitable sequence $b_n \nearrow \infty$, then $S_n/\sqrt{b_n}$ converges in distribution to $\mathcal{N}(0, G)$. In a generic example, σ_n are in the domain of attraction of a symmetric stable law and the sequence $(\sigma_n)_{n\in\mathbb{Z}}$ satisfies certain mixing conditions (see for instance [91]). Limit theorems for S_n of this type can be found in [121]. The special (Gaussian) structure of the sequence ξ_n which is considered in this work, leads to the following result. It is different in essence from the limit theorems obtained in [121].

Theorem 3.14. *Let Assumption 2.22 hold and assume in addition that* $E_P[\mu_0] = 0$ *and* $E_P\big[\sigma_0^2\big] < \infty$. *Then,*

$$\frac{1}{\sqrt{n}} S_n \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{1-\gamma}\,\mathcal{N}\big(0, \Lambda\big)$$

for $\Lambda := E_P[\sigma_0^2]$.

Proof. By the Birkhoff ergodic theorem [39, p. 341],

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_k^2 = E_P[\sigma_0^2], \qquad P-\text{a.s.}$$

Hence, letting $W_n = \sum_{k=1}^{n} \xi_k$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\big[e^{it\frac{W_n}{\sqrt{n}}}\big] &= \lim_{n\to\infty} E_P\Big[E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big] = E_P\Big[\lim_{n\to\infty} E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big] \\ &= E_P\Big[\lim_{n\to\infty} e^{-t\frac{\sum_{k=1}^{n}\sigma_k^2}{2n}}\Big] = e^{-t\frac{\Lambda}{2}}.\end{aligned}$$

Therefore,

$$\frac{W_n}{\sqrt{n}} \stackrel{\mathbb{P}}{\Longrightarrow} \mathcal{N}\big(0, \Lambda\big).$$

It follows now from (3.14) and part (a) of Lemma 3.17 that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\gamma} \frac{W_n}{\sqrt{n}} = \frac{1}{1-\gamma} \mathcal{N}\big(0, \Lambda\big),$$

where the limits in the above identities are understood in terms of the convergence in distribution. The proof of the theorem is completed. □

Remark 3.15. *Under mild additional conditions, the law of large numbers for S_n stated in Theorem 3.13, can be complemented by a large deviation principle and a law of iterated logarithm. See [55] for details. The law of iterated logarithm is discussed also in [121].*

CHAPTER 4. SUMMARY AND DISCUSSION

The thesis is devoted to the study of the linear recursion

$$X_{n+1} = \gamma X_n + \xi_n, \qquad n \in \mathbb{Z},$$

where $\gamma \in (0, 1)$ is a constant and $(\xi_n)_{n\in\mathbb{Z}}$ is a sequence of normal variables modulated by a stationary and ergodic environment. That is

$$\xi_n = \mu_n + \sigma_n \epsilon_n, \qquad n \in \mathbb{Z},$$

for a standard Gaussian sequence $\epsilon = (\epsilon)_{n\in\mathbb{Z}}$ and a stationary and ergodic sequence of random pairs $\omega = (\mu_n, \sigma_n)_{n\in\mathbb{Z}}$, independent of it. We think of ω as an exogenous dynamic environment for the AR(1) process X_n. Correspondingly, we refer to the sequence X_n defined above as a discrete Ornstein-Uhlenbeck process in a stationary dynamic environment.

We propose this framework for modeling time-series as a plausible generalization of the usual AR(1) process. The novel setup belongs to the class of stochastic volatility models, where the variance of the underlying process is assumed to be random itself, due to seasonal change, effect of the news, existence of the business cycles, etc. In particular, when the environment ω is a finite-state Markov chain, our model is a discrete-time counterpart of the Langevin stochastic differential equation with Markov-modulated regime switches. In this model, the current variance σ_n and the short-term average μ_n are determined by a Markov process which represents the current state (regime) of the economy.

The main results of the thesis are presented in Chapter 3. We focus mostly on the general, stationary and ergodic, environment. The sequence X_n forms a Gaussian process when the random environment is fixed. The complete characterization of this process in a given environment is included in Sections 3.1 and 3.3. In Section 3.2 we study the asymptotic behavior of the limiting distribution of X_n when $\gamma \to 1$. In Section 3.4 we focus on the asymptotic behavior of the random walk $S_n = \sum_{k=1}^n X_k$ and carry out to our framework some basic results for the classical random walks. In particular, we compute the asymptotic (long-term) average of the underlying process, prove limit theorems for the fluctuation of the empirical average around its asymptotical value, and study the asymptotic of the extremal values of both the underlying process as well as of its cumulative effect (i.e., the partial sums).

We believe that many of our results, for instance the characterization of the limiting distribution for the sequence X_n as well as limit theorem for their extreme values, are of interest for practitioners working on time-series modeling.

In a paper in preparation [55] we consider in more details an especially interesting in applications case of Markovian environments, where some relatively explicit results are available. We remark that in fact, we consider in [55] a more general model

$$X_{n+1} = \rho_n X_n + \mu_n + \sigma_n \epsilon_n, n \in \mathbb{Z},$$

under the following two conditions on the random sequence $(\rho_n)_{n\in\mathbb{Z}}$:

(i) $\mathbb{P}\big(\rho_n \in (0,1)\big) = 1.$

(ii) $(\rho_n)_{n\in\mathbb{Z}}$ is independent of $(\xi_n)_{n\in\mathbb{Z}}$.

Both, the work done in this thesis as well as the study in [55], continue and complement [121], where a similar model with *heavy-tailed* rather than normal distributions of the innovation term ξ_n is considered. In all this work, the focus is on revealing the probabilistic structure of the underlying random process. Future research that we are planning will consider statistical features of the model. One pressing issue is its predictive power in comparison to closely related stochastic volatility time-series models.

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