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Periodic integral surfaces for periodic systems of differential equations

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PERIODIC INTEGRAL SURFACES FOR PERIODIC SYSTEMS OF DIFFERENTIAL EQUATIONS

by

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TABLE OF CONTENTS

	Page
INTRODUCTION	1
TYPE I	3
EXAMPLE I	10
TYPE II	12
EXAMPLE II	17
LITERATURE CITED	18
ACKNOWLEDGMENTS	19

INTRODUCTION

Fixed point theorems in Banach spaces have, in the recent past, proved quite effective in the study of ordinary differential equations. G. Hufford (4) applied Schauder's fixed point theorem (7) to obtain results concerning the periodic perturbed system of differential equations

$$\dot{x} = X(x) + \epsilon R(\epsilon, t, x), \tag{1}$$

where $R(\epsilon, t, x) = R(\epsilon, t + T, x)$, and the related autonomous system

$$\dot{x} = X(x) \tag{2}$$

has a periodic solution $x = u(t) = u(t + T)$. S. P. Diliberto and G. Hufford (3) investigated the systems (1) and (2) when R has period different from u and applied the Caccioppoli fixed point theorem (2), more recently referred to as the Contraction Mapping Principle (5), to determine conditions under which (1) has almost periodic solutions "near" u(t).

W. T. Kyner (6) proves a fixed point theorem from which the results of S. P. Diliberto and G. Hufford, indicated above, follow. His theorem is based on the Schauder theorem.

In this thesis we consider the system of differential equations

$$\dot{x} = X(t, x) \tag{3}$$

where x is an n-vector and $X(t, x) = X(t + T, x)$. Because of this last periodicity condition, the solutions $\phi(t, \alpha)$ of (3), where $\emptyset(t_0, \alpha) = \alpha$, define a natural mapping of n-space into n-space

$$F(\alpha) = \emptyset(t_0 + T, \alpha).$$

L. Amerio (1) proves, using geometrical arguments, that under certain conditions there is an analytic initial manifold which is fixed under F; that is, there is a set of points M in the plane $t = t_0$ of n+1-dimensional $x \times t$ space such that $F(M) = M$. Solutions of (3) which pass through this initial manifold are almost periodic and form a periodic integral surface for (3). His results, however, leave two important difficulties unresolved. How the points of the fixed initial manifold are rearranged under the mapping F is not evident, and the fixed initial manifold may be trivial, that is, a single point. We apply the Schauder fixed point theorem to show the existence of a non-trivial initial curve $\alpha = f(u)$ and a scalar rearrangement function $\theta(u)$ such that

$$F(f(\theta(u))) = f(u).$$

We obtain our results in two cases: first, when (3) has two distinct periodic solutions and an initial curve connecting their initial points which is fixed under F, and second, when (3) has a simple closed initial curve which is fixed under F.

TYPE I

We consider the system of differential equations

(3) $$\dot{x} = X(t, x)$$

where x is an n-vector, $|x| = \sum_{i=1}^{n} |x_i|$, and the following conditions are satisfied:

(i) there is some open set U_1 in n-space and a constant t_o, such that the matrix $\frac{\partial X(t, x)}{\partial x}$ exists and is continuous for $t \geq t_o$ and x in U_1.

(ii) $X(t + T, x) = X(t, x)$ for $t \geq t_o$ and x in U.

(iii) there is a constant $A > 0$ such that solutions $\phi(t, \alpha)$ of (3), such that $\phi(t_o, \alpha) = \alpha$ and $|\phi(t_o, \alpha)| \leq A$, exist and are continuous for $t \geq t_o$ and $|\alpha| \leq A$.

(iv) the matrix $\frac{\partial \phi(t, \alpha)}{\partial \alpha}$ exists, is continuous for $t_o \leq t \leq t_o + T$, and its elements are non-negative for $t = t_o + T$ and $|\alpha| \leq A$.

