Soil water dynamics

Dan Zaslavsky

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ZASLAVSKY, Dan. SOIL WATER DYNAMICS.

Iowa State University of Science and Technology
Ph.D., 1960
Engineering, agricultural

University Microfilms, Inc., Ann Arbor, Michigan
SOIL WATER DYNAMICS

by

Dan Zaslavsky

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Soil Physics

APPROVED:

Signature was redacted for privacy.

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Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa
1960
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I. INTRODUCTION

No general solution has apparently existed for the differential equation of the stream function in axially symmetric potential flow. In this thesis the problem is solved.

A solution for this problem was first achieved by a simple transformation which was discovered by guess work. Soon thereafter, it was learned that this transformation is only a special case of a general method applying to a wide range of differential equations. Consequently it was possible to find solutions for a variety of problems with nonuniform conductivities. The method applies equally well to two-dimensional flow and axially symmetric flow. Although in essence available in mathematical literature, the sequence of operations needed for obtaining the solution was not at first obvious. Once outlined, the process of solving is straightforward. The number of problems that can be solved conventionally by the new methods developed may very well exceed the number of the saturated flow problems that have been solved to date. The number of solvable problems may be further increased or their complexity reduced by certain transformations which are suggested and which may apply to modern computing techniques.

The second and third sections of this thesis may be considered as special cases of the general theory that is presented in the fourth and fifth sections and Appendix III. In presenting the special and the general theory both the stream function and potential function are treated. Such treatment lends to completeness and convenience.
This thesis also includes under its title "Soil Water Dynamics" some studies on flow-measuring techniques. These studies are incomplete as a deadline had to be met. They are presented as Appendixes I and II. It is expected that the reader who wishes to follow details of the analytical developments of this thesis will be familiar with, and/or have at hand, treatise such as the following:

Churchill's *Fourier Series* [4], Hildebrand's *Advanced Calculus* [11], Ince's *Ordinary Differential Equations* [13], Margenau and Murphy's *Mathematics of Physics and Chemistry* [27], Morse and Feshbach's *Methods of Theoretical Physics* [31], Sommerfeld's *Lectures on Theoretical Physics* [37, 38, 39] and Wylie's *Advanced Engineering Mathematics* [46].

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1 Brackets are used to designate reference numbers in this thesis. Parentheses are used to designate equation numbers. Equation numbers are started anew for each part of this thesis.
II. THEORY OF SEEPAGE FLOW IN AN AXIALLY SYMMETRIC WATER-SATURATED SYSTEMS OF UNIFORM CONDUCTIVITY

A. Derivation of Partial Differential Equations for Axially Symmetric Flow with Uniform Conductivity

1. The potential function

The equation for the flux vector $q$ is assumed to be

$$ q = -k \, \text{grad} \, \phi $$

(1)

where $k$ is the hydraulic conductivity and $\phi$ is the scalar potential. Equation (1) implies that the vector field of the flux is irrotational.

In steady-state flow of an incompressible fluid the equation of continuity, in vector notation is

$$ \text{div} \, q = 0 $$

(2)

The combination of (1) and (2) yields

$$ \text{div}(k \, \text{grad} \, \phi) = 0 $$

(3)

If $k$ is regarded as a scalar function of the coordinates, Equation (3) may be written in the following form

$$ k v^2 \phi + (\text{grad} \, k) \cdot (\text{grad} \, \phi) = 0 $$

(4)

Using conventional cylindrical coordinates $r$, $\theta$ and $z$, Equation (4) then, has the form

$$ k \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] + \frac{\partial \phi}{\partial r} \frac{\partial k}{\partial r} + \frac{\partial \phi}{\partial \theta} \frac{\partial k}{\partial \theta} + \frac{\partial \phi}{\partial z} \frac{\partial k}{\partial z} = 0 $$

(5)

When axial symmetry exists, derivatives with respect to $\theta$ vanish and the equation reduces to
When the conductivity is uniform, that is, when it does not vary with
the space coordinates, the equation is further reduced to

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{s^2} + \frac{\partial \psi}{\partial r} + \frac{\partial \psi}{\partial s} = 0$$  \hspace{1cm} (6)

This is Laplace's equation in cylindrical coordinates for axially
symmetric flow. For later reference, it may be written in the con­
venient form

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial s^2} = 0$$  \hspace{1cm} (7)

Solutions of this equation are well known. The general solution is
quoted below in Section II B 7.

2. The differential equation for the stream function

This equation was first derived by Stokes [40] in 1842. Later
accounts of this derivation may be found in many books on hydrodynamics.
For example, see H. Lamb, [26, p. 428].

The derivation used here will follow closely that of Arnold
Sommerfeld [37, pp. 146-147]. It is required to find a function $\psi$
which describes a family of space surfaces. These surfaces should be
orthogonal, at every point, to the surfaces of equal potential $\psi$. It
is also required that the flux $q$ be expressed in terms of the stream
function derivatives. Furthermore, it is required that $q$ be irrotational.
There exists one obvious solution for this problem. The radial planes \((r,s)\), of constant angle \(\theta\) satisfy the conditions mentioned. The axial symmetry implies that the flow is directed along the radial planes. Consequently, the radial planes \((r,s)\), for \(\theta\) equals a constant, are, at every point, parallel to the velocity vector; and this latter is normal to the equipotentials. Thus the planes \((r,s)\) satisfy the condition of orthogonality. It is easy to see that the planes describe a flow without circulation. Formally, this solution is a consequence of the differential equation

\[
\frac{\partial \phi}{\partial \theta} = 0
\]  

which expresses the condition of axial symmetry.

Another family of surfaces will describe surfaces for \(\phi\). The trace of these surfaces on the radial plane describe a two-dimensional family of curves which will be called the stream lines. The picture of the stream lines does not vary from one radial plane to another. Together with the equipotential lines the stream lines form the flow net.

The first condition of orthogonality of \(\phi\) and \(\psi\) is expressed in vector notation as follows

\[
(\nabla \phi) \cdot (\nabla \psi) = 0
\]  

The second condition requires that the flow be irrotational. In vector terms it is expressed by

\[
\text{curl } \mathbf{q} = 0
\]
Equation (8) suggests, as becomes clear below, that we look for a function \( \psi(r,s) \), such that the following conditions are satisfied

\[
\frac{\partial^2 \psi}{\partial r \partial s} = \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) - \frac{\partial}{\partial s}(r \frac{\partial \psi}{\partial s}) \quad (12a)
\]

\[
\frac{\partial^2 \psi}{\partial s \partial r} = -\frac{\partial}{\partial s}(r \frac{\partial \psi}{\partial s}) = \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) \quad (12b)
\]

After integrating (12a) with respect to \( r \) and (12b) with respect to \( s \), we get, to within constants, the following relations

\[
\frac{\partial \psi}{\partial s} = r \frac{\partial \psi}{\partial r} \quad (13a)
\]

\[
\frac{\partial \psi}{\partial r} = -r \frac{\partial \psi}{\partial s} \quad (13b)
\]

We can now substitute (13a) and (13b) into (10) to check for orthogonality. Noting that (13a) gives the \( s \) component of the stream function gradient and that (13b) gives the \( r \) component of this gradient, one sees that (10) becomes

\[
r(\frac{\partial \psi}{\partial r})(\frac{\partial \psi}{\partial s}) - r(\frac{\partial \psi}{\partial s})(\frac{\partial \psi}{\partial r}) = 0 \quad (14)
\]

which is an identity, proving that \( \psi \) as found in (13a) and (13b), satisfies the orthogonality condition. Thus (13a) and (13b) express the orthogonality of \( \psi \) and \( \phi \) in axially symmetric flow, as do the Cauchy-Riemann conditions in two-dimensional flow.

By the relations (13a) and (13b), the flux \( q \) can be expressed in terms of the stream function. With reference to Equation (1), we get

[Continue the text as needed]
where \( q_r \) and \( q_z \) are the flux components in \( r \) and \( z \) directions, respectively, and \( k \) is the hydraulic conductivity.

We have still to satisfy the condition (11), that the flow be irrotational. It is satisfied by substituting (15a) and (15b) in (11).

By expanding (11) for \( q_r \) and \( q_z \), we get

\[
\text{curl } q = \left( \frac{\partial q_r}{\partial z} \right) - \left( \frac{\partial q_z}{\partial r} \right) = 0 \tag{16}
\]

Substituting for \( q_r \) and \( q_z \) in terms of the stream function derivatives, as expressed in (15), we get, after multiplying through by minus one,

\[
\frac{\partial}{\partial r} \left( k \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial \psi}{\partial z} \right) = 0 \tag{17}
\]

By expanding, for a uniform conductivity \( k \), Equation (17) reduces to

\[
\frac{\partial^2 \psi}{\partial r^2} - \left( \frac{1}{r} \right) \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{18}
\]

This is the required stream function equation.

3. The physical meaning of the stream function

The value of the stream function \( \psi \) at a stream surface is an expression of the total discharge passing between a reference surface and the one in question. \( \psi \) is measured by units of area when the potential \( \phi \) is measured (as we do measure it) in length (head) units. This follows from dimensional considerations of (13a) and (13b).
The total discharge may be computed by integrating the flux over a surface which crosses all the streamlines of the medium. Integration of the radial flux component, (15a), over a cylindrical surface of radius \( r \) between the elevations \( z = z_o \) and \( z = z \) yields

\[
Q_z = \int_{z_o}^{z} (2\pi r q_z) dz = \int_{z_o}^{z} 2\pi k \frac{\partial \psi}{\partial z} dz = -2\pi k (\psi_z - \psi_{z_o})
\]

(19)

If \( \psi_{z_o} = 0 \), \( Q_z \) reduces to

\[
Q_z = -2\pi k \psi_z
\]

(20)

If we use Equation (15b), the discharge from the base of the cylinder in positive \( z \) axis direction is

\[
Q_r = \pi k (\psi_r - \psi_{r_o})
\]

(21)

\((r > r_o)\)

Surfaces or lines with equal values of the stream function are parallel at every point to the flux vector. For further discussion of the stream function see Dryden et al. [7], Prandtl and Tietjens [33] and Lamb [26].
B. Solution of the Differential Equation for Axially Symmetric Flow With Uniform Conductivity

1. Separation of the variables

It is assumed that the stream function \( \phi \) can be expressed as the product

\[
\phi = R(r) \cdot Z(z)
\]  \hspace{1cm} (22)

where \( R \) is a function of \( r \) only and \( Z \) is a function of \( z \) only.

It will be shown in the last part of this section that there exist additional solutions which satisfy the differential equation for \( \phi \) but which cannot be expressed, at least readily, as part of the separated functions \( Z \) or \( R \). The solution derived from the separation of variables is especially convenient if the boundaries are parallel to the axes. The additional solutions will be convenient for special boundaries which are not parallel to the axes in the cylindrical system.

Substituting (22) into the differential Equation (18) and dividing through by \( \phi \), we get

\[
\frac{R''}{R} - \frac{1}{r} \frac{R'}{R} + \frac{Z''}{Z} = 0
\]  \hspace{1cm} (23)

A prime denotes a derivative of \( R \) with respect to \( r \) or derivative of \( Z \) with respect to \( z \). As \( R \) and \( Z \) vary independently, the only nontrivial solution of (23) is expressed by the ordinary differential equations

\[
\frac{Z''}{Z} = -n^2
\]  \hspace{1cm} (24)
(25)

(26a)

(26b)

(26c)

(26d)

(26e)

(26f)
One prime designates a first ordinary derivative, and two primes designate a second-order ordinary derivative.

2. Solution for the ordinary differential equations in $z$

The general solutions for Equations (26a), (26b), and (26c) may be found in treatises on ordinary differential equations. The solution for (26a) is

$$Z_1 = A_1 \sin(mx) + B_1 \cos(mx) \quad (m^2 > 0)$$  \hspace{1cm} (27a)

The solution for (26b) is

$$Z_2 = C_1 \sinh(mx) + D_1 \cosh(mx) \quad (m^2 > 0)$$  \hspace{1cm} (27b)

The solution for (26c) is

$$Z_0 = E_1 s + F_1$$  \hspace{1cm} (27c)

$A$, $B$, $C$, $D$, $E$, and $F$ are constants to be determined for specific boundary conditions; subscripts have been added to the constants for later convenience.

3. Transforming the differential equation in $r$ into a self-adjoint form

To solve Equations (26d), (26e), and (26f), first we perform some transformations. These transformations may be found by guessing. It is preferred, however, to present a systematic method by which equations of this form may be solved. The same transformations will be used in Section IV of this thesis for the more general theory.
The general linear, homogeneous, ordinary differential equation of the second order may be written in the form

\[ L(u) = P_0 \frac{d^2 u}{dx^2} + P_1 \frac{du}{dx} + P_2 u = 0 \]  \hspace{1cm} (28)

where \( P_0, P_1 \) and \( P_2 \) are functions of \( x \), the independent variable.

The adjoint differential equation can be written in the form (see Ince [13, p. 237])

\[ \tilde{L}(v) = (-1)^2 \frac{d^2}{dx^2} (P_0 v) + (-1)^1 \frac{d}{dx} (P_1 v) + P_2 v = 0 \] \hspace{1cm} (29)

After developing and rearranging, we get

\[ P_0 \frac{d^2 v}{dx^2} + (2P_0' - P_1')(dv/dx) + (P_0'' - P_1' + P_2)v = 0 \] \hspace{1cm} (30)

If the coefficients for equal order derivatives in Equations (28) and (30) are identical, it is said that the equations are self-adjoint. The coefficients in (28) and (30) must satisfy the following conditions

\[ 2P_0' - P_1 = P_1 \] \hspace{1cm} (31a)

\[ P_0'' - P_1' + P_2 = P_2 \] \hspace{1cm} (31b)
The first equation yields immediately (or the second by integration)

\[ P_0' = P_1 \]  

(31c)

If condition (31) is satisfied, (28) is said to be self-adjoint, or the operators \( L \) and \( \tilde{L} \) are identical and said to be self-adjoint. Equation (28) can, using (31c), be written in the form

\[ L(u) = P_0 (d^2 u / dx^2) + P_0'(du/dx) + P_2 u = 0 \]

or after rearrangement

\[ L(u) = \frac{d}{dx} \left( P_0 \frac{du}{dx} \right) + P_2 u = 0 \]

(32)

In the general case we can transform any differential operator \( L(u) \) of a linear, homogeneous, ordinary, second-order differential equation to make it self-adjoint.

Equation (28) is multiplied by the factor

\[ \int (P_1 / P_0) \, dx \]

(33)

After some rearrangement the result can be written in the form

\[ \frac{d}{dx} \left( e^{\int (P_1 / P_0) \, dx} \cdot \frac{du}{dx} \right) + (P_2 / P_0) \left( e^{\int (P_1 / P_0) \, dx} \cdot u \right) = 0 \]

(34)

which is self-adjoint and equivalent to the original differential equation. To use the formal transformation just suggested, we identify the coefficients of (26d) with the standard equation (28) as follows
$P_0 = r^2$

$P_1 = -r$

$P_2 = -r^2 n^2$

Consequently, the transformation factor from (33) is found by integration to be

$$\int \left( \frac{P_1}{P_0} \right) dr = -\left( \frac{1}{r} \right) dr = (1/r)$$

Substituting into Equation (34) we obtain the equation

$$\frac{d}{dr} \left( \frac{1}{r} \frac{dR}{dr} \right) - \frac{1}{r} n^2 R = 0$$

which for this single case could have been written down from (26d) by inspection.

4. Transformation of the self-adjoint differential equation into Bessel's Differential Equation

In this section a general method is presented, by which certain forms of Sturm-Liouville equations may be transformed into Bessel differential equations (see Wylie [46, p. 258]).

Consider the general differential equation

$$\frac{d}{dr} \left( r P R' \right) + (ar^q + br^g) R = 0$$

The following transformations are suggested

$$r = t^g \quad g = 2/(2 - p + g)$$

$$R = t^h \quad h = (1 - p)/(2 - p + g)$$
The differential equation (35), to be solved, is compared with the standard form (36), whence we see, using also the equations for \( g \) and \( h \)

\[
\begin{align*}
p &= -1 & q &= -1 & q &= 1 \\
b &= 0 & a &= -n^2 & h &= 1
\end{align*}
\]

Consequently, the proper substitution for our case is

\[
\begin{align*}
r &= t \\
R &= rY
\end{align*}
\]

It is easy to confirm that (39) is useful for both Equations (26d) and (26e). From (39), the following identities are developed

\[
\begin{align*}
(dR/dr) &= rY' + Y \\
(d^2R/dr^2) &= rY'' + 2Y'
\end{align*}
\]

After substitution, Equations (26d) and (26e) transform into the following:

\[
\begin{align*}
r^2Y_1'' + rY_1' - (1 + r^2n^2)Y_1 &= 0 \\
(n^2 > 0) \\
r^2Y_2'' + rY_2' - (1 - r^2n^2)Y_2 &= 0 \\
(-n^2 < n^2 > 0)
\end{align*}
\]

These equations are standard forms of Bessel equations (see Morse and Feshbach [31, pp. 1259, 1321], Jahnke and Emde [20, p. 146]). The solution of (40a) is

\[Y_1 = A_1I_1(nr) + B_1K_1(nr) \quad n^2 > 0\]

\(I_1\) and \(K_1\) are the modified Bessel functions of the first and second kind, respectively, and of the first order. They are sometimes called
hyperbolic Bessel functions. This is because they relate to the unmodified Bessel functions in the same manner as hyperbolic functions relate to trigonometric functions. In fact, Equation (40a) can be derived from (40b) by substituting the imaginary variable (ir) in place of (r). Equation (40b) is similarly found from (40a) by the same substitution.

The solution for (40b) is

\[ Y_2 = C_1 J_1(mr) + D_1 N_1(mr) \quad (-m^2 = n^2 < 0) \quad (42) \]

\( J_1 \) and \( N_1 \) are the Bessel functions of the first and second kind and of the first order. The function \( N_1 \) is sometimes called a Neumann function. In British literature it is often designated by \( Y \).

From solutions (41) and (42) and according to the transformation (39), we derive the required solutions \( R_1 \) and \( R_2 \) as follows

\[ R_1 = r[A_1 J_1(nr) + B_1 K_1(nr)] \quad (n^2 > 0) \quad (43a) \]
\[ R_2 = r[C_1 J_1(nr) + D_1 N_1(nr)] \quad (-m^2 = n^2 < 0) \quad (43b) \]

5. Singular solution when the Eigenvalue \( n^2 \) equals zero

The last equation to be solved is for \( n \) equals zero, as expressed in Equation (26f). This is an Euler type ordinary differential equation. It can be solved by the substitution

\[ r = e^y \quad (44) \]

This is a standard method of solution but, in this case, not the simplest one. Let Equation (26f) be rewritten in the form
\[
\frac{d}{dr} [(1/r)(dR_o/dr)] = 0 \tag{45}
\]

The solution for \( R_o \) is, simply,

\[
R_o = E_2 r^2 + F_2 \tag{46}
\]

6. The general solution for the stream function

The general solution can be written as follows

\[
\phi = Z_0 R_0 + Z_1 R_1 + Z_2 R_2 \tag{47}
\]

The different terms in the solution are defined in Equations (27a), (27b), (27c), (43a), (43b), and (46). The different terms are combined according to Equation (22) and are paired according to the different possible values of the eigenvalues \( n^2 \). The solution can be written in explicit form as follows

\[
\phi = [A_1 \sin(ns) + B_1 \cos(ns)]r[A_2 J_1(nr) + B_2 X_1(nr)] \\
+ [C_1 \sinh(ms) + D_1 \cosh(ms)]r[C_2 J_1(nr) + D_2 X_1(nr)] \\
+ Er^2 s + Fr^2 + Gs + H \tag{48}
\]

7. The general solution for the potential function

The general solution for the potential function in axially symmetric flow is well known. It is found by separating the variables in Equation (7). The ordinary differential equation in terms of \( s \) are identical with Equations (26a), (26b), and (26c), for the stream function. The ordinary differential equations in terms of the radius
render themselves soluble in terms of Bessel functions of the zeroth order, in a straightforward manner.

The general solution for the potential function is

\[ \phi = [A_1 \sin(mx) + B_1 \cos(mx)][A_2 I_0(nr) + B_2 K_0(nr)] + [C_1 \sinh(mx) + D_1 \cosh(mx)][C_2 J_0(nr) + D_2 N_0(nr)] + E' \ln(r/a) + F' \ln(r/a) + G' \phi + H' \]  (49)

8. Formation of new solutions and a test for their orthogonality

The solutions for the stream function in Equation (48) and the solution for the potential function in Equation (49) must satisfy the conditions of orthogonality expressed in Equations (13a) and (13b).

These relations are the following

\[ \left( \frac{\partial \phi}{\partial s} \right)_r = r \left( \frac{\partial \phi}{\partial r} \right) \]  (50a)

\[ \left( \frac{\partial \phi}{\partial r} \right)_r = - r \left( \frac{\partial \phi}{\partial s} \right) \]  (50b)

Performing the differentiations of (50a) and (50b), using \( \phi \) and \( \phi \) of (48) and (49), we derive the following relations between the coefficients of the trigonometric and Bessel functions of (48) and (49)

\[ A_1 = B_1 \quad B_1 = - A_1 \quad A_2 = A_2' \quad B_2 = - B_2' \]  (51a)

\[ C_1 = D_1' \quad D_1 = C_1' \quad C_2 = - C_2' \quad D_2 = - D_2' \]  (51b)

By expanding first the solution of (48) and (49) and then using the orthogonality conditions (50a) and (50b), we get the following
relations, which are equivalent to the relationships in (51)

\[ A_1 A_2 = B_1 A_2 \quad A_1 B_2 = -B_1 B_2 \quad B_1 A_2 = -A_1 A_2 \quad B_1 B_2 = A_1 B_2 \]

(52)

\[ C_1 C_2 = -D_1 C_2 \quad C_1 D_2 = -D_1 D_2 \quad D_1 C_2 = -C_1 C_2 \quad D_1 D_2 = -C_1 D_2 \]

There remains to be considered the singular terms (for \( n = 0 \), that is, the last four terms in (48) and the last four terms in (49)). Considering these terms with (50a) and (50b), as before, yields, in the first place

\[ F = (G'/2) \quad G = F' \]

(53)

But the term

\[ E r^2 \]

(54)

in the stream function, and the term

\[ E' \ln(r/a) \]

(55)

in the potential function do not have the conjugate orthogonal terms to match with. To deal with them we differentiate them according to (50a) and (50b). After integration, we discover two additional solutions.

