INTRODUCTION

A configuration of particular importance in quantitative NDE is the diffraction of transient plane elastic waves by a semi-infinite crack. A related problem of equal importance is scattering by a partially closed crack, since this system may also be encountered in practice.

Various analytical methods are available to solve the forward problem of the diffraction of steady state plane waves by a stress-free half plane (semi-infinite mathematical crack); the two dimensional problem was first treated by Maue[1] and the three dimensional problem was recently presented by Achenbach et al[2]. The mixed boundary value problem is solved by an integral equation and is reduced to algebraic equations using the Weiner-Hopf technique. These works form the canonical solutions for the “high frequency” Geometric Theory of Diffraction (GTD).

Studies of the scattering of elastic waves by semi-infinite cracks, with partially contacting surfaces, have been reported by several investigators. Both analytical and experimental studies have been presented by Thompson and Fiedler[3] and Buck et al[4]. These studies have considered the specular reflection and transmission of plane waves at the contacting faces of cracks.

This paper reports a study, which gives a full solution, both in the near and far field, using numerical modelling, of the two dimensional diffraction of a plane wave by a crack with smooth non-interacting faces. This approach is then extended to consider scattering by partially closed cracks. The diffraction of plane waves by a half plane may be considered as a special case of scattering of plane waves by a partially closed crack. The incident waves treated are compressional (C) and shear-vertical (SV) impulses. The initial-boundary value problem is addressed by an explicit finite difference scheme.

Calculations of the scattered wavefield, for various angles of incidence of either a C or a SV disturbance, are presented as “numerical visualisations”. The time averaged radiation directivity patterns which give the angular distribution of energy of pulsed diffracted waves are also presented and compared with results given by GTD. Comparisons are
presented for the displacement field of an ideal and a partially closed crack for obliquely incident C and SV waves.

GOVERNING ASSUMPTIONS

Consider a semi-infinite mathematical crack which can be described as a vacuous strip and located in a Cartesian plane defined by \( x=0 \) and \( y>g>h>0 \) (see Figure 1), where \( h \) and \( g \) are lengths of the vacuous strip and partial closure, respectively. The medium containing the crack is assumed to be perfectly elastic, isotropic and homogeneous.

![Figure 1. Two dimensional geometry for incidence of a plane transient wave on a crack in an unbounded medium.](image)

The equations of motion for stress waves propagating in a perfectly elastic medium are

\[
\begin{align*}
U_{tt} &= a^2 U_{xx} + (a^2-b^2)W_{xy} + b^2 U_{yy} \\
W_{tt} &= a^2 W_{yy} + (a^2-b^2)U_{xy} + b^2 W_{xx}
\end{align*}
\]

where \( U=U(x,y,t) \) and \( W=W(x,y,t) \) are the horizontal and vertical components of displacement with respect to \( x-y \) axis, and \( a \) and \( b \) are the velocities of C and SV waves, respectively. In equations (1) and (2), the subscripts denote differentiation.

In the following, the massless spring layer approximation used in reference [3] is considered for partial closure at the contact zone, \( g \). Across this interface, the boundary conditions which must be satisfied are

\[
\begin{align*}
T_{xx}^+ &= T_{xx}^-(o^+,y,t) = T_{xx}^-(o^-,y,t) \\
T_{xy}^+ &= T_{xy}^-(o^+,y,t) = T_{xy}^-(o^-,y,t)
\end{align*}
\]

and

\[
\begin{align*}
T_{xx} &= g_x [U(o^+,y,t)-U(o^-,y,t)] \\
T_{xy} &= g_y [W(o^+,y,t)-W(o^-,y,t)]
\end{align*}
\]

where \( T \) is the stress and \( g_x \) and \( g_y \) are the horizontal and vertical interfacial stiffnesses per unit area, and the + and - signs denote the right and left sides of the closed region.
The boundary conditions (5) and (6), in terms of displacements, are

\[
T_{xx}/p = (a^2 - 2b^2)\dot{w} + b^2\dot{u}_x = g_x/p [U^+ - U^-] \quad (7)
\]
\[
T_{xy}/p = b^2(U_y - \dot{w}) = g_y/p [W^+ - W^-] \quad (8)
\]

where \( p \) is the mass density.

