By using dyadic Green's functions the electromagnetic field in an unflawed conducting half-space can be found for a general alternating current source distribution. A crack in the conductor gives rise to a secondary source which may be represented as a surface distribution of current dipoles. An integral equation for the secondary source distribution is derived and the dipole density evaluated numerically for a rectangular subsurface crack.

INTRODUCTION

A small subsurface defect such as a crack or cavity, in an otherwise homogeneous conductor carrying an induced time-harmonic current, produces the same effect as an oscillating current dipole of the site of the flaw. The dipole strength depends on the incident eddy current density and has been calculated by Burrows[1] for a number of different flaw shapes on the assumption that the defect is small compared with the skin depth δ. The aim in the present article is to show how one can determine the eddy current scattering by defects of arbitrary size by finding the equivalent surface distributions of current dipoles. No special restriction will be put on the magnitude of the skin depth but we are mainly concerned with subsurface defects close to a plane interface between air and the conductor.

Our approach is based on boundary integral equation methods which we apply to a half-space conductor. Boundary integral techniques are used extensively in elastodynamics and in electromagnetic theory for calculating scattered radiation. The central problem is to find the discontinuity in the electromagnetic field at the surface of the scatterer[2]. Physically, this discontinuity can be represented by an appropriate source distribution; for a barely open crack in a current carrying conductor, this source distribution is found to be a current dipole layer.

Once the dipole distribution has been evaluated it can be treated as a secondary source of the electromagnetic field, the primary source of excitation being for example, an eddy current probe. By operating on the primary and secondary sources with the correct integral operators, one can get the incident and scattered fields respectively, for both the conducting and non-conducting regions. Half-space Green's functions are available[3] for carrying out these operations but here we are not primarily concerned
with calculating the field everywhere. Nor shall we evaluate the response of the probe to the defect, though this can easily be found from the secondary source distribution using the reciprocity theorem[2]. Instead we shall focus on the problem of finding the discontinuity in the field at the flaw and the associated secondary source distribution.

After setting out the general formulation, the method is applied to the particular case of a rectangular subsurface crack in a plane perpendicular to the interface between the conductor and air. The incident field is arbitrary but the calculation assumes that the unperturbed field is uniform in the plane of the interface, and the skin depth is of the same order of magnitude as the dimensions of the defect.

**ELECTRIC FIELD IN THE CONDUCTOR**

A crack is here modelled as a non-conducting region of negligible thickness. Eddy currents induced by a time-harmonic external source current varying as the real part of $J \exp(-i\omega t)$ will flow around the defect as shown schematically in Figure 1. Assuming the permeability of the conductor and the flaw have the free space value, the tangential component of the magnetic field is continuous across the crack, hence

$$\hat{n} \times \mathbf{H}^+ - \hat{n} \times \mathbf{H}^- = \hat{n} \times \Delta \mathbf{H} = 0 \quad (1)$$

The plus and minus signs represent limiting values each side of the defect. $\Delta \mathbf{H}$ is the difference between the limiting values and $\hat{n}$ is a unit vector normal to the surface of the crack. The tangential component of the eddy current density, and hence the electric field, is not continuous across the crack but the discontinuity can be related to a dipole layer of surface density $\rho$. Thus

$$E^+_{t} - E^-_{t} = \Delta E = \frac{1}{\sigma} \nabla_{t} \rho \quad (2)$$

The suffix $t$ denotes a component normal to $\hat{n}$, $\nabla_{t}$ being the tangential gradient. Defining $\rho$ according to equation (2) entails an assumption because one cannot in general express a surface vector as the gradient of a scalar, specifically it is assumed that $\nabla_{t} \times (E^+_{t} - E^-_{t}) = 0$. This condition is equivalent to requiring that the normal component of the flux density is continuous across the defect, hence the assumption implicit in (2) is valid.

An important property of $\rho$ is the way in which it varies toward the edge of the flaw. The behavior is exemplified by the solution for an infinite crack given by Kahn, Spall and Feldman[4], in which the current on one side of its surface flows in the opposite direction to that on the other side. They note that the eddy current density increases as $(r)^{-1/2}$ as $r \to 0$, $r$ being the distance from the edge of the crack. This implies, according to equation (2), that $\rho$ varies as $r^{-1/2}$ for small $r$ and that it vanishes at the edge of the defect. Quantities analogous to $\rho$ in elastodynamics[5] and radiation scattering theory[6] exhibit similar behavior.