We find that

\[ E \left( \frac{1}{2} r^2 - z^2 \right) \]

(56)

is a possible solution for the potential function; it serves as the orthogonal conjugate of (54). Similarly we find that
is a possible solution for the stream function; it serves as the orthogonal conjugate of (55).

Other systems, besides cylindrical coordinates, have the characteristics of axial symmetry; solutions in these systems may satisfy the differential equation in the cylindrical coordinates. (See Prandtl and Tietjens [33, pp. 138-154], Dryden, et al. [7, p. 79], and Morse and Feshbach [31, pp. 655, and 1252-1320]).

According to relations (50a) and (50b) we can derive the orthogonal conjugate to any such solution.

The following derivation will serve as an example. We make use of the solution of the potential function in spherical symmetry.

\[
\phi = m/(r^2 + z^2)^{1/2}
\]  

(58)

It is easy to verify that it satisfies the differential equation (7) for the potential function in cylindrical coordinates.

Equation (50a), when applied to (58), renders

\[
\frac{\partial \phi}{\partial z} = r \frac{\partial \phi}{\partial r} = -(mr^2)(r^2 + z^2)^{-3/2}
\]  

(59)

By integrating with respect to \(z\), we get

\[
\psi = -(zm)(r^2 + z^2)^{-1/2} + f(r)
\]  

(60)

Differentiating with respect to \(r\), we get
\[
\frac{\partial \psi}{\partial r} = (sr) (r^2 + z^2)^{-(3/2)} + \left( \frac{\partial f(r)}{\partial r} \right)
\]  
(61a)

Differentiation of (58) with respect to \( z \) gives

\[- r \left( \frac{\partial \psi}{\partial z} \right) = (sr) (r^2 + z^2)^{-(3/2)}\]
(61b)

Comparison of (61a) and (61b) according to condition (50b) shows that

\[\left[ \frac{\partial f(r)}{\partial r} \right] = 0\]

In conclusion, we have two new solutions for the potential function and stream function which are mutually orthogonal

\[\psi = \pi (r^2 + z^2)^{-(1/2)}\]
(62a)

and

\[\phi = -(sr) (r^2 + z^2)^{-(1/2)}\]
(62b)

It is easy to verify that (62b) satisfies the differential equation for the stream function (18). In a similar fashion we can derive more terms for \( \phi \) corresponding to other solutions \( \psi \) for flow with axial symmetry, and vice versa. Each term should be specially convenient for special boundary conditions.
C. Summary of Basic Flow Equations with Axial Symmetry and Uniform Conductivity

The orthogonality between the stream function and the potential function is expressed by the following conditions

\[ q_r = - k (\partial \phi / \partial r) = -(k/r) (\partial \psi / \partial s) \]  \hspace{1cm} (63a)

\[ q_s = - k (\partial \phi / \partial s) = + (k/r) (\partial \psi / \partial r) \]  \hspace{1cm} (63b)

Here, \( q_r \) and \( q_s \) are the flux components in \( r \) and \( s \) directions; \( k \) is the hydraulic conductivity; \( \phi \) is the velocity potential; \( \psi \) is the stream function; \( r \) and \( s \) are the radius and elevation in cylindrical coordinates.

The partial differential equations for a uniform conductivity \( k \) are the following

\[ \frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r}) + \frac{\partial}{\partial s} (r \frac{\partial \phi}{\partial s}) = 0 \]  \hspace{1cm} (64a)

\[ \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0 \]  \hspace{1cm} (64b)

The general solution for the potential function is the following

\[ \phi = \left[ A_1^i \sin(ns) + B_1^i \cos(ns) \right] \left[ A_2^i J_0(nr) + B_2^i K_0(nr) \right] \]

\[ + \left[ C_1^i \sinh(ns) + D_1^i \cosh(ns) \right] \left[ C_2^i J_0(nr) + D_2^i K_0(nr) \right] \]

\[ + E_1^i \ln(r/a) + F_1^i \ln(r/a) + G_2^i + L_2^i [(r^2/2) - s^2] \]

\[ + M_2^i (r^2 + s^2)^{-1/2} + H_2^i \]  \hspace{1cm} (65)
The general solution for the stream function is the following

\[
\phi = [A_1 \sin(n \phi) + B_1 \cos(n \phi)] r [A_2 J_1(m \rho) + B_2 K_1(m \rho)] \\
+ [C_1 \sinh(m \phi) + D_1 \cosh(m \phi)] r [C_2 J_1(m \rho) + D_2 K_1(m \rho)] \\
+ E_1 x^2 + F_1 y^2 + G_1 z + L_1 [s^2 + r^2 (1 - \ln \rho)] \\
+ (M_1 z)(x^2 + y^2)^{-1/2} + H_1
\]

(66)

The following are the relations between the coefficients of the orthogonally conjugate terms of the two solutions

\[
A_1 = B_1' \\
B_1 = -A_1' \\
A_2 = A_2' \\
B_2 = -B_2' \\
C_1 = D_1' \\
D_1 = C_1' \\
C_2 = C_2' \\
D_2 = -D_2' \\
E_1 = L_2' \\
F_1 = -(G_2'/2) \\
G_1 = F_2' \\
L_1 = 2E_2' \\
M_1 = -M_2'
\]

(67)

Any pair or several pairs of conjugate terms give the equipotentials and corresponding streamlines for a certain potential flow pattern.
III. APPLICATIONS OF AXIALLY SYMMETRIC FLOW
THEORY TO SPECIFIC PROBLEMS

A. Application of the Orthogonality between
the Stream Function and the Potential Function

1. Statement of the problem

Kirkham and van Bavel [23] solved a boundary value problem for the
seepage into an auger hole. The solution was found in terms of the
potential function $\phi$. The stream lines were drawn by graphical methods.
In this section the same problem will be solved in terms of the stream
function. The two solutions will be compared according to the cor­
relating list, suggested in Section II C, Equation (67).

The geometry of the problem and the boundary condition are
presented below. They are equivalent to the conditions in the original
paper by Kirkham and van Bavel. The boundary conditions are as follows

boundary la: $\frac{\partial \phi}{\partial r} = -a \quad h < z \leq d \quad r = a$
boundary lb: $\frac{\partial \phi}{\partial r} = 0 \quad 0 \leq z \leq h \quad r = a$
boundary 2: $\frac{\partial \phi}{\partial z} = 0 \quad z = d \quad a \leq r \leq b$
boundary 3: $\frac{\partial \phi}{\partial z} = 0 \quad 0 \leq z \leq d \quad r = b$
boundary 4: $\frac{\partial \phi}{\partial r} = 0 \quad z = 0 \quad a \leq r \leq b$

The condition on boundary la will be proved shortly. In addition, the
potential must be stated in two points of the medium. At the boundary
2, the potential is equal to the thickness, d. At boundary lb the
potential is equal to the water head in the auger hole h.

In terms of the potential, the condition on boundary la can be
written as follows
Figure 1. The boundaries for problem of water flow into an auger hole (Section III A 1) with ponded water.
PONDED WATER

IMPERMEABLE LAYER
\[ \phi = z \quad \text{or} \quad \frac{\partial \phi}{\partial z} = 1 \quad (1) \]

This condition is transformed according to Equation (63b) in the last section. We get

\[ (\phi/\partial r) = -r(\partial \phi/\partial s) = -r \quad (2) \]

This is equivalent to the condition at boundary la where \( r \) is equal to \( a \).

2. Solution in terms of the stream function

The detailed process by which the solution can be obtained will be presented in Section III B. The solution suggested in this section may be easily verified against the boundary conditions and the differential equation for the stream function.

The solution suggested for this boundary problem is (\( H \) being a constant)

\[ \phi = H - \sum_{n=1}^{\infty} \left( \frac{8d}{n^2 \pi^2} \right) \cos \left( \frac{nx}{2d} \right) \sin \left( \frac{ny}{2d} \right) r B_1 \left( \frac{nx}{2d} \right) \quad (3) \]

\[
\left( n = 1, 3, 5, \ldots \right)
\]

\[
B_1 \left( \frac{nx}{2d} \right) = \frac{I_1 (nx/2d) - K_1 (nx/2d)}{I_1 (nx/2d) + K_1 (nx/2d)} \quad (4)
\]

This solution is a part of the general solution of the stream function in Equation (66), Section II. For formulas concerning the Bessel functions the reader may refer to Dwight [8, pp. 175-191].
Figure 2. The streamlines for radial flow into an auger hole with ponded water. For basic dimensions and conditions see Figure 1. The specific dimensions are: \( xa/d = 0.2; h = 0 \). The measures on the axes are in units of \( x/d \).
The verification for conditions 2, 3, and 4 can be done in a straightforward manner. Fourier analysis is needed for the verification of conditions 1a and 1b.

3. Calculation of the discharge and the constant $H$

We set the following arbitrary condition

$$\psi = 0 \text{ at } r = a \text{ and } z = d$$

(5)

This condition is substituted in (3).

The following identity is considered

$$- \sin(nx/2) = (-1)^n \quad (n = 1, 3, 5, \ldots)$$

(6)

After substitution in (3) and rearrangement we arrive at the equation for $H$ as follows

$$H = \sum_{n=1}^{\infty} \frac{(8d/n^2 x^2) \cos \left(\frac{nxh}{2d}\right) aR_i \left(\frac{nx}{2d}\right)}{1/n} (-1)^{n-1/2}$$

(7)

$$(n = 1, 3, 5, \ldots)$$

By substituting (7) into (3), we can write the general solution explicitly

$$\psi = \sum_{n=1}^{\infty} \frac{(8d/n^2 x^2) \cos \left(\frac{nxh}{2d}\right) aR_i \left(\frac{nx}{2d}\right)}{1/n} (-1)^{n-1/2}$$

$$- \sum_{n=1}^{\infty} \frac{(8d/n^2 x^2) \sin \left(\frac{nx}{2d}\right) aR_i \left(\frac{nx}{2d}\right)}{1/n}$$

(8)

$$(n = 1, 3, 5, \ldots)$$
The calculation of the discharge $Q$ can be done directly from Equation (21), Section II,

$$Q = 2\pi k[\psi(r = b) - \psi(r = a)]$$

From condition 3 and Equation (5) we find the following

$$\psi(r = b) = H \quad (10a)$$

$$\psi(r = a) = 0 \quad (10b)$$

By substituting these values in Equation (9) we get

$$Q = 2\pi kH \quad (11)$$

The explicit value of $H$ can be taken from Equation (7) to give

$$Q = 2\pi k \sum_{n=1}^{\infty} \frac{8d}{2^2} \cos(\frac{nxh}{2d}) \text{e}_{\frac{n}{2}}^{\frac{\text{e}_{\frac{n}{2}}}{1}} (-1)(n - 1)/2$$

$$(n = 1, 3, 5, \ldots)$$

This result is identical with the one presented by Kirkham and van Bavel [23].

4. Comparison with the solution by the potential function

As already stated, the solution for the potential function was published by Kirkham and van Bavel [23]. With one exception the notation in their article is identical with the notation used here. Where $K$ is used there to denote the hydraulic conductivity, $k$ is instead used here. Capital $K$ is used here to denote the modified Bessel function of the second kind.
The analogy between the two solutions will be made by comparing coefficients of the potential and the stream function with Equations (65) and (66) in Section II, respectively, and checking them against the relation of Equation (67) in the same section.

From the solution in Equation (8) and from Equation (66) Section II, the following identities are found

\[ A_1 = -(8d/n^2 x^2) \cos(mxh/2d) \]  
\[ A_2 = 1/[I_1(mxh/2d)F] \]  
\[ B_2 = -1/[I_1(mxh/2d)F] \]

where \( F \) is the denominator on the right-hand side of Equation (4).

Comparison of Equations (6) and (3) in the original article by Kirkham and van Bavel [23] with Equation (65) in Section II, gives the following identities

\[ B_1' = -(8d/n^2 x^2) \cos(mxh/2d) \]  
\[ A_2' = 1/[I_1(mxh/2d)F] \]  
\[ B_2' = 1/[I_1(mxh/2d)F] \]

Equations (67) of Section II designate the following identities between coefficients

\[ A_1 = B_1' \]  
\[ A_2 = A_2' \]  
\[ B_2 = -B_2' \]
A comparison of (16) with (13), (17) with (14), and (18) with (15) shows that these identities, due to the orthogonality relation, are satisfied.

B. Flow into a Partially Filled Well Penetrating Down to an Impermeable Layer through a Confined Aquifer

1. Presentation of the problem and the boundary conditions

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Condition</th>
<th>Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>la</td>
<td>$\frac{\partial \psi}{\partial r} = -a$</td>
<td>$r = a$, $h &lt; s \leq d$</td>
</tr>
<tr>
<td>lb</td>
<td>$\frac{\partial \psi}{\partial r} = 0$</td>
<td>$r = a$, $0 \leq s \leq h$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\partial \psi}{\partial r} = 0$</td>
<td>$r = b$, $0 \leq z \leq d$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{\partial \psi}{\partial r} = 0$</td>
<td>$a \leq r \leq b$, $a = 0$</td>
</tr>
</tbody>
</table>

The explanation for condition la is brought out in Equations (1) and (2).

The reference level for measuring the potential $\varphi$ is boundary 4, where $z$ equals zero. The potentials are as follows

- at $r = b$, $\varphi = \varphi_2$  \hspace{1cm} (22a)
- at $r = a$, $z \leq h$ $\varphi = \varphi_1$  \hspace{1cm} (22b)

As long as the well is not filled and $h$ is smaller than $d$, $\varphi_1$ is equal to $h$. For all cases it is assumed that

$\varphi_2 \geq d$. 
Figure 3. The boundaries for the problem of flow (Section III B 1) into a partially filled well which penetrates down to an impermeable layer through a confined aquifer.
These boundary conditions cannot be attained in any practical field flow problem in the exact form presented above. It will be shown at the end of this section that at the circular corner \((r = a, z = d)\) the water cannot seep into the well at atmospheric pressure. The theory shows that unless we devise some unusual artificial arrangements over the walls of the well, there must be a region where the water is under tension (pressure lower than atmospheric) at the circular corner \((r = a, z = d)\). However, theory also shows that when the potential \(\phi_2\) is big enough, the tension region at the corner decreases to a negligible size and the following exact theories may be used to correspond to actual problems with a fair degree of accuracy.

2. Fitting for the homogeneous boundary condition

We first make the fitting for the singular terms \((n^2 = 0)\) and for the trigonometric-Bessel function combination \((n^2 \neq 0)\) later.

The general solution for the stream function is presented in Equation (66), Section II.

The linear terms are the following:

\[
H + Br^2 + Fr^2 + Gs + L[s^2 + r^2(1 - \ln \frac{r}{a})] + \frac{Ms}{(r^2 + s^2)^{1/2}}
\]

The derivative of these terms with respect to \(r\) gives the following terms

\[
2Brs + 2Fr + Lr - (Mar)(r^2 + s^2)^{-3/2} - 2Ir \ln \frac{r}{a}
\]
Conditions 2, 3, and 4 imply the following equations

For condition 2, \[ 2E d + 2F + L - (Md)(r^2 + d^2)^{-3/2} = 0 \] (25)

For condition 3, \[ 2E s + 2F + L - (Ms)(b^2 + s^2)^{-3/2} = 0 \] (26)

For condition 4, \[ 2F + L = 0 \] (27)

Condition 4 also requires that \( \psi = 0 \). Thus, the following is implied

\[ E + F + L[1 - \ln(r/a)] = 0 \] (28)

Substitution of (27) into (26) and (25) renders, after proper factoring out of \( d \) and \( s \),

\[ 2E - M(r^2 + d^2)^{-3/2} = 0 \]
\[ 2E - M(b^2 + s^2)^{-3/2} = 0 \]

The only possible solution of these two equations is

\[ E = M = 0 \] (29)

Substitution in (28) renders

\[ F = L = 0 \] (30)

From the original terms in (23) we have left only

\[ Gs + H \] (31)

The fitting for the nonhomogeneous boundaries, as we shall see, is done by determining the coefficients of an infinite series. The nonhomogeneous boundary in this problem is along \( r = a \). Consequently, the terms in the infinite series vary with \( s \). It is convenient to
match the boundary conditions using the Fourier sine or cosine series (rather than using the unmodified Bessel functions and their roots).

In view of this discussion and (31), only the following terms are left from the general solution:
\[ \psi = [A_1 \sin(nz) + B_1 \cos(nz)]r[A_2 I_1(nr) + B_2 K_1(nr)] + Gz + H \] (32)
Both \( I_1 \) and \( K_1 \) are finite in the range
\[ a \leq r \leq b \]
and are possible solutions.

For operations with \( I \) and \( K \) which are Bessel functions, see one of the references [8], [20], [31] or [11].

Differentiation of (32) with respect to \( r \) gives
\[ \frac{\partial \psi}{\partial r} = n[A_1 \sin(nz) + B_1 \cos(nz)]r[A_2 I_0(nr) - B_2 K_0(nr)] \] (33)
The homogeneous boundary conditions 2, 3, and 4 imply the following relations
for condition 2: \[ A_1 \sin(nd) + B_1 \cos(nd) = 0 \] (34)
for condition 3: \[ A_2 I_0(nb) - B_2 K_0(nb) = 0 \] (35)
for condition 4: \[ B_1 \cos(0) = 0 \] (36)
Equation (36) implies
\[ B_1 = 0 \] (37)
As a result, Equation (34) determines the characteristic numbers, the so-called eigenvalues,
\[ nd = \pi m \quad (m = 1, 2, 3 \ldots) \] (38)
\[ n = \pi x/d \]
Equation (35) is satisfied if $A_2$ and $B_2$ have the following values

$$A_2 = \frac{1}{[I_0(nb)](1/F)} \quad (39a)$$

$$B_2 = \frac{1}{[K_0(nb)](1/F)} \quad (39b)$$

$F$ can be any constant. For reasons which will become clear later, we choose for $F$ the following value

$$F = \frac{I_0(nx\alpha/d)}{I_0(nx\alpha/d)} - \frac{K_0(nx\alpha/d)}{K_0(nx\alpha/d)} \quad (40)$$

This is permissible so long as the values of $A_1$ are not yet determined. This factor $F$ would have appeared anyway in the course of the formal development.

Condition (4) requires, in addition, that

$$\phi = 0 \text{ at } s = 0 \text{ and } a \leq r \leq b$$

Thus, from Equation (32) it is found that

$$H = 0 \quad (41)$$

Condition (2) requires, in addition, the following

$$\phi = \phi_2 \text{ at } s = d \text{ and } a \leq r \leq b$$

Thus, from Equation (32) it is found that

$$G = \frac{\phi_2}{d} \quad (42)$$

We substitute (37), (38), (39), (40), (41), and (42) into Equation (32) to give a solution which satisfies all the homogeneous boundary conditions. A summation of all such solutions for different values of $m$ gives
\[ \phi = (\phi_2 z/d) + \sum_{n=1}^{\infty} A_n \sin \left( \frac{mn\pi}{d} \right) r R_{1} \left( \frac{mn\pi}{d} \right) \]  
\[ \quad (n = 1, 2, 3, \ldots) \]  

\( R_{1} \) is defined from (39a), (39b), and (40) as follows  

\[ R_{1} = \frac{I_{1}(mn\pi/d)}{I_{0}(mn\pi/d)} - \frac{K_{0}(mn\pi/d)}{K_{0}(mn\pi/d)} \]  

\[ \left( \frac{\partial}{\partial r} \right) [r R_{1}(mn\pi/d)] = r R_{0}(mn\pi/d) \]  

3. Fitting for the nonhomogeneous boundary conditions  

The conditions at boundary 1 may be expressed as follows  

\[ \frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad 0 \leq z \leq h \]  

\[ \frac{\partial \phi}{\partial r} = -a \quad \text{at} \quad h < z \leq d \]  

Differentiation of (43) with respect to \( r \) at \( r = a \) gives  

\[ \left( \frac{\partial \phi}{\partial r} \right) = \sum_{n=1}^{\infty} B_{n} \sin \left( \frac{mn\pi}{d} \right) A_{n} \sin \left( \frac{mn\pi}{d} \right) \]  

\( B_{0} \) is defined in Equations (44) and (45). From (45) it can be seen that at \( r = a \) we have  

\[ B_{0}(mn\pi/d) = 1 \]
This was the purpose of introducing the factor $F$ as defined in the Equation (40).

Equations (47) and (48) are introduced into (46). After dividing through by $a$, we get the following condition over boundary 1

$$\sum_{n=1}^{\infty} \frac{nx}{d} A_n \sin\left(\frac{nx}{d}\right) = 0 \quad \text{at} \quad 0 \leq s \leq h$$

and $= (-1)$ at $h < s \leq d \quad (49)$

Using Fourier analysis we calculate the $A_n$ from (49) in the usual way, obtaining

$$A_n = -\frac{2}{\pi} \int_{h}^{d} \frac{dx}{nx} \left[ \frac{nx}{d} \right] \left[ \cos\left(\frac{nx}{d}\right) - \cos\left(\frac{nx}{d}\right) \right]$$

By slight rearrangement we get

$$A_n = -(2d/n^2 \pi^2) \left[ \cos\left(\frac{nx}{d}\right) - (-1)^n \right] \quad (50)$$

Substitution into (43) gives us the solution

$$\phi = \left(\frac{\pi}{d}\right) - \sum_{n=1}^{\infty} \frac{2d}{\pi^2} \left[ \cos\left(\frac{nx}{d}\right) - (-1)^n \sin\left(\frac{nx}{d}\right) \right] \quad (n = 1, 2, 3, \ldots) \quad (51)$$

The only undetermined coefficient is $\phi_2$, which is proportional to the total discharge of the well and which we next consider.