In equations (7) and (8), it is implicit that when \( g_x = g_y = 0 \)
stress free boundary conditions are obtained. These conditions are used to describe the displacements at the vacuous strip.

These equations, subject to the initial plane wave conditions, are solved by the finite difference method.

Transient plane waves

It is required that the input motion satisfies the equations of motion in infinite space.

Let transient plane C or SV waves be incident on the semi-infinite crack at an angle \( \theta \). The elastodynamic field due to the incident wave may be described by the two dimensional transient scalar potentials \( \varnothing \) and \( Y \) where

\[
\varnothing = A \delta(t - (x \cos \theta + y \sin \theta)/a) \quad (9)
\]
\[
Y = B \delta(t - (x \cos \theta + y \sin \theta)/b) \quad (10)
\]

where \( \delta(\theta) \) is the Dirac delta function.

Impulse approximation

The form \( \delta(\theta) \) is an improper function. For this function to possess continuous higher order derivatives, the procedure of Ilan et al [5] is applied to obtain a fifth order smoothed wavelet.

Reduction to particle displacements

The incident waves can be described by their corresponding particle displacements with the displacement-potential relation

\[
U(x,y,t) = -d\varnothing/dx + dY/dy \quad (11)
\]
\[
W(x,y,t) = -d\varnothing/dy - dY/dx \quad (12).
\]

NUMERICAL ANALYSIS

The scattering of elastic waves by the semi-infinite crack is simulated using a finite difference method. The finite difference algorithms are obtained by discretizing the equations of motion, (1) and (2), and the boundary conditions, (7) and (8), for both the temporal and spatial derivatives [6,7].

The displacements at the first two time frames are calculated using equations (11) and (12) for a 200x200 spatial grid. For subsequent time frames, the finite difference algorithms of the equations of motion, the stress-free, closure-interface and absorbing boundary conditions, are used in the calculations.
RADIATION PATTERN

It is required that the time averaged angular distribution of energy density, \( E(r,V,\theta) \), of outward going waves is evaluated on a circle centred on the crack tip. The displacements \((U)\) rather than the potentials \((\phi\) and \(V\)) are considered. The angle \((V)\) is defined in an anticlockwise sense from the crack.

The relationship between the Cartesian coordinate system \((x,y)\) and a circular coordinate system \((r,V)\) is

\[
x = r \sin V, \quad y = r \cos V, \quad 0 < V < 2\pi
\] (13).

Equation (13) is used to locate the effective points of observation on the circle. At these points, the time averaged energy density \( E(r,V,\theta) \) is evaluated, and is defined as

\[
E(r,V,\theta) = \frac{1}{T} \int_{T}^{T+\pi} \frac{|U_s(x,y,t)|^2}{|U_i(x,y,t)|^2} dt
\] (14)

where the subscripts \(s\) and \(i\) denote the scattered and incident fields, respectively.

Numerically, the displacements, in equation (14), are determined with bi-cubic spline interpolation because the effective points of observation may not necessarily be at the discrete grid locations.

RESULTS

Figure 2a shows the impulse source in the time domain for waves incident at \(\theta = \pi/2\). The energy density spectrum of the impulse, calculated by application of a discrete Fourier transform technique to the time domain displacement profile, is shown as Figure 2b. This impulse source has been calculated for steel \((a=5900 \text{ m/s} \quad \text{and} \quad b=3150 \text{ m/s})\) with a spatial extent of 20 nodes per wavelength at 5 MHz. The time increment, \(dt\), was chosen to fulfill the von-Neumann stability criterion in the form derived by Alterman and Loewenthal[7]

\[
dt < d(a^2 + b^2)^{-1/2}
\] (15)

where \(d\) is the square grid increment on the \(x-y\) plane, \(d=dx=dy\).

Figure 2. Initial condition; a) smoothed Dirac delta function and b) its energy density spectrum.
Diffraction by a half plane

The theoretically calculated wavefields, which result when a plane wave interacts with a semi-infinite crack, may be represented in various ways for each time frame. A typical sequence of numerical visualisations, calculated at one time frame, is presented as Figure 3. In this figure, a plane transient compressional wave is incident on a semi-infinite crack with $\theta = 20$ degrees. Two plots are presented for each type of visualisation. Figures 3a and 3c show the complete wavefield and Figures 3b and 3d indicate diffraction. Figures 3b and 3d are obtained by subtracting the displacements of the incident wave, propagating in infinite space, from the total scattered field as seen in Figures 3a and 3c. A linear scale factor of ten is applied to the calculated displacements of Figures 3b and 3d so they can be clearly observed.

