I. INTRODUCTION

Quantitative NDE is, by its very nature, a discipline within which inverse source and scattering problems abound. Determining the shape of a scattering obstacle from the obstacle's scattering amplitude or the index of refraction distribution of an inhomogeneous object from scattered field measurements performed in one or more scattering experiments are examples of inverse scattering problems encountered in quantitative NDE. Determining the value of a wavefield (e.g., the pressure of a sound wave) over some surface from measurements of the wave at points removed from the surface is a special case of an inverse source problem. Pulse echo and transmission tomography, holographic imaging and emission tomography are further examples where the inverse source or scattering problems arise.

In this paper we examine certain fundamental limitations that apply to inverse source and scattering problems commonly encountered in NDE. Our goal is not to provide solutions to these problems but, rather, to review the limitations that are imposed by nature on the solutions that can be obtained. We will find that these limitations are of two types: (i) nonuniqueness and (ii) instability. Nonuniqueness refers essentially to the underlying ambiguity of a solution given perfect data while instability refers to the sensitivity of a solution to noise and measurement error. Both of these limitations are important in practice since they dictate an upper bound on the quality of a "solution" to a given inverse source or scattering problem.

The paper is organized as follows. In Section II we review the wave model that will form the basis for our discussion of inverse source and scattering problems. The inverse source problem is discussed in Section III and the inverse scattering problem in Section IV. Finally, Section V presents a summary of the paper.

II. THE WAVE MODEL

There are many types of wavefields that are encountered in NDE. In ultrasound applications the wavefield is either a pressure field or an elastic stress field while in optical, microwave and x-ray applications the wave is electromagnetic in nature. One can even consider cases, such as in quantum mechanical inverse scattering, where the wavefield of interest
is the Schrodinger wave function [1]. The common denominator in all of
these applications is that the underlying physical phenomenon of interest
is a wave that satisfies a wave equation.

In this paper we will limit our attention to wavefields that satisfy
the three-dimensional, scalar wave equation

\[ \nabla^2 - \frac{1}{2} \frac{\partial^2}{\partial t^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u(r, t) = q(r, t) \]  

(2.1)

where \( u(r, t) \) is the wavefield and \( q(r, t) \) the source generating the field. In this equation

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

is the square of the gradient operator with \( x, y, z \) being Cartesian
components of the position vector \( r \) and \( t \) the time. The quantity \( c_0 \) is
the velocity of wave propagation (assumed constant) in the medium in which
the source is imbedded.

The scalar wave equation (2.1) is a good model for many NDE applica­
tions. It can be used, for example, in applications of optical holographic
and tomographic imaging [2,3] and in ultrasonic tomography of non-elastic
objects [4]. Applications where the vector character of the wavefields
is important, such as in elastic wave and microwave tomography, can be
readily treated using vector wave generalizations of Eq. 2.1. Our general
conclusions concerning fundamental limitations in inverse source and
scattering problems that we obtain with the simpler scalar wave equation
(2.1) apply also to the vector wave equations with minor modification.

If we introduce the Fourier transforms of the source and wavefield
according to the definitions:

\[ U(r, \omega) = \int dt \ u(r, t) \ e^{i\omega t} \]  

(2.2a)

\[ Q(r, \omega) = \int dt \ q(r, t) \ e^{i\omega t} \]  

(2.2b)

we find that \( U(r, \omega) \) and \( Q(r, \omega) \) are related via the inhomogeneous
Helmholtz equation

\[ [\nabla^2 + k^2] U(r, \omega) = Q(r, \omega) \]  

(2.3)

where \( k = \omega/c_0 \) is the wavenumber. The scalar wave equation (2.1) and the
Helmholtz equation (2.3) are, of course, completely equivalent, the solu­
tion of one being related to the other through a Fourier transform.

However, although they are mathematically equivalent there are certain
reasons why it is preferable to work with the Helmholtz equation in
investigations into inverse source and scattering problems [4] and, for
this reason we will work in the frequency domain in this paper and, thus,
employ the Helmholtz equation rather than the wave equation as the
fundamental wave model.

