DETERMINISTIC SOURCE INVERSION FOR WAVE PROPAGATION PROBLEMS IN NONDESTRUCTIVE TESTING

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INTRODUCTION

A source of acoustic or elastic waves is of considerable interest for many nondestructive testing methods. During acoustic emission, waves are generated by transient deformation processes in materials or by externally applied excitations. During ultrasonic testing, transducers of several possible types are used to generate waves. Source inversion refers to the process of determining characteristics of the source of waves from measurements of the resulting wave motion.

Source inversion methods fall into two general categories -- deterministic and stochastic. For the deterministic approach, measurements from unknown sources are analyzed using methods that are based on theoretical calculations. Techniques such as source location by triangulation and deconvolution fall into this category. For the stochastic approach, measurements from unknown sources are analyzed using methods that are based on the statistical properties of a large number of previous measurements from known sources. Pattern recognition methods fall into this category. The remainder of this paper will be concerned with deterministic methods.

INVERSION BY MODELING

Regardless of the particular application, the response to a source of waves can be represented as a sum of integrals in time and space over the source region, where each integrand is the product of a Green's function and a source term. The general source inversion problem requires simultaneous solution of a set of these integral equations to determine the unknown source terms. There are typically difficulties with existence and uniqueness that must be dealt with to obtain usable solutions.

One practical way to eliminate many of the difficulties and complexities of the general problem is to first model the source with a finite number of parameters. The source inversion problem may then be formulated as the process of determining these parameters. This is done by selecting a set of features that can be determined both experimentally from measurements and theoretically from the source parameters. The final step is determination of the source parameters such that the theoretically calculated features best fit (in some sense) those obtained from experiment.
With this approach, one can deal with problems of existence and uniqueness by investigating the theoretical dependence of the features on the model parameters. The significance of each model parameter can also be determined by standard statistical tests.

To illustrate this approach, two short examples are briefly presented that might not ordinarily be thought of as source inversion problems.

The Source Location Problem

The general method of locating a point source is to measure the wave arrival time at several locations, and then determine the source coordinates that are most consistent with these times. Since measured times are relative, the time of the source is also unknown. Thus, the source model is:

\[ (x_s, y_s, z_s) = \text{source coordinates} \]
\[ T_s = \text{time of source} \]

The features are the various arrival times,

\[ T_i = \text{arrival time at the } i^{th} \text{ receiver} \]

If the wave speed \( c \) is known (which we assume to be the case), the arrival times can be calculated in terms of the source parameters:

\[
\hat{T}_i = \frac{1}{c} \left[ (x_i - x_s)^2 + (y_i - y_s)^2 + (z_i - z_s)^2 \right]^{1/2} - T_s
\]

where

\[ (x_i, y_i, z_i) = \text{coordinates of the } i^{th} \text{ receiver}. \]

The source parameters are then determined by minimizing the mean square error \( E \) between the measured times and the calculated times.

\[
E = \sum_{i=1}^{N} (T_i - \hat{T}_i)^2
\]

By investigating the dependence of \( E \) on the receiver coordinates, one can determine the optimum number and placement of receivers.

Time-Domain Deconvolution

The problem of deconvolution is that of determining the time history of a source from measured signals when all other parameters are known. Of course, all elements of the source must have a common time history. Thus, the source model consists of \( K \) discrete samples \( s(n) \) of the source time history \( s(t) \),

\[ s(n) = s(n\Delta T) ; \quad n = 1, 2, \ldots, K \]

The measured features are \( N \) discrete samples of the transient wave motion recorded at a receiver.

\[ v(n) = v(n\Delta T) ; \quad n = 1, 2, \ldots, N. \]
This response is the convolution of the source time history with a Green's function $G(t)$ that is assumed to be known.

\[ \hat{v}(t) = \int_0^t G(t-\tau) s(\tau) d\tau = G(t) \ast s(t) \quad (3) \]

In discrete form, the convolution integral becomes a summation,

\[ \hat{v}(n) = \sum_{k=1}^{K} G(n-k+1) s(k) \quad (4) \]

The unknown discrete samples of the time history are determined by minimizing the mean square error $E$ between the measured values and the calculated values.

\[ E = \sum_{n=1}^{N} \left[ v(n) - \hat{v}(n) \right]^2 \quad (5) \]

Minimization of $E$ with respect to the $s(k)$ yields a linear system of equations for the $s(k)$.

