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Coefficient integral domains in commutative algebras

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I. INTRODUCTION

The purpose of this thesis is to find necessary and sufficient conditions for the existence of coefficient integral domains containing the base field in commutative algebras, in particular, coefficient fields containing the base field in quasi-local algebras. The essential method of attack we use is the known theory of integral domain composites and tensor products.

We now give various current definitions of local algebras.

Definition 1.1. A is a quasi-local algebra over a field K if and only if A is a commutative ring with identity containing K and A has a unique maximal ideal N.

It is easily proved that the identities of A and K coincide.

Definition 1.2. A is a weak-local algebra if and only if A is a quasi-local algebra and $\bigcap_{i=1}^{\infty} N^i = (0)$.

Definition 1.3. A is a local algebra if and only if A is a quasi-local algebra and A is noetherian.

Definition 1.4. A is a semi-local ring if and only if A is a noetherian, commutative ring with identity and has a finite number of maximal ideals.

Definition 1.3 implies Definition 1.3, but it is not known whether or not Definitions 1.1 and 1.2 are equivalent.

It is easily shown that in any commutative ring A with identity the non-units of A form an ideal if and only if A has a unique maximal ideal.

We do not restrict ourselves to quasi-local algebras, but consider more general cases. Before continuing, we need the following basic definitions.

Let A be an algebra over a field K where A is a commutative ring with identity such that the identity of A is the identity of K. A always satisfies these conditions throughout this thesis.

The characteristic of the base field K is not restricted except when we assume that a field is pure inseparable over K. Then, of course, K has characteristic $p \neq 0$.

Let f be a function of a set A onto a set B. By a counter image of a set C in B, we mean a set $C' < A$ such that $f(C') = C$ and $f|C'$ (f restricted to C') is one-one.

Definition 1.5. Let N be any prime ideal in A. Let ν be the natural homomorphism of A onto A/N. A set $I \leq A$ is said to be a coefficient integral domain for A if and only if I is an integral domain containing K, $\nu(I) = A/N$, and the mapping $\nu|I$ is one-one.

The mapping $\nu|K$ is an isomorphism of K onto $\nu(K)$ since $K \cap N = (0)$. In A/N, we identify $\nu(K)$ with K and refer to ν as a K-homomorphism.

Definition 1.6. Let N be any prime ideal in A. Let ν be the natural homomorphism of A onto A/N. Then we say that A splits with respect to N if and only if A = I + N (group direct) where I is an integral domain in A containing K.

We should remark that Definitions 1.5 and 1.6 are equivalent. Hence, we refer to both definitions in the course of this work. When we refer to the splitting of a ring other than A, the splitting always takes place with respect to a specified ideal as in Definition 1.6. That is, there is an integral domain (containing K) in this ring which maps one-one onto A/N under a specified mapping. The ideal is the kernel of this mapping.

If N is a maximal ideal in A, then certain obvious modifications of Definitions 1.5 and 1.6 are in order. I becomes a field containing K, and we refer to I as a coefficient field. In this case, $\nu|I$ is automatically one-one.

All rings which are considered in this thesis, unless otherwise specified, are assumed to contain the field K as subring. It is furthermore assumed that the identity of K is also the identity of each of the rings.

If F_0 and F_1 are subrings of a ring F, we denote by $[F_0, F_1]$ the smallest subring of F which contains both rings F_0 and F_1.

Let F_0 and F_1 be algebras over K.

Definition 1.7. By a product of F_0 and F_1 (over K) we mean the composite concept (F, τ_0, τ_1) consisting of an algebra F over K, a K-isomorphism τ_0 of F_0 into F and a K-isomorphism τ_1 of F_1 into F, such that $F = [\tau_0 F_0, \tau_1 F_1]$.

Definition 1.8. Two products (F, τ_0, τ_1) and (F', μ_0, μ_1) of F_0 and F_1 are said to be equivalent if there exists an isomorphism σ of F onto F' such that $\mu_0 = \sigma\tau_0$ on F_0 and $\mu_1 = \sigma\tau_1$ on F_1.

Definition 1.9. A product (F, τ_0, τ_1) of F_0 and F_1 is called a tensor product of F_0 and F_1 (over K) if the rings $\tau_0 F_0$ and $\tau_1 F_1$ are linearly disjoint over K.

Unless otherwise specified, our tensor products are always over K, and we use the notation $F_0 \times F_1$.

A necessary and sufficient condition that a product (F, τ_0, τ_1) of F_0 and F_1 be a tensor product of F_0 and F_1 is that given any two K-homomorphisms h_0 and h_1 of F_0 and F_1 respectively into a ring R there should exist a homomorphism h of F into R such that $h = h_0\tau_0^{-1}$ on $F_0 \times 1$ and $h = h_1\tau_1^{-1}$ on $1 \times F_1$.

This property is referred to as the universal mapping property of tensor products. We always reserve h for this mapping and refer to it as the "universal homomorphism".

Let F_0 and F_1 be integral domains containing K.

Definition 1.10. By a free join of two integral domains F_0 and F_1 (relative to K) we mean the composite concept (F, τ_0, τ_1) consisting of an integral domain F containing K, a K-isomorphism τ_0 of F_0 into F and a K-isomorphism τ_1 of F_1 into F, such that the following conditions are satisfied: (i) $F = [\tau_0 F_0, \tau_1 F_1]$; (ii) the subrings $\tau_0 F_0$ and $\tau_1 F_1$ of F are free over K.

An integral domain composite of two integral domains F_0 and F_1 is the same as the free join of F_0 and F_1 with (ii) above not necessarily holding.

Unless otherwise specified, our composites are always relative to K.

Let (F, τ_0, τ_1) be a composite of F_0 and F_1. By the universal mapping property, there exists a homomorphism f of $F_0 \times F_1$ onto F such that $f = \tau_0$ on F_0 and $f = \tau_1$ on F_1. (Notice that we identify F_0 with $F_0 \times 1$ and F_1 with $1 \times F_1$ here.) We always reserve f for this mapping and refer to it as the "canonical homomorphism" of $F_0 \times F_1$ onto (F, τ_0, τ_1).

For further details, see Zariski and Samuel (5) and Pickert (4).

We conclude the definitions by explaining what we mean by a commutative diagram. Suppose A, B, C, and D are rings. Suppose that there exist isomorphisms a and c such that aA = B and cC = D and that there exist homomorphisms b and d such that bB = D and dA = C. Consider the following diagram:

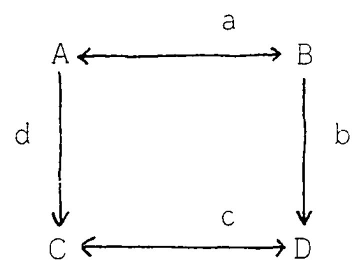

Diagram 1. Example of a commutative diagram

We say that Diagram 1 is commutative if and only if cd = ba. The two-headed arrows represent isomorphisms. When it is unclear what the domain and range of an isomorphism are, we place the isomorphism closer to its range. The one-headed arrows, of course, represent homomorphisms with the arrow head pointing to the range of the homomorphism.

In Chapter II, since A does not in general contain a unique maximal ideal, we consider an arbitrary prime ideal N in A. We also apply the theory developed to the case where N is a maximal ideal of A and A/N is pure inseparable over K. The main result of this chapter is Theorem 2.5 where we prove a necessary and sufficient condition for A to have a coefficient integral domain.

In Chapter III, we form the tensor product A x F_1' of A and an arbitrary subring $F_1' \leq A/N$ containing K. The purpose of this is to see what relationship there is between the existence of a coefficient integral domain in A and the existence of a coefficient integral domain in the tensor product. There is interest in considering this tensor product on its own merit since under certain conditions it is a quasi-local algebra over K. The principal results in this chapter are Theorems 3.6 and 3.10. In Theorem 3.6, we have a relationship between the splitting of A with respect to N and the splitting of $A \times F_1'$ with respect to an ideal $N^* \leq A \times F_1'$ for which $A/N \cong A \times F_1'/N^*$. In Theorem 3.10, we have a sufficient condition for $A \times F_1'$ to split with respect to N^*.

II. COEFFICIENT INTEGRAL DOMAINS FOR A

A. General Commutative Diagram

We begin this section by stating two conditions to which we often refer.

Condition 1. Let A be a commutative ring with identity containing a field K such that the identity of A is the identity of K. Let N be any prime ideal of A, and let ν be the natural homomorphism of A onto A/N.

Condition 2. Consider A/N as a composite of two integral domains F_0 and F_1 containing K, say $(A/N, \tau_0, \tau_1)$. Define T to be the tensor product of F_0 and F_1, say (T, τ_0', τ_1'). Suppose there exist integral domains $F_0', F_1' \leq A$ and containing K which are K-isomorphic to F_0, F_1 respectively under h_0, h_1 say. Let $A' = [F_0', F_1']$ be the ring composite of F_0' and F_1' in A. Let h be the universal homomorphism of $T = F_0 \times F_1$ onto A' (with respect to h_0 and h_1). Let f be the canonical homomorphism of T onto $(A/N, \tau_0, \tau_1)$ and let $g = \nu | A'$.

Hereafter, unless otherwise specified, it is always assumed that homomorphisms are K-homomorphisms, and that fields and integral domains contain K. We should also emphasize that when we consider A/N as a composite, it is clear from the context whether A/N is an integral domain composite or a field composite.

Definition 2.1. Suppose Conditions 1 and 2 hold. Then we say that A' has a commutative diagram if and only if there exists an automorphism α of A/N such that $f = \alpha gh$.

The following diagram is useful in the work that is to follow.

Lemma 2.2. Suppose Conditions 1 and 2 hold. If h_0 and h_1 are such that $\tau_0 = gh_0$ and $\tau_1 = gh_1$, then $f = gh$.