4. Calculation of the coefficient $\phi_2$ and the discharge

The potential change along the radius $r$ is expressed by equation (63a), Section II, as follows

$$\frac{\partial \phi}{\partial r} = \frac{1}{4} \left( \frac{\partial \phi}{\partial s} \right) \quad (52)$$
The potential change is integrated between boundary 1 and boundary 3 and along boundary 4

from \((r = a, z = 0)\) where \(\varphi = \varphi_1\)
to \((r = b, z = 0)\) where \(\varphi = \varphi_2\)

\[
\varphi_2 - \varphi_1 = \int_a^b \frac{1}{r} \left( \frac{\partial \varphi}{\partial z} \right) dr
\]

From Equation (51) the explicit values of the derivative are introduced to the right-hand side of (53) and the integration performed as follows

\[
\varphi_2 - \varphi_1 = \int_a^b \frac{\psi}{r} dr
\]

\[
- \int_a^b \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \cos(\alpha_0) R_1(\alpha r/d) \frac{\alpha}{d} dr
\]

\[
(n = 1, 2, 3, \ldots)
\]

From the first integral on the right hand side we get

\[
[\psi_2 \ln(b/a)/d]
\]

The second integral can be written in the following form

\[
- \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \int_a^b R_1(\alpha r/d) dr
\]

The value of \(R_1\) is introduced from Equation (44)

\[
\int_a^b R_1(\alpha r/d) dr = \int_a^b \frac{I_1(\alpha r/d) - K_1(\alpha r/d)}{I_0(\alpha r/d) - K_0(\alpha r/d)} \\
+ \int_a^b \frac{I_0(\alpha r/d) - K_0(\alpha r/d)}{I_0(\alpha r/d) - K_0(\alpha r/d)} \\
+ \int_a^b \frac{I_0(\alpha r/d) + K_0(\alpha r/d)}{I_0(\alpha r/d) - K_0(\alpha r/d)}
\]

Using formulas 835.4 and 835.6 in Dwight [8] and the definition in
(45), the integration is performed as follows

\[ b \int_{a}^{b} R_{1}(nxr/d) dr = \frac{d}{nx} [R_{0}(\frac{nxb}{d}) - R_{0}(\frac{nya}{d})] = -\frac{d}{nx} \]  

(58)

Introduction of (58) into (56) and, in turn, introduction of (56) and (55) into (54) give the solution for the coefficient \( \phi_{2}^{*} \).

\[ \phi_{2}^{*} - \phi_{1}^{*} = \frac{\phi_{2}^{*}}{d} \ln \frac{b}{d} + \sum_{n=1}^{\infty} \frac{2d}{n^{2}2^{2}} [\cos(\frac{mnh}{d}) - (-1)^{n}] \]  

(59)

\( n = 1,2,3,\ldots \)

The discharge can be found by integrating the flux in the \( r \) direction

\[ -(k/r)(\partial \phi/\partial s) \]  

(60)

over the outside cylinder \( (r = b) \). Consequently we get the following

\[ Q = -\int_{a}^{d} 2\pi k (\partial \phi/\partial s) dz \quad (r = b) \]  

(61)

\[ Q = -2\pi k [\phi(r = b)] = -2\pi k \phi_{2} \]  

(62)

Equation (20), Section II, could be used to give the same results.

In view of Equation (59), Equation (62) can be written in the explicit form

\[ -Q = \{\phi_{2}^{*} - \phi_{1}^{*} - \sum_{n=1}^{\infty} \frac{2d}{n^{2}2^{2}} [\cos(\frac{mnh}{d}) - (-1)^{n}] \} \frac{2\pi k \pi}{\ln(b/a)} \]  

(63)

\( n = 1,2,3,\ldots \)

The minus in front of \( Q \) signifies that the well acts like a sink.
5. Expressing the infinite series in a closed form

The infinite series of Equation (59) is

$$
\sum_{n=1}^{\infty} \left(2d/n^2 x^2\right)[\cos(mx/d) - (-1)^n] \quad (n = 1, 2, 3, \ldots) \quad (64)
$$

It can be summed analytically and expressed in a closed form. Equation (64) is divided into two separate series. The first part is summed as follows

$$
(-2d/x^2) \sum_{n=1}^{\infty} (-1)^n/n^2 = d/6 \quad (65)
$$

The summation is performed according to formula 48.3 in Dwight [8].

The second part of (64) can be transformed in the following fashion

$$
(2d/x^2) \sum_{n=1}^{\infty} (1/n^2)\cos(mx/d) = -2d/x^2 \int \sum_{n=1}^{\infty} (\pi/nd)\sin(mx/d)dh + C \quad (n = 1, 2, 3, \ldots) \quad (66)
$$

In Dwight [8], formula 416.08, we find the following summation

$$
x = x - 2 \sum_{n=1}^{\infty} (1/n)\sin(nx), \quad (n = 1, 2, 3, \ldots) \quad (67)
$$

By rearranging we get

$$
\sum_{n=1}^{\infty} (1/n)\sin(nx) = (x - x)/2 \quad (68)
$$

We identify $x$ with the argument ($mx/d$). By substituting this summation into the right-hand side of (66), we get
After integration, the right-hand side becomes

\[(h^2/2d) - h + C \quad (70)\]

where \(C\) is the constant of integration. It is evaluated by imposing boundary conditions on the original infinite series (left-hand side of (69)). The summation should be valid for all values of \(h\) between zero and \(d\). Substituting into the series \(h = 0\) gives the following summation

\[(2d/\pi^2) \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \right) \cos(mh/d) = \int [1 - (h/d)] dh + C \quad (69)\]

By introducing \((h = 0)\) into (70) and equating to the right-hand side of (71), we find that

\[C = d/3 \quad (72)\]

Using \((h = d)\) as a boundary condition would have given the same result.

Combination of (65), (71), and (72) gives the closed form expression for the infinite series.

\[(2d/\pi^2) \sum_{n=1}^{\infty} \left( \frac{1}{n^2} \right) \cos(mh/d) = (d/3) - [(d/2) + (h^2/2d) - h] \quad (73)\]

\((n = 1, 2, 3, \ldots)\)

The summation can be verified in an alternative form by the use of Fourier analysis. We assume the following

\[\sum_{n=0}^{\infty} A_n \cos(mh/d) = f(h), \quad (n = 1, 2, 3, \ldots) \quad (74)\]
The values of $A_n$ for $n \neq 0$ are given by Equation (64)

$$A_n = \left(\frac{2d}{n^2\pi^2}\right)$$

(75)

From Fourier analysis for the coefficients of the cosine series we have the following integral equations

$$(\frac{2d}{\pi^2 n^2}) = (\frac{2}{d}) \int_0^d f(h) \cos(n\pi h/d) dh \quad n \neq 0$$

(76)

$$A_0 = (\frac{1}{d}) \int_0^d f(h) dh$$

(77)

The first equation is solved for $f(h)$. The following result can be verified

$$f(h) = \left(h^2 / 2d\right) - h$$

(78)

Substitution of this solution in the second equation, (77) gives

$$A_0 = (\frac{1}{d}) \int_0^d \left(h^2 / 2d - h\right) dh = -d/3$$

(79)

The series in (74) can now be written as follows

$$\sum_{n=0}^{\infty} A_n \cos(n\pi h/d) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi h/d)$$

(80)

Consequently, transferring $A_0$ to the other side of the equation, we get

$$2d/\pi^2 \sum_{n=1}^{\infty} \left(1/n^2\right) \cos(n\pi h/d) = f(h) - A_0$$

(81)

$$= (h^2 / 2d) - h + (d/3)$$

This verifies the summation, as found in (73).
6. Summary of the general solution for the stream function

The solution for the boundary value problem is summarized from Equations (44), (51), (59), (62) and (73).

\[
\psi = \frac{\psi_2}{d} - \sum_{n=1}^{\infty} \left( \frac{2d}{\pi^2} \right)^n \left[ \cos \left( \frac{\pi x}{d} \right) - (-1)^n \sin \left( \frac{\pi x}{d} \right) R_1 \left( \frac{\pi x}{d} \right) \right] (82)
\]

\( R_1 \) is defined in Equation (44).

\[ \psi_2 - \psi_1 = \left( \frac{\psi_2}{d} \right) \ln(b/a) + d/2 + h^2/2d - h \] (83)

\[ \psi_2 = (\psi_2 - \psi_1 - d/2 - h^2/2d + h)[d/\ln(b/a)] \] (84)

\[ Q = - 2\pi k \psi_2 \] (85)

Where \( k \) is the hydraulic conductivity. The other symbols are defined in Section III B 1.

Special cases of this boundary value problem will be discussed in the following sections.

7. The case of a partially filled well (0 < h < d)

Let the water level \( h \) be smaller than the thickness of the aquifer \( d \). In this case \( h \) and \( \psi_1 \) may be considered as identical.

Equations (83), (84), and (85) reduce to the following

\[ \psi_2 = \left( \frac{\psi_2}{d} \right) \ln(b/a) + d/2 + h^2/2d \] (86)

\[ \psi_2 = (\psi_2 - d/2 - h^2/2d)[d/\ln(b/a)] \] (87)

\[ Q = - 2\pi k \psi_2 \] (88)
Figure 4. A nomogram for computing the dimensionless discharge \( Q/2x_d^2k \) into a partially filled well. (Section III E 7, Equation (88)). The dimensionless discharge is given by the value assigned to each sloping line where it intersects the right side of the drawing or the top of the drawing. For example, all points \((F, b/a)\) on the line of smallest slope and designated 6.0 at the right of the drawing have the values \( Q/2x_d^2k = 6.0 \). A point \((F, a/b)\) is computed from values of \( a, b, \phi_2, d \) and \( h \).
8. The case of a well full of water \((h \geq d)\)

Let the water level \(h\) be equal to or bigger than the thickness of the aquifer \(d\). In this case, we substitute in (83), (84), and (85), \(d\) for \(h\). Equations (82), (83), (84), and (85) reduce to the following:

\[
\frac{\phi}{(\phi_2/d)z} = \sum_{n=1}^{\infty} \frac{2d/n^2\pi^2}{1 - (-1)^n} \sin\left(\frac{\text{max}}{d}\right) r_{1n} \left(\frac{\text{max}}{d}\right)
\]

(89)

\[
\phi_2 = \frac{(\phi_2 - \phi_1)d}{\ln(b/a)}
\]

(90)

\[
Q = -\frac{(\phi_2 - \phi_1)2\pi d}{\ln(b/a)}
\]

(91)

This is the well-known case of radially symmetric, horizontal flow into a well.

9. The case of an empty well \((\phi_1 = h = 0)\)

By proper substitution, Equation (82) reduces to the following:

\[
\phi = (\phi_2/d)z - \sum_{n=1}^{\infty} \frac{2d/n^2\pi^2}{1 - (-1)^n} \sin\left(\frac{\text{max}}{d}\right) r_{1n} \left(\frac{\text{max}}{d}\right) (n = 1, 2, 3, \ldots)
\]

(92)

The terms in the summation vanish for even values of \(n\). The solution is therefore limited for only the odd values of \(n\) and is

\[
\phi = (\phi_2/d)z - \sum_{m=1}^{\infty} \frac{4d/m^2\pi^2}{\text{max}^2} \sin\left(\frac{\text{max}}{d}\right) r_{1m} \left(\frac{\text{max}}{d}\right) (m = 1, 3, 5, \ldots)
\]

(93)

Equation (83) reduces in this case to the following
\[ \phi_2 = (\psi_2/d)\ln(b/a) + d/2 \]  

(94)

Similarly, (84) and (85) reduce to the following

\[ \phi_2 = [(\phi_2 - d/2)d]/[\ln(b/a)] \]  

(95)

\[ Q = -[(\phi_2 - d/2)2kd]/[\ln(b/a)] \]  

(96)

10. The solution in terms of the potential function \( \phi \)

The solution for the potential function is derived directly from the stream function (82). We make use of the correlation list derived from the orthogonality condition. By reference to this list (67) Section II, and to the stream function (82) we obtain the potential function \( \phi \) as follows

\[ \phi = (\phi_2 - d/2 - h^2/2d)\frac{\ln(r/c)}{\ln(b/a)} + H \]

\[ - \sum_{n=1}^{\infty} \frac{2d/n^2 \pi^2}{\cos(\frac{\pi nh}{d}) + (-1)^n} \cos(\frac{\pi nk}{d}) R_o \left(\frac{\pi k}{d}\right) \]  

(n = 1, 2, 3, ...)

(97)

H and c are two coefficients yet to be determined. \( R_o \) is an expression in terms of Bessel functions. It is defined in Equation (45), Section III.

We impose on (97) the following conditions

\[ \phi = h \quad \text{at} \quad r = a \quad \text{and} \quad s = 0 \]

\[ \phi = \phi_2 \quad \text{at} \quad r = b \]

These conditions when applied to (97) imply the following
\[ h = (\varphi_2 - d/2 - h^2/2d)\frac{\ln(a/c)}{\ln(b/a)} + H - \sum_{n=1}^{\infty} \frac{2d}{2n^2 \pi} (\cos \frac{mn \pi h}{d} - (-1)^n) \]  \hspace{1cm} (98)

\[ \varphi_2 = (\varphi_2 - d/2 - h^2/2d)\frac{\ln(b/c)}{\ln(b/a)} + H \]  \hspace{1cm} (99)

Let \( c = a \), then

\[ H = d/2 + h^2/2d \]  \hspace{1cm} (100a)

Let \( c = b \), then

\[ H = \varphi_2 \]  \hspace{1cm} (100b)

Simple substitution can show that both solutions are compatible with all the other boundary conditions. To make such a verification, one should make use of the following summations

\[ \sum_{n=1}^{\infty} \left( \frac{2d/n^2 \pi^2}{2} \right) = d/3 \]  \hspace{1cm} (101a)

\[ \sum_{n=1}^{\infty} \left( \frac{2d/n^2 \pi^2}{2} \cos(\frac{m \pi h}{d}) - (-1)^n \right) = h^2/2d - d/6 \]  \hspace{1cm} (101b)

In conclusion we may write the solution for the potential function \( \varphi \) in one of the following forms

\[ \varphi = (\varphi_2 - d/2 - h^2/2d)[\ln(r/a)/(b/a)] + d/2 + h^2/2d \]

\[ - \sum_{n=1}^{\infty} \left( \frac{2d/n^2 \pi^2}{2} \cos(\frac{m \pi h}{d}) - (-1)^n \right) \cos(\frac{mx \pi}{d}) \mathcal{R}_0(x \pi/s) \]  \hspace{1cm} (102a)

or

\[ \varphi = (\varphi_2 - d/2 - h^2/2d)[\ln(r/b)/\ln(b/a)] + \varphi_2 \]

\[ - \sum_{n=1}^{\infty} \left( \frac{2d/n^2 \pi^2}{2} \cos(\frac{m \pi h}{d}) - (-1)^n \right) \cos(\frac{mx \pi}{d}) \mathcal{R}_0(x \pi/d) \]  \hspace{1cm} (102b)
Figure 5. Flow net for flow into a partially filled well penetrating down to an impermeable layer, through a confined aquifer. For basic dimensions and boundary conditions, see Fig. 3 of Section III B 1. The flow net is computed for the following dimensions: \( h/d = 1/2; \ \varphi_2/d = 1; \ b/a = 100 \). The measures on the axes are in units of \( \pi/d \). The point designated by S is a stagnation point. As \( \varphi_2 \) increases, S moves towards the well. Values of the potential are given in terms of \( \varphi/d \). Values of the stream function are given in percent of the total discharge.
Figure 6. Flow net for flow into a partially filled well penetrating to an impermeable layer through a confined aquifer. For basic dimensions and boundary conditions, see Fig. 3 of Section III B 1. The flow net is computed for the same dimensions as in Fig. 5. The potential $\phi_2$ was changed to equal 35 d/ft. Note that the stagnation points $S$ could not be resolved here with the degree of accuracy used for computation and drawing. The stagnation point actually moved so close to the well that it became impossible to separate it, in the figure, from the well.
Figure 7. The translation of the stagnation point $S$ towards the well as a result of increased potential $\phi_2$. For basic dimensions and boundary conditions, see Fig. 3 of Section III B 1. The computation of the flow net and the stagnation point in the circular corners at the top of the well was done for the dimensions $h/d = 1/2$ and $b/a = 100$. 
11. The validity of the boundary conditions at the well

From Equations (102a) or (102b) we may calculate the partial derivative with respect to \( r \) of the potential \( \phi \) at the point \( (r = a, z = d) \). It can be shown that this derivative is negative. This implies that at this circular corner the water would flow from the well out, according to the conditions assumed there (condition 1a).

We investigate the problem by asking how big should \( \phi_2 \) be to make this derivative zero.

Equation (102) is differentiated with respect to \( r \) and equated to zero. For simplification we assume the following dimensions to the flow medium:

\[
\begin{align*}
\frac{h}{d} &= \frac{1}{2} \\
\frac{a}{b} &= \frac{1}{100} \\
\frac{a}{d} &= \frac{1}{100}
\end{align*}
\]

A flow problem with these dimensions and with \( \phi_2 = d \) is computed and the flow net drawn. After rearrangement we solve for \( \phi_2 \) with these geometrical dimensions. We find

\[
\frac{\phi_2}{d} = 5/8 - \frac{100}{5\pi^2} \sum_{n=1}^{\infty} \frac{K_1(n/10)}{K_0(n/10)} \left[ \cos\left(\frac{\pi n}{2}\right) (-1)^n - 1 \right]_n
\]

\((n = 1, 2, 3, \ldots)\)

It is easy to verify that the above equation does not converge. We can conclude that \( \phi_2 \) must be infinitely big to prevent the formation of
a tension region at the circular corner \((r = a, z = d)\).

If, however, we make the above test for the sign of \(\partial \theta / \partial r\) not at the very corner, but slightly away at \(r = a + \Delta r\) and \(z = d - \Delta z\), we can find \(\theta_2\) big enough to form a positive derivative \(\partial \theta / \partial r\), or in other words, flow towards the well. By taking large enough values of \(\theta_2\), the region of back flow, or tension, shrinks as much as we want, and \(\Delta s\) and \(\Delta r\) can be as small as we wish.

We conclude that the analysis in the preceding sections and in the following section C, is valid from a practical point of view when \(\theta_2\) is big enough. The advantage in using this theory is in its simplicity, compared with a case where tension regions must be taken into account in the boundary conditions.

The discharge as calculated in this section is probably very close to the true discharge. A discussion of this statement will be found in Section III C 2.

C. Approximation of the Solutions for Large Radial Dimensions

1. The case of large radius of influence \(b\)

The discussion in this section and those following pertain to the boundary value problem that was solved in the preceding section. The procedures used for approximation can also be applied to many other problems.

The exact solutions for the different cases are summarized in Equations (82-97).
The radius of influence $b$ is the distance from the axis of symmetry to the cylindrical envelope which is considered the outside boundary of the flow medium (see Section III B 1). A useful approximation of $R_1$, the term including Bessel functions in the solution (82), can be achieved for large values of $b$ and limited values of the radius $r$.

$R_1$, as defined in Equation (44), can be expressed in the following equations

$$R_1 = \frac{\frac{K_1(y)}{K_0(x)} \frac{I_1(y)}{I_0(x)} \frac{K_0(x)}{K_0(z)}}{\frac{K_0(z)}{K_0(x)} \frac{I_0(z)}{I_0(x)} - 1} \quad \text{(103)}$$

$$x = \frac{\pi b}{d} \quad \text{(104a)}$$

$$y = \frac{\pi r}{d} \quad \text{(104b)}$$

$$z = \frac{\pi a}{d} \quad \text{(104c)}$$

For ease of reference we further abbreviate the expression for $R_1$

$$R_1 = \frac{K_1(y)}{K_0(z)} \left[ \frac{A/B + 1}{C/B - 1} \right] \quad \text{(105)}$$

$$A = \frac{I_1(y)}{K_1(y)} \quad \text{(106a)}$$

$$B = \frac{I_0(x)}{K_0(x)} \quad \text{(106b)}$$

$$C = \frac{I_0(z)}{K_0(z)} \quad \text{(106c)}$$
Figure 8. A diagram for estimating the validity range of the approximation presented in Section III C 1, Equation (107).
Let \((A/B)\) and \((C/B)\) be much smaller than unity. If so, the expression for \(R_1\) can be reduced to the following

\[
R_1 = \frac{K_1(mw/d)}{K_0(mw/d)}
\]  

(107)

Figure 8 is an aid for estimating the validity of this approximation. The following functions are drawn there (Fig. 8)

\[
100A = \frac{100I_1(y)}{K_1(y)}
\]

\[
100C = \frac{100I_0(s)}{K_0(s)}
\]

\[
B = \frac{I_0(x)}{K_0(x)}
\]

When \(100A = B\), the approximation in the numerator of (105) involves an error of about 1 percent. When \(100C = B\), the approximation in the denominator of (105) involves an error of about 1 percent. The total error in the approximation of \(R_1\) is approximately the sum of the errors in the numerator and the denominator. By observing the diagram in Fig. 8 it is realized that if the error is negligible for the first term in the summation \((n = 1)\), it is certainly negligible for greater \(n\) values. The abscissas in Fig. 8 describe in relative terms the permissible ranges of the approximation.

2. The discharge in case of large values of \(b\)

It is remarkable that the calculation of the discharge \(Q\) does not involve series summation (see (84)). Another important feature of the
equation for the discharge is that it is not sensitive to slight changes in the radius \( b \). It can be seen from Equation (102) that the vertical or horizontal variation of the potential becomes negligible for large values of the radius. This characteristic allows us to choose the condition at \( r = b \) with some degree of arbitrariness.

There are several intrinsic inadequacies in the assumed boundary conditions as described in Section III C 1. One such inaccuracy is discussed in Section III B 11. Additional inaccuracies are discussed below.

It is rarely the case that the potential \( \Phi \) would be exactly constant at the cylindrical surface \( r = b \). In field measurements or in theoretical computations, we assume a value for \( b \), where the potential would be expected to vary little in space.

Another inadequacy evolves from the assumption that the flow is in a steady state. This assumption would have been exact if we could consider the soil as a perfectly rigid medium and completely saturated with incompressible fluid. In such a hypothetical case steady state could be achieved at once. Disturbances of elastic nature move at the speed of sound. They undergo a fast damping. Thus, as far as filtration flow is concerned we may neglect these elastic disturbances and assume the existence of a steady state. When the ground water forms a free surface or when the soil is otherwise not perfectly saturated, fluctuation of the water level in the wall will transmit into the soil
disturbances of much slower nature. As we proceed farther from the well, the time necessary for achieving a relative steady state increases. Fortunately, however, the magnitude of the disturbance decreases with the distance at an exponential rate.

For discussion about these problems the reader may refer to Tissis [42] or Jacob [18] and [19].

The mathematical nature of Laplace's equation and the equation of diffusion for soil water flow imply the damping of local variations. It implies that flow conditions at any point depend mostly on the boundaries closest to it. This dependence decreases exponentially with the distance from the boundaries. See Horse and Feshbach [31, pp. 696-706].