POINT SOURCE INVERSE PROBLEM

An acoustic emission source in an elastic body can be modeled as a point source if its extent is small compared to the distance from source to receiver and to the smallest wavelength of interest. A model for such a source can be developed by performing a multipole expansion of the integral representation of the elastodynamic response [1]. In this paper, the point source inverse problem is formulated and the basic approach is outlined. Experimental and numerical results may be found in the references.

Point Source Response

First consider the displacement response to a body force source confined to a volume $V_0$. This response, $u_i(x,t)$, can be written as,

\[ u_i(x,t) = \int_0^t \int_{V_0} dx' G_{ij}(x,t-\tau;x') f_j(x',\tau) \quad (6) \]

In this and subsequent equations, summation over repeated indices is implied. The Green's function $G_{ij}(x,t;x')$ is the displacement in the $i^{th}$ direction at location $x$ and time $t$ due to a unit impulsive force in the $j^{th}$ direction at location $x'$ and time $t = 0$. The quantity $f_j(x,t)$ is the body force density of the source at location $x$ and time $t$. A source generated by a surface discontinuity such as a crack can also be expressed in this form by using the concept of an equivalent body force, as is done by Burridge and Knopoff [2].

To consider a point source, the Green's function in Eq. (6) is expanded in the source variable $\bar{x}'$ about the point $\bar{x}' = \bar{x}_0$.
In this equation, the asterisk refers to the time domain convolution integral. This equation is called the multipole expansion of $u_i$, and the various integral moments of the body force are called the multipole moments.

The zeroth moment is simply the total force distributed throughout the volume $V_0$.

$$F_j(x^0, t) = \int_{V_0} dx' f_j(x', t)$$

(8)

The first moment of the body force density is called the moment tensor.

$$M_{jk}(x^0, t) = \int_{V_0} dx' (x_k' - x_k^0) f_j(x', t)$$

(9)

For a point source, higher order terms can be neglected and the response then becomes,

$$u_i(x, t) = G_{ij}(x, t; x^0) \ast f_j(x^0, t) + \frac{\partial G_{ij}}{\partial x_k} M_{jk}(x^0, t)$$

(10)

This equation is the general form for the response to a point source with non-vanishing zeroth or first moment.

Many point sources of interest are separable in time and space. That is, each source term has the same time dependence $s(t)$. The response is then written as,

$$u_i(x, t) = \{ G_{ij}(x, t; x^0) F_j(x^0, t) \} \ast s(t)$$

(11)

Examples of separable point sources include a force with a time-independent orientation and a crack with planar surfaces.

In general, the receiving transducers do not measure displacement. Here it is assumed that each receiver is characterized by a time function $R_m(t)$ and three coefficients $r_i^m$ ($i=1, 2, 3$) that multiply the three displacement components. Thus, the measured signal $v_m(t)$ at the $m^{th}$ receiver is,

$$v_m(t) = r_i^m [ G_{ij}(x, t; x^0) F_j(x^0) + \frac{\partial G_{ij}}{\partial x_k} M_{jk}(x^0) ] \ast R_m(t) \ast s(t)$$

(12)

In this equation, there is no sum over the receiver index $m$. 

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Point Source Inversion

The inverse problem for a separable point source is the problem of determining the $F_j$, $M_{jk}$, and $s(t)$ in Eq. (12) given the measured signals $v^m(t)$. It is assumed that the Green's functions and receiver parameters are known so that the only unknown quantities in Eq. (12) are the source terms. For the most general case, there are 12 unknown spatial terms -- the 3 force components and the 9 moment tensor components. If $c_q$ designates one of these terms, and $Q$ is the total number of non-vanishing terms, Eq. (12) can be rewritten as,

$$v^m(t) = \left\{ \sum_{q=1}^{Q} c_q g^m_q(t) \right\} * s(t)$$

(13)

where $g^m_q(t)$ is the scaled sum of the Green's functions convolved with the receiver function for the $m$th receiver and $q$th source term.