Proof. Let $\sum_i a_{0i} x a_{1i}$ be any element of T where $a_{0i} \varepsilon F_0$ and $a_{1i} \varepsilon F_1$. Then $gh(\sum_i a_{0i} x a_{1i}) = \sum_i gh(a_{0i} x a_{1i}) =$
$\sum_i gh(a_{0i} x 1)(1 x a_{1i}) = \sum_i gh(a_{0i} x 1) gh(1 x a_{1i}) =$
$\sum_i gh_0 \tau_0'^{-1}(a_{0i} x 1) gh_1 \tau_1'^{-1}(1 x a_{1i}) = \sum_i gh_0(a_{0i}) gh_1(a_{1i}) =$
$\sum_i \tau_0 a_{0i} \tau_1 a_{1i} = \sum_i f(a_{0i} x 1) f(1 x a_{1i}) = \sum_i f[(a_{0i} x 1)(1 x a_{1i})] =$
$\sum_i f(a_{0i} x a_{1i}) = f(\sum_i a_{0i} x a_{1i})$. Hence, $f = gh$.

Lemma 2.3. Suppose Conditions 1 and 2 hold. Let N' be any ideal of T such that $N' \leq \text{Ker } h$. Assume that $\text{Ker } h \leq \text{Ker } f$. Let ν' be the natural homomorphism of T onto T/N', q the natural homomorphism of T/N' onto (T/N')/(Ker f/N'), m the induced isomorphism of (T/N')/(Ker f/N') onto T/Ker f, and n the induced isomorphism of T/Ker f onto $(A/N, \tau_0, \tau_1)$. Let q' be the natural homomorphism of T/N' onto (T/N')/(Ker h/N'), m' the induced isomorphism of (T/N')/(Ker h/N') onto T/Ker h, and n' the induced

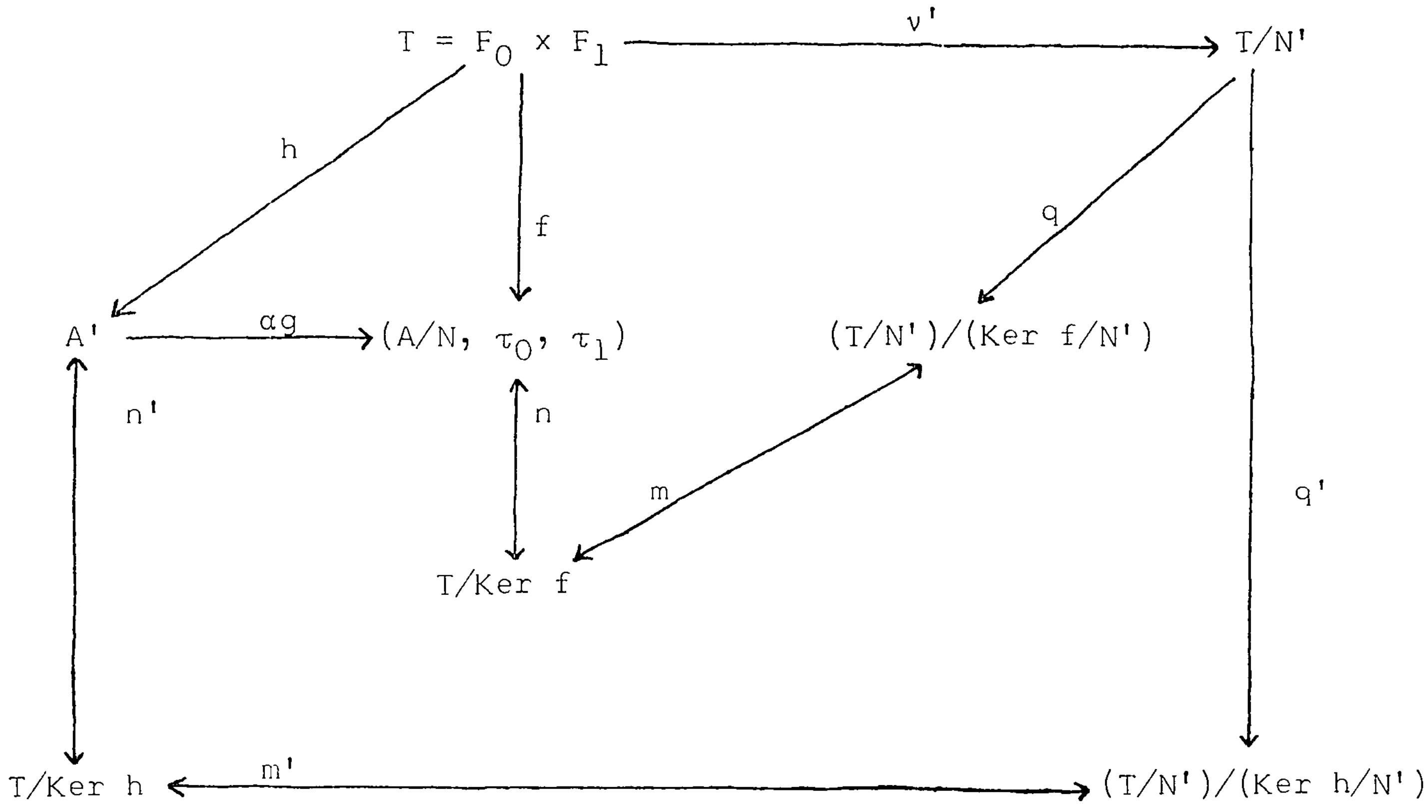

Diagram 2. A commutative diagram for A'

isomorphism of $T/\text{Ker } h$ onto A'. Then $f = nmq\nu'$ and $h = n'm'q'\nu'$.

Proof. See Jacobson (2), page 135, and refer to Diagram 2.

Lemma 2.4. Suppose Conditions 1 and 2 hold. If there exists a commutative diagram for A', then $nmq = \alpha gn'm'q'$.

Proof. Since $f = \alpha gh$, we have $nmq\nu' = \alpha gn'm'q'\nu'$ by Lemma 2.3. Thus, $nmq = \alpha gn'm'q'$.

This brings us to the main result of this section.

Theorem 2.5. Suppose Condition 1 holds. Consider A/N as a composite of two integral domains F_0 and F_1, say $(A/N, \tau_0, \tau_1)$. Define T to be the tensor product of F_0 and F_1, say (T, τ_0', τ_1'). Then A has a coefficient integral domain (containing K) if and only if (i) there exist integral domains F_0', $F_1' \leq A$ which are isomorphic to F_0, F_1 respectively, (ii) there exists a commutative diagram for $A' = [F_0', F_1']$ (see Definition 2.1), and (iii) there exists an ideal $N' \leq \text{Ker } h$ for which T/N' splits with respect to $\text{Ker } f/N'$.

Proof. Suppose A has a coefficient integral domain I'. Then $\nu|I'$ maps I' one-one onto A/N. Since A/N contains τ_0F_0 and τ_1F_1, there exist counter images F_0' and F_1' (with respect to ν) of τ_0F_0 and τ_1F_1 respectively such that F_0' and F_1' are integral domains in I'. That is, there exist integral domains F_0' and F_1' in I' such that $\nu F_0' = \tau_0F_0$ and $\nu F_1' = \tau_1F_1$.

Let A' be the ring composite $[F_0', F_1']$. Then $I' = A'$. Let $g = \nu|I'$. Let h_0 and h_1 be isomorphisms of F_0 and F_1 respectively into I' such that $\nu h_0 = \tau_0$ and $\nu h_1 = \tau_1$. By the universal mapping property, there exists a homomorphism h of T onto $I' = A'$ such that $h = h_0\tau_0'^{-1}$ on $F_0 \times 1$ and $h = h_1\tau_1'^{-1}$ on $1 \times F_1$. If f is the canonical homomorphism of $T = F_0 \times F_1$ onto $(A/N, \tau_0, \tau_1)$, then $f = gh$ by Lemma 2.2 since Condition 2 is now satisfied. Hence, there exists a commutative diagram for A' with α as the identity automorphism of A/N.

Since $h(T) = I'$, $T/\text{Ker } h \cong I'$. Since $g = \nu|I'$ and $f = gh$, Ker f = Ker h. Thus, $T/\text{Ker } h = T/\text{Ker } h + \text{Ker } f/\text{Ker } h$. Hence, we have an ideal $N' \leq \text{Ker } h$ for which T/N' splits with respect to Ker f/N', namely Ker h.

Conversely, suppose Conditions i, ii, and iii hold. In Condition i, let h_0 and h_1 be the isomorphisms such that $h_0(F_0) = F_0'$ and $h_1(F_1) = F_1'$. Since $f = \alpha gh$, Ker $h \leq$ Ker f. Let ν', q, m, n, q', m', and n' be the homomorphisms defined in Lemma 2.3. Since T/N' splits with respect to Ker f/N', $T/N' = I' + \text{Ker } f/N' = I' + \text{Ker } nmq$ where I' is an integral domain in T/N'. Since Ker $h \leq$ Ker f, Ker $h/N' \leq$ Ker f/N'. Thus, Ker $n'm'q'$ = Ker $h/N' \leq$ Ker f/N' = Ker nmq. Therefore, $n'm'q'|I'$ maps I' one-one into A' since $nmq|I'$ is one-one. Thus, $I'' = n'm'q'(I')$ is an integral domain in A'.

By Lemma 2.4, $nmq = \alpha g n'm'q'$. Since $nmq|I'$ and $n'm'q'|I'$ are one-one, $g|I''$ is one-one. We see that

$\alpha g(I'') = A/N$ since $nmq(I') = A/N$. Since α is an automorphism of A/N, $g(I'') = A/N$. Thus, A' has a coefficient integral domain. Since $g = \nu|A'$, $\nu|I''$ maps I'' one-one onto A/N. Therefore, I'' is a coefficient integral domain (containing K) for A.

