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Topics in self-interacting random walks

Reza Rastegar
Iowa State University

Topics in self-interacting random walks

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Reza Rastegar

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Major: Applied Mathematics

Program of Study Committee:

Arka Ghosh, Co-major Professor

Alexander Roiterchtein, Co-major Professor

Wolfgang Kliemann

Anastasios Matzavinos

Paul Sacks

Iowa State University

Ames, Iowa

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DEDICATION

I would like to dedicate this thesis to my wife, Jamie, without whose support I would not have been able to complete this work. I would also like to thank my parents, Hossein and Hakimeh, for their loving guidance during the writing of this work.

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ABSTRACT

The principle focus of this thesis is self-interacting random walks. A self-interacting random walk is a walk on a graph with its past influencing its future. In contrast to the regular random walks, self-interacting random walks are genuinely non-Markovian. Correspondingly, most of the standard tools of the theory of random walks are not directly available for the analysis of these models. Typically, this requires a significant adjustment and novel ad-hoc approaches in order to be applied. In this thesis we study two such processes, namely, excited random walks (ERWs) and directionally reinforced random walks (DRRWs).

ERWs have actively attracted many mathematicians in recent years, and several basic questions regarding these random walks on $\mathbb{Z}^d$ and trees have been answered. Nonetheless, despite all the effort done of late, there are still fundamental questions about ERWs to be answered. Here, we consider a transient ERW on $\mathbb{Z}$ and study the asymptotic behavior of the occupation time of a currently most visited site. In particular, our results imply that, in contrast to the random walks in random environment, a transient excited random walk does not spend an asymptotically positive fraction of time at its favorite (most visited up to a date) sites.

DRRWs were originally introduced by Mauldin, Monticino, and von Weizsäcker. In this thesis, we consider a generalized version of these processes and obtain a stable limit theorem for the position of the random walk in higher dimensions. This extends a result of Horváth and Shao that was previously obtained in dimension one only.

CHAPTER 1. Introduction

This thesis includes different topics in the field of self-interacting random processes. More specifically, the main focus of this work will be on excited random walks (ERWs) and directionally reinforced random walks (DRRWs). These two processes, in contrast to regular random walks, are genuinely non-Markovian. Correspondingly, most of the standard tools of the theory of random walks (such as, for instance, the local time theory or the embedding of discrete-time models into their continuous-time limiting processes) are not directly available for the analysis of these models and it typically requires a significant adjustment and novel ad-hoc approaches in order to be applied. In the remainder of this chapter, we will give a precise description of these two processes. In the second chapter we study the structure of the largest number of visits of an ERW to a single site during the first n steps. In the third chapter, we focus on the DRRW and prove stable limit theorems for this class of random walks in arbitrary dimension $d \geq 1$. In addition, we extend some limit results of [24] to our setting and also complement them by suitable laws of iterated logarithm. In the last chapter of the thesis, we propose several directions for the future research.

1.1 Excited Random Walks

Excited random walks or random walks in a cookie environment on $\mathbb{Z}^d$ is a modification of the nearest neighbor simple random walk such that in several first visits to each site of the integer lattice, the walk's jump kernel gives a preference to a certain direction and assigns equal probabilities to the remaining $(2d-1)$ directions. If the current location of the random walk has been already visited more than a certain number of times, then the walk moves to one of its nearest neighbors with equal probabilities. The model was introduced by Benjamini and Wilson in [10] and extended by Zerner in [49] and [50]. Closely related models were considered in [1, 2, 3, 23, 27].

In this thesis we focus on the excited random walks in dimension one. To define the transition mechanism of the random walk, fix an integer $M \in \mathbb{N}$ and let

$$\Omega_M = \{\omega(z,i)_{i\in\mathbb{N},z\in\mathbb{Z}} : \omega(z,i) \in [0,1], \forall i \in \mathbb{N}, z \in \mathbb{Z} \text{ where } \omega(z,i) = 0.5\ \forall i > M\}.$$

The value of $\omega(z,i)$ determines the probability of the jump from z to $z+1$ upon i-th visit of the random walk to the site $z \in \mathbb{Z}$. The random walk is assumed to be nearest neighbor, and hence the probability of the jump from z to $z-1$ upon its i-th visit to z is given by the complementary probability $1-\omega(z,i)$. The elements of the set Ω_M are called *cookie environments.*

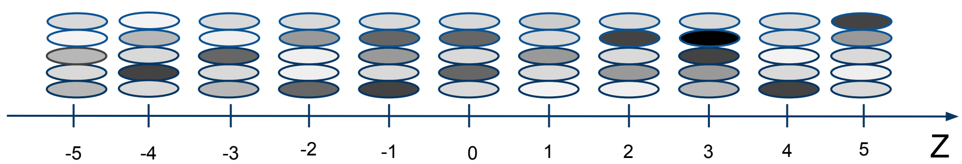

Figure 1.1 A cookie environment with $M = 5$. Various shades of grey represent different transition probabilities associated to different cookies.

For a fixed $z \in \mathbb{Z}$, $\big(\omega(z,i)\big)_{1\le i\le M}$ can be thought of as a sequence of numerical characteristics called “strengths”, associated with a pile of M “cookies” placed at z. Correspondingly, $\omega(z,i)$ is referred to as the *strength of the i-th cookie* at the pile. With a slight abuse of language, using the above introduced jargon we will often identify the strength $\omega(z,i)$ of a cookie with the cookie itself. Transition kernel of the random walk can be informally described as follows: while the supply of the cookies at a given site lasts, the walker eats a cookie upon each visit there and then makes one step in a random direction, such that the probability of moving to the right is equal to the “strength” of the just eaten cookie.

More precisely, the random walk in a cookie environment $\omega \in \Omega_M$ is defined as follows. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let $\Sigma = \mathbb{Z}^{\mathbb{N}_0}$ be the state space of the infinite paths of a discrete-time random walk on $\mathbb{Z}$ ($\mathbb{Z}$ for the location and $\mathbb{N}_0$ for the time), and let $\mathcal{F}$ be its Borel σ-algebra (i. e., the σ-algebra generated by the cylinder sets of the infinite product space Σ). For any $x \in \mathbb{Z}$ and $\omega \in \Omega_M$, an excited random walk (abbreviated in what follows as ERW) starting at $x \in \mathbb{Z}$ in the cookie environment ω, is a sequence of random variables $X = \big(X_n\big)_{n\in\mathbb{N}_0}$ defined in a probability space

$(\Sigma, \mathcal{F}, P_{x,\omega})$ such that $P_{x,\omega}(X_0 = x) = 1$ and

$$P_{x,\omega}(X_{n+1} = X_n + 1 | \mathcal{F}_n) = \omega(X_n, \xi_n),$$

where $\mathcal{F}_n := \sigma(X_i,\, 0 \le i \le n)$ and

$$\xi_n := \#\big\{0 \le i \le n : X_i = X_n\big\}. \tag{1.1}$$

The measure $P_{x,\omega}$ is usually referred to as the *quenched law* of the excited random walk in the cookie environment ω. Note that $(X_n, \xi_n)_{n \ge 0}$ is a Markov chain with respect to $P_{x,\omega}$. Let $\mathbb{P}$ be a probability measure on Ω_M that makes the collection of "piles" $\omega_z := \omega(z,i)_{i\in\mathbb{N}}$ indexed by $z \in \mathbb{Z}$ into an i.i.d. sequence. Notice that we do not insist on the independence of the cookies within a given pile, that is the random variables $\omega(z,i)$ for a fixed $z \in \mathbb{Z}$ can be dependent under $\mathbb{P}$. The (associated with $\mathbb{P}$) *annealed* (average) law P_x of the ERW on $(\Sigma, \mathcal{F})$ is defined by setting $P_x(\,\cdot\,) = \mathbb{E}[P_{x,\omega}(\,\cdot\,)]$, where $\mathbb{E}$ is the expectation induced by the probability law $\mathbb{P}$.

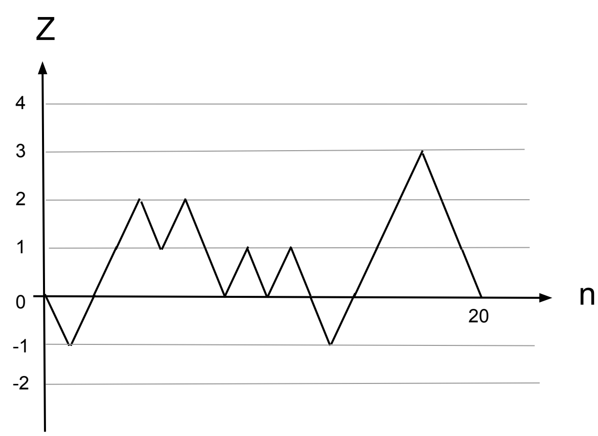

Figure 1.2 The path of an ERW random walk after 20 steps.