The source term \( Q(r, \omega) \) can be of two possible types: (i) a primary
source or (ii) a secondary source. A primary source is characterized by
the property that it is independent of the wavefield \( U(r, \omega) \) that it
generates. Examples of primary sources are thermal radiators, acoustic
noise sources and ultrasound transducers. Secondary sources, on the other
hand, depend directly on the wavefield they generate. An example of a
secondary source is a scatterer characterized by a scattering potential \( V(r,\omega) \). For this case the source term is related to the scattering potential via the equation

\[
Q(r,\omega) = V(r,\omega) U(r,\omega)
\]

where \( U \) is the total wave (incident plus scattered) generated in the scattering experiment.

In this paper we will define the inverse source problem to mean determining a primary source from its radiated wavefield. The inverse scattering problem consists of determining a scattering potential from the wavefield that it scatters in one or more scattering experiments. We address the inverse source problem in the following section and the inverse scattering problem in Section IV.

III. INVERSE SOURCE PROBLEM

In the inverse source problem we have as our goal the determination of a primary source \( Q(r,\omega) \) from measurements of the wavefield that it radiates. Since it is unreasonable to expect that we can make field measurements in the interior of the space region occupied by the source, we will limit the allowable measurements of the radiated field \( U(r,\omega) \) to the region of space exterior to the source region (which we will denote by \( \tau \)). The precise statement of the inverse source problem is then:

Given measurements of the radiated wavefield \( U(r,\omega) \) performed outside the source region \( \tau \) determine the source \( Q(r,\omega) \).

To begin our investigation into the uniqueness question for the inverse source problem it is worthwhile to compare the dimensionality of the unknown source with that of the available data. This dimensionality is simply the total number of independent variables upon which these quantities depend. An obvious requirement for uniqueness of solution to the inverse source problem is that the dimensionality of the data be equal or greater than that of the source. This requirement, which extends to virtually all inverse problems, is of such fundamental importance in the inverse source and scattering problems, that we will call it the Golden Rule:

The dimensionality of the data must exceed or equal the dimensionality of the unknown in order for the solution to and inverse source or scattering problem to be unique.

To apply this rule to the inverse source problem we must compute the dimensionality of the source and data. In general, a source will possess a dimensionality of four: three degrees of freedom associated with the position vector \( r = (x,y,z) \) and a fourth associated with the frequency \( \omega \) (or, equivalently, the time \( t \)). [An exception to this result occurs when the source possesses less than three space dimensions; e.g. when the source is constrained to lie on a place].

The radiated field if specified over all of space would also have a dimension of four. However, we are allowed measurements of this field only over the region of space exterior to the source region \( \tau \). Throughout this restricted region of space the radiated field has dimensionality of only three. This conclusion follows at once from the well known result that a solution to the inhomogeneous Helmholtz equation that obeys the radiation condition is uniquely determined everywhere outside the source region by its boundary value over any closed surface that completely surrounds the source region [5]. Since any such surface can be
characterized by only two independent variables, the wavefield over the surface, and hence throughout the region of space exterior to \( T \), has dimension equal to three (two space dimensions and one for the frequency).

We are then left with the result that the unknown source has a higher dimensionality than the data. It follows immediately from the Golden Rule that the inverse source problem will not possess a unique solution. This result holds even if the data were perfect; i.e., even if we knew the field exactly at every point outside the source region.

Although the inverse source problem does not admit a unique solution we can, nevertheless, seek a solution to the problem. The solution we seek should have the property that it be consistent with the data (e.g., that it radiate a field that is equal to the original field at all the measurement points) and satisfy any a priori constraints known to be satisfied by the source. The solution obtained will often times yield valuable information concerning the true source especially if a number of constraints are available to limit the ambiguity of the solution.