Corollary 2.6. Suppose Condition 1 holds. Consider A/N as a composite of two integral domains F_0 and F_1, say $(A/N, \tau_0, \tau_1)$. Define T to be the tensor product of F_0 and F_1, say (T, τ_0', τ_1'). Then A has a coefficient integral domain if and only if (i) there exist integral domains $F_0', F_1' \leq A$ such that $\nu F_0' = \tau_0 F_0$ and $\nu F_1' = \tau_1 F_1$ and such that $\nu|F_0'$ and $\nu|F_1'$ are one-one, and (ii) there exists an ideal $N' \leq \text{Ker } h$ such that T/N' splits with respect to $\text{Ker } f/N'$ where h is the universal homomorphism of T onto $A' = [F_0', F_1']$.

Proof. If A has a coefficient integral domain, then the conclusion follows immediately from the proof of the theorem. Notice that the homomorphisms h_0 and h_1 are defined in such a way that $\nu h_0 = \tau_0$ and $\nu h_1 = \tau_1$.

Conversely, suppose Conditions i and ii hold. By Lemma 2.2, $f = gh$. Thus, A' has a commutative diagram. Hence, the desired result follows immediately from the theorem.

We conclude this section with some results obtained by imposing further conditions on A/N.

Lemma 2.7. Suppose Conditions 1 and 2 hold. Then A' has a commutative diagram if and only if there exists an automorphism α of A/N such that $\alpha gh_0 = \tau_0$ and $\alpha gh_1 = \tau_1$.

Proof. Suppose A' has a commutative diagram. Then there exists an automorphism α of A/N such that $f = \alpha gh$. Thus, if $a_0 \in F_0$, $\tau_0 a_0 = f(a_0 \times 1) = \alpha gh(a_0 \times 1) = \alpha gh_0(a_0)$. Hence, $\tau_0 = \alpha gh_0$. Similarly, $\tau_1 = \alpha gh_1$.

Conversely, suppose there exists an automorphism α of A/N such that $\alpha gh_0 = \tau_0$ and $\alpha gh_1 = \tau_1$. Let $\sum_i a_{0i} \times a_{1i}$ be any element of $F_0 \times F_1$ where $a_{0i} \in F_0$ and $a_{1i} \in F_1$. Then
$f(\sum_i a_{0i} \times a_{1i}) = \sum_i \tau_0 a_{0i} \tau_1 a_{1i} = \sum_i \alpha gh_0 a_{0i} \alpha gh_1 a_{1i} =$
$\alpha g(\sum_i h_0 a_{0i} h_1 a_{1i}) = \alpha g(\sum_i h(a_{0i} \times a_{1i})) = \alpha gh(\sum_i a_{0i} \times a_{1i})$.
Therefore, $f = \alpha gh$. Hence, A' has a commutative diagram.

Theorem 2.8. Suppose Conditions 1 and 2 hold. Suppose that A/N is a unique composite of F_0 and F_1 in Condition 2. If A/N is not K-isomorphic to proper subring, and if $g|F_0'$ and $g|F_1'$ are one-one, then A' has a commutative diagram.

Proof. Since A/N is an integral domain, $[(g|F_0')h_0F_0, (g|F_1')h_1F_1]$ is an integral domain composite of F_0 and F_1 in A/N. By unicity, there exists an isomorphism α of $[(g|F_0')h_0F_0, (g|F_1')h_1F_1]$ onto $[\tau_0 F_0, \tau_1 F_1] = A/N$ such that $\alpha gh_0 = \tau_0$ and $\alpha gh_1 = \tau_1$. Since A/N is not isomorphic to a proper subring, α is an automorphism of A/N. Therefore, by the preceding lemma, A' has a commutative diagram.

Corollary 2.9. Suppose Conditions 1 and 2 hold. Suppose that A/N is the unique composite of F_0 and F_1 in

Condition 2. If A/N is algebraic over K and if $g|F_0'$ and $g|F_1'$ are one-one, then A' has a commutative diagram.

Proof. Since A/N is algebraic over K, A/N is not K-isomorphic to a proper subring.

Let us notice that if F_0 and F_1 are fields, then $g|F_0'$ and $g|F_1'$ are one-one.

B. Unique Free Join

In this section, we first make use of the free join concept.

Lemma 2.10. Suppose Conditions 1 and 2 hold. Suppose that A/N is a unique free join of F_0 and F_1 in Condition 2. If there exists an automorphism α of A/N such that $f = \alpha gh$, then every element of Ker g is nilpotent.

Proof. By Zariski and Samuel (5, p. 195), Ker f is a nil ideal. Let $\sum_i h_0 a_{0i} h_1 a_{1i}$ be any element of Ker g where $a_{0i} \in F_0$ and $a_{1i} \in F_1$. Then $\sum_i h_0 a_{0i} h_1 a_{1i}$ has a counter image in T, say $\sum_i a_{0i} \times a_{1i}$. Thus, $h(\sum_i a_{0i} \times a_{1i}) = \sum_i h_0 a_{0i} h_1 a_{1i}$. Since $\sum_i h_0 a_{0i} h_1 a_{1i} \in$ Ker g, $\sum_i a_{0i} \times a_{1i} \in$ Ker f. Thus, $\sum_i a_{0i} \times a_{1i}$ is nilpotent. Hence, there exists an integer n such that $(\sum_i a_{0i} \times a_{1i})^n = 0$. Since $0 = h((\sum_i a_{0i} \times a_{1i})^n) = (h(\sum_i a_{0i} \times a_{1i}))^n = (\sum_i h_0 a_{0i} h_1 a_{1i})^n$, we see that $\sum_i h_0 a_{0i} h_1 a_{1i}$ is nilpotent.

Let us consider the case where N is a maximal ideal of A. Then A/N is a field. Let us also assume that A/N is algebraic over K.

Theorem 2.11. Suppose A, N, and A/N are as in the preceding paragraph. If Condition 2 holds with F_0 and F_1 fields, and if A/N is the unique free join of F_0 and F_1, then there exists a commutative diagram for A'.

Proof. The proof is immediate from Corollary 2.9 and the remark following it.

The next three results hold if we replace "Theorem 2.8" by "Theorem 2.11" and "coefficient integral domain" by "coefficient field".

Theorem 2.12. Suppose the hypothesis of Theorem 2.8 (Theorem 2.11) holds. If A' has no nilpotent elements, then A splits into A' + N.

Proof. We have that A' has a commutative diagram. By Lemma 2.10, all the elements of Ker g are nilpotent. Since A' has no nilpotent elements, Ker g = (0). Thus, g maps A' one-one onto A/N. Therefore, A' is a coefficient integral domain (coefficient field in case Theorem 2.11 holds) for A.

Notice, also, that since $f = \alpha gh$ and since α and g are one-one, Ker f = Ker h. Thus, by Theorem 2.5, it follows that A has a coefficient integral domain (coefficient field).

Corollary 2.13. Suppose the hypothesis of Theorem 2.8 (Theorem 2.11) holds. If A has no nilpotent elements, or if

A is an integral domain, then A splits with respect to N.

Proof. The proof is immediate from the theorem.

Remark 2.14. Suppose the hypothesis of Theorem 2.8 (Theorem 2.11) holds. If T has no nilpotent elements, then A has a coefficient integral domain (coefficient field).

Proof. We have that A' has a commutative diagram. Since T has no nilpotent elements, Ker f = (0). Since $f = \alpha gh$, g is one-one. Thus A' is a coefficient integral domain (coefficient field) for A.

We see that we again can apply Theorem 2.5 by noticing that $(0) = \ker f \leq \operatorname{Ker} h = (0)$ and that T/Ker f splits with respect to Ker f/Ker f since T is isomorphic to $(A/N, \tau_0, \tau_1)$.

We conclude this section by stating some cases for which T has no nilpotent elements.

Remark 2.15. Assume F_0 and F_1 are fields. (i) If F_0 or F_1 is a separable extension of K, then $T = F_0 \times F_1$ has no nilpotent elements. (ii) If K is a perfect field, then T has no nilpotent elements. (iii) If K is an algebraically closed field, then T is an integral domain. (iv) If K is q.m.a. in F_0, and if F_1 is a separable extension of K, then T is an integral domain.

Proof. See Zariski and Samuel (5), pages 195-198.

C. p-basis

We now would like to compare Theorems 2.11 and 2.12 with a p-basis argument for the sufficiency of a coefficient field of A containing K where N is a maximal ideal of A and A/N is pure inseparable over K. Let $A/N = F$. Let $p \neq 0$ be the characteristic of K.

Theorem 2.16. Suppose A contains a field F' (containing K) such that $\nu F' = F^p(K)$. Let $G < K - F^p$ and $M < F - F^p(K)$ be selected so that $G \cup M$ is a p-basis for F and $F^p(G) = F^p(K)$. Let $M' < A$ be any counter image (with respect to ν) of M. If $M'^p < F'$, then A has a coefficient field, namely $F'(M')$.

Proof. Well order M'. For $m_1' \in M'$, m_1' satisfies a polynomial equation $x^p - b' = 0$, $b' \in F'$. Suppose $x^p - b'$ is reducible over F'. Then there exists $c' \in F'$ such that $m_1'^p = b' = c'^p$. Thus, $\nu(m_1')^p = \nu(c')^p$. Hence $m_1^p = c^p$ where $\nu(c') = c \in F^p(K)$ and $\nu(m_1') = m_1 \in M$. Thus, $(m_1 - c)^p = 0$. Hence, $m_1 = c$. Thus, $m_1 \in F^p(K)$ which contradicts the p-independence of M over $F^p(G) = F^p(K)$. Thus, $x^p - b'$ is irreducible over F'. Hence, $F'(m_1')$ is a field.