Many important aspects of the asymptotic behavior of excited random walks on $\mathbb{Z}$ are by now well-understood. In particular, Zerner in [49], Basdevant and Singh in [7, 8], Kosygina and Zerner in [32], and Kosygina and Mountford in [31] characterized the recurrence-transience behavior and possible speed regimes of the ERW, and proved limit theorems for the fluctuations of the current location of ERW. The goal of this research is to study the asymptotic dynamics of the occupation time of a currently most visited site for a transient ERW.

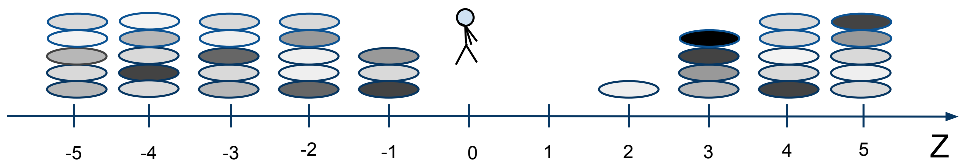

Figure 1.3 The cookie environment shown in Figure 1.1 after the walk shown in Figure 1.2.

Notice that the consumption of a cookie $\omega(z,i)$ results in creating the local drift (i.e., the bias in the conditional on the history $\mathcal{F}_n$ expectation of the subsequent displacement) equal to $E_{\omega,x}\big[X_{n+1}-X_n\big|X_n=z,\xi_n=i\big]=2\omega(z,i)-1$. Let

$$\delta=\mathbb{E}\Big[\sum_{i=1}^{M}\big(2\omega(z,i)-1\big)\Big]$$

be the annealed expectation of the total drift (i.e., the "boost" in the positive direction) available to the random walk at a site $z\in\mathbb{Z}$. Notice that since ω_z is assumed i. i. d., δ is independent of z. Following [32] and [31] (and in contrast to the original version presented in [10, 49]) we do not impose the condition that $\mathbb{P}\big(\omega(z,i)\geq 1/2\big)=1$, that is allowing both "positive" and "negative" cookies. It turns out (see [31, 32]) that the asymptotic behavior of an one-dimensional excited random walk is largely determined by the value of the parameter δ. In particular, under a mild non-degeneracy assumption on the cookie environment, the random walk is transient to the right if and only if $\delta>1$.

Throughout this thesis we will impose the following conditions on the cookie environment.

Assumption 1.1.1. *The following assumptions hold:*

(a) Independence: the sequence of "piles" $\omega_z=(\omega(z,i))_{i\in\mathbb{N}}$ indexed by sites $z\in\mathbb{Z}$ is an i.i.d. sequence under $\mathbb{P}$.

(b) Non-degeneracy: $\mathbb{E}\big[\prod_{i=1}^{M}\omega(0,i)\big]>0$ and $\mathbb{E}\big[\prod_{i=1}^{M}(1-\omega(0,i))\big]>0$.

(c) Transience: $\delta>1$.

The above non-degeneracy condition ensures that, a priori, the random walk can at any given step move to each direction with a positive probability. It is known [32, 49] that under Assumption 1.1.1 the ERW is transient to the right (that is, $P_0\big(\lim_{n\to\infty} X_n = \infty\big) = 1$) and, furthermore, has the asymptotic speed

$$\mathrm{v} := \lim_{n\to\infty} \frac{X_n}{n} \in [0,1), \tag{1.2}$$

which is strictly positive if and only if $\delta > 2$.

1.2 Directionally Reinforced Random Walks

In this thesis, we also study the following *directionally reinforced random walk.* Fix $d \in \mathbb{N}$ and a finite set U of distinct unit vectors in $\mathbb{R}^d$. The vectors in U serve as feasible directions for the motion of the random walk. To avoid trivialities we assume that U contains at least two elements. Let $X_t \in \mathbb{R}^d$ denote the position of the random walk at time t. Throughout this work we assume that $X_0 = 0$. The random walk changes its direction at random times

$$s_1 := 0 < s_2 < s_3 < s_4 < \ldots.$$

We assume that the time intervals

$$T_n := s_{n+1} - s_n, \qquad n \in \mathbb{N},$$

are independent and identically distributed. Let $\eta_n \in U$ be the direction of the walk during time interval $[s_n, s_{n+1})$. We assume that $\eta := (\eta_n)_{n\ge 1}$ is an irreducible stationary Markov chain on U which is, furthermore, independent of $(s_n)_{n\in\mathbb{N}}$. See Figure 1.4 for an example of a path of a DRRW in dimension two.

For $t > 0$, let $N_t := \sup\big\{k \ge 1 : s_k \le t\big\}$ be the number of times that the walker changes direction before time $t > 0$. Then

$$X_t = \sum_{i=1}^{N_t - 1} \eta_i T_i + (t - s_{N_t})\eta_{N_t}. \tag{1.3}$$

Notice that $N_t \ge 1$ with probability one, due to the convention $s_1 = 0$ that we have made. The random walk X_t defined above is essentially the model introduced by Mauldin, Monticino, and von Weizsäcker in [34] and further studied by Horváth and Shao in [24] and by Siegmund-Schultzea and von Weizsäcker in [43]. The technical difference between our model and the variant which has been

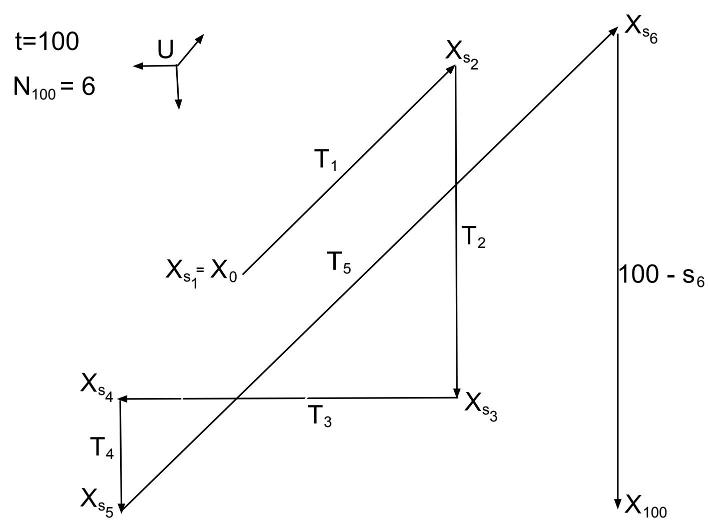

Figure 1.4 A path of a DRRW at time 100 with three feasible directions represented by the set U.

studied in [24] is that in the latter, the next direction of the motion is chosen uniformly from the available set of "fresh directions", while we do not impose any restrictions on the transition kernel of η besides irreducibility.

The original model proposed in [34] was inspired by certain phenomena that occur in ocean surface waves (cf. [47]) and was designed to reproduce the same features within a probabilistic framework. The main topic of [34] and [43] is recurrence-transience criteria. Horváth and Shao in [24] studied scaling limits of the random walk in different regimes, answering some of the questions which have been posed in [34].

We remark that somewhat related random walk models have been considered by Allaart and Monticino in [4, 5] and by Gruber and Schweizer in [22]. In the context of random walks in random environment, a similar in spirit model of *persistent* random walks was introduced by Szász and Tóth in [45, 46]. The common feature of "generic versions" of all models mentioned above is that the underlying random motion has a tendency to persist in its current direction.

Closely related to persistent random walks are recurrent "random flights" models where changes of the direction of the random motion follow a Poisson random clock. These models can be traced back to Pearson's random walk [20, 26] and Goldstein-Kac one-dimensional "telegraph process"

[28, 39] . Random flights have been intensively studied since the introduction of the telegraph process in the early 50's, see for instance [21, 29, 30, 33, 36, 44] and references therein for a representative sample. An introductory part of [30] provides a short authoritative and up to a date survey of the field. We remark that, somewhat in contrary to directionally reinforced random walks, the main focus of the research in this area is on finding explicit form of limiting distributions for these processes.

1.3 Overview of this thesis

In the second chapter we study the structure of ξ_n^*, the largest number of visits of an ERW to a single site during the first n steps. The approach adopted here is based on a reduction of the study of the asymptotic behavior of the ERW to that of a branching process and the subsequent reformulation of the problem in terms of the asymptotic dynamics of the most populated generation of the branching process. Similar questions for transient RWRE have been addressed in the interesting paper by Gantert and Shi [19]. The essential branching processes machinery which enables the implementation of our approach to excited random walks was introduced in [7, 8] and further developed in [31, 32].