In the above discussion we have made no mention of noise or measurement error. These error sources will be present in any practical application and will further limit our ability to solve the inverse source problem beyond that imposed by the inherent non-uniqueness discussed above. The effect of these errors on our ability to solve the inverse source problem can be determined by examining the stability of a solution with respect to minor variations of the data. If a small variation in data causes a large variation in the solution we say the solution is unstable while a comparable (or smaller) variation in the solution for a given variation in the data corresponds to a stable solution. Clearly, we should require that our solutions be stable.

In order to examine the stability question for the inverse source problem we must decide on a measurement configuration. We will take the configuration to be that shown in Fig. 1 consisting of two parallel planes over which the radiated field is measured. Minor variations to this geometry (such as a single measurement plane or a curved measurement surface) will not significantly affect our conclusions concerning the stability of solutions to the inverse source problem.

By employing the so-called angular spectrum representation [6] of the radiated Fourier transform of the radiated field over the measurement planes illustrated in Fig. 1 is related to the three-dimensional spatial Fourier transform of the source \( Q(\mathbf{r}, \omega) \) as follows:

\[
\tilde{Q}(Kx \hat{y} z, \omega) = \gamma e^{-i y z_0} \tilde{U}(Kx \hat{y} z, \omega) ,
\]

where \( K = Kx \hat{x} + Ky \hat{y} \) is a two-dimensional wave vector and

\[
\gamma = \begin{cases} 
\sqrt{k^2 - K^2} & \text{if } |K| \leq k \\
+ i \sqrt{k^2 - K^2} & \text{otherwise}.
\end{cases}
\]

Here, \( \hat{x}, \hat{y}, \hat{z} \), are unit vectors in the \( x,y,z \) directions, and

\[
\tilde{U}(Kx \hat{y} z, \omega) = \int dx dy \, U(\mathbf{r}, \omega) \bigg|_{z = z_0} e^{-iKr}
\]
Fig. 1. Measurement configuration for the inverse source problem. The radiated field is measured over the planes located at $z = \pm z_0$.

is the two-dimensional spatial Fourier transform of the radiated field over the plane $z = \pm z_0$ and

$$\hat{Q}(\mathbf{v}, \omega) = \int d^3 \mathbf{r} \ Q(\mathbf{r}, \omega) \ e^{-i \mathbf{v} \cdot \mathbf{r}} , \tag{3.4}$$

is the three-dimensional spatial Fourier transform of the source.

Equation 3.1 related the two-dimensional spatial Fourier transform of the field over the measurement planes to the three-dimensional Fourier transform of the unknown source distribution. This equation forms the basis for most "solutions" to the inverse source problem and also underlies most inversion methods in linearized inverse scattering and diffraction tomography where it is known as the generalized projection slice theorem (see next section). In terms of this equation the inverse source problem reduces to determining the source Fourier transform for all real values of the three-dimensional spatial frequency vector $\mathbf{v}$ from its boundary value on the surface

$$\mathbf{v} = K \pm \gamma z \ . \tag{3.5}$$

For values of $K$ that are less than or equal to the wavenumber $k = \omega / c_0$, the quantity $\gamma$ is real and the points in $\mathbf{v}$ space that satisfy equation (3.5) lie on the surface of a sphere that is centered at the origin and that has a radius of $k$ (see Fig. 2). These values of $K$ are associated with the so-called homogeneous plane waves in the angular spectrum expansion of the radiated field [6]. For these $K$ values the inverse source problem then consists of determining the source transform for values of $\mathbf{v}$ lying off this spherical surface from the known boundary value of the transform as computed using (3.1). The source is then reconstructed by taking a three-dimensional inverse Fourier transform of the source transform so determined.