In Figure 3, five principal diffracted components are observed: three plane waves and two cylindrical waves. Two of the plane waves are the specularly reflected (C) and mode-converted (SV) wave associated with the specular reflection. The third plane wave is the incident (C) wave.

![Figure 3](image-url)

Figure 3. Transient plane compressional wave diffraction by a semi-infinite crack ($\theta = 20$ degrees; a) and b) displacement field plots, c) and d) isometric projections of the magnitude of the displacement field.
which gives rise to the shadow. The two cylindrical waves (C and SV)
represent diffraction. The secondary components are surface Rayleigh waves
and the head waves trailing the cylindrical SV and C waves, respectively,
at the surfaces of the crack.

The radiation patterns of plane wave diffraction by a half plane,
umerically observed on a circle centred on the tip, are presented in
Figure 4. The angular distribution of energy density is evaluated using
equation (14). The prominent features in this figure are due to the plane
waves. Each of these prominent features show diffraction broadening
because of the superposition of the cylindrical and plane waves.

Comparison with analytical theory

A comparison of data between the present approach and
analytical theory [1,2] is presented in Figure 5. For ease of comparison
the same configuration is used. The data from the present study is obtained
by application of a discrete Fourier transform to a time window (40dt)

![C wave window](image1)

![SV wave window](image2)

![C wave window](image3)

![SV wave window](image4)

Figure 4. The angular distribution of energy density $E(r,V,\theta)$ on a circle,
centred on the tip, for $\theta=45$ degrees. The quadrature limits are defined as
$(t-20dt)<t<(t+20dt)$ and $t=r/c$ where $c=a,b$. a) and b) C wave incidence, c)
and d) SV wave incidence.
on a semi-circle of observation ($\psi=90$ to $270$ degrees) with radius $r=80d$. The steady state response was evaluated at $\lambda_a=20d$, which corresponds to $k_r=8\pi r$ and $k_b=8\pi a/b$. Here, $k$ is the wavenumber and $r$ is the distance from the tip of the semi-infinite crack.

In Figure 5, the comparison of angular displacement diffraction coefficients is for incidence of compressional and shear vertical plane waves at $\theta=45$ degrees. In general, the data from the present study show good agreement with the predictions of analytical theory. However, it should be noted that there are differences between the present study and analytical theory especially at the dips in Figure 5.

Scattering by a partially closed crack

The boundary conditions required to calculate the displacement field for a partially closed crack, which is illustrated in Figure 1, are given as equations (7) and (8). In these equations, the massless spring constant has the limits $0<g/p<\infty$. When these conditions are used in a finite difference method, it is implicit that discrete particle contact is simulated and that an explicit numerical solution is obtained.

Figure 5. Comparison of displacement diffraction coefficients, $|U_c^{c}(kr,\psi,\theta)|$, calculated using finite differences (solid line) and GTD (broken line) for $\theta=45$ degrees. Here $a,b=C,SV$ and $a,b$ denote the incident and diffracted wave respectively. a) $|U_c^c|$, b) $|U_c^{sv}|$, c) $|U_v^{sv}|$ and d) $|U_v^{sv}|$. 
In the present study, a priori information to define the local crack opening displacement, $\Delta u = y - \overline{y}$, is not required; in fact, $\Delta u = 0$ at the initial condition (equations (11) and (12)) and when the incident stress wave is in the vicinity of the partially closed region, $\Delta u$ inherently follows the transient stress cycle. This approach has a significant advantage over analytical methods in that both the near and far field displacement data, as a function of time, are available.

A comparison of the displacement field of an ideal ($g_y = g_y = 0$) and a partially closed crack ($g_y/p = g_y/p = 10^{-1}$ ms$^{-1}$) for C and SV wave incidence ($\theta = 45$ degrees) is presented in Figure 6. In this figure, the partially closed crack gives rise to scattering of the incident plane waves, and the scattering is proportional to the "mark-space" ratio of h.