In the third chapter, we focus on the directionally reinforced random walk and prove stable limit theorems for the directionally reinforced random walk in arbitrary dimension $d \geq 1$. In addition, we extend some limit results of [24] to our setting and also complement them by suitable laws of iterated logarithm. Our proofs can be easily carried over to a setup where the set of feasible directions U is not finite, but is rather supported (under the stationary law of the process) on a general Borel subset of the unit sphere. The non-Gaussian limit theorems for the position of the random walk in higher dimensions, stated in Theorems 3.0.6 and 3.0.7 constitute the main contribution of this chapter.

The last chapter of the thesis includes several possible directions for the future research.

CHAPTER 2. Maximal occupation time of a transient excited random walk

In this chapter, we consider a transient excited random walk on $\mathbb{Z}$ and study the asymptotic behavior of the occupation time of a currently most visited site. In particular, our results imply that, in contrast to the random walks in random environment, a transient excited random walk does not spend an asymptotically positive fraction of time at its favorite (most visited up to a date) sites.

Define the occupation time of the ERW at site $x \in \mathbb{Z}$ as

$$\xi_n(x) := \#\big\{0 \le i \le n \,:\, X_i = x\big\}. \tag{2.1}$$

Thus $\xi_n(x)$ is the number of times that the ERW visits $x \in \mathbb{Z}$ during the first n steps. Let

$$\xi_n^* := \max_{x\in\mathbb{Z}} \xi_n(x) \tag{2.2}$$

be the largest number of visits to a single site during the first n steps. For the sake of notational convenience we will occasionally write $\xi^*(n)$ for ξ_n^*. The asymptotic properties of the process $\xi^* := (\xi_n^*)_{n\in\mathbb{N}}$ can be compared to those of the simple random walk as well as of a random walk in random environment (abbreviated in what follows as RWRE) in dimension one. For a comprehensive up to date review of the latter topics, see a monograph of Révész [42]. For more recent developments on RWRE, we refer to [16, 19] and references therein.

This chapter is devoted to the study of the limit points of the sequence ξ^* for a transient ERW. The approach adopted here is based on a reduction of the study of the asymptotic behavior of the ERW to that of a branching process and the subsequent reformulation of the problem in terms of the asymptotic dynamics of the most populated generation of the branching process. Similar questions for transient RWRE have been addressed in the interesting paper by Gantert and Shi [19]. The essential branching processes machinery which enables the implementation of our approach to excited random walks was introduced in [7, 8] and further developed in [31, 32].

Our first result concerns non-ballistic ERW.

Theorem 2.0.1. *Suppose that Assumption 1.1.1 holds with $\delta \in (1,2)$. Then,*

(i) The following holds:

$$\limsup_{n\to\infty} \frac{\xi_n^*}{n^{1/2}} > 0, \qquad P_0 - \text{a.s.} \tag{2.3}$$

and

$$\liminf_{n\to\infty} \frac{\xi_n^*}{n^{1/2}} < \infty, \qquad P_0 - \text{a.s.} \tag{2.4}$$

(ii) Furthermore, for any $\alpha > \frac{1}{\delta}$ with $\delta \in (1,2]$:

$$\lim_{n\to\infty} \frac{\xi_n^*}{n^{1/2}(\log n)^\alpha} = 0 \qquad \text{while} \qquad \lim_{n\to\infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/2}} = \infty, \qquad P_0 - \text{a.s.}$$

The above theorem implies in particular that unlike RWRE (see [19]), a non-ballistic transient ERW does not spend a positive fraction of time at a favorite site. While the asymptotic behavior of ξ_n^* for transient RWRE seems to be determined by the so called "traps" created by a random potential (cf. [19] and [16]), and is radically different from that of the simple unbiased random walk, the limsup asymptotic of ξ_n^* for a non-ballistic transient ERW turns out to be rather similar to its counterpart for a simple non-biased random walk (cf. Theorem 11.3 in [42]). We also remark that, based on a comparison with the latter, we believe that in fact $\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/2}} = \infty$ and $\liminf_{n\to\infty} \frac{\xi^*(n)}{n^{1/2}} = 0$ under the conditions of Theorem 2.0.1, but were unable to prove this conjecture.

The next theorem deals with the asymptotic behavior of $\xi^*(n)$ for ballistic ERW.

Theorem 2.0.2. *Suppose that Assumption 1.1.1 holds with $\delta > 2$. Then the following holds for any $\alpha > \frac{1}{\delta}$:*

$$\lim_{n\to\infty} \frac{\xi_n^*}{n^{1/\delta}(\log n)^\alpha} = 0 \qquad \text{while} \qquad \lim_{n\to\infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/\delta}} = \infty, \qquad P_0 - \text{a.s.}$$

The rest of the chapter is organized as follows. In Section 2.1, we consider an auxiliary branching process formed by successive level crossings along the random walk path. The proofs of Theorems 2.0.1 and 2.0.2 are contained in Section 2.2.

2.1 Reduction to a branching process

The proofs of our main results which are given in the next section rely on the use of a mapping of the paths of ERW into realizations of a suitable branching process with migration. In this section we discuss the branching process framework and recall some auxiliary results related to it; see [31, 32] and [17] for more details.

For $m \in \mathbb{N}$, let T_m be the first hitting time of site m, that is

$$T_m = \inf\bigl\{n \in \mathbb{N} : X_n = m\bigr\}.$$

Since the ERW is transient to the right under Assumption 1.1.1, the random variables T_m are almost surely finite for all $m \in \mathbb{N}$ under the law P_0. Moreover, it follows from (1.2) (by passing to the random subsequence of indexes $n = T_m$ in (1.2)) that

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{m\to\infty} \frac{T_m}{X_{T_m}} = \mathrm{v}^{-1} \in (0, \infty], \qquad P_0 - \mathrm{a.\,s.} \tag{2.5}$$

Set now $D_m^m := 0$ and for $k \le m-1$ let

$$D_k^m = \sum_{i=0}^{T_m - 1} \mathbf{1}_{\{X_{i+1}=k-1,\, X_i=k\}}$$

be the number of down-crossing steps of the ERW from site k to $k-1$ before time T_m (See Figure 2.1.) Then (see, for instance, [19, 32]),

$$T_m = m + 2\sum_{k\le m} D_k^m = m + 2\sum_{0\le k\le m} D_k^m + 2\sum_{k<0} D_k^m. \tag{2.6}$$

It follows from (2.1) that

$$\xi_{T_m}(k) = \begin{cases} 0 & \text{for} \quad k > m \\ D_k^m + D_{k+1}^m + \mathbf{1}_{\{k\ge 0\}} & \text{for} \quad k \le m, \end{cases}$$

and hence

$$\begin{aligned} \max_{0\le k\le m} D_k^m \;&\le\; \xi^*_{T_m} = \max_{k<m} \bigl(D_k^m + D_{k+1}^m + \mathbf{1}_{\{k\ge 0\}}\bigr) \\ &\le\; 1 + 2\max_{0\le k\le m} D_k^m + 2\max_{k<0} D_k^m. \end{aligned} \tag{2.7}$$

Notice that $\max_{k<0} D_k^m$ is bounded above by the total time spent by the random walk on the

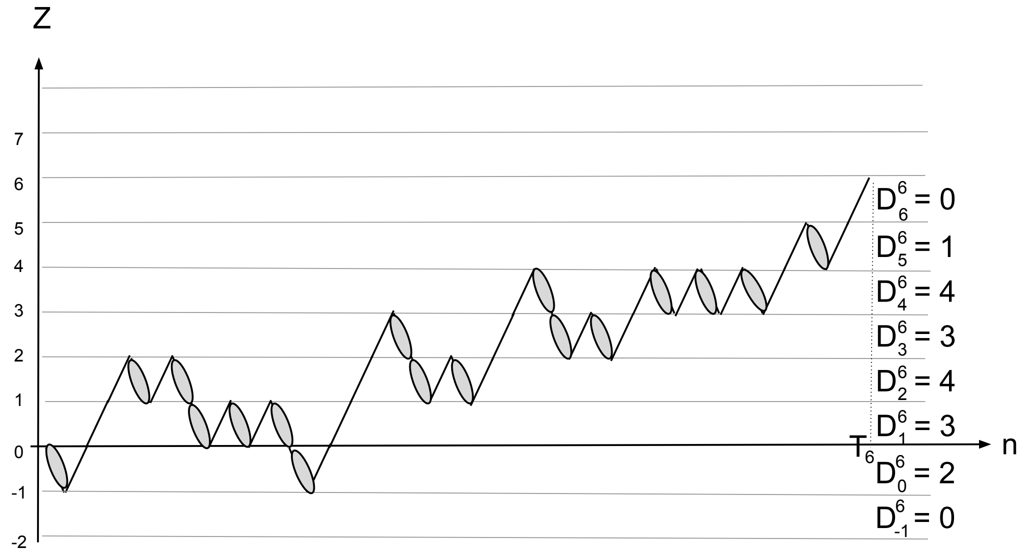

Figure 2.1 Left excursions of the random walk. Down-crossing are marked by tree leafs.

negative half-line. Since the random walk is transient to the right, the latter quantity is $P_0 -$ a. s. finite. Therefore, for any eventually increasing non-negative sequence $\phi : \mathbb{N} \to \mathbb{R}_+$, we have

$$\limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k^m}{\phi(m)} \quad \le \quad \limsup_{m\to\infty} \frac{\xi^*_{T_m}}{\phi(m)} \le 2 \limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k^m}{\phi(m)}.$$

Similar inequalities hold with the lim sup replaced by the lim inf .