(v) there is a constant $\rho > 0$ and positive integers k and ℓ such that $\frac{\partial \phi_\ell (t_o + T, \alpha)}{\partial \alpha_k} > \rho$.

(vi) $|\phi(t_o + T, \alpha)| \leq A$ for $|\alpha| \leq A$.

(vii) there are solutions $\phi(t, \alpha^*)$ and $\phi(t, \bar{\alpha})$, $\alpha^* \neq \bar{\alpha}$, such that $\phi(t + T, \alpha^*) = \phi(t, \alpha^*)$ and $\phi(t + T, \bar{\alpha}) = \phi(t, \bar{\alpha})$ for $t \geq t_o$, $|\alpha^*| \leq A$, $|\bar{\alpha}| \leq A$.

<u>Theorem 1</u>: There exists a periodic integral surface of (3)

with period T; that is, there is a continuous vector function f mapping a closed interval $[a, b]$ into n-space and a scalar function γ mapping $[a, b]$ onto $[a, b]$ such that $\phi(t_0 + T, f(\gamma(u))) = \phi(t_0, f(u)) = f(u)$ and $|f(u)| \leq A$ for u in $[a, b]$, and $f(a) = \bar{\alpha}$, $f(b) = \alpha^*$, $\gamma(a) = a$, $\gamma(b) = b$.

Proof. Without loss of generality we assume that $\bar{\alpha} = 0$, that $\alpha_i^* > 0$, $i = 1, 2, \cdots, n$, and that $t_0 = 0$. We consider the set S of all continuous vector functions f mapping the closed interval $[a, b]$ into n-space. With the usual definitions of addition and scalar multiplication, and with $\|f\| = \max_u |f(u)|$, S is a Banach space over the real number field.

Let $0 < B_1 \leq \frac{1}{b - a} |\alpha^*| \leq B_2$ and let S_B be the set of all functions f in S such that $\|f\| \leq A$, $f'(u)$ exists and is continuous, $f_i'(u) \geq 0$, $i = 1, 2, \cdots, n$, $B_1 \leq |f'(u)| \leq B_2$ for u in $[a, b]$, $f(a) = 0$, and $f(b) = \alpha^*$. S_B is convex since if $0 \leq \beta \leq 1$, and if f and g are in S_B, then

$$\|\beta f + (1 - \beta)g\| \leq \beta \|f\| + (1 - \beta)\|g\|$$
$$\leq \beta A + (1 - \beta)A = A,$$
$$|\beta f'(u) + (1 - \beta)g'(u)| = \beta f'(u) + (1 - \beta)g'(u)$$
$$\leq \beta B_2 + (1 - \beta)B_2 = B_2,$$

and $|\beta f'(u) + (1 - \beta)g'(u)| \geq \beta B_1 + (1 - \beta)B_1 = B_1$.

We next show that the closure of S_B, which we denote by $\bar{S}_B$, is compact. If f is in S_B, then

$$|f(u_2) - f(u_1)| \leq B_2 |u_2 - u_1|$$

for all u_1 and u_2 in $[a, b]$. Therefore, the functions of

S_B are equicontinuous. It follows that the functions of $\bar{S}_B$ are also equicontinuous. Since the functions of $\bar{S}_B$ are in norm less than or equal to A, from Ascoli's lemma we have that $\bar{S}_B$ is compact.

From (3)(iv) and (3)(v)

$$\left|\frac{d\phi(T, f(u))}{du}\right| = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \phi_i(T, f(u))}{\partial \alpha_j} f_j'(u)$$

$$\geq \frac{\partial \phi_\ell(T, f(u))}{\partial \alpha_k} \sum_{j=1}^{n} f_j'(u)$$

$$\geq \rho \, |f'(u)| \geq \rho B_1 > 0$$

for all f in S_B and for u in $[a, b]$.