Figure 6. Comparison of scattered displacement field of an ideal and partially closed crack ($\theta = 45$ degrees). a) and b) C wave incidence c) and d) SV wave incidence.
and g. Furthermore, the transmission of the incident wave into the shadow region from the partially closed zone is clearly observed by comparison with the ideal case.

The comparison of the magnitude of the back-scattered displacements, plotted as a function of time, between the ideal and partially closed case is presented in Figure 7. This comparison indicates the complexity of data which is available when a partially closed case is encountered. It warrants the conclusion that when an experimentalist observes interference in diffraction data, it would be natural to assume that a partially closed crack may be under investigation. Also, if an inverse solution is available for the single scattering ideal case then the partially closed case is clearly a multi-parameter inverse scattering problem.

![Figure 7](image_url)

Figure 7. Time domain traces of the back-scattered diffracted and scattered elastic waves sampled at \(-50d < x < 50d, y = -50d\) (\(\theta = 45\) degrees). a) C wave incidence on ideal crack, b) C wave incidence on closed crack, c) SV incidence on ideal crack and d) SV incidence on closed crack.

CONCLUSIONS

A solution is presented, which uses an explicit finite difference method, for determining the transient diffraction of obliquely incident plane C or SV waves by a semi-infinite crack. General results are presented for the displacements, in the near and far field, and the angular distribution of energy density in the far field.
This solution has been used to give the angular displacement diffraction coefficients, for a time harmonic case, which are compared with the coefficients obtained using an analytical solution and they show good agreement.

It is also demonstrated that the present approach can determine the scattered wavefield arising from the interaction of transient plane waves with a partially closed semi-infinite crack. One of the major advantages of using this approach, using a finite difference method, is that the dynamic local crack opening displacement information is not a prerequisite.

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REFERENCES


DISCUSSION

R. B. Thompson: You may have said this, and I missed it. I would think the features that you had observed would depend very much on the ratio of the wavelength to the dimensions g and h.

M. Punjani: Yes.

R. B. Thompson: In what region of parameter space were these calculations performed?

M. Punjani: What we have done is instead of actually taking the spring constant which depends on both g and h, which I think is what you are referring to, we have taken an arbitrary normalized spring constant value of ten to the power seven metres per second square. This work is just the beginning, and I hope to have further discussions with you about that.

R. B. Thompson: Okay.

From the Floor: Could you describe those boundary conditions in detail? You said "the imperfect and the stress free conditions." I didn't understand what those two boundary conditions were.
M. Punjani: The first boundary condition is the stress free boundary condition and it is used to describe the displacements at the free surface of a half space. The second boundary condition is for an imperfect interface. This boundary condition is used to model closure of two half spaces in contact. In general, a perfect interface has got continuity of stress and displacements, and for the imperfect interface, we have continuity of stresses but not of displacements. The constant of the imperfect boundary condition is a spring coefficient and its value, analytically, depends on g and h, which are the lengths of the open and closed regions of the crack.

R. C. Chivers: The fundamental understanding that calculations of this type produce is obviously invaluable, and they are very elegant pictures. Can I ask whether there are any practical directions of applications? For example, would it help us in our location of our transducers?

M. Punjani: Yes. If I may show a slide. The comparison which I did with the analytical work already shows how we can optimise the data for transducers which are placed on a free surface. For example, if we have a surface-breaking crack and we are interested in looking at the diffraction pattern on the remote free surface, then in that particular slide, we are looking at the two lobes which indicate that if we use a 45 degrees shear vertical wave transducer, we would have a very good experimental setup because most of the energy from the tip of the crack will be directed at 45 degrees. Now, if we have a broken crack, we can also generate these kind of coefficients and then we should be able to understand what transducer angles to use.

R. C. Chivers: Thank you.

R. B. Thompson: If I could just make one more comment. We have been very interested in this problem, as has been evident. Our interest is to use the scattering to study fatigue crack closure effects as a material science tool and there are two papers in this conference relating to that, so that is another possible application of this theory.

E. C. Teague: Your analytical theory, again, was the Geometrical Theory of Diffraction?

M. Punjani: Yes.

E. C. Teague: Relative to the earlier question about the limits of g and h, did that involve your comparison there as well?

M. Punjani: I suppose so, yes.

E. C. Teague: That would certainly set limits upon the validity of the analytical theory.

M. Punjani: I'm not sure if anybody has done enough analytical theory along these lines.

E. C. Teague: Okay.