Suppose the ordinal number of M' is $\delta > 1$. Let $P_1' = F'(m_1')$ and $P_1 = F^p(G)(m_1)$. Let $P_{\beta+1}' = F'(m_1', \ldots, m_{\beta+1}')$ and $P_{\beta+1} = F^p(G)(m_1, \ldots, m_{\beta+1})$ where β is any ordinal number less than δ. Assume P_β' is a field. Then $P_{\beta+1}' =$

$F'(m_1', \ldots, m_{\beta+1}') = P_\beta'(m_{\beta+1}')$. Since $m_{\beta+1}'^p \in F' < P_\beta'$, $m_{\beta+1}'$ satisfies a polynomial equation $x^p - b' = 0$ where $b' \in P_\beta'$. Suppose $x^p - b'$ is reducible over P_β'. Then there exists $c' \in P_\beta'$ such that $m_{\beta+1}'^p = b' = c'^p$. Thus $\nu(m_{\beta+1}')^p = \nu(c')^p$. Hence, $m_{\beta+1}^p = c^p$ where $\nu(c') = c \in \nu(P_\beta') = P_\beta = F^p(G)(m_1, \ldots, m_\beta)$. Therefore, $(m_{\beta+1} - c)^p = 0$. Hence, $m_{\beta+1} = c \in F^p(G)(m_1, \ldots, m_\beta)$. This contradicts the p-independence of M over $F^p(G)$. Thus, $x^p - b'$ is irreducible over P_β'. Hence, $P_{\beta+1}'$ is a field. Let γ be any limit ordinal less than or equal to δ. Let $P_\gamma' = \bigcup_{\beta<\gamma} P_\beta'$ $(= F'(m_1', \ldots, m_\beta', \ldots), \beta < \gamma)$. Clearly, P_γ' is a field. Since $F'(M') = F'(\bigcup_{\beta<\delta} \{m_\beta'\})$, $F'(M')$ is a field by transfinite induction. Therefore, $F'(M')$ is a coefficient field for A since $F'(M') \cong F^p(K)(M) = F^p(G)(M) = F$ under ν.

See Bray (1) for sufficient conditions for the existence of a field $F' \leq A$ such that $\nu F' = F^p(K)$.

Theorem 2.17. Suppose N is a maximal ideal of A and that $A/N = F$ is pure inseparable over K. Consider A/N as a free join of two fields F_0 and F_1, say $(A/N, \tau_0, \tau_1)$. Let $G < K - F^p$ and $M < F - F^p(K)$ be selected so that $G \cup M$ is a p-basis for F and $F^p(G) = F^p(K)$. Then, there exists a field $F' \leq A$ such that $\nu F' = F^p(K)$ and there exists a counter image M' of M such that $M'^p < F'$ if and only if there exist fields $F_0', F_1' \leq A$ such that F_0' and F_1' are isomorphic to F_0 and F_1

respectively and such that $A' = [F_0', F_1']$ has no nilpotent elements.

Furthermore, $F'(M') = A'$.

Proof. Suppose there exists a field $F' \leq A$ such that $\nu F' = F^p(K)$ and there exists a counter image M' of M such that $M'^p < F'$. By the preceding theorem, $F'(M')$ is a coefficient field for A. Thus, $\nu|F'(M')$ is one-one onto $[\tau_0 F_0, \tau_1 F_1] = A/N$. Hence, in $F'(M')$, there exist counter images of $\tau_0 F_0$ and $\tau_1 F_1$, say F_0' and F_1', which are therefore isomorphic to $\tau_0 F_0$ and $\tau_1 F_1$ respectively. Let $A' = [F_0', F_1']$. Then $A' \leq F'(M')$. Hence, A' has no nilpotent elements since $F'(M')$ is a field.

By Theorem 2.12, $\nu|A'$ maps A' one-one onto $(A/N, \tau_0, \tau_1)$. Since $F'(M')$ is not K-isomorphic to a proper subfield (being algebraic over K), $A' = F'(M')$.

Conversely, suppose there exist fields $F_0', F_1' \leq A$ which are isomorphic to F_0, F_1 respectively under h_0, h_1 say. Suppose $A' = [F_0', F_1']$ has no nilpotent elements. Then by Theorem 2.12, A splits into $A' + N$. Thus, $\nu|A'$ maps A' one-one onto A/N. Therefore, there exists a counter image F' of $F^p(K)$ in A' and hence in A. Also, there exists a counter image M' of M in A' such that $M'^p < F'$ since A' is a field isomorphic to A/N under $\nu|A'$.

By Theorem 2.16, $F'(M')$ is isomorphic to A/N under $\nu|F'(M')$. Since $F'(M') \leq A'$ and A' is not K-isomorphic to a proper subfield, $F'(M') = A'$.

Before proceeding to the next chapter, let us comment that although we have not specifically considered A as a quasi-local algebra over K, all the results of this chapter may be applied to this case.

III. COEFFICIENT INTEGRAL DOMAINS FOR $A \times F_1'$

A. Splitting of A and A*

Condition 3. Suppose that Condition 1 holds and that F_1' is any subring of A/N containing K. Let $A^* = A \times F_1'$ (over K).

Condition 4. Suppose that Condition 1 holds. There exist integral domains F_1 and F, $K \leq F_1 \leq F$, for which there exist K-isomorphisms τ and τ_1 such that $\tau F = A/N$ and $\tau_1 F_1 = F_1'$. Consider A/N as a composite of F and F_1 such that $A/N = [\tau F, \tau_1 F_1]$.

Condition 5. Suppose that Condition 4 holds. Suppose further that there exists an integral domain F_0, $K \leq F_0 \leq F$, such that A/N is a composite of F_0 and F_1, $A/N = [\tau_0 F_0, \tau_1 F_1]$ where $\tau_0 = \tau | F_0$.

Let (N, 0) be the ideal in A* generated by $N \times 1$ in $A \times 1$ and 1×0 in $1 \times F_1'$.

Lemma 3.1. Suppose Conditions 3 and 4 hold. Let q_1 be the natural homomorphism of $A \times F_1'$ onto $A \times F_1'/(N, 0)$. Then there exists an isomorphism q_5 of $A \times F_1'/(N, 0)$ onto $A/N \times F_1'$ such that $\nu \times 1 = q_5 q_1$.

Proof. See Zariski and Samuel (5), page 184.

Lemma 3.2. Suppose Conditions 3 and 4 hold. Then there exists an isomorphism g* of $A/N \times F_1'$ onto $F \times F_1$.

Proof. We may consider the tensor product $A/N \times F_1'$ as a tensor product of F and F_1. To see this, let $A/N \times F_1'$ be the tensor product of A/N and F_1', say (C, μ, μ_1) and let $F \times F_1$ be the tensor product of F and F_1, say (C', τ', τ_1'). Since $\tau F = A/N$ and $\tau_1 F_1 = F_1'$, we may consider $A/N \times F_1'$ as the tensor product $(C, \mu\tau, \mu_1\tau_1)$ where $\mu\tau F = A/N \times 1$ and $\mu_1\tau_1 F_1 = 1 \times F_1'$. Since any two tensor products of the same pair of rings are equivalent, there exists an isomorphism g^* of $A/N \times F_1'$ onto $F \times F_1$. See Zariski and Samuel (5), page 182.

It may be of some help to refer to Diagram 3 at this point.

Lemma 3.3. Suppose Conditions 3 and 4 hold. Let f be the canonical homomorphism of $F \times F_1$ onto $(A/N, \tau, \tau_1)$. Let k be the natural homomorphism of $A/N \times F_1'$ onto $(A/N \times F_1')/g^{*-1}$ Ker f. Let N^* be the ideal in $A \times F_1'$ containing $(N, 0)$ for which $N^*/(N, 0) \cong$ Ker $k = g^{*-1}$ Ker f under q_5 where q_5 is the isomorphism defined in the preceding lemma. Let q_2 be the natural homomorphism of $A \times F_1'/(N, 0)$ onto $(A \times F_1'/(N, 0))/(N^*/(N, 0))$. Then there exists an isomorphism g' of $(A/N \times F_1')/$Ker k onto $(A/N, \tau, \tau_1)$ such that $fg^* = g'k$ and an isomorphism q_4 of $(A \times F_1'/(N, 0))/(N^*/(N, 0))$ onto $(A/N \times F_1')/$Ker f such that $kq_5 = q_4 q_2$.

Proof. The existence of g' follows from the fact that Ker $k \cong$ Ker f under g^*, and the existence of q_4 follows from the fact that $N^*/(N, 0) \cong$ Ker k under q_5.

Lemma 3.4. Suppose Conditions 3 and 4 hold. Let q_3 be the induced isomorphism of $(A \times F_1^!/(N, 0))/(N^*/(N, 0))$ onto A^*/N^*. Let ν^* be the natural homomorphism of A^* onto A^*/N^*. Then $\nu^* = q_3 q_2 q_1$.

Proof. See Jacobson (2), page 135.

Lemma 3.5. Suppose Conditions 3 and 4 hold. Then the following diagram is commutative where q_6 is the isomorphism of A^*/N^* onto $(A/N \times F_1^!)/\text{Ker } k$ induced by q_3 and q_4.

Proof. The proof follows directly from Lemmas 3.1-3.4.

Theorem 3.6. Suppose Conditions 3 and 4 hold. Then A splits into I + N if and only if A* splits into I x 1 + N* where I is an integral domain in A (containing K).

Proof. By Lemma 3.5, we have the commutativity of Diagram 3.

Suppose A splits into I + N. Then $\nu \times 1$ maps I x 1 one-one onto A/N x 1. g* maps A/N x 1 one-one onto F x 1. See Zariski and Samuel (5), page 181.