We next show that for every f in S_B there is a differentiable function Υ mapping $[a, b]$ onto $[a, b]$ such that $\Upsilon(a) = a$, $\Upsilon(b) = b$, $\Upsilon'(u) > 0$, and $\left|\frac{d\phi(T, f(\Upsilon(u)))}{du}\right| = \frac{1}{b-a}|\alpha^*|$ for u in $[a, b]$. Since $\left|\frac{d\phi(T, f(\Upsilon(u)))}{du}\right| > 0$, we may define for each f in S_B

$$u(\Upsilon) = a + \frac{\int_a^{\Upsilon} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \phi_i(T, f(s))}{\partial \alpha_j} f_j'(s)ds}{\int_a^{b} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \phi_i(T, f(s))}{\partial \alpha_j} f_j'(s)ds} (b - a).$$

Then $u(a) = a$, $u(b) = b$, and

$$u'(\Upsilon) = \frac{\left|\frac{d\phi(T, f(\Upsilon))}{d\Upsilon}\right| (b - a)}{\sum_{i=1}^{n} \int_a^b d\phi_i(T, f(s))}$$

$$= \frac{\left|\frac{d\phi(T,\ f(\Upsilon))}{d\Upsilon}\right| (b - a)}{|\phi(T,\ f(b)) - \phi(T,\ f(a))|}$$

$$= \frac{\left|\frac{d\phi(T,\ f(\Upsilon))}{d\Upsilon}\right| (b - a)}{|f(b) - f(a)|}$$

$$= \left|\frac{d\phi(T,\ f(\Upsilon))}{d\Upsilon}\right| \frac{b - a}{|\alpha^*|} > 0$$

so we may solve for Υ in terms of u, and it follows that

$$\left|\frac{d\phi(T,\ f(\Upsilon(u)))}{du}\right| = \left|\frac{d\phi(T,\ f(\Upsilon(u))}{d\Upsilon}\right| \frac{d\Upsilon(u)}{du} = \frac{|\alpha^*|}{b - a}$$

for u in $[a, b]$.

We define a function F from S_B into S_B by $F(f) = \phi(T, f(\Upsilon))$. From (3)(vi) it follows that $\| F(f) \| \leq A$ for all f in S_B. $\frac{d\phi_i(T,\ f(u))}{du} > 0$, $i = 1, 2, \cdots, n$, since $\frac{d\phi_i(T,\ f(u))}{du}$ is a solution of the equation

$$\dot{y} = \frac{\partial X(t,\ \phi(t,\ f(u))}{\partial x_i}\, y,$$

and therefore, if $\frac{d\phi_i(T,\ f(u_o))}{du} = 0$, $i = 1, 2, \cdots, n$ for some u_o in $[a, b]$, then $f_i'(u_o) = 0$. But by the definition of S_B, $|f'(u)| > 0$ so that $\frac{dF(f)_i(u)}{du} = \frac{d\phi_i(T,\ f(\Upsilon(u)))}{d\Upsilon}\Upsilon'(u) > 0$.

Since $\left|\frac{dF(f)(u)}{du}\right| = \left|\frac{d\phi(T,\ f(\Upsilon(u)))}{du}\right| = \frac{1}{b - a}|\alpha^*|$, $B_1 \leq \left|\frac{dF(f)(u)}{du}\right| \leq B_2$. Also $F(f)(a) = \phi(T, f(\Upsilon(a)))$

$= \phi(T, f(a)) = \phi(0, f(a)) = f(a) = 0$ and $F(f)(b)$
$= \phi(T, f(\Upsilon(b))) = f(b) = \alpha^*$. Therefore, F maps S_B into S_B.