To elucidate the probabilistic structure of the sequence D_k^m, it is convenient to exploit the following alternative definition of the random walk $(X_n)_{n\ge 0}$. Assume that the underlying probability space is enlarged to include a sequence of, conditionally on ω, independent Bernoulli random variables ("coins") $\big(B(z,i)\big)_{i\in\mathbb{N},z\in\mathbb{Z}}$ such that

$$P_{0,\omega}\big(B(z,i)=1\big)=\omega(z,i) \qquad \text{and} \qquad P_{0,\omega}\big(B(z,i)=-1\big)=1-\omega(z,i). \tag{2.8}$$

Then the ERW X can be alternatively defined by specifying the jump sequence recursively, as follows:

$$X_{n+1} = X_n + B\big(X_n, \xi_n\big), \tag{2.9}$$

where ξ_n is introduced in (1.1). We adopt the terminology of [31, 32] and refer to the event $\{B(z,i)=1\}$ as a "success" and to the event $\{B(z,i)=-1\}$ as a "failure". For $z \ge 0$, denote by $F_m^{(z)}$ the number of failures before the m-th success in the sequence $B^{(z)} := \big(B(z,i)\big)_{i\in\mathbb{N}}$. Let

$V := (V_k)_{k\geq 0}$ be a Markov chain on $\mathbb{N}_0$ with transition kernel defined (under the law P_0) by means of the following recursion:

$$V_{k+1} = F^{(k)}_{V_k+1}, \qquad k \geq 0.$$

The process V can be thought of as a branching process with the following properties:

1. There is has exactly 1 immigrant in each generation and the immigration happens before the reproduction.

2. The number of offspring of m-th individual in generation $k \in \mathbb{N}$ is equal to $F^{(k)}_m - F^{(k)}_{m-1}$.

For non-negative reals $x \geq 0$, denote by P^V_x the law of the process V that starts with $[x]$ individuals in the generation zero. It turns out that for every $n \geq 0$, the distribution of $(V_0, V_1, ..., V_n)$ under P^V_0 coincides with the distribution of the array $(D^n_n, D^n_{n-1}, ..., D^n_0)$ associated with a transient to the right ERW (see, for instance, Section 2 in [31]).

Under Assumption 1.1.1, X is transient to the right, and hence there exists an infinite sequence of times when the random walk moves forward to a "fresh point", i. e. to a site which has been never visited before and to which it will never return afterwards [32, 49]. It turns out [7, 32] that for the branching process V this implies that the following random times are finite with probability one under the law P^V_0 :

$$\sigma_{-1} := 0 \quad \text{and} \quad \sigma_k := \inf\{i > \sigma_{k-1} : V_i = 0\}, \quad k \geq 0.$$

Thus $(\sigma_k)_{k\geq 0}$ is the sequence of renewal times in which the extinction occurs and the branching process starts afresh due to the immigration. Notice that while the immigrants serve as "founders of dynasties" of descendants, they themselves are not counted in the population of the branching process. In what follows we refer to the part of the branching process evolving between two successive extinction times as a *life cycle* of the process. The difference $\sigma_k - \sigma_{k-1}$, $k \geq 0$, represents therefore the duration of the $(k+1)$-th life cycle. We remark that, although under our assumptions the event $\sigma_k - \sigma_{k-1} = 1$ can happen with a positive probability for any $k \geq 0$, the branching process does not get absorbed at zero, and is eventually revived in a future generation with a strictly positive number of immigrants. Let

$$\varrho_m := \min\{k \geq 0 : \sigma_k \geq m\} \tag{2.10}$$

denote the number of the life cycles completed by the first m generations.

Further, let

$$S_k := \sum_{i=\sigma_{k-1}}^{\sigma_k - 1} V_i, \qquad k \geq 0,$$

be the total population present during the $(k+1)$-th cycle, and let

$$M_k := \max_{\sigma_{k-1} \leq i < \sigma_k} V_i, \qquad k \geq 0,$$

be the size of the most populated generation in the $(k+1)$-th cycle. Notice that the sequence $(\sigma_k - \sigma_{k-1})_{k\geq 0}$ as well as the sequence of pairs $(S_k, M_k)_{k\geq 0}$ are i.i.d. under P_0^V. The following asymptotic results hold under Assumption 1.1.1 (see, for instance, Theorem 2.1, Theorem 2.2, and Lemma 8.1 in [31], respectively):

$$\lim_{n\to\infty} n^{\delta/2} P_0^V(S_0 > n) = K_0 \in (0, \infty), \tag{2.11}$$

$$\lim_{n\to\infty} n^{\delta} P_0^V(\sigma_0 > n) = K_1 \in (0, \infty), \tag{2.12}$$

$$\lim_{n\to\infty} n^{\delta} P_0^V(M_0 > n) = K_2 \in (0, \infty). \tag{2.13}$$

2.2 Proof of the main results

In this section we prove Theorem 2.0.1 and Theorem 2.0.2. We begin with a brief outline of the proof. First, using properties of the regular variation and the tail asymptotic of the renewal times which is given by (2.12), we reduce the study of ξ_n^* to that of $\xi_{T_n}^*$. The bounds for the latter sequence, stated in (2.7), enable us then to exploit the connection between the random walk and the branching process V introduced in Section 2.1. We remark that a similar strategy has been used for RWRE in [19]. The implementation of this approach (which at most stages is technically fairly different in this thesis from the one presented in [19]) is based on the existence of the renewal structure (life cycles) for the branching process and the asymptotic results for the distribution tails of the key random variables stated in (2.11)-(2.13) above.

We start the proof with the following $0-1$ law for the maximal occupation time of the random walk. A similar statement for one-dimensional random walks in random environment is given in [19, Proposition 3.1].

Lemma 2.2.1. *Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be an unbounded eventually increasing function. Then*

$$\limsup_{n\to\infty} \frac{\xi_n^*}{\phi(n)} = K_4 \in [0,\infty] \qquad P_0 - \text{a. s.}$$

Proof. Fix any constant $c \geq 0$ and for any realization X of the random walk let

$$g_c(X) := \mathbf{1}_{\{\limsup_{n\to\infty} \frac{\xi_n^*}{\phi(n)} = c\}},$$

where the random variables ξ_n^* are computed along the infinite path X of the random walk.

Following [49] (see the paragraph right before Lemma 3 there), we next define recursively a sequence of random times $(\eta_{k,m})_{m\in\mathbb{N},k\in\mathbb{Z}}$ by setting

$$\eta_{k,0} := -1 \qquad \text{and} \qquad \eta_{k,m+1} := \inf\{n > \eta_{k,m} : X_n \geq k\}.$$

For a fixed $k \in \mathbb{N}$, the sequence $\eta_{k,m}$ represents the successive times when the random walk is located on the right-hand side of the site $k-1$. Thus the sequence $X^{(k)} = (X\eta_{k,m})_{m\geq 0}$ extracts from the path of the random walk the fragments which are included in the half-line $\{x \in \mathbb{N} : x \geq k\}$. Notice that for a fixed $k \in \mathbb{N}$, random variables $\eta_{k,m}$ are stopping times with respect to the natural filtration of X under the law $P_{0,\omega}$.

Let G_k be the σ-algebra generated by the sequence $\big(X^{(n)}\big)_{n\geq k}$, that is

$$G_k = \sigma\big(X^{(k)}, X^{(k+1)}, \ldots\big).$$

It is formally shown in [49, Lemma 3] that the quenched distribution of $X^{(k)}$ in a fixed cookie environment ω is independent of $(\omega_i)_{i\leq k-1}$. Recall now the coin-tossing construction of the excited random walks which is described in (2.8) and (2.9) above (see [32, Section 4] or [31, Section 2] for more details). By virtue of (2.9), in terms of the "coin variables" introduced in (2.8), we have

$$G_k \subset \sigma\big(B^{(k)}, B^{(k+1)}, B^{(k+2)}, \ldots\big), \qquad k \geq 0, \tag{2.14}$$

where $B^{(k)} = \big(B(k,i)\big)_{i\in\mathbb{N}}$.