Since f is the canonical homomorphism of $F \times F_1$ onto $(A/N, \tau, \tau_1)$, $f|F \times 1 = \tau$ on F (identifying F with F x 1). Thus, $f|F \times 1$ maps F x 1 one-one onto $\tau F = A/N$. Therefore, $q_6^{-1} g'^{-1} f g^*(\nu \times 1)|I\times 1$ maps I x 1 one-one onto A*/N*. However, this composition of mappings is equal to $\nu^*|I\times 1$. Thus, ν^* maps I x 1 one-one onto A*/N*. That is, A* splits into I x 1 + N*.

Conversely, suppose A* splits into I x 1 + N*. Then

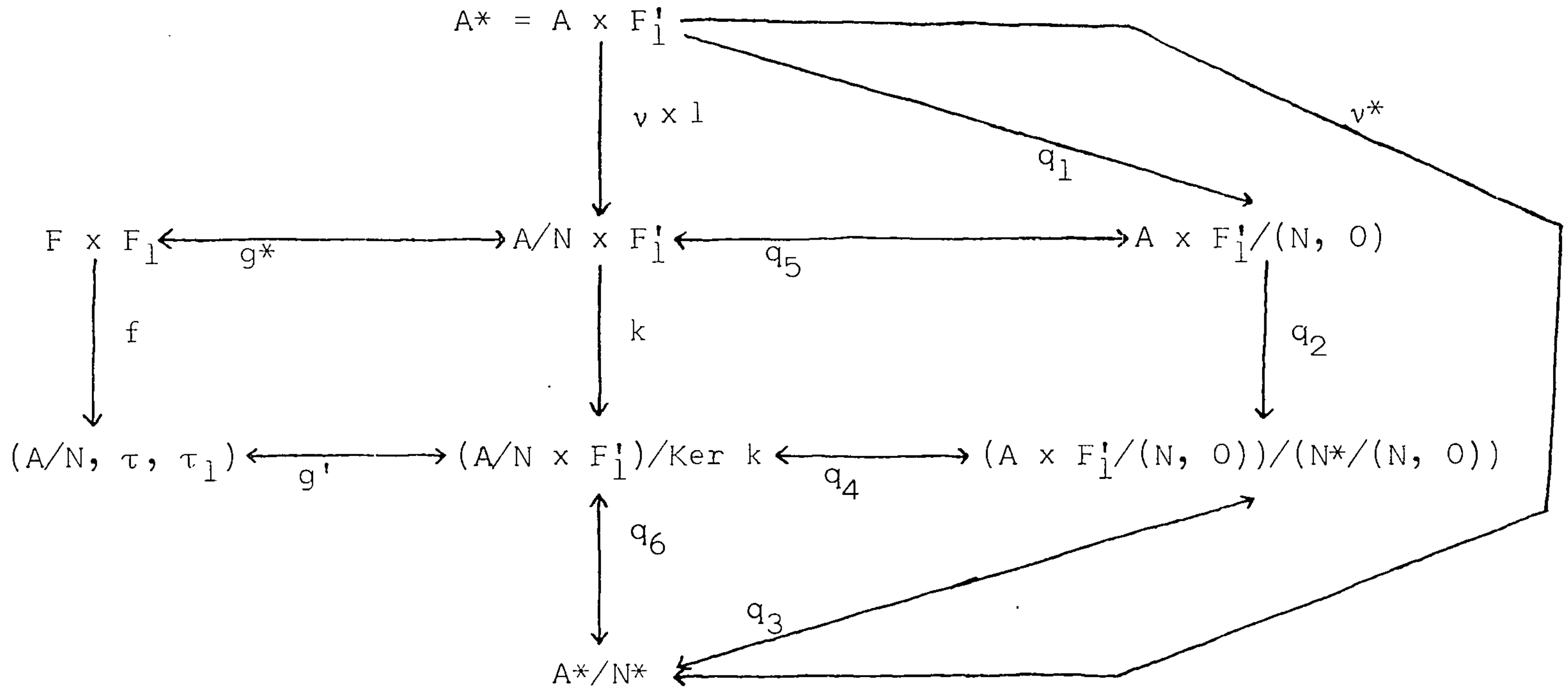

Diagram 3. A commutative diagram for A*

$g'q_6\nu^*|I \times 1$ maps $I \times 1$ one-one onto A/N. However, $fg^*(\nu \times 1)|I \times 1 = g'q_6\nu^*|I \times 1$. Hence, $\nu \times 1|I \times 1$ is one-one since $g'q_6\nu^*|I \times 1$ is one-one. If $\nu \times 1$ maps $I \times 1$ properly into $A/N \times 1$, then g^* maps $\nu I \times 1$ properly into $F \times 1$ and f maps $g^*(\nu \times 1)(I \times 1)$ properly into $\tau F = A/N$ which is impossible since $I \times 1$ goes onto A/N by $g'q_6\nu^*|I \times 1 = fg^*(\nu \times 1)|I \times 1$. Therefore, $\nu \times 1$ maps $I \times 1$ one-one onto $A/N \times 1$. Thus, ν maps I one-one onto A/N. Hence, A splits into $I + N$.

Notice that if A^* splits into $J + N^*$ where J is an integral domain in A^* containing K, and if this splitting implies A^* splits into $I \times 1 + N^*$, then A splits with respect to N. A natural question to ask is that under what conditions does a splitting of A^* into $J + N^*$ imply a splitting of A^* into $I \times 1 + N^*$. This question goes unanswered in this thesis.

Remark 3.7. Suppose Conditions 3 and 4 hold. (i) If A is a quasi-local algebra over K with unique maximal ideal N, then N^* is a maximal ideal of A^* and is unique in the sense that it contains $(N, 0)$. (ii) If A^* is a quasi-local algebra over K, then N^* is the unique maximal ideal of A^*. In this case, A and A^* have the same residue field (in the sense of isomorphism).

Proof. The proof of both cases is immediate from the commutativity of Diagram 3 and the definition of N^*.

In section C, we give some sufficient conditions for A^*

to be quasi-local algebra over K.

B. Splitting of A* Implied

Before giving the next lemma, let us note that $(N, 0) = (A \times F_1')(N \times 1) + (A \times F_1')(1 \times 0) = AN \times F_1' = N \times F_1'$.

Lemma 3.8. Suppose Condition 3 holds. If $F_0' \leq A$ is any integral domain such that $F_0' \cap N = (0)$, then $(F_0' \times F_1') \cap (N \times F_1') = (0)$.

Proof. Since $F_0' \cap N = (0)$, $F_0' + N$ is a group direct sum as K-modules. Thus, $(F_0' + N) \times F_1' = F_0' \times F_1' + N \times F_1'$ is a group direct sum as K-modules. Therefore, $(F_0' \times F_1') \cap (N \times F_1') = (0)$.

Lemma 3.9. Suppose that Conditions 3-5 hold. Suppose there exists an integral domain $F_0' \times 1 \leq A \times 1$ such that $g^*(\nu \times 1)(F_0' \times 1) = F_0 \times 1 \leq F \times 1$. Let $q_1(F_0' \times F_1') = J_1 \leq A \times F_1/(N, 0)$, $q_2(J_1) = J_2 \leq (A \times F_1'/(N, 0))/(N^*/(N, 0))$, $q_3(J_2) = J_3 \leq A^*/N^*$, $\nu \times 1(F_0' \times F_1') = J_1' \leq A/N \times F_1'$, and $k(J_1') = J_2' \leq F_0' \times F_1'/\text{Ker } k$. Then there exist isomorphisms g'', q_6', and q_3' such that $g''(J_2') = (A/N, \tau_0, \tau_1)$, $q_6'(J_2') = A^*/N^*$, and $q_3'(J_3) = A^*/N^*$. (Refer to Diagram 3).

Furthermore, the following diagram is commutative.

The appropriate restrictions of the mappings occurring in Diagram 4 are, of course, meant.