A conclusion of utmost importance and of great usefulness can be drawn here about the techniques of approximate solutions. Let us be interested in the flow solution at point A. For the convenience of the solution we may alter the boundary conditions at a point B. The error involved in this arbitrary alteration will be smaller, as the distance between A and B will be larger, compared with the distance to other boundaries. For the calculation of the discharge it may suffice to solve exactly only for a limited subregion out of the whole flow medium. Consequently, if we are not interested in plotting the flow net for the whole region but only in estimating the discharge,
alteration of the boundaries may be made that would simplify considerably the solution.

The consequences to the boundary value problem presented here were already discussed. The solution close to the well and the discharge are insensitive to variations at \( r = b \) if \( b \) is large. Similarly, as discussed in Section III B 11, the discharge is well approximated, according to our boundary conditions, when \( b \) and \( \varphi_2 \) are large. It should be noted that \( b \) cannot be extended to infinity, as in that case for any nonzero discharge the potential must be infinitely large.

3. The case where both the radius of the well \( a \) and the radius of influence \( b \) are large

We make use of the asymptotic representation of the hyperbolic Bessel functions (modified Bessel functions) for large arguments. (See Hildebrand [11, p. 161], Morse and Feshbach [31, p. 1324] or Dwight [8, pp. 181-182]).

For large arguments we can express the following approximations

\[
I_p(n \pi r/d) \approx e^{n \pi r/d} (2n \pi r/d)^{-1/2} \tag{108a}
\]

\[
K_p(n \pi r/d) \approx e^{-n \pi r/d} (2n \pi r/d)^{-1/2} \tag{108b}
\]

Substituting these approximations in the expressions for \( R_1 \) and \( R_0 \), as defined in Equations (44) and (45), we get the following
After rearrangement, these approximations can be written in terms of hyperbolic functions as follows

\[ R_1 = -(a/r)^{1/2} \frac{\cosh((nx/d)(b-r))}{\sinh((nx/d)(b-a))} \]  

(110a)

\[ R_0 = +(a/r)^{1/2} \frac{\sinh((nx/d)(b-r))}{\sinh((nx/d)(b-a))} \]  

(110b)

Substitution of \( R_1 \) in the solution for the stream function and \( R_0 \) into the solution of the potential function, gives

\[ \psi_c = \psi_2 \frac{z}{d} + \]

\[ + \sum_{n=1}^{\infty} \frac{2d}{n^2 \pi^2} \cos \frac{nxh}{d} \cos \frac{nxz}{d} \frac{r}{r}^{1/2} \frac{\cosh[\frac{nx(b-r)}{d}]}{\sinh[\frac{nx(b-a)}{d}]} \]  

(111a)

and

\[ \varphi_c = (\psi_2/d) \ln(r/b) + \varphi_2 \]

\[ - \sum_{n=1}^{\infty} \frac{2d}{n^2 \pi^2} \cos \frac{nxh}{d} \cos \frac{nxz}{d} \frac{r}{r}^{1/2} \frac{\sinh[\frac{nx(b-r)}{d}]}{\sinh[\frac{nx(b-a)}{d}]} \]  

(111b)

where

\[ \psi_2 = (\varphi_2 - d/2 - h^2/2d)[d/\ln(b/a)] \]  

(111c)

The subscript \( c \) designates that these solutions are in terms of
cylindrical coordinates in an axially symmetric flow.

It is interesting to note the similarity between these approximate solutions and a solution for a truly two-dimensional flow.

The terms in infinite sine and cosine series are, to save a "correction factor," the same as the expressions found for a two-dimensional flow with the same boundary conditions. The "correction factor" for the stream function is obviously

\[ r \left( \frac{a}{r} \right)^{1/2} \]  

(112a)

The correction factor for the potential function is

\[ \left( \frac{a}{r} \right)^{1/2} \]  

(112b)

The velocity potential has the same dimensions and the same physical meaning both in the two-dimensional flow and in the axially symmetric flow. In both cases the solution is derived from Laplace's equation. It is clear that as \( r \) and \( a \) increase and their difference is constant, the correction factor approaches unity. The logarithmic terms in the solution for \( \varphi \) (111b) can be approximated by expansion in Taylor's series and neglecting all but the first term, as follows

\[
(\varphi_2 - d/2 - h^2/2d) \frac{\ln(r/a)}{\ln(b/a)} = - (\varphi_2 - d/2 - h^2/2d) \frac{b - r}{b - a} 
\]  

(113)

We make the following translation of the coordinate system in the \( r \) direction
\[ \rho = (b - r) \]
\[ B = (b - a) \]

Consequently for very large radial dimensions, the solution for the potential reduces to the exact two-dimensional form in coordinates \( z \) and \( \rho \):

\[ \varphi = \varphi_2 - \left[ \varphi_2 - \frac{\alpha}{2} - \frac{h^2}{2d} \right] \frac{\rho}{B} \]

\[ - \sum_{n=1}^{\infty} \frac{2d}{2 \pi n^2} \left[ \cos \frac{\pi n h}{d} - (-1)^n \right] \cos \frac{\pi n x}{d} \frac{\sinh(\pi n \rho/d)}{\sinh(\pi n B/d)} \]  

This reduction should not be a surprise. It is obvious that as the internal and external radiuses of a cylindrical ring increase, any segment of this ring becomes in the limit a straight domain with uniform cross-section. The flow becomes truly two-dimensional.

The stream functions in the axially symmetric flow and the two-dimensional flow do not have the same dimensions and have somewhat different physical meaning. This becomes obvious by observing the correction term (112a). The cylindrical stream function reduces to the two-dimensional stream function only after it is divided through by \( r \).

\textbf{A. Differences between the two-dimensional and the axially symmetric stream functions}

It was already recognized in Section II of this thesis that the differential equation for the stream function in cylindrical coordinates
is not a Laplace equation. The conditions for orthogonality between the stream function and the potential function were found to be the following

\[ \frac{\partial \psi_c}{\partial r} = -r \frac{\partial \phi}{\partial z} \]  \hspace{1cm} (115a)

\[ \frac{\partial \psi_c}{\partial z} = r \frac{\partial \phi}{\partial r} \]  \hspace{1cm} (115b)

As before, c designates the fact that the stream function is defined in terms of cylindrical coordinates. In the two-dimensional flow the orthogonality between the stream function and the potential function is secured by Cauchy Riemann conditions.

\[ \frac{\partial \psi_p}{\partial x} = -\frac{\partial \phi}{\partial y} \]  \hspace{1cm} (116a)

\[ \frac{\partial \psi_p}{\partial y} = \frac{\partial \phi}{\partial x} \]  \hspace{1cm} (116b)

The subscript p designates the fact that the stream function is defined in terms of two-dimensional (planar) cartesian coordinate system. Let the potential be defined in the same way for both two-dimensional and radial flow; then we find the following relations between the stream functions

\[ \frac{\partial \psi_p}{\partial x} \text{ corresponds to } \frac{1}{r}(\frac{\partial \psi_c}{\partial r}) \]  \hspace{1cm} (117a)

\[ \frac{\partial \psi_p}{\partial z} \text{ corresponds to } \frac{1}{r}(\frac{\partial \psi_c}{\partial z}) \]  \hspace{1cm} (117b)

Thus the observations in the preceding section are explained. It is clear why the two-dimensional stream function is measured by length
units and why the axially symmetric stream function is measured by area units.

An instructive explanation can be obtained by comparing the calculation of the discharge in the two systems.

In the axially symmetric cylindrical case the discharge (between \( \psi_c = 0 \) and \( \psi_c = \psi_0 \)) is calculated as follows (see (20) Section II).

\[
Q_c = - 2 \pi k \psi_c \tag{118}
\]

This discharge penetrates through a cylinder of radius \( r \) and circumference \( 2 \pi r \), or an arc of the central angle \( \Theta_0 \) equal to \( 2 \pi \). The discharge through a cylindrical section of any other central angle \( \Theta \) is

\[
Q_{c \Theta} = - 2 \pi k \psi_c \tag{119}
\]

The angle \( \Theta \) can be expressed by the arc length \( L \) and its radius \( r \).

\[
\Theta = \frac{L}{r} \tag{120}
\]

By substitution into (119) we obtain

\[
Q_{c \Theta} = Q_L = \left( \frac{L}{r} \right) k \psi_c \tag{121}
\]

We compare this expression with the discharge as calculated in a two-dimensional flow along a straight segment of the length \( L \) with a uniform cross section.

\[
Q_{p \Theta} = L k \psi_p \tag{122}
\]

Again, the relation between the two stream functions is demonstrated.
In making the approximation suggested in the preceding section, we can draw the following rules

a. A correction term for the case of large $b$ and $a$ must be applied to the term including trigonometric and hyperbolic functions in the form $(a/r)^{1/2}$.

b. The stream function in the axially symmetric system must be divided first by the radius $r$, if we want to use it in the two-dimensional sense.

c. For the purpose of calculating the discharge, the radius $r$ is measured to the arc along which the length of the circular arc segment $L$ is measured.

B. The General Theory for Flow through Slightly Curved, Two-Dimensional Segments with Uniform Conductivity

When seepage occurs through slightly curved media, the equations of axially symmetric flow may be used, but it is more convenient to consider such problems as two-dimensional ones to which a correction factor is applied. The methodology for this correction factor is described below for uniform conductivity. The methodology for non-uniform conductivity will be described in Sections IV 2 C f and IV 3 C f.

1. The general solution for the potential $\phi$

The differential equation for the velocity potential in the axially symmetric flow is

$$\frac{d^2 \phi_c}{dr^2} + \frac{1}{r} \frac{d \phi_c}{dr} + \frac{\partial^2 \phi_c}{\partial s^2} = 0$$

(123)
The subscript $e$ designates that it is the potential in terms of the cylindrical coordinates for an axially symmetric flow.

We suggest the following transformation

$$
\Phi_e = \left(\frac{r}{c}\right)^{1/2} \Phi_p = \left(\frac{r}{c}\right)^{1/2} R(r) \cdot Z(s) \quad (124)
$$

$R$ is a function of the radius $r$ only and $Z$ is a function of the elevation $s$ only. The coefficient $c$ has the dimensions of length and will be determined by the boundary conditions. Obviously, both $\Phi_e$ and $\Phi_p$ are measured by units of length. By substitution of (124) and by separation of the variables, we obtain the following two ordinary differential equations

$$
d^2E/dr^2 + R[1/4\pi^2 + n^2] = 0 \quad (125a)
$$

$$
d^2Z/ds^2 - n^2 Z = 0 \quad (125b)
$$

This transformation is found useful in solving more complex cases of non-uniform conductivities in Section IV B 4.

Let $n^2$ be different from zero and let $r$ be large, in which case Equation (125a) is reduced to

$$
d^2E/dr^2 + n^2 R = 0 \quad (126)
$$

By defining $\Phi_p$ as below, Equations (125b) and (126) may be combined to give Laplace's equation for two-dimensional velocity potential in the form

$$
\frac{\partial^2 \Phi_p}{\partial x^2} + \frac{\partial^2 \Phi_p}{\partial z^2} = 0 \quad (127)
$$
Let \( n = 0 \), then we cannot neglect the term \( 1/4r^2 \) in (125a). For this singular case we must use the original equation in cylindrical coordinates

\[
d^2R/dr^2 + R/4r^2 = 0 \quad (n = 0)
\]

The general solution for the nonzero \( n^2 \) is found from (125b) and (126) as follows

\[
\varphi_p (n \neq 0) = \left[ A_1 \sin(nz) + B_1 \cos(nz) \right] \left[ A_2 \sinh(nr) + B_2 \cosh(nr) \right] + \left[ C_1 \sinh(nz) + D_1 \cosh(nz) \right] \left[ C_2 \sin(nr) + D_2 \cos(nr) \right]
\]

The solution when \( n^2 = 0 \) is found from (129).

\[
\varphi_p (n = 0) = r^{1/2} \left[ E \ln(r/a) + F \right] \left[ Gs + H \right]
\]

Combination of (130) and (131) gives, in view of transformation (124), the result

\[
\varphi_c = (a/r)^{1/2} \varphi_p
\]

\[
= \left[ A_1 \sin(nz) + B_1 \cos(nz) \right] \left[ A_2 \sinh(nr) + B_2 \cosh(nr) \right] (a/r)^{1/2} + \left[ C_1 \sinh(nz) + D_1 \cosh(nz) \right] \left[ C_2 \sin(nr) + D_2 \cos(nr) \right] (a/r)^{1/2} + \left[ E' \ln(r/a) + F' \right] \left[ G's + H' \right]
\]

As it may be observed in the developments of the preceding section, both \( \varphi_p \) and \( \varphi_c \), for the singular case \( n^2 = 0 \), will approach the two-dimensional solution as \( r \) increases. It is therefore possible, when
the detailed solution is not known explicitly, to apply the same correction \((\alpha/r)^{1/2}\) for the whole solution. Even if we cannot find the complete solution, it may sometimes be possible to determine the singular terms for \(n = 0\) separately. The above methods of approximation could then be applied in an exact form.

2. The general solution for the stream function

The differential equation for the stream function was found in Section II of this thesis. It is the following

\[
\frac{\partial^2 \psi_c}{\partial r^2} - (1/r)\frac{\partial \psi_c}{\partial r} + \frac{\partial^2 \psi_c}{\partial z^2} = 0
\] (133)

In Section II we made the following substitution

\[
\psi_c = rY(r)Z(z)
\] (134)

For our discussion here we suggest a further transformation as follows

\[
Y = (\alpha/r)^{1/2}U(r)
\] (135)

\(\alpha = \text{a constant}\)

By substituting (135) into (134) and in turn (134) into (133), we can separate the variables and obtain the following ordinary differential equations

\[
\frac{d^2 Z}{ds^2} - n^2 Z = 0
\] (136a)

\[
\frac{d^2 U}{dr^2} + U[n^2 - 3/4r^2] = 0
\] (136b)

Let \(n^2\) differ from zero and \(r^2\) be large, in which case (136b) reduces to the following:
Equations (136a) and (137), when combined, give a two-dimensional
Laplace equation, as we shall see. Let us define $\phi_p$ and $\phi_a$ as follows

$$\phi_p = (a/r)^{1/2} \phi_a = (a/r) U \cdot Z = (1/r)\psi_c$$  \hspace{1cm} (138)

Then the partial differential equation becomes, in terms of $\phi_a$,

$$\frac{\partial^2 \phi_a}{\partial r^2} + \frac{\partial^2 \phi_a}{\partial s^2} = 0$$  \hspace{1cm} (139)

When $n^2 = 0$, we must use the original Equation (136b) which reduces to

$$\frac{d^2 U}{dr^2} - (3/4r^2)U = 0$$  \hspace{1cm} (140)

Its solution in this case is

$$U(n = 0) = Er^{3/2} + Fr^{-3/2}$$  \hspace{1cm} (141)

where $E$ and $F$ are constants.

The general solution can be expressed in terms of $\phi_p$ in the form

$$\phi_p = (a/r)^{1/2} [A_1 \sin(ns) + B_1 \cos(ns)] [A_2 \sinh(nr) + B_2 \cosh(nr)]$$
$$+ (a/r)^{1/2} [C_1 \sinh(ns) + D_1 \cosh(ns)] [C_2 \sin(nr) + D_2 \cos(nr)]$$
$$+ a^{1/2} (E/r) [G_2 + H]$$  \hspace{1cm} (142)

where $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$ are constants.

This result corroborates the discussion in the preceding section.

It is obvious that
where $\phi_p$ is the corrected two-dimensional stream function and $\phi_c$ is the axially symmetric stream function in cylindrical coordinates.

3. Generalization of the asymptotic solution of flow equations

The discussion in this section may be considered as an illustration for a much more general flow case. In Section IV of this thesis we transform all flow equations in two dimensions or with axial symmetry into a form which resembles Schrodinger's equation, much the same as Equation (125a). Other asymptotic solutions can be found which would simplify a wide variety of problems (see Sections IV C 2 f and IV C 3 f).

E. Notes About Use of Models

In concluding this Section III we note some problems in connection with axially symmetric flow and the use of models.

The main problem in obtaining axially symmetric flow in models stems from the fact that axially symmetric flow is a three-dimensional flow. Any resistance network analogue becomes complicated and expensive. A flow of fluid between two non-parallel plates cannot be used because of the adherence of the fluid to the boundaries which does not allow an axially symmetric flow. We can use an actual porous medium in the form of a cylindrical section. The boundary effect in this case is negligible. By tilting a flat-bottomed electrolytic bath we can form
a cylindrical sector between the flat bottom and the water surface. Curvature of the water surface due to capillarity may have a significant effect very near the axis of symmetry. The potential function can be easily measured in such an electrolytic bath analogue by recording the electrical potential through the electrolyte.

There appears to be no simple way to measure the stream function directly. An interesting suggestion in this connection was introduced by Beaver [1]. He devised a method to measure the stream function in an electrolytic bath. It is achieved by transformation into some kind of hyperbolic curvilinear coordinates. Except in extremely simple cases, building a model in the transformed system would be very complex and expensive. Most workshop tools are set to measure and work along straight lines or circles, but not hyperbolic surfaces. Before making models for stream function determinations, one should consider the advisability of obtaining equipotential surfaces and plotting graphically the streamlines on these surfaces. It seems that in many cases it would be easier to build the complete cylinder, rather than a single sector of it. The measurements in a sector are suggested only because most electrolytic baths are adapted for two-dimensional flow modeling work. They have a relatively big area and a relatively small depth.

In some special cases axially symmetric flow can be simulated with a two-dimensional model and vice versa. Such cases for nonuniform
conductivity are demonstrated in Section IV C 2a and 3a. Other cases of variable conductivity can be simulated by two-dimensional models when in Schrödinger's equation the potential vanishes or almost vanishes (see IV C 2e, 2f, 3e and 3f).
IV. THE GENERAL THEORY OF SATURATED, STEADY FLOW WITH NONUNIFORM CONDUCTIVITY

A. Introduction

1. Idealized solutions and the actual flow problems

The solutions employed for flow problems in soil are at best reasonable approximations of the actual flow conditions. The accuracy of these approximate solutions is limited by several factors.

Every solution for a flow problem must be tied with actual field conditions. At least three quantities must be measured in the field. These are the conductivity, the velocity potential and/or flow at the boundaries, and the geometry of the flow medium. Some of these quantities cannot be measured in the field within the accuracy of more than two significant figures.

In actual flow problems, there are different fluctuations in the geometry of the medium. For example, the surface of an impermeable layer, which is regarded as an horizontal plane, may actually be sloping, wavy, or discontinuous. The conductivity will almost always vary with time and space. In an attempt to find a practical solution, it is common practice to idealize the flow conditions in a way which one believes will give the best approximation of the actual flow. Each solution should serve for a wide range of different flow problems. As there are no two cases in the field which are exactly alike, we must admit some degree of approximation if we want to have solutions of reasonable usefulness.
One finds that even for relatively simple, idealized flow conditions, the exact solution may become very complex. The application of such solutions to any one special set of dimensions may involve a very complex and tedious computational work (see Kirkham [21]). If the theory of filtration flow is to be of practical value, it should be within the operational ability of an average engineer to apply it in his practice.

It must be admitted that most flow problems solved to date are idealized almost to a degree of being impractical. For example, as far as we know, all the solutions of a two-dimensional flow into a ditch assume that the walls are perfectly vertical. In reality most ditches are dug with sloping embankments. In existing literature, most flow problems are solved for uniform and isotropic conductivity. There are very few actual field problems which can enter this category. A tenfold variation in conductivity, due to biological activity or mechanical, chemical or thermal changes, is almost unavoidable. Unless these variations are of local nature, tending to average evenly over the medium, we are not justified in using the uniform isotropic solution.

2. Continuous and discontinuous variation of conductivity

There are two possible ways to account for the varying conductivity. In the first way the domain may be divided into several subregions. Each subregion is assigned a constant conductivity. First, a solution is found for each subregion. Next, a condition is imposed, that the
potential function or the stream function and its normal derivative should have the same value on both sides of the interfaces between two subregions. The alternative approach is to approximate the conductivity distribution in space by a continuous function of the coordinates. It may also be approximated by a power series or its equivalents.

Abrupt changes in conductivity are in most cases man-made. The first approach of subregioning is then justified, if the number of subregions does not exceed two or three. Beyond that the solution may become very involved.

Unless we find some simple analytic methods to solve problems with nonuniform conductivity, it is better to compute the solution for each specific case directly from the differential equation by some of the modern computing techniques.

3. Vertical and horizontal variations of conductivity

The soil profile is generally formed in layers parallel to the soil surface. Variations in conductivity will be more common in the vertical direction than the horizontal.

In the following sections variations in both horizontal and vertical direction will be analyzed. These variations will be applied to a two-dimensional flow and to an axially symmetric flow. Both the potential function and the stream function will be treated. In all, eight groups of problems are considered. It is of significance to
note that six out of these eight combinations render themselves to a completely identical analysis. The two cases which lead to different solutions are for the stream function and potential function with axial symmetry and for the conductivity varying horizontally.

It is very unlikely that the conductivity will vary horizontally in such a way as to yield a problem of axial symmetry, but there are some such practical problems and these will be presented in one of the following sections.

4. The scope of presentation of the nonuniform flow analysis

The main purpose of this part of the thesis is to demonstrate solutions of flow equations for a wide variety of conductivity patterns. The solutions are formed in terms of well-known functions. It is not much harder to handle them than the solutions for a uniform conductivity. The functions used for the solutions are in common use in mathematical physics. Their behavior is well known, and in most cases they are tabulated. Of great importance is the fact that these flow equations can be transformed into some standard equations for which many analytical and numerical methods of solutions are known. For example, it will be shown that all steady flow differential equations can be transformed into a Sturm-Liouville equation and in turn into Schrödinger's equation. For the last several decades, physicists and mathematicians have devoted themselves to the study of these two equations and have obtained a host of exact and approximate methods
for their solution. Among the methods we find names like "Variational method", "Series solution" and "Perturbation theory", to name only a few.

This wide range of methods, when combined with modern techniques of computation, may very well offer new perspectives for the soil physicist in trying to solve new, more complex flow problems. We may then analyze the more complex cases with greater accuracy and practicality.

The discussion in this part of the thesis extends primarily to the steady state flow of an incompressible homogeneous fluid.

B. The Differential Equations of the Steady State Flow with Nonuniform Conductivity

1. The partial differential equations for two-dimensional flow

The partial differential equations for the potential function $\phi$ and the stream function $\psi$ in two-dimensional flow are the following

$$ \text{div}(k \text{ grad } \phi) = 0 \tag{1} $$

$$ \text{curl}(k \frac{\partial \psi}{\partial y} \mathbf{l}_x - k \frac{\partial \phi}{\partial x} \mathbf{l}_y) = 0 \tag{2} $$

Here, $k$ is the hydraulic conductivity, $x$ and $y$ are the two cartesian coordinates and $\mathbf{l}_x$ and $\mathbf{l}_y$ are unit vectors in the $x$ and $y$ directions.