Under Assumption 1.1.1, both X_n and the location of the most visited by time n site (say, the right-most one if there are several such equally visited sites) are transient to the right. In particular, since $\max_{x<k} \xi_n(x)$ is bounded above by the total time spent by the random walk on the half-line $\{z \in \mathbb{Z} : z < k\}$, the random variable $g_c(X)$ is measurable with respect to

$\sigma\big(B^{(k)}, B^{(k+1)}, B^{(k+2)}, \ldots\big)$ for any $k \geq 0$. Since $B^{(k)}$ are independent under $P_{0,\omega}$, Kolmogorov's $0-1$ law for independent sequences implies that for $\mathbb{P}$-almost every fixed cookie environment $\omega \in \Omega_M$, the random variable $g_c(X)$ considered as a function of the random walk's path is an $P_{0,\omega}$-almost sure constant (compare with the first step in the proof of [19, Proposition 3.1]).

Therefore, we can without loss of generality consider $g_c(X)$ as a function of ω only (but not of the outcome of the "coin tossing" procedure) and correspondingly denote it by, say, $g_c(\omega)$. Specifically, the values of $g_c(\omega)$ can be chosen in such a way that $P_{0,\omega}\big(g_c(X) = g_c(\omega)\big) = 1$ for $\mathbb{P}$-almost every ω. In view of (2.14), this implies that for any $k \geq 0$,

$$g_c(\omega) \in \sigma\big(\omega_k, \omega_{k+1}, \ldots\big)$$

In other words, $g_c(\omega)$ is translation-invariant with respect to the shift operator $\theta : \Omega_M \to \Omega_M$ such that $(\theta\omega)_n = \omega_{n+1}$, $n \in \mathbb{Z}$. By virtue of condition (a) of Assumption 1.1.1, Kolmogorov's $0-1$ law implies then that $g_c(\omega)$ is a $\mathbb{P}-\text{a.s.}$ constant function for any $c \geq 0$. Since $g_c(\omega)$ are indicator functions taking values 0 and 1 only, this completes the proof of the lemma. □

2.2.1 Non-ballistic regime: Proof of Theorem 2.0.1

Part (i). We first prove (2.3). In order to prove (2.3), it suffices to show that for all $\delta \in (1,2)$ and any constant $c > 0$ which is small enough we have

$$\liminf_{n\to\infty} P_0\big(\xi_n^* \geq c\sqrt{n}\big) > 0. \tag{2.15}$$

Indeed, (2.15) together with the reverse Fatou's lemma imply that

$$\begin{aligned} P_0\Big(\limsup_{n\to\infty} \frac{\xi_n^*}{\sqrt{n}} > c\Big) &\geq P_0\Big(\limsup_{m\to\infty} \frac{\xi_{T_m}^*}{T_m^{1/2}} > c\Big) = E_0\Big(\limsup_{m\to\infty} \mathbf{1}_{\{\xi_{T_m}^* > cT_m^{1/2}\}}\Big) \\ &\geq \limsup_{m\to\infty} E_0\big(\mathbf{1}_{\{\xi_{T_m}^* > cT_m^{1/2}\}}\big) = \limsup_{m\to\infty} P_0\big(\xi_{T_m}^* > cT_m^{1/2}\big) > 0. \end{aligned}$$

By virtue of Lemma 2.2.1, this yields the claim.

We now turn to the proof that (2.15) holds for any sufficiently small $c > 0$. Observe that according to (2.6) and (2.7), we have

$$\begin{aligned} P_0\big((\xi_{T_m}^*)^2 \geq c^2 T_m\big) &= P_0\Big(\big(\max_{k\leq m} D_k^m\big)^2 \geq c^2\Big(m + 2\sum_{k\leq m} D_k^m\Big)\Big) \\ &\geq P_0^V\Big(\max_{0\leq i<\varrho_m} M_i^2 \geq c^2\Big(m + 2\sum_{i=0}^{\varrho_m} S_i\Big)\Big). \end{aligned}$$

Denote by $\mu := E_0^V[\sigma_0]$ the annealed (i. e., taken under the law P_0^V) expectation of σ_0, choose an arbitrary $\delta' \in (1,\delta)$, and define

$$a_m^{\pm} = \lfloor \mu^{-1}(m \pm m^{1/\delta'}) \rfloor, \tag{2.16}$$

where we use the notation $\lfloor x \rfloor$ to denote the integer part of a real number x. It follows from the last inequality stated above that

$$\begin{aligned} &P_0\big((\xi_{T_m}^*)^2 \geq c^2 T_m\big) \\ &\quad \geq P_0^V\Big(\max_{1\leq i\leq a_m^-} M_i^2 \geq c^2\Big(m + 2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big) - P_0^V(\varrho_m > a_m^+) - P_0^V(\varrho_m < a_m^-). \end{aligned}$$

Since $\{\varrho_m \geq a_m^+\} = \{m \geq \sigma_{a_m^+}\} = \{m \geq \sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1})\}$, then

$$P_0^V(\varrho_m > a_m^+) \leq P_0^V\Big(\frac{\sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1} - \mu)}{(a_m^+)^{1/\delta}} \leq -\frac{m^{1/\delta'}}{(a_m^+)^{1/\delta}}\Big). \tag{2.17}$$

Hence (2.12) and a stable limit theorem for i.i.d. random variables $\sigma_i - \sigma_{i-1}$ (see, for instance, Theorem 1.5.1 in [14]) imply that $\lim_{m\to\infty} P_0^V(\varrho_m > a_m^+) = 0$. Similarly, one can show that $\lim_{m\to\infty} P_0^V(\varrho_m < a_m^-) = 0$. Therefore, in order to prove (2.15), it suffices to show that the following strict lower bound holds:

$$\liminf_{m\to\infty} P_0^V\Big(\max_{1\leq i\leq a_m^-} M_i^2 \geq c^2\Big(m + 2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big) > 0. \tag{2.18}$$

Toward this end, fix any positive constants $\beta, \gamma > 0$ such that

$$\frac{1}{2}\cdot\big(K_2\cdot\beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}\big) - K_2^2\cdot\beta^{-\delta} > 0, \tag{2.19}$$

and observe that

$$\begin{aligned} &P_0^V\Big(\max_{1\leq i\leq a_m^-} M_i^2 \geq c^2\Big(m + 2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big) \geq \\ &\quad \geq P_0^V\Big(\max_{1\leq i\leq a_m^-} M_i^2 \geq \beta m^{2/\delta},\, m + 2\sum_{i=1}^{a_m^+ +1} S_i \leq \frac{\beta m^{2/\delta}}{c^2}\Big) \geq P_0^V\Big(\bigcup_{i=1}^{a_m^-} A_{i,m}\Big), \end{aligned}$$

where

$$A_{i,m} := \Big\{M_i^2 \geq \beta m^{2/\delta},\ S_i \leq \gamma m^{2/\delta},\ m + 2\sum_{\substack{1\leq j\leq a_m^+ +1,\\ j\neq i}} S_j < (\beta/c^2 - \gamma)m^{2/\delta}\Big\}. \tag{2.20}$$

Therefore, the inclusion-exclusion formula yields

$$\begin{aligned} P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m+2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big) &\ge P_0^V\Big(\bigcup_{i=1}^{a_m^-} A_{i,m}\Big) \\ &\ge \sum_{i=1}^{a_m^-} P_0^V(A_{i,m}) - \sum_{i=1}^{a_m^-}\sum_{j=i+1}^{a_m^-} P_0^V\Big(A_{i,m}\bigcap A_{j,m}\Big) \\ &\ge a_m^- P_0^V(A_{1,m}) - (a_m^-)^2 P_0^V\big(M_1^2\ge \beta m^{2/\delta},\, M_2^2 \ge \beta m^{2/\delta}\big). \end{aligned} \tag{2.21}$$

Using the independence of the life-cycles of the underlying branching process, we obtain that

$$P_0^V(A_{1,m}) = P_0^V\big(M_1^2 \ge \beta m^{2/\delta},\, S_1 \le \gamma m^{2/\delta}\big)\cdot P_0^V\Big(m+\sum_{j=2}^{a_m^+ +1} S_j < (\beta/c^2-\gamma)m^{2/\delta}\Big).$$

Taking into account (2.11) and (2.13), one can deduce from the following inequality:

$$P_0^V(M_1^2 \ge \beta m^{2/\delta},\, S_1 \le \gamma m^{2/\delta}) \ge P_0^V\big(M_1^2 \ge \beta m^{2/\delta}\big) - P_0^V\big(S_1 > \gamma m^{2/\delta}\big),$$

that

$$\liminf_{m\to\infty} a_m^- \cdot P_0^V\big(M_1^2 \ge \beta m^{2/\delta},\, S_1 \le \gamma m^{2/\delta}\big) \ge K_2\cdot \beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}.$$