To set up an integral equation for $\rho$, the dyadic Green's theorem[7] is applied to a conducting half-space with the exclusion of a region enclosed by a surface $S$ surrounding the flaw. $S$ contains $S^+$ and $S^-$ on each side of the defect and a semicylindrical surface $S_e$ around the edge (Figure 1). From Green's theorem the electric field in the conductor is given by[7]

$$E(r) = E^i(r) - \int_S [\hat{n} \times E'(r')] \cdot \mathbf{v} \times \mathbf{G}(r, r') + i \omega \mu_0 [\hat{n} \times \mathbf{H}(r')] \cdot \mathbf{G}(r, r') dS' \quad (3)$$
$E^i(r)$ represents the field in the absence of the crack, in other words the incident field. The dyadic Green's function $G(r,r')$ satisfies

$$\nabla \times \nabla \times G(r,r') - k^2 G(r,r') = \mathbb{1} \delta(r,r')$$

(4)

where $k^2 = i\omega \mu_0$ and $\mathbb{1}$ is the unit dyad. $G(r,r')$ is conveniently expressed as the sum of two terms,

$$G(r,r') = G_o(r,r') + F(r,r')$$

(5)

$G_o$ is the dyadic Green's function for an unbounded domain given by

$$G_o(r,r') = \left(\mathbb{1} + \frac{1}{k^2} \nabla \nabla\right) \phi(r,r')$$

(6)

with

$$\phi(r,r') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

(7)

$F$ represents the partially reflected field due to the interface and must be included if the solution of (3) is to satisfy the proper interface conditions at the surface of the conductor.

Suppose we let $S$ collapse onto the open surface of crack $S_o$ (Figure 1). Since $\hat{n} \times H$ is finite at the edge of the crack, the integral over $S_o$ contain-
ing $\mathbf{n}\times\mathbf{H}$ vanishes in the limit. Taking note of the direction of $\mathbf{n}$ on $S^+$ compared with its direction on $S^-$, the remaining part of the $\mathbf{n}\times\mathbf{H}$-integral also vanishes in the limit because of the continuity condition, equation (1). $\mathbf{n}\times\mathbf{E}$ is singular at the crack edge but it is a $(r)^{-1/2}$ singularity, therefore the contribution from the $\mathbf{n}\times\mathbf{E}$-integral over $S_e$ again vanishes in the limit as the radius of $S_e$ goes to zero. Thus (3) reduces to

$$E(r) = E^i(r) - \int_{S_0} [\mathbf{n}\times\mathbf{E}(r')] \cdot \nabla' \times \mathbf{G}(r, r') dS'$$  \hspace{1cm} (8)

After substituting for $\Delta \mathbf{E}$ from (2), the integrand in (8) can be written as

$$\hat{\mathbf{n}} \times (\mathbf{\nabla p}) \cdot \nabla \mathbf{G} = \hat{\mathbf{n}} \cdot \nabla' \times (p \mathbf{\nabla G}) - p \hat{\mathbf{n}} \cdot \nabla' \times \mathbf{G}$$

Applying Stokes theorem and the condition that $\mathbf{p}$ vanishes at the edge shows that the integral of the first term is zero. Transforming the remaining term using (4) gives, for $r \neq r^*$,

$$E(r) = E^i(r) + i \omega \mu \int_{S_0} \mathbf{G}(r, r') \cdot \mathbf{G}(r, r') dS'$$  \hspace{1cm} (9)

where $\mathbf{p} = \mathbf{\hat{n}} \cdot \mathbf{p}$. Although the tangential component of $\mathbf{E}$ is discontinuous at the crack its normal component is continuous. In fact in the limit as the crack is approached the normal component of the electric field is zero since the defect under consideration does not allow current to flow across it. Hence

$$\hat{\mathbf{n}} \cdot \mathbf{E} = 0 = \hat{\mathbf{n}} \cdot E^i(r_0) + i \omega \mu \lim_{r \rightarrow r^*} \int_{S_0} \mathbf{G}^m(r, r') \mathbf{p}(r') dS'$$  \hspace{1cm} (10)

where $r_0 \in S$ and $\mathbf{G}^m = \hat{\mathbf{n}} \cdot \mathbf{G} \cdot \hat{\mathbf{n}}$. (10) is a convenient scalar equation for the determination of $\mathbf{p}$.