Proof. By Lemma 3.8, $q_1|F_0' \times F_1'$ is one-one, and thus

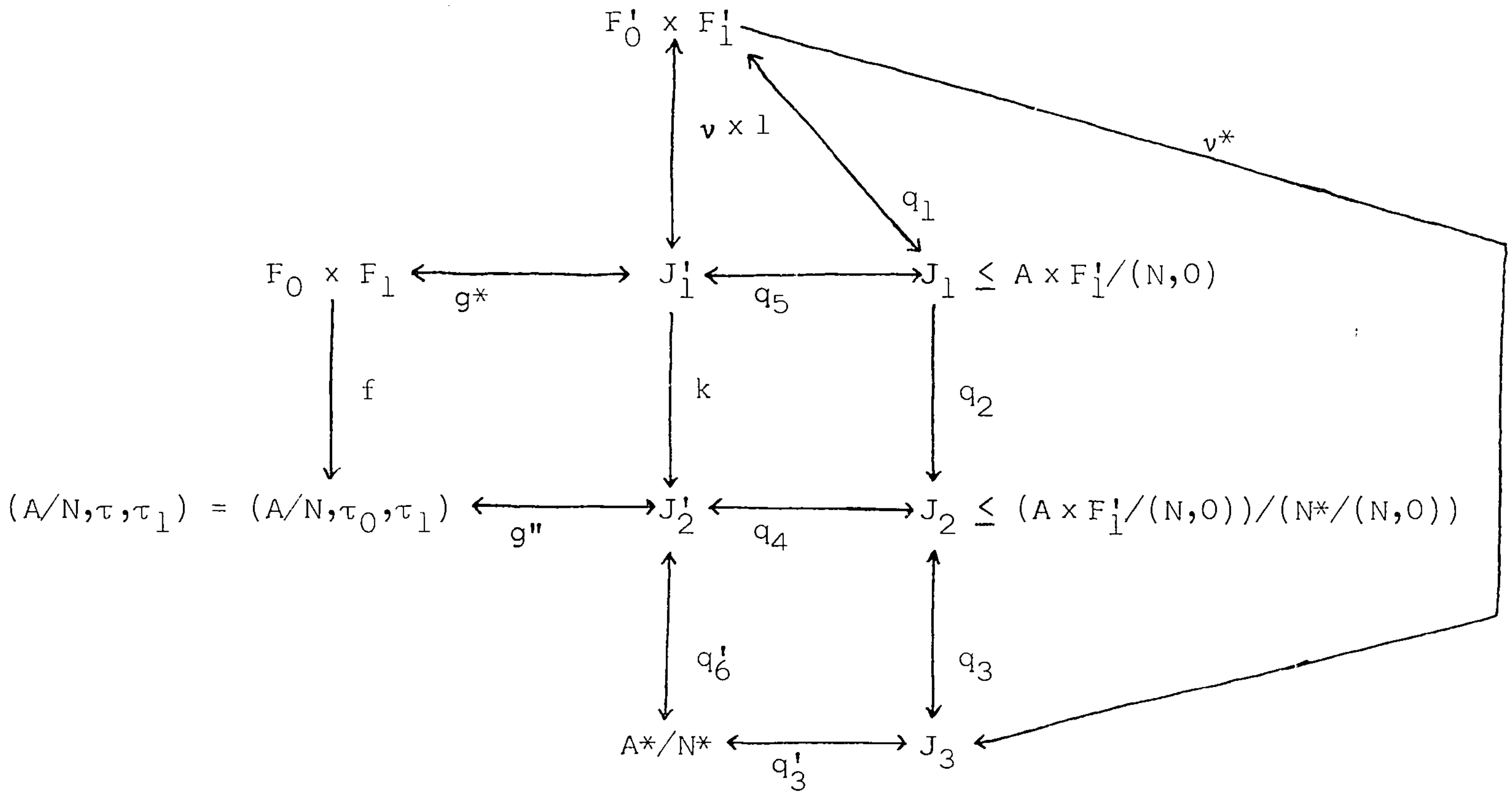

Diagram 4. A commutative diagram for $F_0' \times F_1'$

by the commutativity of Diagram 3, $\nu \times 1|F_0' \times F_1'$ is one-one and $\nu \times 1|F_0' \times F_1' = (q_5|J_1)(q_1|F_0' \times F_1')$. The factorization of $\nu^*|F_0' \times F_1'$ into $(q_3|J_2)(q_2|J_1)(q_1|F_0' \times F_1')$ is immediate from Lemma 3.5 as is that part of the diagram involving $q_5|J_1$ and $q_4|J_2$. Since Ker k $\cong$ Ker f under g^*, Ker $k|J_1' \cong$ Ker $f|F_0' \times F_1'$ under $g^*|J_1'$. Thus, there exists an isomorphism g'' of J_2' onto $(A/N, \tau_0, \tau_1)$ which fulfils its duty in the above diagram since, also, $J_1' \cong F_0 \times F_1$ under $g^*|J_1'$.

The following subdiagram of Diagrams 3 and 4 is helpful for the remainder of the proof.

Let q_6' be the isomorphism of J_2' onto A^*/N^* induced by g'', g', and q_6. Then, let q_3' be the isomorphism of J_3 onto A^*/N^* induced by q_6', $q_4|J_2$, and $q_3|J_2$.

We now come to the principal result of this section.

Theorem 3.10. Suppose that Conditions 3-5 hold. Suppose that A/N is algebraic over K. If $F_0 \times F_1$ splits with respect to Ker $f|F_0 \times F_1$, then A^* splits with respect to N^*.

Proof. By Lemma 3.9, we have the commutativity of Diagram 4. Since A/N is algebraic over K and A/N is isomorphic to A^*/N^*, A^*/N^* is algebraic over K. Since $J_3 \leq A^*/N^*$ and J_3 is K-isomorphic to A^*/N^*, $J_3 = A^*/N^*$, i.e., A^*/N^* cannot be K-isomorphic to a proper subring since it is algebraic over K. Since $F_0 \times F_1$ splits with respect to Ker $f|F_0 \times F_1$, $F_0 \times F_1 = J +$ Ker $f|F_0 \times F_1$ where J is an integral domain in $F_0 \times F_1$ (containing K). Thus, J is mapped one-one onto A/N

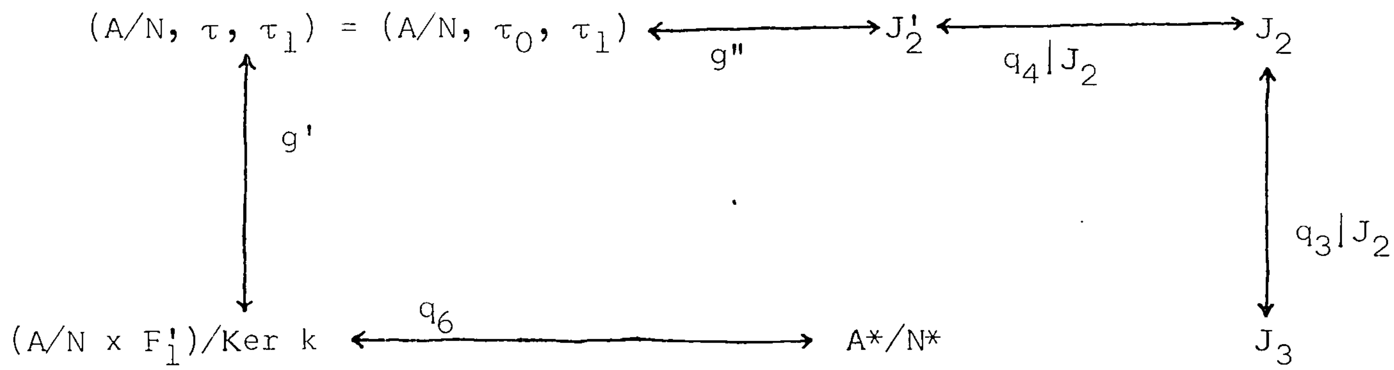

Diagram 5. A subdiagram of Diagrams 3 and 4

by $f|J$. J is also mapped one-one into $F_0' \times F_1'$ by $(\nu \times 1|F_0' \times F_1')^{-1}(g^*|J_1')^{-1}$. Let $J' \leq F_0' \times F_1'$ be the image of J by this mapping. Then by the commutativity of Diagram 4, ν^* maps J' one-one onto $J_3 = A^*/N^*$. Thus, J' is a coefficient integral domain for $F_0' \times F_1'$ and hence for A^*. That is, A^* splits into $J' + N^*$.

Corollary 3.11. *Suppose Conditions 3-5 hold. Suppose A/N is algebraic over K. If $F_0 \times F_1$ is an integral domain, then A^* splits with respect to N^*.*

Proof. Since A/N is algebraic over K, F_1 is algebraic over K. Thus, A/N is a free join of F_0 and F_1. See Zariski and Samuel (5), page 117. Since $f = \tau$ on F, $f|F_0 \times 1 = \tau F_0 = \tau_0$. Since, also, $f = \tau_1$ on F_1, we have that $f|F_0 \times F_1$ is the canonical homomorphism of $F_0 \times F_1$ onto $(A/N, \tau_0, \tau_1)$. Hence, all the elements of Ker $f|F_0 \times F_1$ are zero divisors. See Zariski and Samuel (5), page 189. Since $F_0 \times F_1$ is an integral domain, Ker $f|F_0 \times F_1 = (0)$. Thus, $F_0 \times F_1$ is mapped one-one onto A/N by $f|F_0 \times F_1$. By the commutativity of Diagram 4, ν^* maps $F_0' \times F_1'$ one-one onto $J_3 = A^*/N^*$. Thus, A^* splits into $F_0' \times F_1' + N^*$.

Corollary 3.12. *Suppose Conditions 3-5 hold. Suppose that A/N is algebraic over K and that A/N is the unique free join of F_0 and F_1 in Condition 5. If $F_0 \times F_1$ has no nilpotent elements, then A^* splits with respect to N^*.*

Proof. Since $f|F_0 \times F_1$ is the canonical homomorphism of

$F_0 \times F_1$ onto $(A/N, \tau_0, \tau_1)$, we have that Ker $f|F_0 \times F_1$ is a nil ideal. This follows since A/N is a unique free join of F_0 and F_1. See Zariski and Samuel (5), page 195. Since $F_0 \times F_1$ has no nilpotent elements, $f|F_0 \times F_1$ maps $F_0 \times F_1$ one-one onto A/N. Thus, by the commutativity of Diagram 4, ν^* maps $F_0' \times F_1'$ one-one onto $J_3 = A^*/N^*$. Thus, $F_0' \times F_1'$ is a coefficient integral domain for A*.

Recall that Remark 2.15 gives some sufficient conditions for $F_0 \times F_1$ to have no nilpotent elements or to be an integral domain.

C. Conditions for A* to Be a Quasi-local Algebra

We conclude this chapter by giving some sufficient conditions for A* to be a quasi-local algebra over K. It is assumed in this section that K has characteristic $p \neq 0$.

Lemma 3.13. Suppose that Condition 1 holds and that N is a maximal ideal of A. If N is nil, then N is the unique maximal ideal of A.

Proof. Suppose there exists a maximal ideal N' of A such that $N' \neq N$. Then $1 = n + n'$ where $n \in N$ and $n' \in N'$. Since N is a nil ideal, there exists an integer e such that $n^{p^e} = 0$. Thus, $1^{p^e} = n^{p^e} + n'^{p^e}$ or $1 = n'^{p^e} = n'n'^{p^e-1}$. Hence, n' has an inverse which is impossible since $N' < A$. Thus, N is the unique maximal ideal of A.

Remark 3.14. Suppose that Condition 3 holds and that N is a maximal ideal of A. If A/N is pure inseparable over K, and if N is a nil ideal, then A* is a quasi-local algebra over K.