Furthermore, it follows from (2.11) and a stable limit theorem for i.i.d. variables S_i (see, for instance, Theorem 1.5.1 in [14]) that the following limit exists and is strictly positive:

$$\lambda(c,\beta,\gamma) := \lim_{m\to\infty} P_0^V\Big(m+2\sum_{j=2}^{a_m^+ +1} S_j < (\beta/c^2-\gamma)m^{2/\delta}\Big) > 0.$$

Moreover, given $\beta,\gamma>0$, we can choose $c>0$ so small that $\lambda(c,\beta,\gamma)>1/2$. For such a constant $c>0$, (2.21) along with (2.13) yield

$$\begin{aligned} \liminf_{m\to\infty} P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m+2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big) \\ \ge \lambda(c,\beta,\gamma)\cdot\big(K_2\cdot\beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}\big) - K_2^2\cdot\beta^{-\delta} \\ \ge \frac{1}{2}\cdot\big(K_2\cdot\beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}\big) - K_2^2\cdot\beta^{-\delta}. \end{aligned}$$

In view of (2.19), this implies (2.18), and hence (2.15) for the indicated above values of the parameter $c>0$. Now we prove (2.4). In order to prove (2.4), it suffices to show that for all $\delta\in(1,2)$ and any constant $b>0$ which is large enough we have

$$\liminf_{n\to\infty} P_0\big(\xi_n^* > b\sqrt{n}\big) < 1. \tag{2.22}$$

Indeed, (2.22) together with the Fatou's lemma imply that

$$P_0\Big(\liminf_{n\to\infty}\frac{\xi_n^*}{\sqrt{n}}>b\Big)\le P_0\Big(\liminf_{m\to\infty}\frac{\xi_{T_m}^*}{T_m^{1/2}}>b\Big)=E_0\Big(\liminf_{m\to\infty}\mathbf{1}_{\{\xi_{T_m}^*>bT_m^{1/2}\}}\Big)$$
$$\le\liminf_{m\to\infty}E_0\big(\mathbf{1}_{\{\xi_{T_m}^*>bT_m^{1/2}\}}\big)=\liminf_{m\to\infty}P_0\big(\xi_{T_m}^*>bT_m^{1/2}\big)<1.$$

By virtue of Lemma 2.2.1, this yields the claim.

We now turn to the proof that (2.22) holds for any sufficiently large $b>0$. Observe that according to (2.6) and (2.7), we have

$$\begin{aligned}P_0\big((\xi_{T_m}^*)^2\ge b^2T_m\big) &= P_0\Big(4\big(\max_{k\le m}D_k^m\big)^2\ge b^2\Big(m+2\sum_{k\le m}D_k^m\Big)\Big)\\ &\le P_0^V\Big(\max_{0\le i\le\varrho_m}M_i^2\ge\frac{b^2}{4}\Big(m+2\sum_{i=0}^{\varrho_m}S_i\Big)\Big).\end{aligned}$$

Recall (2.16). It follows from the last inequality stated above that

$$P_0\big((\xi_{T_m}^*)^2\ge b^2T_m\big)$$
$$\le P_0^V\Big(\max_{1\le i\le a_m^+}M_i^2\ge c^2\Big(m+2\sum_{i=1}^{a_m^-+1}S_i\Big)\Big)+P_0^V(\varrho_m>a_m^+)+P_0^V(\varrho_m<a_m^-).$$

Since $\lim_{m\to\infty}P_0^V(\varrho_m>a_m^+)=0$ and $\lim_{m\to\infty}P_0^V(\varrho_m<a_m^-)=0$, then, in order to prove (2.22), it suffices to show that the following strict upper bound holds:

$$\liminf_{m\to\infty}P_0^V\Big(\max_{1\le i\le a_m^+}M_i^2\ge\frac{b^2}{4}\Big(m+2\sum_{i=1}^{a_m^-+1}S_i\Big)\Big)<1. \tag{2.23}$$

Toward this end, observe that

$$P_0^V\Big(\max_{1\le i\le a_m^+}M_i^2\ge\frac{b^2}{4}\Big(m+2\sum_{i=1}^{a_m^-+1}S_i\Big)\Big)\le$$
$$\le P_0^V\Big(\max_{1\le i\le a_m^+}M_i^2\ge\beta m^{2/\delta}\Big)+P_0^V\Big(m+2\sum_{i=1}^{a_m^-+1}S_i\le\frac{4\beta m^{2/\delta}}{b^2}\Big).$$

Since

$$P_0^V\Big(\max_{1\le i\le a_m^+}M_i^2\ge\beta m^{2/\delta}\Big)=1-\Big(1-P_0^V\Big(M_1\ge\beta m^{1/\delta}\Big)\Big)^{a_m^+},$$

we first choose a large value of $\beta>0$ such that the right hand side is less than half. Then, in virtue of (2.11), we can find a large $b\gg\beta$ such that

$$P_0^V\Big(m+2\sum_{i=1}^{a_m^++1}S_i\le\frac{4\beta m^{2/\delta}}{b^2}\Big)<\frac{1}{2},$$

which finishes the proof of (2.23). □

Part (ii). First, we will show that for any constant $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{\xi_n^*}{n^{1/2}(\log n)^\alpha} = 0, \qquad P_0 - \text{a. s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^\alpha$. Recall the alternative notation $\xi^*(n)$ for ξ_n^*. Observe that in order to prove the above claim, it suffices to show that

$$\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \qquad P_0 - \text{a. s.} \tag{2.24}$$

Indeed, let k_m, $m \in \mathbb{N}$, be the (uniquely defined) non-negative random integers such that

$$T_{k_m} < m \le T_{k_m+1}, \qquad m \in \mathbb{N}. \tag{2.25}$$

Then, since $\phi(n)$ is an eventually increasing sequence,

$$\begin{aligned}\limsup_{m\to\infty} \frac{\xi^*(m)}{\phi(m)} &\le \limsup_{m\to\infty} \frac{\xi^*(T_{k_m+1})}{\phi(m)} \le \limsup_{m\to\infty} \frac{\xi^*(T_{k_m+1})}{\phi(T_{k_m})} \\ &\le \limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \qquad P_0 - \text{a. s.}\end{aligned} \tag{2.26}$$

Now, we will establish certain large deviation type estimates for the distribution tails of the random variables ϱ_m. To this end, observe that the inequality stated in (2.17) remains true for any $\delta > 1$, in particular for $\delta = 2$. Furthermore, (2.12) and the large deviation estimate stated, for instance, in [14, Theorem 3.4.1] imply that the following holds under Assumption 1.1.1 (with arbitrary $\delta > 1$) for a suitable constant $c_1 = c_1(\delta) > 0$:

$$P_0^V(\varrho_m > a_m^+) \le \frac{c_1 a_m^+}{m^{\delta/\delta'}}. \tag{2.27}$$

Similarly, using the following inequality instead of (2.17):

$$P_0^V(\varrho_m \le a_m^-) = P_0^V(m \le \sigma_{a_m^-}) \le P_0^V\Big(\frac{\sum_{i=0}^{a_m^-}(\sigma_i - \sigma_{i-1} - \mu)}{(a_m^-)^{1/\delta}} \ge \frac{m^{1/\delta'}}{(a_m^-)^{1/\delta}}\Big), \tag{2.28}$$

one can deduce from (2.12) and [14, Theorem 3.4.1] that the following holds under Assumption 1.1.1 (with arbitrary $\delta > 1$) for some constant $c_2 = c_2(\delta) > 0$:

$$P_0^V(\varrho_m < a_m^-) \le \frac{c_2 a_m^-}{m^{\delta/\delta'}}. \tag{2.29}$$

We remark that, in the course of proving (2.27), in order to formally meet the lower tail conditions of Theorem 3.4.1 in [14] one can, for instance, use in (2.17) the following “unpolarized” version of $\sigma_i - \sigma_{i-1} - \mu$ which has the same structure of upper and lower distribution tails:

$$(\sigma_i - \sigma_{i-1} - \mu)' := U_i \cdot (\sigma_i - \sigma_{i-1} - \mu),$$

where $U = (U_i)_{i\geq 0}$ is a sequence of i.i.d. Bernoulli random variables, independent of “anything else” (i. e., such that the probability law P^U of U is independent of the measure P^V in the enlarged probability space), and such that

$$P^U(U_i = 1) = P^U(U_i = -1) = \frac{1}{2}.$$

Notice that (cf. (2.17))

$$P_0^V\Big(\sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1} - \mu) \leq -m^{1/\delta'}\Big) \leq P_0^V\Big(\sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1} - \mu)' \leq -m^{1/\delta'}\Big).$$