Since $u'(\Upsilon) = \left|\frac{d\phi(T, f(\Upsilon))}{d\Upsilon}\right| \frac{b - a}{|\alpha^*|} \geq \rho B_1 \frac{b - a}{|\alpha^*|}$ for all f in S_B, the set of functions $\{\Upsilon'(u)\}$ is uniformly bounded for u in $[a, b]$. Let f be in $\bar{S}_B$. Then there is a sequence $\{f^{(n)}\}$ such that $f^{(n)}$ is in S_B and $f^{(n)} \longrightarrow f$ as $n \longrightarrow \infty$. Let $g^{(n)} = F(f^{(n)})$. Then $g^{(n)}$ is in S_B. Since the sequences $\{|\Upsilon_n(u)|\}$ and $\{\Upsilon_n'(u)\}$ are uniformly bounded, there is a subsequence $\{\Upsilon_m'(u)\}$ of $\{\Upsilon_n'(u)\}$ and a continuous function $\Upsilon(u)$ such that $\Upsilon(a) = a$, $\Upsilon(b) = b$, and $\Upsilon_m \longrightarrow \Upsilon$ as $n \longrightarrow \infty$. Let $g(u) = \phi(T, f(\Upsilon(u)))$. Then

$$\begin{aligned} \|g - g^{(m)}\| &= \|\phi(T, f(\Upsilon)) - \phi(T, f^{(m)}(\Upsilon_m))\| \\ &\leq M_1 \| f(\Upsilon) - f^{(m)}(\Upsilon_m) \| \end{aligned}$$

where $M_1 = \max \left| \frac{\partial \phi(T, \alpha)}{\partial \alpha} \right|$, $|\alpha| \leq A$. Hence
$\|g - g^{(m)}\| \leq M_1(\|f(\Upsilon) - f^{(m)}(\Upsilon)\| + \|f^{(m)}(\Upsilon) - f^{(m)}(\Upsilon_m)\|)$,
and therefore, $g^{(m)} \longrightarrow g$ as $m \longrightarrow \infty$ since $\|f(\Upsilon) - f^{(m)}(\Upsilon)\| = \|f - f^{(m)}\|$, and the functions $f^{(m)}$ are equicontinuous. Thus g is in $\bar{S}_B$, and therefore we may extend the domain of F such that F maps $\bar{S}_B$ into $\bar{S}_B$.

To show that F is continuous on $\bar{S}_B$ let f and h be in $\bar{S}_B$ and let $f^{(n)} \longrightarrow f$, $h^{(n)} \longrightarrow h$, $\Upsilon_n \longrightarrow \Upsilon$, and $\theta_n \longrightarrow \theta$ as $n \longrightarrow \infty$ where

$$F(f) = \phi(T,\ f(\gamma)),$$

$$F(f^{(n)}) = \phi(T,\ f^{(n)}(\gamma_n)),$$

$$F(g) = \phi(T,\ g(\theta)),$$

$$F(g^{(n)}) = \phi(T,\ g^{(n)}(\theta_n)).$$

Then

$$\begin{aligned} \| F(f) - F(h) \| &\leq M_1 \| f(\gamma) - h(\theta) \| \\ &\leq M_1 (\| f - h \| + \| h(\gamma) - h(\theta) \|) \\ &\leq M_1 (\| f - h \| + \| h(\gamma) - h^{(n)}(\gamma_n) \| \\ &\quad + \| h(\theta) - h^{(n)}(\theta_n) \| + \| h^{(n)}(\gamma_n) - h^{(n)}(\theta_n) \|). \end{aligned}$$

Thus it follows that F is continuous if we can show that $\| h^{(n)}(\gamma_n) - h^{(n)}(\theta_n) \|$ is small provided $\| f - h \|$ is small. But

$$\begin{aligned} \| h^{(n)}(\gamma_n) - h^{(n)}(\theta_n) \| &\leq B_1 \max_u | \gamma_n(u) - \theta_n(u) | \\ &\leq B_1 \max_u \big| [\gamma_n(\theta_n^{-1}(\theta_n(u)))] \\ &\qquad - \gamma_n(\gamma_n^{-1}(\theta_n(u))) \big| \\ &\leq B_1 K \max_u | \theta_n^{-1}(\theta_n(u)) \\ &\qquad - \gamma_n^{-1}(\theta_n(u)) | \end{aligned}$$