Proof. There exist fields F_1 and F, $K \leq F_1 \leq F$, for which there exist isomorphisms τ and τ_1 such that $\tau F = A/N$ and $\tau_1 = F_1'$. Since A/N is pure inseparable over K, A/N is a free join (A/N, τ, τ_1,) of F and F_1. Hence, the hypothesis of Lemma 3.5 is satisfied. Furthermore, since A/N is pure inseparable over K, A/N is the unique free join of F and F_1. Thus, Ker f is nil and hence, Ker k is nil. Since N*/(N, 0) $\cong$ Ker k under q_5 (see Diagram 3) and since Ker k is nil, we have for any $n^* \in N^*$ that there exists an integer j such that $n^{*j} \in (N, 0)$, i.e., $n^{*j} = \sum_i n_i \times a_{1i}$ where $n_i \in N$ and $a_{1i} \in F_1'$. There exists an integer e such that $n_i^{p^e} = 0$ for all i in the above sum since N is a nil ideal. Thus, $(n^{*j})^{p^e} = \sum_i n_i^{p^e} \times a_{1i}^{p^e} = \sum_i 0 \times a_{1i}^{p^e} = 0$. Hence, n* is nilpotent. Thus, N* is a nil ideal. Since A/N $\cong$ A*/N* (see Diagram 3) and since A/N is a field, N* is a maximal ideal of A*. Thus, by a similar argument as in Lemma 3.13, N* is the unique maximal ideal of A*. Hence A* is a quasi-local algebra over K.

Remark 3.15. Suppose that Condition 3 holds and that A is a quasi-local algebra over K with unique maximal ideal N.

If A/N is pure inseparable over K, and if A is algebraic over K, then A* is a quasi-local algebra over K.

Proof. Let n be any element of N. Since A is algebraic over K, $k_m n^m + \ldots + k_1 n + k_0 = 0$ for some integer $m \geq 1$ and some elements $k_i \in K$, $i = 0, \ldots, m$. However, $k_0 = 0$ since otherwise n has an inverse. This follows since if $k_0 \neq 0$, then $n(k_m n^{m-1} + \ldots + k_1)(-k_0)^{-1} = 1$. Thus, n satisfies the equation $k_m n^m + \ldots + k_j n^j = 0$ where $k_j \neq 0$ and $1 \leq j \leq m$. Therefore, $n^j(k_m n^{m-j} + \ldots + k_j) = 0$. Clearly, $k_m n^{m-j} + \ldots + k_j \notin N$ since $k_j \notin N$. Therefore, $k_m^{m-j} + \ldots + k_j$ has an inverse. Hence, $n^j = 0$. Therefore, N is a nil ideal. Thus, the desired result follows from the preceding remark.

Remark 3.16. Suppose Conditions 3 and 4 hold. If N is a maximal ideal in A, and if A* has only one proper prime ideal, then A* is a quasi-local algebra over K.

Proof. Consider the ideal $N^* < A^*$ in Diagram 3. Since A/N is a field and $A/N \cong A^*/N^*$, N* is a maximal ideal in A*. Since N* is thus a prime ideal, it is unique. Therefore, A* is a quasi-local algebra over K.

IV. EXAMPLES

In this chapter, we give some examples illustrating Theorem 2.5. Throughout this chapter, J_p stands for the integers mod p, a prime, K denotes the field $J_p(s, t)$ where s and t are independent indeterminates over J_p, and $F_0 = K(a)$, $F_1 = K(c)$ where $a^{p^2} = s$ and $c^{p^2} = st^p$. Since $K(a^p) < K(a)$ is K-isomorphic to $K(c^p/t) < K(c)$, we can identify these two fields when convenient. When this identification is made, we always denote a^p and c^p/t by d and K(d) by K_0.

Lemma 4.1. The tensor product $K_0(a) \times K_0(c)$ over K_0 is a field. Furthermore, $K_0(a) \times K_o(c)$ is K_0-isomorphic to $K_0(a, c)$.

Proof. Consider $K_0(a, c)$ as a field composite (relative to K_0) of $K_0(a)$ and $K_0(c)$, say $(K_0(a, c), \tau_0, \tau_1)$. Let f be the canonical homomorphism of $K_0(a) \times K_0(c)$ onto $(K_0(a, c), \tau_0, \tau_1)$. Since $K_0(a) \times 1$ and $1 \times K_0(c)$ are linearly disjoint over K_0, the dimension of $K_0(a) \times K_0(c)$ over K_0 is p^2. Clearly, the dimension of $K_0(a, c)$ over K_0 is p^2. Thus, considering $K_0(a) \times K_0(c)$ and $K_0(a, c)$ as finite dimensional vector spaces over K_0, we see that f is one-one. Therefore, $K_0(a) \times K_0(c)$ is a field.

Lemma 4.2. The tensor product $K(a) \times K(c)$ over K is a quasi-local algebra over K with unique maximal ideal

$N = (a^p \times 1 - 1 \times c^p/t)$.

Proof. Consider K(a, c) as a field composite (relative to K) of K(a) and K(c), say $(K(a, c), \tau_0, \tau_1)$. $(K(a, c), \tau_0, \tau_1)$ is a unique free join since K(a, c) is pure inseparable over K. Thus, K(a) x K(c) has only one maximal ideal. See Pickert (4). Hence, K(a) x K(c) is a quasi-local algebra over K. However, Ker f, where f is the canonical homomorphism of K(a) x K(c) onto $(K(a, c), \tau_0, \tau_1)$ is a maximal ideal in K(a) x K(c). Thus, Ker f is the unique maximal ideal of K(a) x K(c).

Let us show that N = Ker f. Clearly, $N \leq$ Ker f. The dimension of K(a) x K(c) over K equals the dimension of K(a) x K(c)/Ker f over K plus the dimension of Ker f over K. The dimension of K(a) x K(c) over K is p^4 and the dimension of K(a) x K(c)/Ker f over K is p^3. Thus, the dimension of Ker f over K is $p^4 - p^3$. Therefore, if we show that the dimension of N over K is $p^4 - p^3$, then N = Ker f.

Consider the collection of elements $\{(a \times 1)^i(1 \times c^j)n^m\} \leq N$ where $n = a^p \times 1 - 1 \times c^p/t$ and where $i = 0, \ldots, p^2 - 1$, $j = 0, \ldots, p - 1$, and $m = 1, \ldots, p - 1$. There are $p^4 - p^3$ elements in this collection. Suppose this collection is linearly dependent over K. Then there exist elements k_{ijm}, not all zero, in K such that $\sum_{ijm} k_{ijm}(a \times 1)^i(1 \times c)^j n^m = 0$. The collection $\{(a \times 1)^i(1 \times c)^j\}$ is linearly independent over K mod Ker f since K(a) x K(c)/Ker f $\cong$ K(a, c) and since the collection $\{a^i c^j\} \leq K(a, c)$ is linearly independent over K. Thus, there exists an integer m_0 in the above sum such that $\sum_{ij} k_{ijm_0}(a \times 1)^i(1 \times c)^j \neq 0$. This follows since if $\sum_{ij} k_{ijm}(a \times 1)^i(1 \times c)^j = 0$, $m = 1, \ldots, p - 1$, then $k_{ijm} = 0$ for all i, j, m contrary to assumption. Let m_0 be the first such integer. Multiplying both sides of the equation, $\sum_{ijm} k_{ijm}(a \times 1)^i(1 \times c)^j n^m = 0$, by n^{p-m_0-1}, we have $(\sum_{ij} k_{ijm_0}(a \times 1)^i(1 \times c)^j)n^{p-1} = 0$. Since $n^{p-1} \neq 0$, $\sum_{ij} k_{ijm_0}(a \times 1)^i(1 \times c)^j$ is a zero divisor. Every zero divisor of $K(a) \times K(c)$ is nilpotent since $(K(a, c), \tau_0, \tau_1)$ is a unique free join of K(a) and K(c). Thus, $\sum_{ij} k_{ijm_0}(a \times 1)^i(1 \times c)^j \in \text{Ker } f$. Therefore, $k_{ijm_0} = 0$ for all i and j. Thus, $\sum_{ij} k_{ijm_0}(a \times 1)^i(1 \times c)^j = 0$. Hence, we have a contradiction. Therefore, $\{(a \times 1)^i(1 \times c)^j n^m\}$ is linearly independent over K. Thus, the dimension of N over K is $p^4 - p^3$. Hence, N = Ker f.

Lemma 4.3. The tensor product $K(a) \times K(c)$ (over K) does not split with respect to $N = (a^p \times 1 - 1 \times c^p/t)$.

Proof. Suppose that $K(a) \times K(c)$ splits with respect to N. Then $K(a) \times K(c) = F + N$ where F is a field containing K. Thus, there exist elements $a', c' \in F$ such that $a \times 1 = a' + n_1$ and $1 \times c = c' + n_2$ where $n_1, n_2 \in N$. Thus, $a^p \times 1 = a'^p$ and

$1 \times c^p = c'^p$. Hence, $a^p \times 1 - 1 \times c^p/t = a'^p - c'^p/t \in F$. This is impossible since $a^p \times 1 - 1 \times c^p/t$ is nilpotent and F is a field. Thus, $K(a) \times K(c)$ does not split with respect to N.

For another proof, see Montgomery (3).