It is found that the equation for the stream function (2) can be written in an equivalent form like the equation for the potential function (1), that is,
Let $k$ be a function of the coordinates $x$ and $y$. Equations (1) and (3) can be expanded in the following fashion:

$$\nabla \cdot (k \nabla \phi) = 0 \quad (3)$$

$$k \nabla^2 \phi + (\nabla k) \cdot (\nabla \phi) = 0 \quad (4a)$$

$$k \nabla^2 \psi + (\nabla k) \cdot (\nabla \psi) = 0 \quad (4b)$$

which are the generalized differential equations for two-dimensional steady flow.

2. The differential equations for axially symmetric flow

Equation (1), for the potential, applies to all orthogonal systems of coordinates. The gradient and the divergence should be properly interpreted for each set of coordinates. For axially symmetric flow in cylindrical coordinates Equation (1) reduces to the following:

$$\frac{\partial}{\partial r}(kr \frac{\partial \psi}{\partial r}) + \frac{\partial}{\partial z}(kr \frac{\partial \psi}{\partial z}) = 0 \quad (5)$$

The orthogonality conditions between the potential function and the stream function were found in Section II, Equations (12a), (12b). They are valid for the nonuniform case and may be written here as:

$$\frac{\partial \psi}{\partial r} = (1/r) \frac{\partial \phi}{\partial z} \quad (6a)$$

$$\frac{\partial \psi}{\partial z} = -(1/r) \frac{\partial \phi}{\partial r} \quad (6b)$$
Accordingly, the flux $q$ can be expressed (Wylie [46, p. 454, formula 12]) as follows

$$q = -(k/r)(\partial \psi / \partial z)l_r + (k/r)(\partial \psi / \partial r)l_z \tag{7}$$

$l_r$ and $l_z$ are unit vectors in the $r$ and $z$ directions. The condition that $q$ is solenoidal ($\text{div} \ q = 0$) is automatically satisfied. This can be proved by showing that the divergence of (7) vanishes identically.

To insure that the flow be irrotational the curl of the flux must also vanish. Application of this condition to (7) gives the differential equation for the stream function.

$$\frac{\partial}{\partial r} \left( \frac{k \partial \phi}{r \partial r} \right) + \frac{\partial}{\partial z} \left( \frac{k \partial \psi}{r \partial z} \right) = 0 \tag{8}$$

Expansion of (5) and (8) gives the following equations for the potential function and the stream function, respectively,

$$\frac{k \partial^2 \psi}{r^2} + \frac{k \partial \psi}{r \partial r} + \frac{\partial k}{\partial r} \frac{\partial \psi}{\partial r} + \frac{k \partial^2 \psi}{r \partial z^2} + \frac{\partial k}{\partial z} \frac{\partial \psi}{\partial z} = 0 \tag{9a}$$

$$\frac{k \partial^2 \psi}{r^2} - \frac{k \partial \psi}{r \partial r} + \frac{\partial k}{\partial r} \frac{\partial \psi}{\partial r} + \frac{k \partial^2 \psi}{r \partial z^2} + \frac{\partial k}{\partial z} \frac{\partial \psi}{\partial z} = 0 \tag{9b}$$

which are the equations for axially symmetric steady flow.

3. The separation of variables

In the two-dimensional cases the following is assumed

$$\psi_p = X_1 Y_1 \tag{10a}$$

$$\phi_p = X_2 Y_2 \tag{10b}$$

The subscript $p$ designates planar or two dimensional flow. $X$ and $Y$ are
functions of x only or y only, respectively.

For the axially symmetric flow the following is assumed

\[ \psi_c = R_1 Z_1 \tag{10c} \]

\[ \phi_c = R_2 Z_2 \tag{10d} \]

R and Z are functions of r and z only, respectively. The subscript c designates the axial symmetry of this flow and the use of cylindrical coordinates.

In addition, it is assumed that the conductivity k can be separated in the following way

In two-dimensional flow, \( k = k_{xy} \) \( \tag{10e} \)

In the axially symmetric flow, \( k = k_{rz} \) \( \tag{10f} \)

Each separate term of the conductivity is a function of its subscript only.

Consider the two-dimensional case. By substituting Equations (10a) and (10e) into (4a) and substituting Equations (10b) and (10e) into (4b), we can separate the variables and arrive at the following four ordinary differential equations. For the potential function we get

\[ k_{x_1}^n + k_{x_1}^{1'} + n^2 k_{x_1} = 0 \tag{11a} \]

\[ k_{y_1}^n + k_{y_1}^{1'} - n^2 k_{y_1} = 0 \tag{11b} \]

and for the stream function we get
One prime designates first order ordinary derivative. Two primes designate second order ordinary derivative. The value of \( n^2 \) can be positive, negative or zero.

For the axially symmetric flow we substitute Equations (10c) and (10f) in (9a) and Equations (10d) and (10f) in (9b). Consequently we can separate the variables and arrive at four ordinary differential equations. For the potential function we get

\[
\frac{1}{x_2} \frac{d}{dx_2} \left( k_x x_2^2 + k_x' x_2 + n^2 k_x x_2 \right) = 0 \tag{11c}
\]

\[
\frac{1}{y_2} \frac{d}{dy_2} \left( k_y y_2^2 + k_y' y_2 - n^2 k_y y_2 \right) = 0 \tag{11d}
\]

One prime designates first order ordinary derivative. Two primes designate second order ordinary derivative. The value of \( n^2 \) can be positive, negative or zero.

For equal functional form of the conductivity terms, and if all values of \( n^2 \) are considered, the general solutions of Equations (11a), (11b), (11c), (11d), (11f) and (11h) are identical, except for the
notation of the independent variable.

Consequently only three prototype equations will be discussed in the following sections. They are

\[ kX'' + k'X' + n^2kX = 0 \]  \hspace{1cm} (12a)

which stands for (11a), (11b), (11c), (11d), (11f) and (11h); and

\[ kR_1'' + (k' + k'/r)R_1' + k'R_1^2 = 0 \]  \hspace{1cm} (12b)

which stands for (11e); and

\[ kR_2'' + (k' - k'/r)R_2' + k'R_2^2 = 0 \]  \hspace{1cm} (12c)

which stands for (11g).

Each of the above three equations will have three types of solutions for the three possible values of \( n^2 \) (positive, negative or zero).

The general solution for any one of the partial differential equations (4a), (4b), (9a) or (9b), is found by a proper combination of the solutions of (12). For example, the general solution of the differential equation (4a), for the potential in two-dimensional flow, will be found by summing the following combinations

a. A product of the solution of (12a) for positive values of \( n^2 \) with a solution of the same equation for negative value of \( n^2 \); the first is in terms of the coordinate \( x \), and the second in terms of the coordinate \( y \);

b. The same product is employed, but the order is changed; the
first part of the product, for positive values of $n^2$, is in terms of the coordinate $y$; the second part of the product, for negative values of $n$, is in terms of $x$;

c. A product of two solutions, one in terms of $x$ and the other in terms of $y$; both solutions of Equation (12a) when $n^2$ is zero.

4. Useful transformations and standard forms of the differential equations

It is easy to verify the equivalence of the following equations with (12a), (12b) and (12c), respectively

$$\frac{d}{dx} (k \frac{dx}{dx}) + n^2 k_x x = 0 \quad (13a)$$

$$\frac{d}{dr} (k_r \frac{dr}{dr}) + n^2 r k R_1 = 0 \quad (13b)$$

$$\frac{d}{dr} (k \frac{dr}{dr}) + n^2 (k_r/r) R_2 = 0 \quad (13c)$$

These equations are of the Sturm-Liouville type. For general discussion of these equations see Ince [13, pp. 204-253], Margenau and Murphy [27, pp. 267f.], Morse and Feshbach [31, pp. 719f.] and Hildebrand [11, pp. 95, 225]. We shall relate Equations (13a), (13b) and (13c) to the following general equation

$$\frac{d}{dx} (P \frac{dF}{dx}) + (Q + \lambda S)F = 0 \quad (14)$$

This is the standard form of Sturm-Liouville equation; in it $P$, $Q$ and $S$ are known functions of $x$ and $F$ is the unknown function of $x$. By
comparison we find the following identities, valid for all three Equations (13)

\[ Q = 0 \]

\[ \lambda = n^2 \]

The fact that the separated flow equations can be written in the Sturm-Liouville form is of great significance. For our discussion we mention only that they can be solved by expansion of the unknown function in infinite series with orthogonal terms, and that all possible solutions are real.

The same three equations, (13a), (13b) and (13c), may be written in another standard form. Although of smaller importance, it may lead to some solutions, as we shall see in the following sections. We get the following from (13) in a straightforward manner

\[ x'' + \left(k_x^2/k_x\right)x' + n^2x = 0 \] (15a)

\[ \mathcal{E}_1'' + [(k_x^2/k_x) + (1/r)]\mathcal{E}_1' + n^2\mathcal{E}_1 = 0 \] (15b)

\[ \mathcal{E}_2'' + [(k_x^2/k_x) - (1/r)]\mathcal{E}_2' + n^2\mathcal{E}_2 = 0 \] (15c)

Next, the three equations will be transformed into a form which resembles Schrödinger's wave equation. Relating to the standard Sturm-Liouville equation, (14), the new form is achieved by the following transformation, in which \( Y \) (not to be confused with the \( Y \) in (10) and (11)) becomes a new unknown function of a new independent variable \( u \).
By comparing (14) with (13a), (13b) and (13c), one finds for all three flow equations, the identity

\[ P = S \]  \hspace{1cm} (17)

Consequently, (16) reduces to the following

\[ Y = P^{1/2} \]  \hspace{1cm} (18a)
\[ u = x \]  \hspace{1cm} (18b)

Performing the transformation on (14), we arrive, after some manipulations, at the following standard equation

\[ \frac{d^2 Y}{dx^2} + [\lambda - V(x)] Y = 0 \]  \hspace{1cm} (19)

where

\[ -V(x) = \frac{1}{4} \left( \frac{P'}{P} \right)^2 - \frac{1}{2} \left( \frac{P''}{P} \right) \]  \hspace{1cm} (20)

which simulates the potential in Schrödinger's wave equation.

By comparing again (14) with (13a), (13b) and (13c) we obtain the following identities

- in (13a) \[ P = k_x; \]
- in (13b) \[ P = r k_x; \]
- in (13c) \[ P = k_x/r. \]

The specific transformations become according to (18a)
For (12a) or (13a) \[ I_1 = k_x^{1/2} x_j \]  

(21a)

For (12b) or (13b) \[ I_2 = (rk_x)^{1/2} x_j \]  

(21b)

For (12c) or (13c) \[ I_3 = (k_x/r)^{1/2} r_2^2 \]  

(21c)

Substituting (21) into (20) yields the following three values for the simulated potential \( V \) of Schrodinger's equation

\[
-V_1(x) = \frac{1}{4} \left( \frac{k_x'}{k_x} \right)^2 - \frac{1}{2} \left( \frac{k_x''}{k_x} \right) 

(22a)

-V_2(r) = \frac{1}{4} \left( \frac{k_x'}{k_x} + \frac{1}{r} \right)^2 - \frac{1}{2} \left( \frac{k_x''}{k_x} + 2 \frac{k_x'}{rk_x} \right) 

(22b)

-V_3(r) = \frac{1}{4} \left( \frac{k_x'}{k_x} - \frac{1}{r} \right)^2 - \frac{1}{2} \left( \frac{k_x''}{k_x} - 2 \frac{k_x'}{rk_x} + 2/r^2 \right) 

(22c)

In this section on transformations we have presented the original separated differential equations in the three additional standard forms in (13), (15), and (19). The coefficients in these equations are different functions of the independent variables. In the following sections we shall suggest different solutions of the flow equations, as the coefficients take different functional forms.

C. Solutions for the Separated Equations of Flow with Nonuniform Conductivity

1. Solutions for the general case

In the general case it is assumed that the conductivity can be expressed from experimental data in the following fashion

\[
k = k_x k_y 

(23a)

k_x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n 

(23b)
\[ k_y = b_0 + b_1 y + b_2 x^2 + \ldots + a_n x^n \] \hspace{1cm} (23c)

It is possible to introduce also terms with negative integers as exponents. The coefficients \( a_i \) and \( b_j \) are determined by the least square method from the experimental data. We should try to keep in the \( k \) series as few terms as possible for the purpose of simplicity.

From Equation (23) for two-dimensional flow, or similar ones for axially symmetric flow, we substitute the expanded values of \( k \) into any of the three groups of standard differential equations, (12), (15) or (19). Consequently the coefficients of the latter equations are expressed as finite power series of the coordinates or as a ratio between two power series. To find some of the coefficients we will have to calculate the derivatives of (23).

Rather than expanding \( k_x \) or \( k_z \) individually in series, for using them and their derivatives in Equations (12), it may be simpler to expand \( (k_x/k_x) \) and \( (k_y/k_y) \) into power series for use in (15). In the latter event graphical or numerical differentiation of the field conductivity data would need to be employed. It would be important to find out what, if any, influence such an inverted procedure would have on the accuracy of the approximation.

In all actual flow cases, the unknown functions in (12), (15) or (19) can be expanded in an infinite power series and be solved.
The procedure of solving the general case is described in detail by Hildebrand [11, p. 132]. The method is named after Frobenius. In the general case it is not easy to determine all the characteristic numbers $n^2$ and the coefficients in the infinite series. Unless we can express the infinite series in a closed form or relate them to some familiar series, the task of fitting to the boundary conditions may not be easy, although formally possible. Several numerical methods are available, like the one mentioned by Hildebrand [11, pp. 218-232] and named after Stodola and Vianello. As we are dealing with the separated ordinary differential equation, the characteristic numbers $n^2$ may be determined by either one of the two equations comprising the complete flow equation. It may sometimes be possible to determine the characteristic numbers by one separated part of the solution which does not involve variable conductivity (for example, in a solution in terms of $x$ when $k_x$ is constant).

It is not intended to demonstrate here in detail the general method of solution. In the following sections we shall demonstrate specific solutions of several special patterns of the conductivity. It seems that they can cover a wide range of practical cases. Unless the solution can be expressed in a compact, easy-to-handle form, we might better revert to the use of modern computers and derive each specific boundary solution directly from the differential equation.
2. Solutions for the separated equations of the two-dimensional flow and for the z-dependent equation in the axially symmetric flow

By separating the variables we obtain eight ordinary differential equations (Equation 11). Six of these equations were found to be identical, except for the notation of the independent variable. These six equations are represented by the following prototype

\[ k_x x'' + k_x' x' + n^2 k_x x = 0 \]  \hspace{0.5cm} (24a)

as was already recorded in (12a). For purpose of reference we copy here the other forms of this equation as was shown in (13a), (15a), (19), (20) and (22a)

\[ \frac{d}{dx} \left( k_x \frac{dx}{dx} \right) + n^2 k_x x = 0 \]  \hspace{0.5cm} (24b)

\[ x'' + \left( \frac{k_x'}{k_x} \right) x' + n^2 x = 0 \]  \hspace{0.5cm} (24c)

\[ \frac{d^2 y_1}{dx^2} + [n^2 - v_1(x)] y_1 = 0 \]  \hspace{0.5cm} (24d)

\[-v_1(x) = \left( 1/4 \right) \left( k_x'/k_x \right)^2 - \left( 1/2 \right) \left( k_x''/k_x \right) \]  \hspace{0.5cm} (24e)

\[ y_1 = k_x^{1/2} \]  \hspace{0.5cm} (24f)

a. Solutions with Bessel functions. Let the conductivity have the following general form

\[ k_x = a(x + b)^\alpha \]  \hspace{0.5cm} (25)

The coefficients a and b can be any positive or negative real number,
or zero. The exponent, \( \alpha \) can be a positive or negative integer or a noninteger. When the exponent is zero the problem reduces to the uniform case. We define a new variable \( u \) as follows

\[
u = x + b \quad (26)
\]

Equation (25) then reduces to

\[
k_x = a u^\alpha \quad (27)
\]

Substitution in (24b) gives the following equation

\[
\frac{d}{du}(u^\alpha \frac{dx}{du}) + n^2 u^\alpha x = 0 \quad (28)
\]

Next, we follow the general method suggested in Section II B 4. (See also Wylie [46, p. 258] and Hildebrand [11, p. 165].) The following transformation for (28) is suggested

\[
x = (u)^{(1 - \alpha)/2u} \quad (29)
\]

Equation (28) then reduces to the following Bessel differential equation

\[
u^2 u'' + uu' + \{-[(1 - \alpha)/2]^2 + n^2 u^2\} u = 0 \quad (30)
\]

The general solution of this equation is

\[
U = Z_p(nu) \quad (31a)
\]

\[
p = |(1 - \alpha)/2| \quad (31b)
\]

where \( Z_p \) is a combination of Bessel functions of the order \( p \) which may be expressed as follows.

\[
Z_p = A_p J_p(nu) + B_p H_p(nu) \quad \text{for } n^2 > 0 \quad (32a)
\]
\[ Z_p = A_2 I_p(\nu) + B K_p(\nu) \quad \text{for } n^2 < 0 \quad (32b) \]

For definition of \( J, N, I \) and \( K \) see Section II B 4 or any one of the general references [8], [11], [20], [27], [35] or [46]. The solution of (28) for nonzero values of \( n^2 \) is in general found from (29) and (31a) and is

\[ I = u^p Z_p(\nu) \quad (n \neq 0) \quad (33) \]

The singular solution for \( n^2 = 0 \) is easily found by integration from (28). It is

\[ x_n = 0 = [c/k_x]u + D \quad (34) \]

By integrating we obtain, after introducing the value of \( k_x \) from (27), for all values of the exponent \( \alpha \), except unity,

\[ x_n = 0 = [cu^1 - \alpha/(\alpha - 1)] + D \quad (35) \]

When the exponent \( \alpha \) in (27) is unity, we have

\[ k_x = au = a(x + b), \quad (36) \]

When \( \alpha \) is unity, Equation (28), the prototype for the two-dimensional separated equations, is identical with the prototype (13b) for the potential in axially symmetric flow with uniform conductivity. Thus, we can simulate two-dimensional flow with conductivity varying as in (36) by axially symmetric flow with uniform conductivity.

Similarly, when in (27) \( \alpha = -1 \), we have

\[ k_x = a/u = a/(x + b), \quad (37) \]
and the two-dimensional flow can be simulated by the stream function in axially symmetric flow with uniform conductivity (compare (28) with (13c)).

The solutions of the nonuniform two-dimensional flow are not harder to handle than the usual solutions in axial symmetry and with uniform conductivities, as both involve trigonometric, hyperbolic and Bessel functions and both require one to be careful at singular points like at the axis of symmetry or where the conductivity vanishes.

Whenever the variable

$$u = x + b \quad (38)$$

as defined in (26) ranges only on large values, the solution is subject to the approximation suggested in Section III D, which further simplifies the solution.

As for the exponent $\alpha$ in Equation (27), we adjust it to the closest convenient value of $p$ ($p$ being the order of the Bessel function in (31)). If we limit $p$ to integers only, then $\alpha$ must be an odd positive or negative number. As some Bessel functions are tabulated for orders of halves, quarters and thirds, the exponent $\alpha$ may take also even integral values and values containing halves and thirds. Together with the two coefficients $a$ and $b$ we should be able to apply the general form of $k_x$ in (25) to a wide variety of conductivity patterns with a fair degree of approximation. In most cases, anyhow,
we will not have enough permeability data to determine more than two coefficients with any degree of confidence or significance.

b. Solution with Hermite polynomials. Let the conductivity vary in the following manner

\[ k_x = ae^{-x^2} \]  \hspace{1cm} (39)

where \( x \) is expressed in some dimensionless form. The differential equation may then be solved by use of Hermite polynomials (see the standard texts [11], [31] and [27]).

c. Solution with Hermite orthogonal functions. If in Equation (24d) we have the conductivity expressed through (24e) such that

\[ -V_1(x) = x^2 - 1 \]  \hspace{1cm} (40)

the differential equation may be solved in terms of Hermite's orthogonal functions (see [11], [31] or [27]).

d. Solution with Mathieu functions. If in Equation (24d) we have the conductivity expressed through (24e) such that

\[ V_1(x) = 16 b \cos 2x \]  \hspace{1cm} (41)

the solution may be found in terms of Mathieu functions.

e. Special solutions by the Schrodinger equation. Let the following special relation exist in (24e)

\[ (1/4)(k^l_x/k_x) - (1/2)(k^{"}_x/k_x) = 0 \]  \hspace{1cm} (42)
Then we solve (24d) like the Schrodinger equation for a free mass point in a potential-free medium. Equation (24d) reduces to the following

\[ \frac{d^2y}{dx^2} + n^2y = 0 \]  

(43)

Its solutions are

\[ y_0 = Ex + F \quad \text{for } n^2 = 0 \]  

(44a)

\[ y_1 = A_1 \sin(nx) + B_1 \cos(nx) \quad \text{for } n^2 > 0 \]  

(44b)

\[ y_2 = A_2 \sinh(nx) + B_2 \cosh(nx) \quad \text{for } n^2 < 0 \]  

(44c)

By solving (42) we find the conductivity pattern to be

\[ k_x = (ax + b)^2 = a^2x^2 + 2abx + b^2 \]  

(45)

Instead of (42) let us take the general case where we write for \( V_1(x), f(x) \). Then we have

\[ \frac{1}{4} \left( \frac{k_x'}{k_x} \right)^2 - \frac{1}{2} \left( \frac{k_x''}{k_x} \right) = f(x) \]  

(46)

Depending on the form of \( f(x) \) we may find different solutions for (24d).

If in (46) we take \( f(x) \) equal to a constant \( C \)

\[ f(x) = C \]  

(47)

We get by integrating (46) (see below) that

\[ k_x = \frac{B}{2C} [1 - \cos \frac{C}{2}(x + D)] \]  

(48)

Where \( B, C \) and \( D \) are constant coefficients which may be shown to fit the field data for many cases.

When we use Equation (47) we notice that (24d) reduces to the famous Schrodinger equation for a constant potential. If the constant
C changes abruptly in the middle of the flow medium, we have the famous problem of the potential barrier or "square well". Equation (46) was solved for both (43) and (47) by making the following substitutions

\[
\frac{dk}{dx} = w \tag{49}
\]

\[
\frac{d^2k}{dx^2} = w \frac{dw}{dk} \tag{50}
\]

Another example is when in (46)

\[
f(x) = \frac{Ax^2}{2} \tag{51}
\]

This is the famous Schrödinger problem of the harmonic oscillator.