**CALCULATION OF THE DIPOLE DENSITY**

A numerical solution of (10) has been found by using the method of moments[8] applied to a rectangular subsurface defect, Figure 2. The dipole density is expanded in terms of basis functions $\psi_\beta$, thus

$$\mathbf{p} = \sum_\beta \mathbf{p}_\beta \psi_\beta$$  \hspace{1cm} (11)

To keep the algorithm relatively simple, square pulse functions were used as a basis and delta functions $\delta(r-r_\alpha)$, were used as weights. This choice of weighting functions implies that a discretized version of equation 10;

$$E^i_\alpha + \sum_\beta G_{\alpha \beta} p_\beta = 0$$  \hspace{1cm} (12)

is to be satisfied at discrete points $r_\alpha$ on the surface of the flaw, the matching points being at the center of each square patch or pulse. In (12)

$$E^i_\alpha = \hat{\mathbf{n}} \cdot E^i(r_\alpha)$$  \hspace{1cm} (13)

and the matrix elements are given by an integral over the patch area $S_\beta$:

$$G_{\alpha \beta} = i \omega \mu \int_{S_\beta} \mathbf{G}^m(r_\alpha, r') dS'$$  \hspace{1cm} (14)

To calculate the self patch matrix element $G_{\alpha \alpha}$, part of the integration over the patch was done numerically and, in order to correctly take account of the singularity in the Green's function, part was done analytically. The analytical part $I_{\alpha \alpha}$, was calculated for a field coordinate displaced from the center of the $\alpha$-patch by $\mathbf{n}c$ and the integration taken over a small square region $S_s$, of side $2d$, such that $|kd|<<1$, (Figure 3). Thus
where
\[ C_{nn}^{nn}(r, \mathbf{r}) = \hat{n} \cdot C_{nn} \hat{n} = \left( 1 + \frac{1}{k^2} \frac{\alpha^2}{\beta^2} \frac{e^{ik|m-r'|}}{4\pi|m-r'|} \right) \]
expanding the exponential as a Taylor's series, integrating and taking the limit, gives
\[ I_{aa} = \frac{1}{\pi} \left( \sqrt{\frac{2}{k^2 d}} + d \ln \left( \sqrt{2} + 1 \right) + \frac{2ikd^2}{3} \right), \] (16)
to second order in \( d \). Adding \( I_{aa} \) to the result of integrating \( C_{nn}^{nn} \) over the remainder of the \( \alpha \)-patch gives a result which is insensitive to the choice of \( d \), as indeed it should be[9].

For this initial calculation the magnetic field above the conductor \( H_0 \), was assumed to be uniform and \( \hat{n} \cdot E_0 \) given by
\[ \hat{n} \cdot E_0^i = -\frac{\omega \mu_0 H_0}{k} \exp(-ikz) \] (17)
The dipole density at a flaw with \( 2a = 2b = \delta \) (Figure 2) was computed from equation (12) for a 400 x 400 matrix giving \( p \) on a 20 x 20 grid. The top edge of the defect was taken to be a distance \( \delta \) from the interface and at such a depth, \( \mathbf{E} \) has a negligible effect on dipole distribution (<1%). In fact \( \mathbf{E} \) was neglected in computing the results shown in Figure 4. It is found that the amplitude of the dipole distribution, Figure 4a, tends to zero, not quite at the edge of the crack, but at a point just beyond the edge. This form is discretization error is usual in this type of calculation[10]. It is interesting to note that the phase of \( p \), Figure 4b, follows very closely in antiphase to the incident field \( \mathbf{E}_i^0 \).
Figure 4: (a) Amplitude of the dipole density
(b) Phase in radians
The dipole density is, in itself, of theoretical rather than of experimental interest, but finding $p$ is a very important step in calculating the observable effects of cracks. As pointed out in the introduction, calculating the probe response to a defect is a simple matter once the dipole density is known.

CONCLUSION

It has been shown that the boundary integral method may be used to calculate the effect of subsurface defects on the distribution of eddy currents in conductors. This is a very powerful technique because it may be easily adapted to deal with a variety of defect shapes. The incident field is arbitrary and can be found from a completely independent calculation, such as a numerical finite element model, or it may be determined from experimental measurements on an eddy current probe. In addition, integral equation methods can be used to formulate inverse as well as forward problems and it may prove useful to define the inverse problem in terms of finding the dipole distribution $p$, or a closely related quantity.

ACKNOWLEDGEMENTS

The author would like to thank Dr. M. G. Jones for helpful discussions and Dr. D. Harrison for suggesting the use of dyadic Green's functions.

The work was carried out with the support of the Procurement Executive, Ministry of Defence, U.K.

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