where K is a uniform bound for the sequence $\{ \gamma_n'(u) \}$. Hence by the definition of θ_n^{-1} and γ_n^{-1},

$$\begin{aligned} \| h^{(n)}(\gamma_n) - h^{(n)}(\theta_n) \| &\leq B_1 K \frac{b - a}{|\alpha^*|} \| \phi(T,\ h^{(n)}(\theta_n(u))) \\ &\qquad - \phi(T,\ f^{(n)}(\theta_n(u))) \| \end{aligned}$$

$$\leq B_1 KM_1 \frac{b - a}{|\alpha^*|} \| h^{(n)}(\theta_n(u)) - f^{(n)}(\theta_n(u)) \|$$

$$\leq B_1 KM_1 \frac{b - a}{|\alpha^*|} \| h - f \| .$$

The Schauder fixed point theorem states that if $\overline{S}_B$ is a compact convex subset of a Banach space S, and if F is a continuous function mapping $\overline{S}_B$ into $\overline{S}_B$, then F has a fixed point in $\overline{S}_B$. It follows, therefore, that there is a continuous function f in $\overline{S}_B$ and a continuous function Υ such that $|f(u)| \leq A$ for u in $[a, b]$, $f(a) = \overline{\alpha} = 0$, $f(b) = \alpha^*$, $\Upsilon(a) = a$, $\Upsilon(b) = b$, and $\phi(T, f(\Upsilon(u))) = \phi(0, f(u)) = f(u)$ for u in $[a, b]$.

EXAMPLE I

$$(4)\qquad \begin{aligned} \dot{x}_1 &= (x_1 - \sin \sigma t)(x_1 - 1 - \sin \sigma t) + \sigma\cos t \\ \dot{x}_2 &= -x_2 \end{aligned}$$

where $\sigma = \dfrac{2\pi}{\log 2}$.

Then

$$\phi_1(t, \alpha_1, \alpha_2) = \frac{\alpha_1}{\alpha_1 + (1 - \alpha_1)e^t} + \sin \sigma t$$

$$\phi_2(t, \alpha_1, \alpha_2) = \alpha_2 e^{-t},$$

and

$$\phi_1(\log 2, \alpha_1, \alpha_2) = \frac{\alpha_1}{2 - \alpha_1}$$

$$\phi_2(\log 2, \alpha_1, \alpha_2) = \frac{\alpha_2}{2}.$$

Let $A = 1$. Then if $|\alpha| \leq 1$, then $|\phi(\log 2, \alpha)| \leq 1$. Also

$$\frac{\partial \phi_1(\log 2, \bar{\alpha}_1, \alpha_2)}{\partial \alpha_1} = \frac{2}{(2 - \alpha_1)^2} > 1,$$

$$\frac{\partial \phi_2(\log 2, \alpha_1, \alpha_2)}{\partial \alpha_2} = \frac{1}{2},$$

$$\frac{\partial \phi_1(\log 2, \alpha_1, \alpha_2)}{\partial \alpha_2} = \frac{\partial \phi_2(\log 2, \alpha_1, \alpha_2)}{\partial \alpha_1} = 0.$$

Let $\alpha^* = (1, 0)$ and $\bar{\alpha} = (0, 0)$. Then $\phi_1(t, 1, 0) = \phi_1(t, 0, 0) = \phi_2(t, 1, 0) = \phi_2(t, 0, 0) = \sin \sigma t$. Thus $\phi(t, \alpha^*) = \phi(t + \log 2, \alpha^*)$, and $\phi(t, \bar{\alpha}) = \phi(t + \log 2, \bar{\alpha})$.

Consider $f(u) = (u, 0)$, $\gamma(u) = \dfrac{2u}{u + 1}$ for u in $[0, 1]$. We have that $f(0) = (0, 0)$, $f(1) = (0, 1)$, $\gamma(0) = 0$, $\gamma(1) = 1$, and $\emptyset(\log 2, f(\Upsilon(u)) = \frac{f(\Upsilon(u))}{2 - f(\Upsilon(u))} = (\frac{\Upsilon(u)}{2 - \Upsilon(u)}, 0) = (u, 0)$.