If the source has no net forces or moments, the force components vanish and the moment tensor is symmetric. Then, $Q = 6$ in Eq. (13). If there is a net force, the force term in Eq. (10) will typically dominate the moment tensor term, and $Q = 3$.

For the inverse problem, the model parameters are the $c_q$ and $s(t)$ in Eq. (13), where the $c_q$ are the non-vanishing force and moment tensor components and $s(t)$ is their common time history. The features are the time samples of the measured response at each receiver. The parameters are determined by minimizing the mean square error $E$ between the measured response and the calculated response.

$$E = \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ v^m(n) - \sum_{q=1}^{Q} c_q \sum_{k=1}^{K} g^m_q(n-k+1) s(k) \right]^2$$

(14)

This is a non-linear problem because the $c_q$ and $s(k)$ appear in the form of a product. An iterative time-domain method was developed by Michaels and Pao [3] to solve for the $c_q$ and $s(k)$ at each receiver. Experimental results are shown in Ref. 4. A frequency domain method was developed by Chang [5] that involves dividing the spectra at different receivers to eliminate the common time function and then solving for the $c_q$.

By examining the form of Eq. (14), several statements can be made concerning the solution of this problem. First, the time history and spatial components can be determined only to within a scale factor since they appear as a product. Second, the length $N$ of the measured signals must be the same or longer than the length $K$ of the source time history ($N \geq K$). Third, the total number of measurements must equal or exceed the total number of unknowns ($N \times M \geq K + P - 1$). Fourth, the time history can only be recovered within the bandwidth of the Green's functions.

TRANSDUCER CHARACTERIZATION BY INVERSION

The process of characterizing an ultrasonic transducer can be thought of as an inverse source problem. The goal is to record a set of measured signals from a transducer, and from these signals extract transducer parameters that can be used to predict future measurements.
Transducer Model

Considered here are axially symmetric immersion transducers that are modeled by specifying an appropriate pressure or velocity distribution over a region of a fluid half-space that is otherwise either free or rigid. This type of model is reviewed by Harris for several different velocity distributions \([6,7]\). It is extended to pressure distributions by Greenspan \([8]\). The geometry is illustrated in Figure 1.

![Diagram of a transducer model](image)

Figure 1. Axisymmetric half-space model for immersion transducer

In this paper, the particular model considered is that of a circular piston embedded in a rigid half-space. That is, the normal velocity \(w(t)\) of a circular region of radius \(a\) is specified for a half-space that is otherwise rigid (zero normal velocity). For this problem, the response of a point pressure receiver located at coordinates \((x,y,z)\) is given by,

\[
p(x,y,z,t) = \frac{3}{2\pi} \int_{S_a} \frac{w(t-R/c)}{2\pi R} \delta(t-t') \frac{dS'}{2\pi R} = w(t) \frac{3}{2\pi} \int_{S_a} \frac{\delta(t-R/c)}{2\pi R} \frac{dS'}{2\pi R} \tag{15}
\]

In this equation, \(S_a\) is a circular region in the \(xy\) plane of radius \(a\) and centered at the origin, and \(R\) is the distance from a point in the source region to the receiver.

\[
R = \left[ (x-x')^2 + (y-y')^2 + z^2 \right]^{1/2}
\]

Let the integral in Eq. (15) be \(\phi(x,y,z,t)\) and let the polar coordinates of the receiver be \((r,\theta,z)\). Since the source is axisymmetric, both \(p\) and \(\phi\) have no \(\theta\) dependence. The definition of \(\phi\) is,

\[
\phi(r,z,t) = \frac{3}{2\pi} \int_{S_a} \frac{\delta(t-R/c)}{2\pi R} \frac{dS'}{2\pi R} \tag{16}
\]

This expression can be exactly evaluated \([6]\), but here the derivation is omitted. The times \(t_1, t_2\) and \(t_3\) are defined to be,
\[
t_1 = \frac{z}{c}
\]
\[
t_2 = \frac{[z^2 + (r-a)^2]^{1/2}}{c}
\]
\[
t_3 = \frac{[z^2 + (r+a)^2]^{1/2}}{c}
\]