Again, we can apply the Golden Rule to see that a continuation of the source Fourier transform from its boundary value specified over the spatial frequencies defined in Eq. (3.5) is highly non-unique, i.e., for any given frequency $\omega$ there will exist an infinite number of source transforms that will reduce to the known boundary value over this set of
Fourier Space

Fig. 2. Spherical surfaces over which the two-dimensional spatial Fourier transform of the field on the measurement surfaces equals the three-dimensional spatial Fourier transform of the source.

spatial frequencies. This non-uniqueness continues to hold even if we impose the constraint that the source be contained in a finite volume $v$. This nonuniqueness is, of course, to be expected in view of our previous discussion.

The above discussion applies only for $K$ values for which $|K| = K < k$. For values of $K$ for which $K > k$ the quantity $\gamma$ defined in Eq. (3.4) is pure imaginary and the values of $\nu$ defined by Eq. (3.5) are no longer real and, thus, do not lie on the surface of a real sphere in $\nu$ space. These $K$ values correspond to the well-known evanescent plane waves in the angular spectrum expansion of the radiated field [6]. One is then led to consider the importance of these $K$ values to the inverse source problem. We should keep in mind that including these values of $K$ will not affect the uniqueness of the solution. The values of $\nu$ satisfying Eq. (3.5) have a magnitude equal to the wavenumber $k$ independent of the value of $K$. This set of points thus possesses only two degrees of freedom (for fixed $k$) which, by the Golden Rule, is not sufficient to guarantee uniqueness.

The use of the $K$ values corresponding to evanescent plane waves as an aid in solving the inverse source problem is severely limited for a number of reasons. First, is the mathematical question of how to incorporate information concerning the Fourier transform of the source evaluated at a set of complex spatial frequencies into the inversion process. A second limitation, and the one of prime importance to us here, concerns the stability of the computation of the source transform in Eq. (3.1) when $\gamma$ is pure imaginary.

To examine the stability of the computation of the source transform for values of $K > k$ let us replace $\gamma$ by $i|\gamma|$ in Eq. (3.5) where $|\gamma| = \sqrt{K^2 - k^2}$. We obtain

$$\nu = K \pm i|\gamma|^2$$

(3.6)

for the corresponding set of complex spatial frequencies over which the source transform is related to the transform of the field via Eq. (3.1). On replacing $\gamma$ by $i|\gamma|$ in Eq. (3.1) we then find that for these spatial frequencies

$$\tilde{Q}(\nu,\omega) = i|\gamma| e^{+|\gamma|z_0} \tilde{U}(K, \pm z_0, \omega)$$

(3.7)
We conclude from (3.7) that small variations in the field amplitude resulting from noise or measurement error become magnified by the factor $\exp(\gamma z_0)$ in the computation of the source transform by means of this equation. This factor grows exponentially with $\gamma$ and $z_0$ so that this computation becomes increasingly unstable as the distance of the measurement planes increase from the source and as the magnitude of the spatial frequency vector $K$ increases. This result is not surprising in view of the well known limitations of measuring evanescent plane wave components of a propagating wavefield [6].

The above discussion establishes that the source Fourier transform can be stably determined from the radiated field only for real spatial frequencies lying on the surface of a sphere, centered at the origin in spatial frequency space and having a radius equal to the wavenumber $k$. We should note, however, that if the location of the measurement planes is sufficiently close to the source then it is possible to measure at least some of the evanescent wave components of the field and, in so doing, determine values of the source transform at complex spatial frequencies given in Eq. (3.6). The use of this information in the actual inversion process for three-dimensional sources has not, as yet, been investigated and remains an open area for future research. However, for two-dimensional sources, e.g., planar sources, this information is readily incorporated into the inversion process as shown by Williams and Maynard [7].

We conclude this section with a brief discussion on nonradiating sources [8,9]. These sources, which we will denote by $Q_{nr}(r, \omega)$, have the property that they radiate fields that vanish everywhere outside the source region $T$. A necessary and sufficient condition for the vanishing of the field everywhere outside $T$ is that it vanish at all points on the two measurement planes illustrated in Fig. 1. This is equivalent to requiring that the spatial Fourier transform of the field over these two planes vanish. We then conclude from Eq. (3.1) that a necessary and sufficient condition for a source to be nonradiating is that

$$Q_{nr}(\gamma, \omega) = 0,$$  (3.8)

for all spatial frequencies $\gamma$ satisfying Eq. (3.5).