Hence the integral surface formed by solutions of (4) such that $\alpha = (\alpha_1, 0)$, $0 \leq \alpha_1 \leq 1$ is a periodic integral surface for (4).

TYPE II

We first define the following notation. Let a be an m-vector and g(w) be a differentiable n-vector function. Then we denote the $n \times m$ matrix $\frac{\partial g(w)}{\partial w} = \left(\frac{\partial g_i(w)}{\partial w_j} \right)$ by $g'(w)$, and we let $|w| = \left(\sum_{j=1}^{m} w_j^2 \right)^{\frac{1}{2}}$ and $|g'(w)|$ $= \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{g_i(w)}{w_j} \right)^{\frac{1}{2}}$.

We consider the system of differential equations

$$\dot{x} = X(t, x) \tag{5}$$

where x is an n-vector and the following conditions are satisfied:

(i) there is some open set U_2 and a constant t_o, which we again assume to be zero, such that the matrix $\frac{\partial X(t, x)}{\partial x}$ exists and is continuous for $t \geq 0$ and x in U_2.

(ii) $X(t + T, x) = X(t, x)$ for $t \geq 0$ and x in U_2.

(iii) there are constants $0 < A_1 < A_2$ such that if $\phi(t, r, u)$ is the solution of (5) such that

$$\phi_i(0, r, u) = |\phi(0, r, u)| \cos u_i,$$

$i = 1, 2, \cdots, n$, and if R is the set $\left\{ (r, u) \middle| A_1 \leq r \leq A_2, \sum_{i=1}^{n} \cos^2 u_i = 1, 0 \leq u_i \leq 2\pi \right\}$, then the vector $\frac{\partial \phi(t, r, u)}{\partial r}$ and the matrix $\frac{\partial \phi(t, r, u)}{\partial u}$ exist and are continuous for $t \geq 0$ and (r, u) in R.

(iv) $A_1 \leq |\phi(T, r, u)| \leq A_2$ for (r, u) in R.

(v) we let

$$\delta(r, u) = \frac{\phi(T, r, u)}{|\phi(T, r, u)|},$$

$$S(u) = \begin{pmatrix} \sin u_1 & & & \\ & \sin u_2 & & \\ & & \ddots & \\ & & & \sin u_n \end{pmatrix},$$

$$a(r, u) = \left|\frac{\partial \delta}{\partial r}\right|,$$

$$b(r, u) = \left|\frac{\partial \delta}{\partial u}\right| - \left|\frac{\partial |\phi(T, r, u)|}{\partial r}\right| (n - 1)^{\frac{1}{2}},$$

$$c(r, u) = \frac{\partial |\phi(T, r, u)|}{\partial u}(n - 1)^{\frac{1}{2}},$$

and we assume that $b \geq 0$, that $b^2 - 4ac \geq 0$, and that there is a number $B \geq 0$ such that the roots ρ_1 and ρ_2 of $a\rho^2 - b\rho + c = 0$ satisfy $\rho_1 \leq B \leq \rho_2$ and $\left|\frac{\partial \delta}{\partial u}\right| > \left|\frac{\partial \delta}{\partial r}\right| B$ for (r, u) in R.

(vi) we let P_B be the set of all continuous functions f, periodic with period 2π, such that $A_1 \leq f(u) \leq A_2$, $f'(u)$ exists and is continuous, and $|f'(u)| \leq B$ for all u, and for each f in P_B we assume that the equations $\delta_i(f(v), v) = \cos u_i$, $i = 1, 2, \cdots, n$, define a continuously differentiable vector function $v(u)$ such that

$$v(u_1 + 2\pi, \cdots, u_n + 2\pi) = (v_1(u) \pm 2\pi, \cdots, v_n(u) \pm 2\pi)$$

for all u.