The patterns of the conductivity \(k\) in (45) and (48) may be adapted to a wide variety of cases. Especially versatile is the expression in (48). This is the only solution, from those presented in this thesis, where nonmonotonous variation of \(k\) with the coordinates can be accounted for.

**f. Asymptotic solutions from the Schrödinger equation.** When \(V_1(x)\), as defined in (24a), is small, it may be discarded from (24d). We may then find simplified approximate solutions as was done in Section III D for slightly curved media.

Other asymptotic solutions can be achieved if only the second derivative of \(k_x\) is small, and can be ignored in (24a).

If values of \(k_x\) are substituted from (27) into (24e) we realize that the approximation can be done for very large or very small values of \(u\), depending on the exponent \(\alpha\).
For further information of cases b, c, d and e of this section, see the general texts [11], [13], [27], [31], [46], and [20].

Let us pause for a moment to recapitulate. We have seen how variable conductivity in the x and y directions may be accounted for in basic differential equations separated out from (4a) and from (4b) for the two-dimensional potential function and stream function. We have also seen how variable conductivity may be accounted for in the z-direction for three-dimensional flow when the z-direction corresponds to the axis of symmetry. We have yet to take the horizontal variation of conductivity into account for the latter case of axially symmetric flow, which is done in the next part. We must also show how to satisfy the boundary conditions simultaneously with the satisfying of variable conductivity. This will be done in Section V of the thesis.

3. Solutions for the separated equations, involving radial changes of the conductivity, for the potential and stream functions in axially symmetric flow

The procedure here is identical with the one used in the preceding part. It will be briefly repeated in a similar order.

We shall concern ourselves here with the solution of the following equations. For the potential function in axial symmetry we have from equations (12b), (13b), (15b), (19) and (22b)

\[ k_R \frac{d^2 R_1}{dR_1^2} + (k_R + kr/r) R_1' + k_R n^2 R_1 = 0 \]  

or

\[ \frac{d}{dr} \left( r k_R \frac{dR_1}{dr} \right) + n^2 k_R R_1 = 0 \]
or

\[ R_1'' + \left( \frac{k_r'}{k_r} + \frac{1}{r} \right) + n^2 R_1 = 0 \]  \hspace{1cm} (52c)

or

\[ Y_2'' + [n^2 - V_2(r)]Y = 0 \]  \hspace{1cm} (52d)

where

\[ Y_2 = (rk_r)^{1/2} R_1 \]  \hspace{1cm} (52e)

and

\[ -V_2(r) = \frac{1}{4} \left( \frac{k_r'}{k_r} + \frac{1}{r} \right)^2 - \frac{1}{2} \left( \frac{k_r''}{k_r} + \frac{2k_r'}{rk_r} \right) \]  \hspace{1cm} (52f)

For the stream function in axial symmetry we have from (12c), (13c), (15c), (19) and (22c)

\[ k_r R_2'' + \left( \frac{k_r'}{k_r} - \frac{1}{r} \right) R_2' + k_r^2 R_2 = 0 \]  \hspace{1cm} (53a)

or

\[ \frac{d}{dr} \left( k_r \frac{dR_2}{dr} \right) + k_r^2 R_2'/r = 0 \]  \hspace{1cm} (53b)

or

\[ R_2'' + \left( \frac{k_r'}{k_r} - \frac{1}{r} \right) R_2' + n^2 R_2 = 0 \]  \hspace{1cm} (53c)

or

\[ Y_3'' + [n^2 - V_3(r)]Y_3 = 0 \]  \hspace{1cm} (53d)

where

\[ Y_3 = (k_r/r)^{1/2} R_2 \]  \hspace{1cm} (53e)

and

\[ -V_3(r) = \frac{1}{4} \left( \frac{k_r'}{k_r} - \frac{1}{r} \right)^2 - \frac{1}{2} \left( \frac{k_r''}{k_r} - \frac{2k_r'}{rk_r} + \frac{2}{r^2} \right) \]  \hspace{1cm} (53f)
a. Solutions in terms of Bessel functions. Let the conductivity vary radially with the following pattern

\[ k_r = ar^\alpha \]  \hspace{1cm} (54)

This pattern will not be proper when the axis of symmetry \((r = 0)\) is within the flow medium, since the \(k_r\) would be zero and water could not flow. The coordinate translation used in the two-dimensional case (26) cannot be used here without changing the nature of the differential equation.

Equation (52b) transforms into a Bessel differential equation by the following substitution

\[ R_1 = r^{-\alpha/2}U \]  \hspace{1cm} (55)

It then reduces to the following equation

\[ U'' + (1/r)U' + \left[ n^2 - (\alpha/2)^2 / r^2 \right]U = 0 \]  \hspace{1cm} (56)

This is a Bessel equation and the unknown function \(U\) is given by \(Z_p(nr)\), where \(Z_p\) are Bessel functions of order \(p\) given either by (59a) or (59b) below. We now find \(R_1\) of (55) to be given by

\[ R_1 = r^{-\alpha/2}Z_p(nr) \hspace{1cm} (n \neq 0) \]  \hspace{1cm} (57)

where

\[ p = \left| \frac{\alpha}{2} \right| \]  \hspace{1cm} (58)

and

\[ Z_p(nr) = A_1J_p(nr) + B_1H_p(nr) \hspace{1cm} \text{for} \ n^2 > 0 \]  \hspace{1cm} (59a)

and
\[ Z_p(nr) = A_2 p_2(nr) + B_2 K_2(nr) \quad \text{for } n^2 < 0 \quad (59b) \]

The singular solution (for \( n^2 = 0 \)) is found from (52b) by a straightforward integration.

\[ B_1(n = 0) = \int \frac{C}{k r} dr + D \quad (\alpha \neq 0) \quad (60) \]

By substituting (54) into (60) and performing the integration, we get

\[ B_1(n = 0) = \int \frac{C}{ar^\alpha + 1} dr + D = -\frac{C}{(a r^\alpha)} + D; \quad \alpha \neq 0 \quad (61) \]

When the exponent \( \alpha \) is zero, the usual case of uniform conductivity results, as presented in Section II of this thesis.

Let the exponent \( \alpha \) be

\[ \alpha = -1 \quad (62) \]

Then we see, remembering (54), by comparing (52b) with (24b) that the axially symmetric flow may be simulated by a two-dimensional flow with uniform conductivity. In other words, (52b) with \( k_r = a r^{-1} \) becomes the same as (24b) with \( k_r \) is constant.

To solve the equation for the stream function (53b), when the conductivity varies radially, as in (54), we make the following substitution

\[ B_2 = r^{(1 - \alpha/2)} W \quad (63) \]

Equation (53b) then transforms into a Bessel differential equation of the following form

\[ W'' + \left( \frac{1}{r} \right) W' + \left[ \alpha^2 - \frac{(1 - \alpha/2)^2}{r^2} \right] W = 0 \quad (64) \]
The solutions for non-zero $n$ values are

$$ R_2 = r^p Z_p(nr) $$    \hspace{1cm} (65a)

where

$$ p = |1 - a/2| $$    \hspace{1cm} (65b)

$$ Z_p = A_1^p(nr) + B_1^p(nr) \quad \text{for } n^2 > 0 $$    \hspace{1cm} (66a)

or

$$ Z_p = A_2^p(nr) + B_2^p(nr) \quad \text{for } n^2 < 0 $$    \hspace{1cm} (66b)

The singular solution, for $n^2 = 0$, is found from (53b) by a straightforward integration

$$ R_2(n = 0) = \int [c/(ar^\alpha - 1)]dr $$

$$ = \{c/[a(a - 2)r^\alpha - 2]\} + B $$    \hspace{1cm} (67)

When the exponent $\alpha$ equals two, the stream function in axially symmetric flow can be simulated (compare (53b) for $k = ar^2$ with (52b) for $k_p = \text{constant}$) by a potential function of axially symmetric flow in a uniform medium.

Let the conductivity pattern be

$$ k_p = ar $$

in which case the stream function can be simulated by a two-dimensional flow in a uniform medium (compare (53b) when $k = ar$ with (24b) when $k = \text{constant}$).
b. Solutions in terms of Hermite polynomials. When in

Equation (52b)

$$k_r = (a/r)e^{-r^2}$$

(68)

where $r$ is expressed in some dimensionless form; or when in Equation (53b)

$$k_r = ar^{-r^2}$$

(69)

then (52b) or (53b), respectively, can be solved in terms of Hermite polynomials.

c. Solution in terms of Hermite orthogonal functions. When in

(52d)

$$V_2(r) = 1 - r^2$$

(70a)

or in (53d)

$$V_3(r) = 1 - r^2,$$

(70b)

(52d) or (53d), respectively, can be solved in terms of Hermite orthogonal functions.

d. Solution in terms of Mathieu functions. These solutions can be found for (52d) and (53d) if

$$V_2(r) = 16 b \cos 2r$$

(71a)

or

$$V_3(r) = 16 b \cos 2r$$

(71b)

respectively.
Special solutions of the Schrödinger equation. Equation (52d) and (53d) may be solved when

\[ V_2(r) = f(r) \] (72)

or

\[ V_3(r) = f(r) \] (73)

for special values of \( f(r) \). Especially useful are the cases where

\[ f(r) = 0 \]
\[ f(r) = C \]

and

\[ f(r) = A r^2 / 2 \]

which correspond to well-known solutions of Schrödinger's equation.

For more details and for reference texts, see preceding sections.

Asymptotic solutions from the Schrödinger equation. When the derivatives of \( k_r \) are negligibly small, \( V_2 \) and \( V_3 \) in (52f) and (53f) simplify. The problem then reduces to an homogeneous axially symmetric flow with a correction factor which is expressed in (52e) and (53e), respectively. Furthermore, if the radial dimension is large, the problem is further simplified by reducing to a two-dimensional case with the same correction factor. For a detailed illustration of such approximate treatment, see Section III D, on the flow through slightly curved media. A simplification can be achieved also if only the radial
dimensions are large or if only the second derivative of $k_r$ is small.

D. Special Examples of Axially Symmetric Flow with Horizontally Varying Conductivity

1. Introduction

As we have already stated in Section IV A 3, it would be unusual to find cases of varying conductivity with an axially symmetric pattern in undisturbed media. For disturbed media, axial variability in conductivity may be encountered. Several possibilities of this nature arise when a well is dug in the soil and water motion takes place. The conductivity may vary due to physical and chemical changes in the soil. It may also vary due to the hydrodynamic nature of filtration flow.

2. Changes in conductivity due to physical, chemical and biological effects

The drilling of a well and radial water flow into it may cause changes of conductivity in an axially symmetric pattern. Such changes may be caused by several different processes. Drilling a well may cause a release of the natural pressure in soil. Depending on the method of drilling, it may also add new pressure to the soil. A decrease in the pore pressure of the water may effect a consolidation of the soil near the well.

The flow of water may cause a translocation of fine soil particles which in turn will change the conductivity according to a radial pattern.
The water flow may impose chemical and physico-chemical changes in the soil. Leaching by relatively pure water may cause a dispersion of the clay particles and then reduce conductivity. Penetration of salts from a substratum may be caused by the lowering of the ground water level. Cation exchange may take place and, depending on the kind of cations and their concentrations, such an exchange will increase or decrease the conductivity.

Due to changes in aeration conditions and due to chemical changes in the soil, different biological activities may take place. Such activities are capable of changing the soil conductivity, at least temporarily.

3. Changes in conductivity due to hydrodynamic effects

It is an observed fact that Darcy's law of filtration fails for Reynolds numbers bigger than unity. When the law fails, a modified flow equation in one dimension may be written (see references below) as follows

\[-d\phi/dr = Aq_r + Bq_r^2\]  

(74)

Here, \(q\) is the flux in one-dimensional flow, in the \(r\) direction; \(\phi\) is the potential; \(A\) and \(B\) are constants of resistivity, depending on the soil micro-characteristics and the kinematic viscosity of the water. The second term in the right-hand side of (74) becomes negligible for small Reynolds numbers. Reynolds number is defined as follows
where $D$ is a typical soil particle diameter or a typical length measure of the soil pore; $q$ is the average velocity of water; $\mu$ is the dynamic viscosity (Newton's viscosity); and $\rho$ is the fluid density. When the second term on the right-hand side of (74) is negligible, we can identify the coefficient of resistivity, $A$ with the reciprocal of the hydraulic conductivity $k$ in Darcy's law

$$q_r = -k_r \frac{d\phi}{dr} \quad (75)$$

Equation (74) was found as early as 1901 by Forchheimer [9]. It was corroborated by many others [24], [34, pp. 56-69], and [36] and also by the writer [47]. The latter performed his experiments with air flow through cracked clay cores. The flow data from each core was plotted according to the following equation

$$-(d\phi/dr)/q = A + Bq \quad (76)$$

which is equivalent to (74). A perfectly straight line was found. From the slope and the intercept of this line the coefficients $A$ and $B$ were calculated. The non-linear equation (74) was derived by a dimensional analogy with formulas for the drag force of fluid on immersed bodies. Solution for the drag force on a sphere with Reynolds numbers smaller than 1 has been found by Stokes. Solution for slightly bigger Reynolds numbers has been found by Osselem (see Dryden et al. [7, pp. 295-332]). Solution for large Reynolds numbers was found by Goldstein [10].
It should be emphasized that the fact that the flow is not linear does not imply that it is not laminar. In this sense, the attempt to simulate the flow in porous media by a bundle of straight capillary pipes is very misleading. The linearity of the laminar flow equation in a straight capillary tube (Poiseuille's law) arises from an exact solution of Navier-Stokes equations (see Drydan et al. [7, p. 177]). The linearity of filtration flow evolves from the approximation of Navier-Stokes equation when the inertia terms are neglected for small Reynolds numbers [7, p. 295]. Irmay [17] used the Navier-Stokes equation to study the filtration law. Instead of using dimensional analysis and comparisons with specific solutions of Navier-Stokes equation, he derived Equation (74) by averaging the different terms of Navier-Stokes equation in the soil matrix. Actually, without admitting it, he neglects in the process of averaging some correlation terms which are not exactly zero. Irmay's derivation is elegant and instructive, especially for those who doubt the rigorousness of a dimensional analysis.

From all the above it is clear that the equivalent conductivity near a well may decrease due to increased flux concentration.

4. Solution for nonlinear, horizontal flow into a well

In the case of horizontal or almost horizontal flow, into a well, the flux $q$ at any point may be expressed as follows

$$q = Q/(2\pi rz)$$

(77)
Here, $Q$ is the total discharge into the well; and in the case of confined flow, $z$ is the thickness of the aquifer and is a constant. By substituting the value of $q$ from (77) into (74), we obtain the following differential equation

$$\frac{d\phi}{dr} = \frac{AQ}{2\pi z r} + \frac{Bq^2}{4\pi z^2 r^2}$$

(78)

This equation is readily soluble by integration. Let $a$ be the radius of the well where $\phi = \phi_1$ and let $b$ denote the radius of influence where the potential is $\phi_2$. We obtain by integrating (78) from $a$ to $b$

$$\phi_2 - \phi_1 = (AQ/2z)\ln(b/a) + \frac{Bq^2}{4\pi z^2} (1/a - 1/b)$$

(79)

By integrating (78) from $a$ to any radius $r$ we obtain

$$\phi_2 - \phi_1 = (AQ/2z)\ln(r/a) + \frac{Bq^2}{4\pi z^2} (1/a - 1/r)$$

(80)

To put the explicit value of $Q$ in (80) we may solve from (79) utilizing special field measurements. The constants $A$ and $B$ can be derived from (80) by data of pumping experiments.

An approximate solution of (79) can be obtained by ignoring first the second term in the right-hand side and solving the resulting equation for $Q$. The value of $Q$ so obtained is then substituted in the second term in the right-hand side of (79) and the resulting equation solved for $Q$. The result is

$$Q = 2xz\left\{ \left[ (\phi_2 - \phi_1)/(A \ln \frac{b}{a}) \right] - \frac{B(\phi_2 - \phi_1)^2}{(A \ln \frac{b}{a})^3} \left( \frac{1}{a} - \frac{1}{b} \right) \right\}$$

(81)
This is an excellent approximation. It has the advantage that the first term on the right of (81) describes the solution for exact Darcy law flow when $A = 1/k$. The second term gives the effect of departure from Darcy's law. We shall not discuss this equation further.

Instead of (78) we get a more complicated equation if the water flows with a free surface which descends toward the well. It is customary in such a case to make the following approximation

$$\frac{d\psi}{dr} = \frac{dz}{dr}$$  \hspace{1cm} (82)

Consequently (78) transforms to

$$\frac{dz}{dr} = \frac{AQ}{2\pi r^2} + \frac{BQ}{4\pi r^2} + \frac{2}{r}$$ \hspace{1cm} (83)

where $z$ is one of the variables. The further we get from the well, the better is the approximation achieved by this equation, and the smaller is the weight of the non-linear term. We shall not attempt to solve (83) here.

E. Solutions for Nonisotropic Media

Without going into many details we should note that all the solutions suggested in this thesis can be applied to nonisotropic media. This is done by simple transformation as suggested in reference [32]. This substitution should be applied before the separation of variables. In view of this possibility and the methods suggested in the preceding sections we can find, at least formally, the general solution for saturated potential flow in two dimensions and in axial
symmetry in almost any kind of a porous medium. However, the matching of boundary conditions may be impractically cumbersome. The reader will find a more extensive discussion about nonisotropic flow problems in references [29], [15], [2, pp. 5-6] and [35, pp. 63-68]. A comprehensive review of this problem is in [28, pp. 216-285].
V. STEADY FLOW SOLUTIONS WITH NONUNIFORM CONDUCTIVITY
FOR SPECIFIC BOUNDARY CONDITIONS

A. Introduction

In the preceding section we presented the general theory of potential flow in media of nonuniform conductivity. Solutions were found for a wide variety of conductivity patterns. The methods of adoption of these solutions to specific boundary conditions do not vary in principle, as we shall show, from the usual methods of boundary fitting for flows with uniform conductivities. In this section we shall review some of the procedures and generalize them for all possible cases of saturated steady flow problems.

The mathematical techniques in the following sections are by no means original. They can be found in many texts on mathematical physics, ordinary differential equations and advanced calculus for engineers. The problems with which we concern ourselves in this section appear in texts dealing with subjects like Hermitian operators, Sturm-Liouville equations and boundary conditions; also orthogonality and completeness of series, Fourier analysis or approximation in the mean.

It is our purpose to demonstrate, by means of general treatment, the relative simplicity and generality of the boundary fitting procedures as applied to the flow problems, whether of uniform or non-uniform conductivity.
In defining an expression for the variable conductivity, care must be taken that the expression will not lead to negative values of conductivity. Some problems may arise in computation if the conductivity vanishes at the boundary. In fitting infinite series of Bessel function one should carefully examine if zero eigenvalue (or characteristic number) is a possible root of the eigenvalue problem. If this is the case the fitting is done like with the Fourier cosine series.

Finally we shall treat a specific solution with specific boundary conditions. A problem with the same geometry but with uniform conductivity was solved by Kirkham and van Bavel [23] and is discussed in Section III A of this thesis.

B. Hermitian Operators and Orthogonal Solutions

It was shown in the preceding sections that all partial differential equations of saturated flow of incompressible fluid can be separated and represented by three ordinary differential equations (Equations (12), Section IV). It was also shown that these three equations can be presented in a Sturm-Liouville equation form (Equation (13), Section IV). All three of these equations can be presented in the following standard form

\[ L(U) = (pU')' + \alpha^2 pU = 0 \]  

(1)

where \( L \) is an operator, \( p \) is a known differentiable function of the independent variable (in our case one of the coordinates) and \( U \) is the unknown function of the independent variable (coordinate). For the
following discussion we use \( x \) as the independent variable of (1).

Let us assume that \( U_i \) and \( U_j \) are two possible solutions of (1) for two different values of \( n \), namely \( i \) and \( j \), respectively. From Equation (1) we can write

\[
L(U_i) = (pU_i')' + i^2 pU_i = 0
\]

\[
L(U_j) = (pU_j')' + j^2 pU_j = 0
\]

We now multiply (2) by \( U_j \) and (3) by \( U_i \) and subtract the result of (3) from the result of (2). We get

\[
U_j L(U_i) - U_i L(U_j) = U_j (pU_i')' - U_i (pU_j')' + (i^2 - j^2) pU_i U_j = 0
\]

By integrating (4) between the limits \( a \) and \( b \) along the axis \( x \) and by rearranging we get

\[
\int_a^b U_j (pU_i')' \, dx - \int_a^b U_i (pU_j')' \, dx = (j^2 - i^2) \int_a^b pU_i U_j \, dx
\]

\( i \neq j \)

We define the following operator \( M \)

\[
M(U) = (pU')'
\]

It will be proved in general that the left-hand side of (5) vanishes. In such a case the operator \( M \) in (6) is said to be Hermitian and the solutions \( U_i \) and \( U_j \) in (2), (3) and (5) are shown to be orthogonal with the weight function \( p \). Thus we have the following relations between two solutions for different eigenvalues \( n \)
The proof is achieved in the following manner. We integrate one of the terms on the left-hand side of (5) twice, through integration by parts. We get the following
\begin{equation}
\int_a^b U_j(pU_j') \, dx = [U_j(pU_j')]_a^b - [U_j'pU_j]_a^b + \int_a^b U_j(pU_j')' \, dx
\end{equation}

The limits of integration a and b are actually boundaries of the unknown function U along the axis x. In most of the cases at least one of the three terms \(U_j', U_j, \) and \(p\) vanishes at each of the boundaries. Consequently the two terms on the right-hand side of (8) vanish. In case that each solution \(U_n\) is separately fitted at the boundaries (a) and (b), the above conclusion is achieved also when none of the functions \(U, U_j, \) or \(p\) vanish at the boundaries a and b. In any case the two first terms on the right-hand side of (8) are identical and cancel each other. Thus we proved that
\begin{equation}
\int_a^b U_j(pU_j') \, dx = \int_a^b U_j(pU_j')' \, dx
\end{equation}

Because we assumed that
\begin{equation} j^2 - i^2 \neq 0 \end{equation}
we can see from (5) that (7) is an identity.