The condition (3.8) can be used to derive a simple expression for the most general nonradiating source [8]. These sources have been employed as models for elementary particles [8] and, of course, play a very important role in the inverse source problem [2,9]. Indeed, it is readily verified that the difference of any two solutions to an inverse source problem is a nonradiating source [2,9]. The nonradiating sources thus lie at the heart of the nonuniqueness issue for the inverse source problem. We will not discuss these sources further here but refer the interested reader to the literature [2,8,9] for a more detailed account.

IV. THE INVERSE SCATTERING PROBLEM

As discussed in Section II both primary and secondary wave sources are encountered in NDE applications. The inverse source problem deals with primary sources while the inverse scattering problem deals with secondary sources. Secondary sources are induced by a wavefield through the scattering process. The goal of the inverse scattering problem is then the characterization of these induced sources from the measurement of scattered field data obtained in one or more scattering experiments.

The simplest type of secondary (induced) sources are those that are characterized by a scattering potential according to Eq. (2.4). Although these are by no means the only types of induced sources that occur in
applications they form a very important class. Indeed, by setting the scattering potential

\[ V(\mathbf{r}, \omega) = k^2 \left[ 1 - n^2(\mathbf{r}, \omega) \right], \tag{4.1} \]

where \( n(\mathbf{r}, \omega) \) is a complex index of refraction, we see that potential scattering describes the usual volume scattering that occurs in electromagnetic, optical, and acoustic applications.

In this paper we will limit our attention to weak, potential scattering. We thus employ Eq. (2.4) as our model for the induced source where, in addition, we will assume that the scattering potential is sufficiently weak to admit the Born approximation [6]. This approximation consists simply of approximating the total wavefield in the expression (2.4) for the induced source by the wavefield incident to the scatterer. This approximation removes the scattered wave component of the wave generated in a scattering experiment as an unknown quantity and results in a linear relationship between the scattering potential and the scattered field data. For a further discussion on the use of the Born approximation in inverse scattering the reader is referred to references 4 and 10 and the references therein.

We will assume, for simplicity, that the incident wavefields employed in the set of experiments are the plane waves

\[ U^{(i)}(\mathbf{r}, s_0) = e^{i k \mathbf{s}_0 \cdot \mathbf{r}}, \tag{4.2} \]

where \( s_0 \) is the unit propagation vector of the plane wave. Our conclusions concerning uniqueness and stability are in no way affected by limiting the discussion to scattering experiments employing incident plane waves. Moreover, many NDE applications employ incident plane waves and most of the inversion methods also make this assumption [3,4]. On substituting Eq. (4.2) into Eq. (2.4) we obtain the following expression for the induced source for potential scattering within the Born approximation

\[ Q(\mathbf{r}, \omega; s_0) = V(\mathbf{r}, \omega) e^{i k s_0 \cdot \mathbf{r}}, \tag{4.3} \]

where we have included the wavevector \( s_0 \) in the argument of the source to indicate its dependence on the direction of propagation of the incident wave.

The inverse scattering problem for potential scattering within the Born approximation is equivalent to a set of inverse source problems, with each inverse source problem arising out of a different scattering experiment and where the set of sources are related to a single scattering potential via Eq. (4.3). All of the results obtained for the inverse source problem apply to the inverse scattering problem if we restrict our attention to any single given scattering experiment. It is only when we consider the totality of experiments that our conclusions will differ from those obtained in Section III for the inverse source problem.

To begin our investigation into the uniqueness question for the inverse scattering problem we will apply the Golden Rule. For any given scattering experiment the scattered field can be considered to be simply the field that is radiated by the induced source (4.3) and hence possesses three degrees of freedom (has dimension of three). This dimensionality is not changed when we perform any finite number of experiments; i.e., in order to increase the dimensionality we would require that an infinite number of experiments be performed. We then conclude from the Golden Rule that:
The inverse scattering problem for potential scattering within
the Born approximation does not possess a unique solution
given scattered field data obtained in any finite number of
scattering experiments.