For the receiver within the aperture of the transducer \((r \leq a)\), we have,

\[
\phi = \begin{cases} 
0 & \text{; } t < t_1 \text{ and } t > t_3 \\
\frac{c}{\pi} \cos^{-1} \left\{ \frac{r^2 + c^2 t^2 - z^2 - a^2}{2r \left( c^2 t^2 - z^2 \right)^{1/2}} \right\} & \text{; } t_1 \leq t \leq t_2 \\
\frac{c}{\pi} \cos^{-1} \left\{ \frac{r^2 + c^2 t^2 - z^2 - a^2}{2r \left( c^2 t^2 - z^2 \right)^{1/2}} \right\} & \text{; } t_2 \leq t \leq t_3
\end{cases}
\]

(17a)

If the receiver is outside of the transducer aperture \((r > a)\), we have,

\[
\phi = \begin{cases} 
0 & \text{; } t < t_2 \text{ and } t > t_3 \\
\frac{c}{\pi} \cos^{-1} \left\{ \frac{r^2 + c^2 t^2 - z^2 - a^2}{2r \left( c^2 t^2 - z^2 \right)^{1/2}} \right\} & \text{; } t_2 \leq t \leq t_3
\end{cases}
\]

(17b)

Thus, pressure signals may be determined by calculating \(\phi\) according to Eq. (17), numerically differentiating, and convolving with the velocity time function. If the receiver is sensitive to pressure, the received signals \(v(t)\) can be calculated by performing an additional convolution with a receiver transfer function \(R(t)\).

\[
v(r,z,t) = R(t) \ast w(t) \ast g(r,z,t;a)
\]

(18)

\[
g(r,z,t;a) = \frac{3}{a} \phi(r,z,t;a)
\]

(19)

In this equation, \(g\) is the Green's function for the transducer and is a function of receiver location, time and transducer radius.

Inversion Procedure

The goal of the source inversion procedure is to recover the transducer radius and velocity time function from the measured signals at several receivers. Thus, the model parameters are the radius \(a\) and the \(K\) samples of the velocity time function, \(w(k)\), \(k = 1, \ldots, K\). The features are the \(N\) time samples of the measured signals at \(M\) receivers, \(v^m(n)\), \(n = 1, \ldots, N\) and \(m = 1, \ldots, M\). The parameters are determined by minimizing the normalized mean square error \(E\) between the measured signals and the calculated signals.

\[
E = \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{[v^m(n) - \hat{v}^m(n)]^2}{\sum_{n=1}^{N} [v^m(n)]^2}
\]

(20)

For the remainder of this example, it is assumed that the receiver is a perfect pressure sensor; that is,

\[
R(t) = \delta(t)
\]

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Then,

\[ \hat{V}^m(n) = \sum_{k=1}^{K} g(r^m, z^m, n-k; a) w(k) \]  

(21)

Thus, for a given radius and receiver location, the Green's function \( g \) can be calculated according to Eqs. (17) and (19), and the response is determined by convolution with \( w(t) \) as given by Eq. (21).

If the radius is known, the velocity time function can be recovered by linear least squares deconvolution. However, the dependence upon radius is non-linear, as shown by Eq. (17). Thus, the inversion procedure shown here is an iterative one, and is done as follows:

1. Select initial guess for radius and desired tolerance.
2. Calculate \( w(k) \) by linear least squares using current radius.
3. Calculate error \( E \) by Eq. (20).
4. Done if \( E \) minimum for radius determined within desired tolerance.
5. Otherwise, update radius via a two-sided binary search algorithm.
6. Go to step 2.

It is assumed that there are no local minima so that this algorithm will always converge to the global minimum within the desired tolerance.

Inversion Results

To illustrate the inversion procedure, we consider a specific example whose geometry is illustrated in Figure 2. The transducer has a radius of 0.25 inches, and there are seven receivers located 2 inches in front of the transducer separated from each other by 0.2 inches. Note that the receivers at \( x = 0.2, 0.4 \) and 0.6 inches theoretically record the same signals as those at \( x = -0.2, -0.4 \) and -0.6 inches because of the symmetry about the \( z \) axis.