Recall now (2.27) and (2.29). Let $\psi_\varepsilon(m) = m^{2/\delta}(\log m)^{-\varepsilon}$ with $\varepsilon \in (0, 2\alpha)$ and let $m_i = 3^i$ for $i \in \mathbb{N}$. Then, for any constant $c > 0$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} P_0\big(T_{m_i} < c\psi_\varepsilon(m_{i+1})\big) &\leq \sum_{i=1}^{\infty}\Big[P_0^V\Big(2\sum_{k=0}^{a_{m_i}^- -1} S_k < c\psi_\varepsilon(m_{i+1})\Big) + P_0^V\left(\varrho_{m_i} < a_{m_i}^-\right)\Big] \\
&\leq \sum_{i=1}^{\infty}\Big[P_0^V\Big(2\max_{0\leq k<a_{m_i}^-} S_k < c\psi_\varepsilon(m_{i+1})\Big) + P_0^V\left(\varrho_{m_i} < a_{m_i}^-\right)\Big] \qquad (2.30) \\
&\leq \sum_{i=1}^{\infty}\Big[\Big(1 - \frac{c_5}{(c\psi_\varepsilon(m_{i+1}))^{\delta/2}}\Big)^{a_{m_i}^-} + \frac{c_2 a_{m_i}^-}{(m_i)^{\delta/\delta'}}\Big] \leq \sum_{i=1}^{\infty} c_6 \cdot \left[e^{-c_7(i+1)^{\varepsilon\delta}} + e^{-c_8 i}\right] < \infty,
\end{aligned}$$

where $c_5, c_5, c_7, c_8 > 0$ are suitable positive constants and c_2 is the constant which appears at (2.29). Thus, by the Borel-Contelli lemma,

$$\liminf_{i\to\infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \qquad P_0 - \text{a. s.}$$

Since for each $n \in \mathbb{N}$,

$$m_i < n \leq m_{i+1} \quad \text{for some } i \in \mathbb{N} \text{ which is uniquely determined by } n, \qquad (2.31)$$

then

$$\liminf_{n\to\infty} \frac{T_n}{\psi_\varepsilon(n)} \geq \liminf_{i\to\infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \qquad P_0 - \text{a. s.}, \qquad (2.32)$$

and hence $P_0\big(T_m < \psi_\varepsilon(m)\text{i. o.}\big) = 0$. This implies that for any constant $b > 0$,

$$P_0\Big(\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\Big) \le P_0\Big(\limsup_{i\to\infty} \frac{\xi^*(T_{m_{i+1}})}{\phi(T_{m_i-1})} > b\Big)$$
$$\le P_0\Big(\limsup_{i\to\infty} \frac{\max_{0\le k\le m_{i+1}} D_k^{m_{i+1}}}{\phi(\psi_\varepsilon(m_i-1))} > \frac{b}{2}\Big). \tag{2.33}$$

On the other hand for each $m \in \mathbb{N}$,

$$P_0\Big(\max_{0\le k\le m} D_k^m > \frac{b}{2}\phi(\psi_\varepsilon(m-1))\Big) \le P_0^V\Big(\max_{0\le k\le \varrho_m} M_k > \frac{b}{2}\phi(\psi_\varepsilon(m-1))\Big)$$
$$\le P_0^V\Big(\max_{0\le k\le m} M_k > \frac{b}{2}\phi(\psi_\varepsilon(m-1))\Big), \tag{2.34}$$

where in the last but one step we used the inequality $\varrho_m \le m$. It follows from (2.13), (2.31), and (2.34) that if $\varepsilon \in (0, 2\alpha)$ is chosen in such a way that in fact $\alpha - \varepsilon/2 > 1/\delta$, then

$$\sum_{i=1}^\infty P_0\Big(\max_{0\le k\le m_{i+1}} D_k^{m_{i+1}} > \frac{b}{2}\phi(\psi_\varepsilon(m_i-1))\Big) \le \sum_{i=1}^\infty P_0^V\Big(\frac{\max_{0\le k\le m_{i+1}} M_k}{m_i^{1/\delta}(\log m_i)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big)$$
$$= \sum_{i=1}^\infty 1 - \big(1 - P_0^V(M_1 > m_i^{1/\delta} > \frac{b\delta^\alpha}{2^{1+\alpha}}(\log m_i)^{\alpha-\varepsilon/2} >)\big)^{m_{i+1}+1}$$
$$= \sum_{i=1}^\infty c_9 \frac{m_{i+1}}{m_i(\log m_i)^{\alpha\delta-\varepsilon\delta/2}} < \infty.$$

Thus the Borel-Cantelli lemma combined with (2.33) completes the proof of (2.24).

We now turn to the proof that for any constant $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/2}} = \infty, \qquad P_0 - \text{a. s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^{-\alpha}$. In order to prove the claim, it suffices to show that

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} = \infty, \qquad P_0 - \text{a. s.} \tag{2.35}$$

Indeed, in view of (2.25), it follows from (2.35) that

$$\infty = \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} \le \liminf_{m\to\infty} \frac{\xi^*(T_{k_m})}{\phi(T_{k_m+1})} \le \liminf_{m\to\infty} \frac{\xi^*(m)}{\phi(m)}, \qquad P_0 - \text{a. s.}$$

We will next show that (2.35) indeed holds true. Fix any $\varepsilon \in (0, \alpha\delta - 1)$ and define

$$\psi_\varepsilon(m) := m^{2/\delta}(\log m)^{2/\delta+\varepsilon}, \qquad m \in \mathbb{N}.$$

By a counterpart of the law of the iterated logarithm for i.i.d. random variables in the domain of attraction of a stable law and with infinite variance, we have (see, for instance, Theorem 1.6.6. in [14]):

$$\limsup_{m\to\infty} \frac{\sum_{k=1}^{m} S_k}{\psi_\varepsilon(m)} = 0, \qquad P_0^V - \text{a. s.}$$

Combining this result with the fact that $\lim_{m\to\infty} P_0^V\big(\frac{\varrho_m}{m} = \mu^{-1}\big) = 1$ (which is an implication of the renewal theorem applied to the renewal sequence σ_k), we obtain that

$$P_0\big(T_m > \psi_\varepsilon(m) \qquad \text{i. o.}\big) = 0.$$

Therefore,

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(\psi_\varepsilon(m+1))} \le \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \qquad P_0 - \text{a. s.}$$

Thus it suffices to prove that the left-hand side of the above inequality is infinity. Recall (2.16) and (2.29). Then, for $m_i = 3^i$ and any constant $c > 0$, similarly to (2.30), we have:

$$\begin{aligned}\sum_{i=1}^{\infty} P_0\Big(\xi^*(T_{m_i}) < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) &\le \sum_{i=1}^{\infty} P_0^V\Big(\max_{0\le k<\varrho_{m_i}} M_k < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) \\ &\le \sum_{i=1}^{\infty}\Big[P_0^V\Big(\max_{0\le k<a^-_{m_i}} M_k < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) + P_0^V(a^-_{m_i} > \varrho_{m_i})\Big] \\ &\le \sum_{i=1}^{\infty} c_9 \cdot \Big[e^{-c_{10}(i+1)^{\alpha\delta-1-\varepsilon}} + \frac{a^-_{m_i}}{(m_i)^{\delta/\delta'}}\Big] < \infty, \end{aligned} \tag{2.36}$$

where $c_9 > 0$ and $c_{10} > 0$ are some appropriate positive constants. Therefore, the Borel-Contelli lemma yields (recall that the value of the parameter ε is chosen from the interval $(0, \alpha\delta - 1)$, and hence $\alpha\delta - 1 - \varepsilon > 0$):

$$\liminf_{i\to\infty} \frac{\xi^*(T_{m_i})}{\phi(\psi_\varepsilon(m_{i+1}+1))} = \infty, \qquad P_0 - \text{a. s.}$$

This completes the proof of (2.35) by using a suitable variation of (2.32). □

2.2.2 Ballistic regime: Proof of Theorem 2.0.2

Part (i). Suppose first that $\phi(n) = n^{1/\delta}(\log n)^{\alpha}$ for a fixed $\alpha > 1/\delta$. Then, a slight modification of (2.33) and (2.34) (namely, formally replacing there the composition of two functions $\phi \circ \psi_\varepsilon$ by the "new" $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$) along with the Borel-Contelli lemma completes the proof of the first half of part (i) of Theorem 2.0.2. □

Part (ii). Let $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ for a fixed constant $\alpha > 1/\delta$. In order to prove the claim, it suffices to verify (2.35) (see the next two lines below (2.35)). Toward this end, observe that according to the law of large numbers for T_n stated in (2.5) we have $P_0(T_m > 2m\mathrm{v}^{-1}\mathrm{i.\,o.}) = 0$, and hence