The above result may seem a bit surprising in view of the fact that
the inverse scattering problem, unlike the inverse source problem, allows
us to perform multiple experiments to determine the scattering potential.
Clearly, more information is generated in a set of experiments than in a
single experiment. However, it simply turns out that the total amount of
information generated in any finite number of experiments is still not
sufficient to guarantee uniqueness of solution. This does not imply,
however, that nothing is gained by performing multiple experiments. It
simply means that the solution obtained from, say, N experiments although
"better" than the solution obtained from M experiments (M < N) will still
not be unique.

In order to study the stability question for the inverse scattering
problem we will adopt the measurement configuration that we employed in
the inverse source problem and that is illustrated in Fig. 1. We then
have the situation where a sequence of scattering experiments are performed
using different incident plane waves and the scattered field is measured
over the planes \( z = \pm z_0 \) in each experiment. If we then formally replace
the source \( Q(x, \omega) \) with the induced source \( Q(x, \omega; s_0) \) and the radiated field
\( U(x, \omega) \) with the scattered field \( u(x) \) we conclude from Eq. (3.5)
that

\[
\tilde{Q}(k \gamma z, \omega; s_0) = \gamma e^{-i \gamma z_0} \tilde{u}(s)(k, \pm z_0, \omega; s_0). \tag{4.4}
\]

We have, on using Eq. (4.3), that

\[
\tilde{Q}(k \gamma z, \omega; s_0) = \gamma \tilde{V}(k \gamma z, \omega; s_0) \tag{4.5}
\]

where

\[
\tilde{V}(v, \omega) \equiv \int d^3 r \, V(x, \omega) \, e^{-iv \cdot r}, \tag{4.6}
\]

is the three-dimensional spatial Fourier transform of the scattering
potential. Substituting Eq. (4.5) into Eq. (4.4) then leads to the result
that

\[
\gamma \tilde{V}(k \gamma z, ka_0, \omega) = \gamma e^{-i \gamma z_0} \tilde{u}(s)(k, \pm z_0, \omega; s_0). \tag{4.7}
\]

Eq. (4.7) relates the two-dimensional spatial Fourier transform of the
scattered field over the measurement planes to the three dimensional
spatial Fourier transform of the scattering potential. This equation is
the scattering problem counterpart to Eq. (3.5) and, as mentioned in
Section III, forms the basis for most treatments of the inverse scattering
problem within the Born approximation [6,10]. It is of prime importance
in the field of diffraction tomography where it is referred to as the
generalized projection-slice theorem [3]. Its origins go back to the early
use in inverse scattering and diffraction tomography date from the landmark

Eq. (4.7) allows us to determine the scattering potential's Fourier
transform for all values of the spatial frequency vector \( v \) lying on the
surfaces defined by

\[
v = k \gamma z - ka_0. \tag{4.8}
\]
Fourier Space

Fig. 3. Generalized projection-slice theorem. The two-dimensional spatial Fourier transform of the scattered field over the measurement planes illustrated in Fig. 1 equals the three-dimensional spatial Fourier transform of the scattering potential over the surface of a sphere (the Ewald sphere) centered at \(-kz_0\) and of radius \(k\).

For real \(\gamma\) these are the surfaces of spheres, centered at \(-kz_0\) and having a radius equal to the wavenumber \(k\). We show in Fig. 3 a typical such surface. As in the inverse source problem real values of \(\gamma\) result from \(K\) values for which \(K<k\) and correspond to homogeneous plane waves in the angular spectrum decomposition of the scattered field. Pure imaginary values of \(\gamma\) result when \(K>k\) and are associated with evanescent plane waves in the angular spectrum decomposition of the scattered field. Again, as in the inverse source problem it is readily verified that the computation of the scattering potential's Fourier transform using Eq. (4.7) is stable only for the homogeneous plane wave components; i.e., only for real values of \(\gamma\) corresponding to \(K\) values for which \(K<k\).