We are now prepared for our first example. Let $A = (K_0(a) \times K_0(c))[x]$, a polynomial domain with indeterminate x, where the tensor product is over K_0. Let $N = (x)$, the ideal in A generated by x. Clearly, $K_0(a) \times K_0(c) < A$ maps one-one onto A/N under the natural homomorphism of A onto A/N. However, let us use the concepts used in Theorem 2.5. There exist K-isomorphisms h_0, h_1 of K(a), K(c) respectively onto $K_0(a) \times 1$, $1 \times K_0(c) < A$. Let $A' = [(K_0(a) \times 1, 1 \times K_0(c)]$ $(= K_0(a) \times K_0(c))$. Let h be the universal homomorphism of $T = K(a) \times K(c)$ (over K) onto A'. By Lemma 4.1, $A/N \cong K_0(a, c)$. Thus, there exist K-isomorphisms τ_0, τ_1 of K(a), K(c) respectively into A/N such that A/N is the composite of K(a) and K(c) (relative to K). Let f be the canonical homomorphism of T onto $(A/N, \tau_0, \tau_1)$. Since A/N is pure inseparable over K, A' has a commutative diagram by Corollary 2.9. Since A/N is a field, Ker f is a maximal in T. Hence, Ker f must be the unique maximal ideal of T by Lemma 4.2. Thus, T does not split with respect to Ker f by Lemma 4.3, but there exists an ideal $N' \leq \text{Ker } h$ such that T/N' splits with respect to Ker f/N', namely N' = Ker h =

Ker f. That is, we have an example of an algebra A over K which splits with respect to a prime ideal N (maximal in this case). We have the existence of an $A' < A$, $K_0(a) \times K_0(c)$, for which there exists a commutative diagram, and we have the existence of an ideal N' such that T/N' splits with respect Ker f/N' although T itself does not split with respect to Ker f.

The next example is one of an algebra A over a field K which does not split with respect to a given ideal, but for which there exists an $A' \leq A$ which has a commutative diagram. As a preliminary, we need the following lemma.

Lemma 4.4. Let $A = K(a) \times K(c)$ over K. Then $A/N \cong K(a, c)$ where $N = (a^p \times 1 - 1 \times c^p/t)$.

Proof. Consider K(a, c) as a composite of K(a) and K(c) (relative to K), say $(K(a,c), \tau_0, \tau_1)$. In the proof of Lemma 4.2, we have that $K(a) \times K(c)/\text{Ker } f \cong (K(a, c), \tau_0, \tau_1)$ and that Ker f = N. Thus, $K(a) \times K(c)/N \cong (K(a,c), \tau_0, \tau_1) = K(a, c)$.

For the example, let $A = K(a) \times K(c)$. Let $N = (a^p \times 1 - 1 \times c^p/t)$. Since $A/N \cong K(a, c)$ by the preceding lemma, there exist K-isomorphic images of K(a) and K(c) in A/N. The field composite of K(a) and K(c) (relative to K) in A/N, say $[\tau_0 K(a), \tau_1 K(c)]$ must equal A/N since $A/N \cong K(a, c) \cong [\tau_0 K(a), \tau_1 K(c)]$ and since A/N is not isomorphic to a proper subfield. Hence, let f be the canonical homomorphism of $T = K(a) \times K(c)$ onto A/N. Let $A' = K(a) \times K(c) = A$. Since Ker $f = (a^p \times 1 - 1 \times c^p/t) =$ Ker g, $g = \nu$ where ν is the natural homomorphism of A onto A/N, there exists an automorphism α of A/N such that $f = \alpha gh$ where h is the universal homomorphism of T onto A'. (Note that α is the identity since A/N is pure inseparable over K and that h is the identity since $T = A'$.) Thus, we have a commutative diagram for A', but A does *not* split with respect to N. See Lemma 4.3.

It is reasonable to ask the question that is it possible to have an algebra A over a field K which splits with respect to a prime ideal N, but such that there exists an $A' \leq A$ for which there exists a commutative diagram and there does not exist an ideal $N' \leq$ Ker h for which T/N' splits with respect to Ker f/N'. Our final example gives an affirmative answer to the question. Before beginning, we need the following lemmas.

Lemma 4.5. Let $A = K(a) \times K(c)$ over K. Then $A/N \cong K_0(a) \times K_0(c)$ over K_0 where $N = (a^p \times 1 - 1 \times c^p/t)$.

Proof. By Lemma 4.1, $K_0(a) \times K_0(c) \cong K_0(a, c)$. Since $K_0(a, c) = K(a, c)$ and $K(a) \times K(c)/(a^p \times 1 - 1 \times c^p/t) \cong K(a, c)$ (see the proof of Lemma 4.4), $A/N \cong K_0(a) \times K_0(c)$.

Lemma 4.6. Let $B = (K_0(a) \times K_0(c)) \times (K_0(a) \times K_0(c))$ over K_0, K, K_0. Then B is a quasi-local algebra over K and $B/M' \cong K_0(a) \times K_0(c)$ where M' is the unique maximal ideal of

B.

Proof.

The proof that B is a quasi-local algebra is similar to the proof of Lemma 4.2 if we consider the field $K_U(a) \times K_0(c)$ as a composite of $K_0(a) \times K_0(c)$ and $K_U(a) \times K_0(c)$. Then, if f is the canonical homomorphism of B onto $K_0(a) \times K_0(c)$, $M' = \text{Ker } f$ as before. Hence, $B/M' \cong K_0(a) \times K_0(c)$.

For the example, let $A = (K(a) \times K(c)) \times (K_0(a) \times K_0(c))$ over K, K, K_0. Let $M = ((a^p \times 1 - 1 \times c^p/t) \times (1 \times 1), ((1 \times 1) \times 0))$. Then $A/M \cong (K_0(a) \times K_0(c)) \times (K_0(a) \times K_0(c)) = B$ by Zariski and Samuel (5, page 184), and by Lemma 4.5. Let M' be the unique maximal ideal of B. See Lemma 4.6. Then there exists an ideal $N > M$ in A such that $N/M \cong M'$. Thus, $A/N \cong (A/M)/(N/M) \cong B/M'$. By Lemmas 4.1 and 4.6, $B/M' \cong K_0(a) \times K_0(a, c) \cong K_0(a, c) = K(a, c)$. Thus $A/N \cong K(a, c)$. Hence, there are isomorphic images of K(a) and K(c) in A/N. Therefore, we may consider A/N as a composite (relative to K) of K(a) and K(c). Let $T = K(a) \times K(c)$. Let h be the universal homomorphism (isomorphism in this case) of T onto $(K(a) \times K(c)) \times (1 \times 1) = A' < A$. Let f be the canonical homomorphism of T onto A/N. Let $g = \nu | A'$ where ν is the natural homomorphism of A onto A/N. $\text{Ker } g = A' \cap N \geq A' \cap M \geq (a^p \times 1 - 1 \times c^p/t) \times (1 \times 1)$. Since $(a^p \times 1 - 1 \times c^p/t)$

is the unique maximal ideal of $K(a) \times K(c)$, Ker $g \leq (a^p \times 1 - 1 \times c^p/t) \times (1 \times 1)$. This follows since Ker $g \leq A'$ is isomorphic to an ideal in $K(a) \times K(c)$ which is contained $(a^p \times 1 - 1 \times c^p/t)$. Thus, Ker $g = (a^p \times 1 - 1 \times c^p/t) \times (1 \times 1)$. Therefore, $A'/\text{Ker } g \cong K(a) \times K(c)/(a^p \times 1 - 1 \times c^p/t) \cong K_0(a) \times K_0(c) \cong A/N$. Thus, g maps A' onto A/N. Since Ker $f = (a^p \times 1 - 1 \times c^p/t) \cong$ Ker g, there exists an automorphism α of A/N such that $f = \alpha g h$. Thus, A' has a commutative diagram, but A' does not split with respect to Ker g by Lemma 4.3. Clearly, there does not exist an ideal $N' \leq$ Ker $h = (0)$ such that T/N' splits with respect to Ker f/N'.

Consider the field $F = (1 \times 1) \times (K_0(a) \times K_0(c)) < A$. Clearly, $F \cong K_0(a) \times K_0(c) \cong A/N$. Thus, $\nu|F$ must map F one-one onto A/N since A/N is not K-isomorphic to a proper subfield. Therefore, A splits with respect to N.

V. CONCLUSION

Diagram 2 clarifies the relationship between the three essential elements in our method of attack to test the splitting of A. These elements are (i) the pair of integral domains F_0, F_1 of which we consider A/N to be the composite, (ii) the tensor product $T = F_0 \times F_1$, and (iii) the collection of ring composites in A which are generated by isomorphic images of F_0 and F_1.

Once we decide upon which pair of integral domains F_0, F_1 of which we consider A/N to be the composite, this pair remains fixed. The tensor product $T = F_0 \times F_1$ is then formed and it remains fixed. In A, there may be many isomorphic images of F_0, F_1 which we may choose from in order to generate a ring composite A'. The correct choice of images is one which leads to a commutative diagram for A'.

With respect to Theorem 2.5, the splitting of A is invariant under the choice of integral domains F_0 and F_1 of which we consider A/N to be the composite, for suppose we detect the splitting of A through a particular choice of integral domains. Then using the result that A splits, Conditions i, ii, and iii in Theorem 2.5 hold for any other choice of integral domains of which we consider A/N to be composite (by Theorem 2.5 itself).

Following Theorem 2.5, we have some sufficient conditions for the splitting of A. Thus, it may be advantageous to consider A/N as a particular composite of two integral domains in order to detect the splitting of A by the theory developed following Theorem 2.5.

In Chapter III, the relation between the splitting of A and the splitting of A* is a relation between split exact sequences of modules. Theorem 3.6 says that if the short exact sequence

$$(0) \to N \to A \to A/N \to (0)$$

is split exact, then the short exact sequence

$$(0) \to N^* \to A^* \to A^*/N^* \to (0)$$

is split exact. Theorem 3.10 says that if the short exact sequence

$$(0) \to \mathrm{Ker}(f|F_0 x F_1) \to F_0 x F_1 \to F_0 x F_1/\mathrm{Ker}(f|F_0 x F_1) \to (0)$$

is split exact, then the short exact sequence

$$(0) \to N^* \to A^* \to A^*/N^* \to (0)$$

is split exact.

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