Theorem 2: There exists a periodic integral surface of (5) with period T whose cross section for any t is a closed n - 1 dimensional surface, that is, there is a continuous function f(u), periodic with period 2π, and a continuous vector function v(u) such that $v(u_1 + 2\pi, \cdots, u_n + 2\pi) = (v_1(u) \pm 2\pi, \cdots, v_n(u) \pm 2\pi)$,

$$\phi(T, f(v(u)), v(u)) = \phi(0, f(u), u),$$

and $A_1 \leq f(u) \leq A_2$ for all u.

Proof. We let P be the set of all continuous functions f which are periodic with period 2π. With the usual definitions of addition and scalar multiplication, and with $\|f\| = \max_u |f(u)|$, P is a Banach space over the real number field. It follows that P_B is a subset of P.

If f is in P_B, then

$$\left(\frac{\partial\delta(f(v(u)), v(u))}{\partial r} f'(v(u)) + \frac{\partial\delta}{\partial u}\right) v'(u) = - S(u).$$

It follows that

$$\left(\left|\frac{\partial\delta}{\partial u}\right| - \left|\frac{\partial\delta}{\partial r}\right| B\right) |v'(u)| \leq |S(u)| = (n - 1)^{\frac{1}{2}},$$

and, therefore,

$$|v'(u)| \leq \frac{(n - 1)^{\frac{1}{2}}}{\left|\frac{\partial\delta}{\partial u}\right| - \left|\frac{\partial\delta}{\partial r}\right| B} .$$

We define a function F from P_B into P by $F(f) = |\phi(T, f(v), v)|$. From (5)(iv) $A_1 \leq F(f)(u) \leq A_2$.

Also

$$F(f)'(u) = \left(\frac{\partial|\phi(T, f(v(u)), v(u))|}{\partial r} f'(v(u)) + \frac{\partial|\phi|}{\partial u}\right) v'(u)$$

so that

$$|F(f)'(u)| \leq \left(\left|\frac{\partial|\phi|}{\partial r}\right| B + \left|\frac{\partial|\phi|}{\partial u}\right|\right) |v'(u)|$$

$$\leq \frac{\left|\frac{\partial|\phi|}{\partial r}\right| B + \left|\frac{\partial|\phi|}{\partial u}\right|}{\left|\frac{\partial\delta}{\partial u}\right| - \left|\frac{\partial\delta}{\partial r}\right| B}(n - 1)^{\frac{1}{2}} - B + B$$

$$\leq \frac{aB^2 - bB + c}{\left|\frac{\partial\delta}{\partial u}\right| - \left|\frac{\partial\delta}{\partial r}\right| B} + B \leq B$$

since $aB^2 - bB + c \leq 0$. Thus F maps P_B into P_B.

Since $\left|\frac{\partial\delta}{\partial u}\right| > \left|\frac{\partial\delta}{\partial r}\right| B$, the set of functions $\{|v'(u)|\}$ for f in P_B is bounded above. Therefore, using arguments similar to those used in Theorem 1, we may extend the definition of F to $\overline{P}_B$ such that F maps $\overline{P}_B$ continuously into $\overline{P}_B$. Since P_B is convex and $\overline{P}_B$ is compact, from the Schauder theorem F has a fixed point in $\overline{P}_B$, that is, there is a continuous periodic function f in $\overline{P}_B$ with period 2π and a continuous function v(u) such that $v(u_1 + 2\pi, \cdots, u_n + 2\pi) = (v_1(u) \pm 2\pi, \cdots, v_n(u) \pm 2\pi)$, $A_1 \leq f(u) \leq A_2$, and $\phi(T, f(v(u)), v(u)) = \phi(0, f(u), u)$ for all u. This completes the proof of Theorem 2.