Radius = 0.25 inches

![Figure 2. Geometry for numerical example of transducer inversion.](image)
The pressure Green's functions for this geometry are shown in Figure 3. They were obtained by calculating $\phi$ by Eq. (17), applying a cubic spline running average, and then numerically differentiating according to Eq. (19). The spline smoothing was needed to minimize errors caused by discretizing the time variable.

For this example, a 1 Mhz broadband pulse was used for the velocity time function, as shown in Figure 4. The received signals, which were obtained by convolving the Green's functions in Figure 3 with this velocity time function, are shown in Figure 5.

**Figure 3.** Pressure Green's functions for geometry of Figure 2.

**Figure 4.** Velocity time function for numerical example.

**Figure 5.** Received signals for numerical example.
Figure 6 illustrates the effect of radius on inversion error. The curve in this figure was obtained by assuming various radii, deconvolving for the velocity time function, and calculating the error according to Eq. (20). Thus, when the radius is equal to the correct value of 0.25 inches, the error is zero. As the radius is changed from this value, the error smoothly increases. The inversion procedure is the process of determining the minimum of this curve.

![Graph showing the effect of radius on inversion error.](image)

Figure 6. Effect of radius on inversion error.

To test the procedure, Gaussian white noise was added to the received signals shown in Figure 4. The peak noise amplitude was approximately 10% of the peak amplitude of the signals recorded at $x = \pm 0.6$ inches. Four of the noisy signals are shown in Figure 7. The radius and velocity time function were determined by initially setting the radius to an arbitrary value of 0.18 inches and searching for the minimum of the error vs. radius curve with a tolerance of 0.005 inches. The final radius was 0.2488 inches, and the final velocity time function is shown in Figure 8. Its noise level is about the same as the average noise level of the received signals.

**SUMMARY AND CONCLUSIONS**

In this paper the source inversion problem has been formulated as a parameter determination problem. That is, the source is modeled by a finite number of parameters, and these parameters are determined by fitting calculated features to measured features. This approach to source inversion has been successfully applied to many problems in nondestructive testing.

The key factor in using this approach is understanding the source mechanism well enough to develop a model. If the model does not explain the data, the inversion results are meaningless. There is, in general, no method for improving the model other than increasing its complexity by adding parameters. It is straightforward to investigate significance of model parameters as well as existence and uniqueness of solutions.

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REFERENCES


Chairman Rose: We will take questions now. Do you use absolute arrival times of your signals or do you only use relative times?

Dr. Michaels: For some experimental situations, we can't even use relative arrival times because different instruments may have slightly different trigger points. We never depend upon absolute arrival times, and, at best, can get accurate relative arrival times.

Dr. Dudley (University of Arizona): How high are your model orders for those deconvolution problems? That is, how many parameters do you typically have?

Dr. Michaels: The spatial terms are relatively low in number; say, 3 for a force and 6 for a moment tensor. But in terms of time samples, the model may have 50 to 100 sample points. The measured signals typically have 200 to 1000 time samples, so the problem is usually quite overdetermined.

Dr. Dudley: Have you considered incorporating the transducer radius parameter and any other geometrical parameters into your original model and using a non-linear optimization approach?

Dr. Michaels: All parameters are directly incorporated into the model. However, only the time samples can be handled linearly. All the other parameters are obtained non-linearly, but one at a time. As you have suggested, the next logical step is to use a global optimization approach in which all parameters are iterated simultaneously.

Dr. Dudley: Your current approach of doing one parameter at a time will become more and more difficult as you get more parameters because the searches are going to get very complicated.

Dr. Michaels: That's right. As the models get more complicated, I'm confident that we can develop whatever numerical methods are required to run on our small computers.

Dr. K. L. Langenberg (University of Kassel, Germany): Did I understand correctly that you specified your piston transducer with a homogeneous aperture distribution?

Dr. Michaels: Yes.

Dr. Langenberg: How do you reconcile this with the real-life situation of a non-homogeneous aperture distribution?

Dr. Michaels: In practice, I believe that the homogeneous model will not work for many transducers, and that I will have to consider a non-homogeneous model. This change will not affect the mathematical method except for the calculation of the Green's functions. As long as they can be calculated, a non-homogeneous aperture can be handled.