In most cases the final fitting to the boundary conditions is done by determining the coefficients in an infinite series which is made of the solutions \(U_n\) for different values of the eigenvalue \(n\). The
calculations of these coefficients is made possible by the fact that
these different solutions \(U_n\) are orthogonal to each other with the
weight function \(p\), as in (7). \(p\) is the coefficient defined in (1).

Comparison of (1) with (11), "the separated flow equations",
Section IV, gives the following identities for the weight functions.
For the potential function and the stream function in two-dimensional
flow we get

\[ p = kx \quad \text{or} \quad p = ky \]  

(11)

It is clear that for the uniform case \(p\) reduces to unity, as the
solutions reduce to sine and cosine terms.

For the solutions of the stream function and the potential function
in terms of the elevation \(z\) we get

\[ p = kz \]  

(12)

For the solutions of the potential in terms of the radius \(r\) we get

\[ p = rk_r \]  

(13)

For the solutions of the stream function in terms of the radius \(r\) we
get

\[ p = k_r/r \]  

(14)

Thus we have identified the weight functions for the orthogonal solutions
in all cases of saturated steady flow with uniform or nonuniform
conductivities.
C. Approximation in the Mean

The subject "approximation in the mean" is considered in some treatises under the subject "expansions in series of orthogonal functions".

To solve the partial differential equations of flow we first separated the variables and thereby reduced each partial differential equation to two ordinary differential equations (see Section IV B 3). To make the discussion simpler we shall refer here to the stream function in axially symmetric flow. By changing only the notation the discussion can be extended to other cases. Due to the separation of variables we arrive at three pairs of separated solutions as follows

\[ \phi = Z_0 B_0 + Z_1 R_1 + Z_2 R_2 \]  

(see Section IV B 3). Here \( Z_0, Z_1, \) and \( Z_2 \) are solutions of the ordinary differential equation (IIa), Section IV, in terms of the coordinate \( z \) for the eigenvalues \( n^2 \) being of the value zero, or of positive values, or of negative values, respectively. Similarly, \( B_0, B_1, B_2 \) are solutions of the ordinary differential equation (IIe), Section IV, in terms of the radius \( r \), for the eigenvalues \( n^2 \) being of the value zero, or of positive values, or of negative values, respectively.

In general the \( Z_0 B_0 \) includes only four possible terms. The solutions \( Z_1 R_1 \) and \( Z_2 R_2 \) may have an infinite number of terms for all possible values of the eigenvalue \( n \), and so they can form infinite series.
In fitting the general solution to the boundary conditions we first try to choose those terms which will conform individually to the boundary values. One of the boundary conditions will generally determine the eigenvalues \( n \). Thus, finally we can express the general solution of \( \psi \) in terms of infinite series. Each term in the infinite series has a different eigenvalue \( n \) and generally satisfies all boundary conditions but one. The fitting to this last boundary condition is done by adjusting the coefficients in the infinite series (Fourier series are the most common), so that the series will describe the condition function on this last boundary. This last fitting of the infinite series is called "an approximation in the mean" [4, p. 40]. It is with this procedure that we are concerned in this section. For the following a convenient reference is [4, pp. 34-52 and 11, pp. 229-239].

Let the boundary condition be
\[
\psi = f(z) \quad \text{along} \quad r = a \quad 0 \leq z \leq d
\]
or
\[
\frac{\partial \psi}{\partial r} = g(z) \quad \text{along} \quad r = b \quad 0 \leq z \leq d
\]
Let the general form of \( \psi \) be
\[
\psi = \sum_{n} A_n Z(n,z) R(n,r)
\]
It is obvious that for both forms of conditions (for the function itself or for its normal derivative) that the \( R \) function is a constant
at \( r = a \) or \( r = b \), varying only with \( n \); and that the \( Z \) function varies with \( z \) and \( n \). Otherwise the boundary condition may be given along the \( r \) coordinate for a given \( z \)-value; then the function \( Z \) becomes constant and the function \( R \) varies.

At any rate, it is obvious that in deriving the coefficients of the infinite series one of the functions, \( R \) or \( Z \), can be lumped with the coefficient \( A_n \). Thus we shall study the following equation

\[
\sum_{n} A_n Z(n,z) = f(z) \quad 0 \leq z \leq d \tag{16}
\]

In this equation \( f(z) \) varies continuously or piece-wise continuously and \( A_n \) is to be determined.

To comply with the notation in the preceding section we rewrite (16) as follows.

\[
\sum_{n} A_n U(n,x) = f(x) \quad a \leq x \leq b \tag{17}
\]

The determination of the coefficients \( A_n \) is achieved by the least square method. We express the series in (17) for a limited number of terms with the eigenvalues ordered according to their size with \( N \) the biggest value of \( n \) as follows

\[
S_N = \sum_{n=1}^{N} A_n U(n,z), \quad n = n_1, n_2, n_3, \ldots \tag{17}
\]

By taking a finite number of terms, \( S_N \) will approximate \( f(z) \) at the boundary with an error \( e_n^2 \) such that
We now require that the error square \( e_N^2 \) will be a minimum by adjusting the coefficients \( A_n \).

It is easy to see that (17) will remain an equality if we multiply it through by the square root of the weight function \( [p(x)]^{1/2} \). Accordingly, instead of (18) we can write for an error \( E \), the square of which is to be minimized

\[
E_N^2 = \int_{a}^{b} \left\{ [p(x)]^{1/2} f(x) - \sum_{n=1}^{N} A_n [p(x)]^{1/2} U(n,x) \right\}^2 dx \tag{19}
\]

If \( E_N^2 \) is minimized in (19), so is \( e_N^2 \) in (18). The condition of minimizing the error is met by taking the partial derivative of the error with respect to each \( A_n \) and equating to zero. There result \( N \) equations of the following form

\[
\int_{a}^{b} \left\{ [p(x)]f(x) - \sum_{n=1}^{N} A_n [p(x)]^{1/2} U(n,x) \right\} [p(x)]^{1/2} U(m,x) dx = 0 \tag{20}
\]

Equation (20) is arrived at by taking the partial derivative of (19) with respect to one coefficient \( A_m \) and equating to zero. By rearranging (20) we get

\[
\int_{a}^{b} [p(x)]^{1/2} U(m,x) \sum_{n=1}^{N} A_n [p(x)]^{1/2} U(n,x) dx = \int_{a}^{b} p(x) U(m,x) f(x) dx \tag{21}
\]

In view of the orthogonality of the eigenfunctions \( U_n \) with the weight function \( p(x) \) (see (4)), the integrated terms on the left-hand side of (21) vanish, unless \( m = n \). Thus (21) reduces to the following
By rearranging (22) we find the general expression for the coefficients $A_n$ as follows

$$A_n = \frac{\int_a^b p(x)U(n,x)f(x)\,dx}{\int_a^b p(x)[U(n,x)]^2 \,dx}$$

The different weight functions $p$ were explained and derived in the preceding section. $U(n,x)$ are the different eigenfunctions for different values of the eigenvalues $n$. Each varies along the coordinate $x$ (or any other) between $a$ and $b$. The function $f(x)$ is the boundary value along a boundary in the $x$ direction (other coordinate constant).

The above developments apply to boundary values given along one coordinate at a constant value of other coordinates. When the boundaries are not this simple, perturbation methods may be considered [31, pp. 1038-1062]. We shall not discuss such methods.

D. Example of a Specific Flow Problem in Axial Symmetry and Nonuniform Conductivity

1. Presentation of the problem

We assume the conductivity to vary as follows

$$k = k_z k_r$$

where

$$k_r = 1$$

and

$$k_z = (az + b)^2$$
and \( \alpha \) and \( \beta \) are constants, and where we emphasize that \( k_r \) is not a component of conductivity in the \( v \)-direction and \( k \) is not a component of conductivity in the \( z \)-direction; both \( k_r \) and \( k_z \) are scalar functions of its subscript. The breaking of \( k \) into its two factors of (24) is purely a mathematical artifice. In (24-26) the coordinates are the radius \( r \) and the elevation \( z \), in cylindrical coordinates and with axial symmetry.

In the example we solve the boundaries and the boundary conditions are identical with those in the case solved by Kirkham and van Bavel [23] (also discussed in Section III A of this thesis). For convenience of reference, the boundary conditions will be registered below in terms of the stream function \( \psi \).

boundary la: \( \frac{\partial \psi}{\partial r} = a \quad h < z \leq d \quad r = a \)
boundary lb: \( \frac{\partial \psi}{\partial r} = 0 \quad 0 \leq z \leq h \quad r = a \)
boundary 2: \( \frac{\partial \psi}{\partial z} = 0 \quad z = d \quad a \leq r \leq b \)
boundary 3: \( \frac{\partial \psi}{\partial z} = 0 \quad 0 \leq z \leq d \quad r = b \)
boundary 4: \( \frac{\partial \psi}{\partial r} = 0 \quad z = 0 \quad a \leq r \leq b \)

The partial differential equation for the stream function is presented in Section IV, Equation (9b). In Section IV B 3, we separated the variables according to Equations (10d) and (10f) there. From the separation we obtained the following two ordinary differential equations, (see (11g) and (11h) of Section IV)
Figure 9. Boundaries for the flow problem discussed in Section V D 1.
PONDED WATER

IMPERMEABLE LAYER

\[ z \]

\[ d \]

\[ h \]

\[ b \]

\[ a \]

\[ 2 \]

\[ 3 \]

\[ 4 \]

\[ r \]
In view of our assumption of (25), Equation (27) reduces to the simple case of uniform conductivity. This case is solved in Section II of this thesis. The solutions of (27) for \( k_r = 1 \) are as follows:

\[
R_1 = r[A_2 J_1(nr) + B_2 K_1(nr)] \quad (n^2 > 0) \tag{29}
\]

\[
R_2 = r[C_2 J_1(mr) + D_2 K_1(mr)] \quad (-m^2 = n^2 < 0) \tag{30}
\]

\[
R_0 = Er^2 + F \quad (n^2 = 0) \tag{31}
\]

where the coefficients \( A_2, B_2, C_2, D_2, E \) and \( F \) are real constants. The solution for (28) in terms of the coordinate \( z \) are suggested in Section IV C 3 e. As suggested there we first transform (28) to the Schrodinger equation form. The transformation needed is

\[
Y(z) = p^{1/2} Z(z) \tag{32}
\]

where in our case

\[
p^{1/2} = k_z^{1/2} = \alpha z + \beta \tag{33}
\]

After the transformation, equation (28) reduces to the following:

\[
Y'' + [n^2 - V(z)]Y = 0 \tag{34}
\]
where

\[-V(z) = \frac{1}{4}(k_z^r/k_z) - \frac{1}{2}(k_z^n/k_z)\tag{35}\]

By substituting the value of \(k_z\) from (26) to (35) we find that in our case \(V(z)\) vanishes. Thus equation (28) that was reduced to (34) is further reduced to give

\[Y'' + n^2 Y = 0\tag{36}\]

This last equation is solved in a straightforward manner to give

\[Y_1 = A_1 \sin(nz) + B_1 \cos(nz) \quad (n^2 > 0)\tag{37}\]

\[Y_2 = C_1 \sinh(nz) + D_1 \cosh(nz) \quad (n^2 < 0)\tag{38}\]

\[Y_0 = Gz + H \quad (n^2 = 0)\tag{39}\]

where \(A_1, B_1, C_1, D_1, G\) and \(H\) are real constants.

Considering again the transformation (32) and the explicit value of \(k_z\) (26), the solutions of (28) can be written as follows

\[Z_1 = (az + \beta)^{-1}[A_1 \sin(nz) + B_1 \cos(nz)] \quad (n^2 > 0)\tag{40}\]

\[Z_2 = (az + \beta)^{-1}[C_1 \sinh(mz) + D_1 \cosh(mz)] \quad (-m^2 = n^2 < 0)\tag{41}\]

\[Z_0 = (az + \beta)^{-1}[Gz + H] \quad (n^2 = 0)\tag{42}\]

The general solution of our problem is found by combining Equations (29), (30), (31), (40), (41), and (42) in the general combination

\[\psi = Z_0 R_0 + Z_1 R_1 + Z_2 R_2\tag{43}\]

for which the boundary conditions need to be satisfied as below.
3. Fitting to the boundary conditions

It may be easily verified that the following solutions satisfy conditions 3 and 4

\[ \phi = H + \sum \left( a_0 + \beta \right) A_n \sin(nz) r R_1(nr) \]  

(44)

where

\[
R_1(nr) = \frac{I_1(nr) - K_1(nr)}{I_1(nb) - K_1(nb)}
\]

\[
I_1(nb) + K_0(nb)
\]

\[ I_0(na) + K_0(na) \]

(45)

K_1 and I_1 are modified Bessel functions (see Section II of the thesis).

We have yet to satisfy conditions 2, la, and lb, and to find the eigenvalues \( n \) and the coefficients \( A_n \).

Condition 2 when applied to the solution in (44) implies, after differentiating, equating to zero, and factoring out the proper terms, that

\[ \sin(nd) = \left( \frac{n}{a} \right) (ad + \beta) \cos(nd) \]  

(46)

or in a form more convenient for computation

\[ \tan(m) = m(1 + \beta/\omega d) \]  

(47)

where

\[ m = nd \]

Thus the values of \( nd \) or of \( n \) are determined.

Conditions la and lb when applied to (44) imply the following

\[ \sum_{n} n A_n \left( a_0 + \beta \right)^{-1} \sin(nz) = \begin{cases} 0 & (0 \leq z < h) \\ -A & (h < z \leq d) \end{cases} \]  

(48)
Now we make use of the discussion about the approximation in the mean in Section C of this part of the thesis. Comparison of (48) with (44) shows that $(az + \beta)^{-1}\sin(nz)$ is one of the solutions in $z$ (see Equation (40)). A direct application of (23) gives

$$nA_n = \frac{\int_0^d (-a)p(z)(az + \beta)^{-1}\sin(nz)dz}{\int_0^d p(z)(az + \beta)^{-2}\sin^2(nz)dz} \quad (49)$$

As in Equation (12), the weight function $p(z)$ is identified here with $k_z$. It was also so identified in (33). By introducing the explicit value of $k_z$ from (26) into (49) and by rearranging we get

$$A_n = \frac{1}{n} \frac{\int_0^d (az + \beta)\sin(nz)dz}{\int_0^d \sin^2(nz)dz} \quad (50)$$

The integral in the denominator is found as follows

$$\int_0^d \sin^2(nz)dz = d/2 - (1/2n)\sin(2nd) = d/2 - (1/2n)\sin(nd)\cos(nd) \quad (51)$$

We may substitute the values of $\sin(nd)$ from (46) and express the integral in the denominator in still another form as follows

$$d/2 - (1/2n)\sin(nd)\cos(nd) = d/2 - [(ad + \beta)/2\alpha]\cos^2(nd) \quad (52)$$

The integral in the numerator of (50) is found as follows

$$\int_0^d (az + \beta)\sin(nz)dz = \left[-(az/d)\cos(nz) - \frac{d}{h} \left(\frac{a}{n^2}\sin(nz)\right)\right] - \left[\left(\frac{\beta}{n}\cos(nz)\right)\right] \quad (53)$$
After we expand the right-hand side of (53) and introduce the value of $\sin(nd)$ from (46), we get for the numerator of (50)

$$(l/n)(oh + p)\cos(nh) - (a/h^2)\sin(nh)$$

(54)

We can now express the coefficients $A_n$ explicitly by introducing to (50) the expression in (51) for the denominator, and the expression from (54) for the numerator. After slight rearrangement we get

$$A_n = -\frac{1}{n^2} \frac{(ah + p)\cos(nh) - (a/n)\sin(nh)}{d/2 - (1/4n)\sin(2nd)}$$

(55)

4. Calculation of the constant $H$ and the discharge

The constant $H$ in (44) is determined by assuming that $\phi = 0$ at $(4 = a)$ and $(z = d)$. We get in a straightforward manner that

$$H = -\sum A_n \sin(nd)aR_1(na)$$

(56)

The discharge $Q$ can be calculated by integrating the flux component $q_z$ over the soil surface at ($z = d$)

$$Q = \int_a^b 2\pi r(k/r)(\partial\phi/\partial r)dr$$

(57)

The value of the conductivity $k$ at ($z = d$) is obtained from (26) as

$$k = k_x k_z = (ad + p)^2$$

(58)

Thus we get from (57)

$$Q = -(ad + p)^2 2\pi[\phi(r = b, z = d) - \phi(r = a, z = d)]$$

(59)
But as we have already assumed

$$\phi(r = a, z = d) = 0$$  \hspace{1cm} (60)

and as can be found from the solution of (44) and (45)

$$\phi(r = b, z = d) = H$$  \hspace{1cm} (61)

Thus from (59) and (56) we get

$$Q = (\alpha d + \beta)^{-1} 2\pi \sum_n A_n \sin(n\delta) \alpha R_1(n\delta)$$  \hspace{1cm} (62)

and

$$A_n = -\frac{1}{n^2} \frac{(\alpha h + \alpha)\cos(nh) - (\alpha/n)\sin(nh)}{d/2 - (1/2n)\sin(2nd)}$$  \hspace{1cm} (63)

5. An independent solution in terms of the potential

One way to solve for the potential in our special example (and other examples) is to start from the basic differential equation and use exactly the same method employed in the preceding sections for the stream function. The variables are separated, and we get two ordinary differential equations. The one in terms of the radius $r$ is identical with the equation in $r$ for uniform conductivity. Its solutions are

$$R_1 = A_2 J_0(nr) + B_2 K_0(nr) \quad (n^2 > 0)$$  \hspace{1cm} (64)

$$R_2 = C_2 J_0(nr) + D_2 N_2(nr) \quad (-m^2 = n^2 < 0)$$  \hspace{1cm} (65)

$$R_o = E \ln(r/c) + F \quad (n^2 = 0)$$  \hspace{1cm} (66)

The ordinary differential equation in terms of $z$ is identical with the same one for $\phi$ (see Equation (28)). Therefore, its solutions are
given in (40), (41), and (42). The general solution for $\phi$, the potential, can be expressed as follows

$$\phi = R_0 z_0 + R_1 z_1 + R_2 z_2$$

(67)

where the different terms are defined in Equations (40), (41), (64), (65), and (66). The solution has now to be fitted to the boundary conditions of the potential corresponding to the boundary conditions on the stream function (Section V D 1). The two sets of conditions, for the stream function and the potential function, relate to each other according to the orthogonality relations in Section II C.

It can be verified that the specific solution of $\phi$ in our case can be expressed as follows

$$\phi = d + \sum_n [A_n \sin(nz) + B \cos(nz)] R_0(nr)(dz + \beta)^{-1}$$

(68)

where

$$R_0(nr) = \frac{I_0(nr) K_0(nr)}{I_1(nb) + K_1(nb)}$$

(69)

This solution satisfies condition 3 on the potential. Condition 4 on the potential requires that

$$-\pi \beta^{-2} B_n + \beta^{-1} a A_n = 0$$

(70)

Thus

$$A_n = \pi^{-1} \beta^{-1} B_n$$

(71)
Equation (68) may now be written as follows

$$
\varphi = d + (az + \beta)^{-1} \sum B_n \left[ a(n\beta)^{-1} \sin(nz) + \cos(nz) \right] R_0(nr) \tag{72}
$$

Condition 2 on the potential implies that

$$
(ad + \beta)^{-1}[an^{-1}a^{-1} \sin(nd) + \cos(nd)] = 0 \tag{73}
$$
or

$$
\sin(nd) = nax^{-1} \cos(nd) \tag{74}
$$

This equation determines the eigenvalues $n$. It may be written in the more convenient form

$$
\tan(m) = m \beta(ad)^{-1} \tag{75}
$$

Where

$$
m = nd \tag{76}
$$

Conditions 1a and 1b require that

$$
\sum B_n \left[ a(n\beta)^{-1} \sin(nz) + \cos(nz) \right] \left\{ \begin{array}{c}
h, \\
z, 
\end{array} \right. \begin{array}{c}
(0 \leq z \leq h) \\
(h < z \leq d)
\end{array} \tag{77}
$$

Consequently, we calculate $B_n$ in a straightforward manner, obtaining

$$
B_n = a_n/b_n \tag{78a}
$$

where

$$
a_n = \int_0^h h(az + \beta)[(d/\beta n) \sin(nz) + \cos(nz)] dz \\
+ \int_h^d z(az + \beta)[(a/\beta n) \sin(nz) + \cos(nz)] dz \tag{78b}
$$

and
The integrals in (78) can be easily solved, and (74) can be used to simplify the expressions. It is noted that the solution for $\phi$ does not have the same eigenvalues as the solution for $\psi$.

6. Remark on orthogonality between the stream function and the potential function

It has been shown in several places that the stream lines and potential lines are orthogonal both in the cases of uniform and non-uniform flow. In fact, in Section II of this thesis we derived the stream function by assuming such orthogonality.

By using the orthogonality relations we should be able, at least formally, to derive $\psi$ from $\phi$ and vice versa. Alternatively, it should be formally possible to demonstrate the orthogonality between the independent solutions of $\phi$ and $\psi$. Such a procedure is demonstrated for a case of uniform conductivity in Section III A.

In our specific example, solved above, it is not easy to transform from solution in terms of the potential $\rho$ to solution in terms of the stream function $\psi$. Nevertheless, the orthogonality exists and in the flow net the streamlines should be normal to the potential lines.

Should we attempt to calculate the discharge from the expression for the potential (72), we would have to differentiatate with respect to one variable and integrate with respect to the other. This is still possible in our specific example. It may, however, become very hard
in some other problems. In any rate the advantage in using the stream function for solution is obvious in our example.
VI. APPENDIX I. NOTES ON THE MEASUREMENT OF SOIL WATER VAPOR PRESSURE

A. Introduction

We assume that a proper free energy function has been defined by using the Legendre transformation on the first law of thermodynamics (see Sommerfeld [39, p. 42]). The formulation of the first law of thermodynamics should include all types of work such as that included in Sommerfeld [39, p. 49]. If the water in the liquid phase is in equilibrium with the water in the vapor phase, then a properly defined free energy (per mole of water) must be the same in the vapor and liquid phases. Assuming that the soil gaseous phase can be considered as an ideal gas, the only two intensive variables in the free energy expression of the water vapor phase are the partial pressure and the temperature. It is certainly easier, then, to measure the free energy in the vapor phase than in the liquid phase. When measuring any entity by direct contact with the liquid phase, we cannot be sure that we achieve an equilibrium without any constraints.