We conclude from the above that the inverse scattering problem then reduces to estimating the Fourier transform of the scattering potential throughout Fourier space from its specification over a set of spherical surfaces (one surface for each experiment; i.e., each \(z_0\)). It is apparent from this formulation that this problem does not have a unique solution since there are an infinite number of ways that the potential's transform can be continued throughout Fourier space from this set of surfaces. We should note, however, that as was the case for the inverse source problem, a unique solution is possible if sufficiently strong constraints are imposed on the solution to rule out all but one of the infinite number of scattering potentials that are consistent with the data. An example of such a constraint is to demand that the solution possess minimum \(L_2\) norm; i.e., that \(V(r,\omega)\) minimize the functional

\[
E = \int d^3 r \left| V(r,\omega) \right|^2
\]

among all possible scattering potentials consistent with the data. This constraint is employed in diffraction tomography and leads to a unique stable solution for any finite number of scattering experiments [3].

We conclude this Section with a brief discussion of non-scattering potentials [10]. These potentials are the scattering counterparts of the nonradiating sources discussed in the preceding section. We define a
non-scattering potential to be a scattering potential that scatters no radiation in one or more scattering experiments. These potentials are thus invisible in this set of experiments. Because the scattered field generated by one of these potentials must vanish outside the potentials support volume we conclude from Eq. (4.7) that a necessary and sufficient condition for a potential to be non-scattering is that

\[ \nabla(\nu, \omega) = 0 \]  

(4.10)

for all spatial frequencies satisfying Eq. (4.8).

The condition (4.10) can be used as a starting point to develop a simple expression for the most general non-scattering potential. As one might expect the non-scattering potentials play an important role in the inverse scattering problem. Indeed, in analogy to what we found in the inverse source problem it is possible to show that the difference between any two solutions to an inverse scattering problem within the Born approximation is a non-scattering potential. We will not discuss further the non-scattering potentials here but refer the reader to the literature for a more detailed account [9,10].

V. CONCLUDING REMARKS

We have in this paper reviewed some fundamental limitations associated with certain inverse source and scattering problems that occur in quantitative NDE. A Golden Rule was established that provides a very simple test to determine whether or not a solution to an inverse source or scattering problem is unique. This test consists simply of comparing the dimensionality of the data with the dimensionality of the unknown. The solution will be unique only if the dimensionality of the data exceeds or equals that of the unknown.

Application of the Golden Rule to the inverse source and scattering problems led to the conclusion that these problems do not, in general, admit unique solutions. However, by imposing constraints either of a mathematical nature (such as minimum L2 norm) or arising out of a priori information, it is possible to obtain a unique solution.

The stability of the inverse source and scattering problems was examined within the context of the measurement system illustrated in Fig. 1. It was shown that field data resulting from homogeneous plane wave components of the radiated or scattered field will lead to stable "solutions" to the inverse source and scattering problems while field data resulting from evanescent plane wave components will generate unstable solutions.

Finally, we discussed the role of nonradiating sources and non-scattering potentials within the context of the inverse source and scattering problems. These sources (scattering potentials) generate a vanishing radiation (scattered) field outside the support volume of the source (scatterer) and thus are invisible in any radiation (scattering) experiment. We showed that they have a simple mathematical definition and arise as the difference between any two solutions to an inverse source or scattering problem.

Our goal in this paper has not been to provide solutions to inverse source and scattering problems but, rather, to review some fundamental limitations that are associated with these solutions. A great deal of research is currently being carried out in the development of reconstruction algorithms for inverse source and scattering problems. It is extremely
important that this research be carried out with the full understanding of these limitations.