When n = 2, using the polar representation $\phi(t, r, \theta)$ where

$$\phi_1(0, r, \theta) = |\phi(0, r, \theta)| \cos\theta = r\cos\theta$$
$$\phi_2(0, r, \theta) = |\phi(0, r, \theta)| \sin\theta = r\sin\theta,$$

if we let

$$W_r(r, \theta) = \det \begin{pmatrix} \phi_1(T, r, \theta) & \phi_2(T, r, \theta) \\ \dfrac{\partial \phi_1(T, r, \theta)}{\partial r} & \dfrac{\partial \phi_2(T, r, \theta)}{\partial r} \end{pmatrix},$$

$$W_\theta(r, \theta) = \det \begin{pmatrix} \phi_1(T, r, \theta) & \phi_2(T, r, \theta) \\ \dfrac{\partial \phi_1(T, r, \theta)}{\partial \theta} & \dfrac{\partial \phi_2(T, r, \theta)}{\partial \theta} \end{pmatrix},$$

$$a(r, \theta) = |W_r(r, \theta)|,$$
$$b(r, \theta) = |W_\theta(r, \theta)| - \left|\frac{\partial |\phi(T, r, \theta)|}{r}\right| |\phi(T, r, \theta)|^2,$$
$$c(r, \theta) = \left|\frac{\partial |\phi(T, r, \theta)|}{\partial \theta}\right| |\phi(T, r, \theta)|^2,$$

and assume that $b \geq 0$, that $b^2 - 4ac \geq 0$, and that there is a number $B \geq 0$ such that the roots ρ_1 and ρ_2 of $a\rho^2 - b\rho + c = 0$ satisfy $\rho_1 \leq B \leq \rho_2$ and $|W_\theta| > |W_r| B$ for (r, θ) in R, then it follows that the equations

$$\frac{\phi_1(T, f(v), v)}{|\phi(T, f(v), v)|} = \cos\theta$$

and

$$\frac{\phi_2(T, f(v), v)}{|\phi(T, f(v), v)|} = \sin\theta$$

define, for each continuously differentiable periodic function f such that $|f'(\theta)| \leq B$, a continuously differentiable function $v(\theta)$ such that $v(\theta + 2\pi) = v(\theta) \pm 2\pi$. Thus when $n = 2$ assumption (5)(vi) is unnecessary.

EXAMPLE II

$$\text{(6)}\qquad \begin{aligned} \dot{x}_1 &= x_2 + 2\pi\cos 2\pi t - \sin 2\pi t \\ \dot{x}_2 &= -x_1 + 2\pi\cos 2\pi t + \sin 2\pi t \end{aligned}$$

and

$$\phi_1(t, r, u_1, u_2) = r\cos (u_1 - t) + \sin 2\pi t$$
$$\phi_2(t, r, u_1, u_2) = r\cos (u_2 - t) + \sin 2\pi t.$$

Then $T = 1$ and $|\phi(1, r, u_1, u_2)| = r$ so that

$$\begin{aligned} \delta_1(r, u_1, u_2) &= \cos (u_1 - 1), \\ \delta_2(r, u_1, u_2) &= \cos (u_2 - 1), \\ a(r, u_1, u_2) &= 0, \\ b(r, u_1, u_2) &= 1, \\ c(r, u_1, u_2) &= 0. \end{aligned}$$

Therefore $B = 0$, and if we consider the set P_0^- of all constant functions $f(u) = r_0$ where $A_1 \leq r_0 \leq A_2$ and A_1 and A_2 are any constants such that $0 < A_1 < A_2$, then $v_1(u_1, u_2) = u_1 + 1$ and $v_2(u_1, u_2) = u_2 + 1$.

Thus the integral surfaces formed by solutions $\phi(t, r_0, u)$ of (6) are periodic with period 1 since

$$\begin{aligned} \phi_i(1, r_0, v(u_1, u_2)) &= \phi_i(1, r_0, u_1 + 1) \\ &= r_0\cos u_1 \\ &= \phi_i(0, r_0, u_1, u_2) \end{aligned}$$

for $i = 1$ and $i = 2$.

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