The problem in measuring the soil vapor pressure is that its relative change is very small. Let us regard the measurements of water "tension" on a pressure plate as the osmotic pressure that compensates for the reduction in soil moisture content (see Taylor [41]). The wilting point of plants occurs around a tension of 15 atmospheres. This is equivalent to about a 1 percent reduction in the vapor pressure of water.
All the methods that were devised, to date, for measuring directly the vapor pressure are based on measuring the difference between the ambient temperature and the dew point. (Korven and Taylor [25], Richards and Gen Ogata [34]). As these differences are extremely small, temperature control within $10^{-3}$ to $10^{-4}$ centigrade is essential. Even then the sensitivity and accuracy of these methods is very limited and only several points can be measured in the interesting range of vapor pressure variations (98 to 100 percent relative humidity).

Three methods were considered as possible ways to solve this problem. One of them was tried and was found inadequate. The other two did not pass the stage of planning and small preparatory work. The three methods will be described here very briefly without many technical or theoretical details.

B. Unsuccessful Use of Humidity Sensing Elements

The humidity sensing elements are made as printed circuits on an absorptive surface. The change in humidity is measured by the change in resistance across gaps in the circuit. Such elements are available commercially\(^1\). As a rule they are made of some plastic material with or without some sensitive coating. It was generally found that sensitivity decreased when close to air saturation. Slight

\(^1\) American Instrument Co., Inc. - Solvet Spring Maryland in metropolitan Washington, D.C.; Warren Components Division, Kl-Tronics, Inc., P.O. Box 479, Warren, Pa.; Minneapolis-Honeywell Regulator Company, Minneapolis 8, Minn., Toronto 17, Ontario, Canada.
changes in ambient temperature or CO$_2$ content caused big variations in reading. The 1 percent range between 99 percent and 100 percent relative humidity, is equal to or smaller than the experimental error of the measurement.

C. The Use of Osmotic Pressure Transducer

Let a cell with constant volume contain a standard solution whose vapor pressure is by several percent smaller than vapor pressure of pure water. Through a semi-permeable membrane, inserted as one of the cell walls, the solution comes in contact with vapor atmosphere of the soil. If soil vapor pressure is bigger, the cell will build up an osmotic pressure without adding a great amount of water. The relation between the osmotic pressure and the difference in vapor pressure can be calculated from van't Hoff's equation. It can otherwise be found by calibrating at only a few points and interpolating by the theoretical formula. One percent difference in vapor pressure will give rise to a pressure of 14 to 15 atmospheres.

In building such an instrument, the following technical problems are involved. It is necessary to measure pressure accurately with very small signals. It has to be remembered that to detect pressure differences, some change in volume is required, which should be kept to a minimum. This problem may be solved by using straingage pressure transducers. They consist of straingages attached to a metal cylinder. They detect pressure change by the slight elastic deformation of the cylinder under pressure variations. Such transducers are available
commercially\(^2\). The working range of their different types range from measurements of blood pressure to the weighing of heavy trucks.

The second problem is of finding a good semi-permeable membrane combined with a proper solute. The membrane must be permeable enough so that equilibrium will be established relatively fast. It should also be reasonably stable chemically and biologically and capable of withstanding high pressures. In case that too long a period is needed to achieve equilibrium, an alternative method can be devised with some sacrifice of accuracy. A cell which is in a given level of free energy is put in contact with soil atmosphere. The rate of pressure build-up may serve as a measure of the soil water vapor pressure.

D. The Use of Krypton (Kr\(^{85}\)) for Laboratory Measurement of Vapor Pressure

In the space within the plastic bag (see Figure 10) there is a (gaseous) Krypton radioisotope emitting beta particles and water vapor above a standard water solution. In the space surrounding the plastic bag there exists water vapor in equilibrium with water in a dish of wet soil. The concentration of the standard solution is so adjusted that the vapor pressure inside the plastic bag is slightly less than the vapor pressure of the soil water.

By choosing a thin enough plastic film that will bend under the slightest pressure difference, we can obtain an equality of the total

Figure 10. Schematic arrangement for the use of Kr$^{85}$ for measuring vapor pressure.
a. GEIGER COUNTER TUBE IN A SHIELDING CASE
b. EXPANDABLE PLASTIC BAG
c. STANDARD WATER SOLUTION
d. WET SOIL
pressure in the inside and outside of the plastic bag. The following equation then holds

\[ P_s = P_v + P_k \]

where

- \( P_s \) = the soil water vapor pressure
- \( P_v \) = the water vapor pressure in the plastic bag
- \( P_k \) = the Krypton pressure.

We assume that \( P_k \) is known. \( P_k \) can be measured by assaying, by means of a Geiger counter, the Krypton activity within a shielding base that provides a permanent geometry and allows, through ducts, free passage of gas. The count will be almost proportional to \( P_k \). Changes in counting due to variations in self-absorption should be negligible, as the total absolute pressure of the krypton and water vapor is small.

The vapor pressure of the solution \( P_v \) is a known standard for a given temperature. The absolute partial pressure of the Krypton \( P_k \) can be determined in proportion to the radioactive counting rate, once a calibration measurement has been made. A change in the pressure difference between \( P_v \) the known water vapor pressure, and \( P_s \) the unknown soil water vapor pressure, will cause a movement of the plastic bag and compression or expansion of the krypton. Change in the krypton pressure will, in turn, be detected by a change in the
activity count rate. The equation \( P_s = P_v + P_k \) will under each circumstance give \( P_s \). We can reach any degree of sensitivity of the measurement, by making \( P_v \) and \( P_k \) as close as possible and by extending the radioactivity counting period.

Several other arrangements are possible where compression of Krypton would compensate for slight differences in vapor pressure. Only the principle of the idea is presented here.
Figure 11. Schematic arrangement for unsaturated flow experiment.
Evaporation

Constant head supply of saturated solution

Column

Soil

Burette

$x$

$L$
Several transfer mechanisms effect the movement of water in the soil. It is important to be able to distinguish between them and to measure how each one is related to specific driving forces and mobility coefficients.

Water transfer mechanisms may be divided into two groups. One group includes mass flow and diffusional processes in the liquid phase or in the liquid-gas interface. The second group includes mass flow and diffusional flow which occurs in the gaseous phase, at least over part of the flow path.

A simple suggested experiment is described which offers a possibility of distinguishing between these two groups (see Figure 11). The soil core is dried from the top. We let the processes of evaporation from the top, and water supply from the bottom, reach a steady state.

The water flux in the liquid phase is designated by \( q_w \), and the flux through the gaseous phase is designated by \( q_g \). In the steady state we have for any level in the soil column the following equation of continuity

\[
q_g + q_w = q = \text{constant};
\]

or alternatively

\[
\text{div } q = 0;
\]

or in one dimension \( x \) (the coordinate \( x \) is measured upward with the origin located where the soil is water-saturated at its bottom)
\[
\frac{\partial q}{\partial x} = 0
\]

Consequently
\[
\frac{\partial q_g}{\partial x} = - \frac{\partial q_w}{\partial x}
\]

The physical interpretation of the left-hand side is the following. The divergence of the gaseous phase is an expression of the net evaporation within the infinitesimal volume (here, length) element. It is expected that the divergence of the gaseous flux in our experiment has the same sign (positive) at every point of the soil core. In other words, the gaseous flux pattern along the x axis is expected to be monotonically increasing. This expectation seems to be achieved in the suggested experiment.

We have the following boundary conditions on the flux.

\[
q_w = q \quad \text{and} \quad q_g = 0 \quad \text{at} \quad x = 0
\]

\[
q_w = 0 \quad \text{and} \quad q_g = q \quad \text{at} \quad x = L
\]

A radioactive salt is introduced into the water that feeds into the core to form a saturated solution. This salt should satisfy the following requirements

a. The radioactivity must originate in the anion. In this way, adsorption and exchange of radioactive material on the soil particles will be kept to a minimum.

b. The radioactive material should be a gamma emitter, so that self-absorption in the soil will be kept to a minimum.
c. The salt must be of small solubility in water but soluble enough to give a measurable activity.

An increase of the gaseous flux over a length element $dx$ implies net evaporation within this element. As the water is saturated with the radioactive salt, the local evaporation will cause a proportional sedimentation of active salt. Assuming a steady state flow and monotonic flux pattern, there is no other reason for change in local radioactivity.

The following equation will relate the change in activity to the change in gaseous flux at a point $x$

$$\frac{dc}{dt} dt = A \frac{dq}{dx} dx$$

(6)

where

- $t$ is the time
- $A$ is a proportionality coefficient
- $c$ is the activity in terms of counts per unit time ($c$ is a function of $x$ and $t$ and its units will depend on $A$; $c$ may be, for example, counts per minute per unit volume of moist bulk soil).

The constant $A$ can be found by integration of (6) over time and length. Thus

$$\int_{t_1}^{t_2} \int_{0}^{L} \frac{(dc/dt)dx}{dt} dt = A \int_{t_1}^{t_2} \int_{0}^{L} \frac{(dq/dx)dx}{dx} dt$$

(7)
Integrating the left-hand side with respect to t and the right-hand side with respect to x and then t, we find upon taking the boundary conditions into account the result

\[
\int_0^L c_{t_2} \, dx - \int_0^L c_{t_1} \, dx = A(t_2 - t_1)q
\]

(8)

where \( c_{t_2} \) and \( c_{t_1} \) are the values of c at a distance x at times \( t_2 \) and \( t_1 \) respectively (where q is the total flux passing through the column).

The remaining integration on the left-hand side of (8) is performed numerically from values which are to be measured, of \( c_{t_2} \) and \( c_{t_1} \) along the x axis. Consequently we can calculate A. By rearrangement of (8) we obtain

\[
A = \frac{\int_0^L (c_{t_2} - c_{t_1}) \, dx}{(t_2 - t_1)q}
\]

(9)

The term in the denominator is simply the total amount of water vapor that passed through the core during the time period \( t_2 - t_1 \).

It is possible now to calculate the vapor or the liquid flux at every point through the medium. By integration of (6) from zero to \( x \) we get

\[
q_g(x) = \frac{1}{A} \int_0^x \frac{dc}{dt} \, dx
\]

(10)

In a steady state we have at some value \( x = x^* \)

\[
\frac{dc}{dt} = \frac{c_{t_2} - c_{t_1}}{t_2 - t_1}
\]

(11)
On substituting into (10) and considering the value of $A$ from (9), we obtain

$$q_g(x) = \frac{\int_0^x (c_{t2} - c_{t1})dx}{\int_0^L (c_{t2} - c_{t1})dx} q$$

(12)

The right-hand side of this equation consists of experimentally measurable data. The fraction consists of the count change integrated from the bottom of the core to any point $x$ divided by the integrated count change along the complete core.

VIII. APPENDIX III. THE GENERAL DERIVATION OF THE STEADY STATE STREAM FUNCTION

A. The Differential Equation and its Complete Integral

In Section II of this thesis we derived the stream function in axially symmetric flow. We first assumed a function form of a velocity potential and from it derived the stream function by assuming orthogonality between the two.

We now observe that the deviation of the stream function should be independent of the existence or validity of the physical concept of a potential in filtration flow. Actually the only fundamental observable phenomenon is that there exists a stream of water. We shall therefore attempt here to derive the different stream functions using only kinematical considerations. We shall show that the stream functions for different flow cases can be derived all in a standard formal manner. The general differential equation of a stream line is (see Dryden et al. [7, pp. 31-35, 41])

\[
\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt
\]

(1)

where dx, dy and dz are the cartesian components of an infinitesimal portion of the path line ds, and u, v and w are the cartesian components of the flux along the axes s, y and z; and dt is the time increment needed for a mass "point" in our simulated continuum to sweep through the portion of the path line ds.

The general solution of (1) can be expressed (see Webster [44, pp. 58-63], Hildebrand, [11, pp. 368-378]) by the following three
scalar functions

\[ f(x, y, z, t) = a \quad (2a) \]
\[ g(x, y, z, t) = b \quad (2b) \]
\[ h(x, y, z, t) = c \quad (2c) \]

This type of solution is called the complete integral, to distinguish from the general integral (Webster [44, pp. 63-64]).

The functions \( f, g \) and \( h \) must be functionally independent; that is, in terms of the Jacobian we must have

\[ \frac{\partial (t, g, h)}{\partial (x, y, z)} \neq 0 \quad (3) \]

An equivalent way to express the independence of \( f, g \) and \( h \) in vector operator terms is as follows

\[ (\nabla f) \cdot (\nabla g \times \nabla h) \neq 0 \quad (4a) \]

or

\[ (\nabla h) \cdot (\nabla f \times \nabla g) \neq 0 \quad (4b) \]

or

\[ (\nabla g) \cdot (\nabla h \times \nabla f) \neq 0 \quad (4c) \]

(see Hildebrand [11, p. 363]). The geometrical meaning of equations (4) is that the normals to \( f, g \) and \( h \) at any given instant are not all three coplanar and no two are collinear.

In a case of a steady state flow it should be possible (at least formally) to eliminate the time \( t \) from equation (2). The complete
integral then reduces to the general form

\[ f(x, y, z) = a \]  
\[ g(x, y, z) = b \]  

(5a) \hspace{1cm} (5b)

The function \( h \) must then be functionally dependent on \( f \) and \( g \) or

\[ (\nabla h) \cdot (\nabla f \times \nabla g) = 0 \]  

(6)

It will be shown later in (13) that

\[ (\nabla f \times \nabla g) \neq 0 \]  

(7)

and therefore we must have

\[ (\nabla h) = 0 \]  

(8)

or by integration we get that

\[ h = h(t) \]  

(9)

In a similar way we consider the case where the flow is two-dimensional or axially symmetric. For example, in a cartesian system where the flow is independent of the coordinate \( z \), we have

\[ w = 0 \]  

(10)

This in turn implies that equation (1) reduce to the following

\[ ds = 0 \]  

(11a)

\[ \frac{dx}{u} = \frac{dy}{v} \]  

(11b)

The complete solution, then, reduces to
Equations (2), (5), (12) give implicit solutions for the stream function in the forms of families of surfaces. We showed also how the solution can be reduced when symmetry exists. In the following sections of this appendix we shall impose conditions to find the explicit expressions for the flux and the stream function.

B. The General Integral and the Flux Vector

The general integral $F$ of (1) may be expressed (Hildebrand [11, p. 371], Webster [44, p. 63]) by any differentiable function of $f$, $g$ and $h$ of (2) in the form

$$F(f, g, h) = 0$$

The flux vector, at a given instant, is parallel to the instantaneous curve, where the two surfaces $f$ and $g$ intersect. Consequently we may express the flux vector $q$ in the following form

$$q = S(Vf \times Vg)$$

(13)

where $S$ is some scalar function of the coordinates. We may replace $f$ and $g$ in (13) by any two differentiable functions $F_1(f)$ and $F_2(g)$. We can then express $q$ in an alternative form with a different scalar function $S$.
As $F_1$ and $F_2$ are arbitrary, $S$ may also be arbitrary, as long as $f$ and $g$ have not yet been specifically determined.

Several values of $S$ are of interest. In the case of a steady, incompressible, homogeneous fluid flow we may choose to express the flux by putting into (13) $S = 1$ in the following form

$$ q = \mathbf{Vf} \times \mathbf{Vg} $$

(15)

where $f$ and $g$ are two scalar functions of the coordinates.

In the case of compressible fluid flow we may put

$$ S = \rho $$

(16)

where $\rho$ is the density of the fluid or another expression of concentration. The general expression for the flux is then

$$ q_\rho = q_0 = \rho[\mathbf{Vf} \times \mathbf{Vg}] $$

(17a)

In the case of saturated flow in porous media, as in soil, we may choose

$$ S = k $$

(17b)

where $k$ is the hydraulic conductivity.

C. Equation of Continuity

The condition of conservation of matter is already implied in (15) and (17). It is easy to verify that in (15)

$$ \nabla \cdot q = 0 $$

(18)
For aid in vectorial operations, see Wylie [46, p. 454 formulas (13) and (16)] or any other standard text.

Similarly in (17), remembering that we have a steady state flow, we have

$$\nabla \cdot (\rho q) = 0$$  \hspace{1cm} (19)

(see Wylie [46, p. 454, formulas (11), (13) and (16)]). By developing (19) according to (17) in a cartesian system we get, using Jacobian notation

$$\frac{\partial (f, g, \rho)}{\partial (x, y, z)} = 0$$  \hspace{1cm} (20)

All this means is that $\rho$ is a function of $f$ and $g$ which are in turn functions of the space coordinates, as we have already assumed. In the nonsteady state the continuity equation will yield the Jacobian form

$$\frac{\partial (f, g, \rho)}{\partial (x, y, z)} = -\frac{\partial \rho}{\partial t}$$  \hspace{1cm} (21)

Here the density may be considered as functionally independent or as an independent coordinate by itself.

D. The Condition of Irrotationality

By applying the condition of irrotationality we finally can derive the explicit forms for the stream function differential equations.
1. The general case

The condition of irrotationality is expressed vectorially as follows

\[ \nabla \times \mathbf{q} = 0 \]  \hspace{1cm} (22)

For the steady state case we get from (15) and (17) for an incompressible fluid

\[ \nabla \times [\nabla f \times \nabla g] = 0 \]  \hspace{1cm} (23)

and we get for a compressible fluid

\[ \nabla \times [\rho \nabla f \times \nabla g] = 0 \]  \hspace{1cm} (24a)

and we get for a porous medium as soil

\[ \nabla \times [k \nabla f \times \nabla g] = 0 \]  \hspace{1cm} (24b)

Each of equations (22), (23), (24a) and (24b) can be expanded into three separate equations for each one of the three components, with each component vanishing. These equations are the differential equations for the stream function in three-dimensional space.

2. Two-dimensional flow in cartesian coordinates

In this case the complete solution for the steady state is

\[ f(x,y) = a \]  \hspace{1cm} (25a)

\[ g(z) = z \]  \hspace{1cm} (25b)

as we have found in (12)

Equation (23) can now be written explicitly as follows
\[ \nabla \{(\frac{\partial f}{\partial x})_y + (\frac{\partial f}{\partial y})_x \times (l_z)\} = 0 \]  
(26)

or

\[ \nabla \times \{-(\frac{\partial f}{\partial x})_y + (\frac{\partial f}{\partial y})_x\} = 0 \]  
(27)

or by further expanding

\[ (\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) = 0 \]  
(28)

This is of course the familiar Laplace equation in two dimensions, which is applicable to an incompressible, homogeneous, uniform, steady flow as Darcy law flow in soil.

\[ \nabla \times \{\rho[-(\frac{\partial f}{\partial x})_y + (\frac{\partial f}{\partial y})_x]\} = 0 \]  
(29)

or

\[ \rho \nabla \times \{-(\frac{\partial f}{\partial x})_y + (\frac{\partial f}{\partial y})_x\} 
+ (\nabla \rho) \times \{- (\frac{\partial f}{\partial x})_y + (\frac{\partial f}{\partial y})_x\} = 0 \]  
(30)

or by further expanding

\[ \rho \left[ (\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) \right] - \frac{\partial \rho}{\partial x} \frac{\partial f}{\partial x} - \frac{\partial \rho}{\partial y} \frac{\partial f}{\partial y} = 0 \]  
(31a)

This result is also well known. For nonuniform conductivity \( k \) we replace in (29), (30) and (31) the concentration \( \rho \) by the conductivity \( k \) (see Equations (16) and (18)). Thus, we get the equation for the stream function in two-dimensional steady flow with nonuniform conductivity as follows.
3. Axially symmetric flow

From the three orthogonal axes, in a right-handed system, the second one is taken to be the azimuth \( \theta \). We have several choices for the other two axes. For analogy with the deviation of the stream function, in Section II of this thesis, we shall use the cylindrical system where \( r \) is the radius and \( z \) the elevation. Any other axially symmetric system may be used. (See Morse and Feshbach [31, pp. 508-523 and 655-666]).

The complete solution in axial symmetry and in cylindrical coordinates is similar to Equation (12)

\[
\begin{align*}
  f(r,z) &= a \\ 
  g(\theta) &= \theta
\end{align*}
\] (32a)

Equation (32b)

\[
\nabla \times \left\{ \left[ \left( \frac{\partial f}{\partial r} \right)_r + \left( \frac{\partial f}{\partial z} \right)_z \right] \left( \frac{l_\theta}{r} \right) \right\} = 0
\] (33)

for noncompressible fluid.

Equation (24) for compressible fluid reads

\[
\rho \nabla \times \left[ \left( \frac{\partial f}{\partial r} \right)_r + \left( \frac{\partial f}{\partial z} \right)_z \right] \times \left( \frac{l_\theta}{r} \right)
\] (34)

\[
+ (\nabla \rho) \times \left[ \left( \frac{\partial f}{\partial r} \right)_r + \left( \frac{\partial f}{\partial z} \right)_z \right] \times \left( \frac{l_\theta}{r} \right) = 0
\]

By introducing to Equation (14)
where \( k \) is the hydraulic conductivity, we get from (24)

\[
k \nabla \times [(\frac{\partial f}{\partial r} \frac{1}{r} + \frac{\partial f}{\partial z} \frac{1}{z}) \times (1/e/r)]
\]

\[+ (\nabla k) \times [(\frac{\partial f}{\partial r} \frac{1}{r} + \frac{\partial f}{\partial z} \frac{1}{z}) \times (1/e/r)] = 0\]

Equations (33), (34) and (36) reduce after expansion to

\[
\frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} = 0
\]

for the noncompressible uniform case; to

\[
k \left( \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} \right) + \frac{\partial k}{\partial r} \frac{\partial f}{\partial r} + \frac{\partial k}{\partial z} \frac{\partial f}{\partial z} = 0
\]

for nonuniform conductivity; and to

\[
\rho \left( \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} \right) + \frac{\partial \rho}{\partial r} \frac{\partial f}{\partial r} + \frac{\partial \rho}{\partial z} \frac{\partial f}{\partial z} = 0
\]

for compressible fluid in a steady state. In case of nonuniform conductivity and compressible fluid we would get

\[
\rho k \left( \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} \right) + \frac{\partial f}{\partial r} \left( k \frac{\partial \rho}{\partial r} + \rho \frac{\partial k}{\partial r} \right)
\]

\[+ \frac{\partial f}{\partial z} \left( k \frac{\partial \rho}{\partial z} + \rho \frac{\partial k}{\partial z} \right) = 0
\]

These results validate in an independent manner the derivations of the basic differential equations for the stream functions throughout this thesis and extend them.
IX. REFERENCES


42. Theis, C. V. The relation between the lowering of piezometric surface and the rate and duration of discharge of a well using ground water storage. Trans. of Amer. Geophysical Union. 16:519-524. 1935.