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DISCUSSION

R.B. Thompson (Ames Laboratory): I think I need a lesson in the Golden Rule. My friends often tell me that. In the potential scattering problem I believe you argued that there were three dimensions of information required to define the scattering potential. You have data on two dimensions?

A.J. Devaney: Right.

R.B. Thompson: If I do a broad band, time domain experiment, it would seem that then I have three dimensions of information?

A.J. Devaney: Yes. However, the problem is that in general, the scattering potential will also depend explicitly on the frequency in an unknown way.

R.B. Thompson: Okay.

A.J. Devaney: So I was saying frequency by frequency one has three degrees of freedom, but really, if you include frequency, there's four degrees of freedom in your knowns and three degrees of freedom in your data.

R.B. Thompson: So if I had a class of scattering objects that were non-dispersive I would only have three degrees of freedom. In optics, things are often very frequency dependent. Elastic waves exhibit some frequency dependence, but in some cases, not much.

A.J. Devaney: Of course, you can make an approximation, but always remember another thing, the Kramers-Kronig dispersion relationship cannot be
violated, and if you have attenuation, for example, you know that you are going to have dispersion in your velocity. So you've got to be very careful doing this. But I agree, in many cases, to a good approximation, you can neglect dispersion and this is precisely what's behind some of the time domain inversion procedures.

R.B. Thompson: Thank you.

From the Floor: The criterion of minimal energy seems to be very reasonable from a physical point of view, but also it seems to me that it implies an assumption that non-radiating source or non-scattering scatterers are not physical. Is it correct?

A.J. Devaney: No. I can give you an example of a non-scattering scatterer. When I was a student at the University of Rochester, one of the standard questions asked in the Ph.D. qualifying exam is: What object can you imagine that if you illuminate with monochromatic laser light is completely invisible? The answer is a Fabry Perot interferometer.

If I take a Fabry Perot interferometer--two parallel perfectly conducting mirrors--and I put them an integral number of half-wavelengths apart, light will go right through both, absolutely no reflection whatsoever. This is a nonscattering potential. But there is a much larger class of objects that are physical and nonradiating or non-scattering. I don't know if you are familiar with some of the work done in acoustic noise cancellation, but these are non-radiating sources. What one tries to do is surround the source by small microphones and speakers, and if you are very careful, you can generate an out-of-phase signal that, in principal, completely nullifies the sound field. So nonradiating sources and scatterers are not non-physical.

From the Floor: But then how do you justify the criterion of minimal energy?

A.J. Devaney: We justify it from a signal processing viewpoint. We say, okay, we know we are dealing with a non-unique problem. Let's find one solution, and at least we know which one we found, and it turns out it's very easy to find that one.

J.H. Rose (Ames Laboratory): Let's take two more questions.

B. DeFacio (University of Missouri): I have a question about your minimum energy. Are you considering only homogeneous flaws?

A.J. Devaney: No. This is in general.

From the Floor: Yes. Why isn't there a gradient term involved in the functional to be minimized. I would expect a wave energy functional to have a modulus square term like yours and then a gradient term.

A.J. Devaney: What he's asking is on the minimum energy condition that we use, why doesn't it have a gradient of the wave field in there somewhere. Most functionals (Lagrangians) dealing with waves, have such a term.

And it turns out it is there. What that turns out to be is a joint minimization. The minimization of the Lagrangian with respect to the perturbations in the wave, that gives you the wave equation and perturbations with respect to the scattering potential give you the minimum energy solution.
T. Derkacs (TRW, Inc.): The example you gave of a non-scattering scatterer was at a fixed wavelength, a monochromatic wavelength. Is there such a thing over a broad range of wavelengths?

A.J. Devaney: That's a very interesting question, of course, and the answer is I can concoct those by just making that Q function independent of K.

The important aspect of that question is: can you physically realize these over a large band of frequencies or are these somehow non-physical, and I don't know. But let me emphasize that the reason we are interested in the non-radiating source has nothing to do with physical realization of those. They appear as solutions basically of integral equations with zero eigenvalue, so they are very important from that view.