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(2\mathrm{v}^{-1}(m+1))} \leq \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \qquad P_0 - \mathrm{a.\,s.}$$

Thus it suffices to show that the left-hand side of the above inequality is infinity. Let a_m^- be as defined in (2.16) with the only exception that this time we will use an arbitrary constant $\delta' \in (1,2)$. In view of (2.12) and (2.28), Chebyshev's inequality implies

$$P_0^V(a_m^- > \varrho_m) \leq \frac{a_m^- + 1}{m^{2/\delta'}} \cdot E_0^V\big[(\sigma_0 - \mu)^2\big]. \tag{2.37}$$

Therefore, a slight modification of (2.36) (namely, formally replacing there the composition of two functions $\phi \circ \psi_\varepsilon$ by the "new" $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ and also using (2.37) instead of (2.29)) along with the Borel-Contelli lemma imply that $\liminf_{i\to\infty} \frac{\xi^*(T_{m_i})}{\phi(m_{i+1}+1)} = \infty$, $P_0 - \mathrm{a.\,s.}$ This completes the proof of (2.35) by using an appropriate variation of (2.32). □

CHAPTER 3. Limit laws for a directionally reinforced random walk

We consider a generalized version of a directionally reinforced random walk, which was originally introduced by Mauldin, Monticino, and von Weizsäcker in [34]. Our main result is a stable limit theorem for the position of the random walk in higher dimensions. This extends a result of Horváth and Shao [24] that was previously obtained in dimension one only (however, in a more stringent functional form).

We first introduce a few notations. For a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $\|x\| = \max_i |x_i|$. For (possibly, random) functions $f, g : \mathbb{R}_+$ (or $\mathbb{N}$) $\to \mathbb{R}$, write $f \sim g$ and $f(t) = o(g(t))$ to indicate that, respectively, $\lim_{t\to\infty} f(t)/g(t) = 1$ and $\lim_{t\to\infty} f(t)/g(t) = 0$, a. s. Let $\pi = (\pi_v)_{v\in U} \in \mathbb{R}^{|U|}$ be the unique stationary distribution of the Markov chain η and let

$$\mu = \sum_{v\in U} \pi_v v. \tag{3.1}$$

Thus $\mu = E(\eta_n) \in \mathbb{R}^d$ for each $n \in \mathbb{N}$.

The following theorem shows that a strong law of large numbers holds for X_t and that, under suitable second moment condition, the sample paths of the random walk are uniformly close to the sample paths of a drifted Brownian motion. We have:

Theorem 3.0.2.

(a) Suppose that $E(T_1^p) < \infty$ for some constant $p \in (1, 2)$. Then,

$$\big\|X_t - \mu t\big\| = o\big(t^{1/p}\big).$$

(b) If $E(T_1^p) < \infty$ for some constant $p > 2$, then (in an enlarged, if needed, probability space) there exist a process $\widehat{X} = \big(\widehat{X}_t\big)_{t\geq 0}$ distributed as X and a Brownian motion $(W_t)_{t\geq 0}$ in $\mathbb{R}^d$, such that,

$$\sup_{0\leq t\leq T} \big\|\widehat{X}_t - \mu t - W_t\big\| = o\big(T^{1/p}\big).$$

Remark 3.0.3. *The results stated in Theorem 3.0.2 as well as in Theorem 3.0.4 below are essentially due to [24]. In fact, the original proofs can be adapted to our more general setup. However, the proofs we give in Section 3.1 are shorter and somewhat simpler than the original ones. Furthermore, our proves can easily be seen working for the general Markov chain setup described in Remark 3.0.9 below.*

The second part of Theorem 3.0.2 implies the invariance principle for $(X_{nt} - \mu nt)$ with the usual normalization $\sqrt{n}$. We next state an invariance principle and the corresponding law of iterated logarithm under a slightly more relaxed moment condition. Let $D(\mathbb{R}^d)$ denote the set of $\mathbb{R}^d$-valued càdlàg functions on $[0,1]$ equipped with the Skorokhod J_1-topology. We use notation $\Rightarrow$ to denote the weak convergence in $D(\mathbb{R}^d)$. We have:

Theorem 3.0.4. *For $n \in \mathbb{N}$, define a process S_n in $D(\mathbb{R}^d)$ by setting*

$$S_n(t) = \frac{X_{nt} - \mu nt}{\sqrt{n}}, \qquad t \in [0,1]. \tag{3.2}$$

If $E(T_1^2) < \infty$, then

(a) $S_n \Rightarrow W$, where $W = (W_t)_{t \geq 0}$ is a (possibly degenerate, but not identically equal to zero) d-dimensional Brownian motion.

(b) For every $x \in \mathrm{Span}(U) \subset \mathbb{R}^d$, there is a constant $K(x) \in (0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{(X_t - \mu t)\cdot x}{\sqrt{t \ln\ln t}} = K(x).$$

Furthermore, a similar statement holds for the $\liminf$.

We next consider the case when $E(T_1^2) = \infty$ and T_1 is in the domain of attraction of a stable law. Namely, for the rest of our results we impose the following assumption. Recall that a function $h : \mathbb{R}_+ \to \mathbb{R}$ is said to be *regularly varying of index* $\alpha \in \mathbb{R}$ if $h(t) = t^\alpha L(t)$ for some $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. We will denote the set of all regularly varying functions of index α by $\mathcal{R}_\alpha$.

Assumption 3.0.5. *There is $h \in \mathcal{R}_\alpha$ with $\alpha \in (0,2]$ such that* $\lim_{t\to\infty} h(t)\cdot P(T_1 > t) \in (0,\infty)$.

For $t > 0$ let

$$a_t = \begin{cases} \inf\{s > 0 : t \cdot P(T_1 > s) \le 1\} & \text{if} \quad \alpha < 2, \\ \\ \inf\{s > 0 : ts^{-2} \cdot E\big(T_1^2; T_1 \le s\big) \le 1\} & \text{if} \quad \alpha = 2 \end{cases} \tag{3.3}$$

If $h(t) \in \mathcal{R}_\alpha$ with $\alpha \in (1,2]$ (and hence $E(T_1) < \infty$), one can obtain the following analogue of Theorem 3.0.4. It turns out that also in this case the functional limit theorem and the law of iterated logarithm for X_t inherit the structure of the corresponding statements for the partial sums of i.i.d. variables $\sum_{k=1}^n T_k$.

Theorem 3.0.6. *Let Assumption 3.0.5 hold with $\alpha \in (1,2]$. Let*

$$S_t := \frac{X_t - \mu t}{a_t}, \qquad t > 0.$$

We have:

(a) If $\alpha \in (1,2)$, then

(i) S_t converges weakly to a non-degenerate multivariate stable law in $\mathbb{R}^d$.

(ii) For every $x \in \mathrm{Span}(U) \subset \mathbb{R}^d$ such that $x \cdot u > 0$ for some $u \in U$,

$$\limsup_{t\to\infty} \frac{(X_t - \mu t)\cdot x}{a_t \cdot (\ln t)^{1/\alpha+\varepsilon}} = \begin{cases} 0 & \text{if} \ \ \varepsilon > 0, \\ \infty & \text{if} \ \ \varepsilon < 0 \end{cases} \qquad \text{a. s.}$$

In particular, for some constant $c(x) > 0$,

$$\limsup_{t\to\infty} \Big\{\frac{(X_t - \mu t)\cdot x}{a_t}\Big\}^{1/\ln\ln t} = c(x) \qquad \text{a. s.}$$

(b) If $\alpha = 2$ and $E(T_1^2) = \infty$, then S_t converges weakly to a non-degenerate multivariate Gaussian distribution in $\mathbb{R}^d$.

For $\alpha \in (0,1)$ we have the following limit theorem.

Theorem 3.0.7. *Let Assumption 3.0.5 hold with $\alpha \in (0,1)$. Then $\frac{X_t}{t}$ converges weakly in $\mathbb{R}^d$ to a non-degenerate limit.*

Remark 3.0.8. *The limiting random law in the statement of Theorem 3.0.7 is specified in* (3.28) *below. The stable limit laws for X_t stated in Theorems 3.0.6 and Theorem 3.0.7 are extensions of corresponding one-dimensional results in [24]. The latter however are obtained in [24] in a more stringent functional form. The law of iterated logarithm given in Theorem 3.0.6 appears to be new even for $d = 1$.*

Remark 3.0.9. *Recall Markov chain $\eta = (\eta_n)_{n\geq 0}$ which records successive directions of the random walk. Let $\mathcal{S}^{d-1}$ denote the d-dimensional unit sphere and let $\mathcal{T}_d$ denote the σ-algebra of the Borel sets of $\mathcal{S}^{d-1}$. Denote by $H(x, A)$ transition kernel of η on $(\mathcal{S}^{d-1}, \mathcal{T}_d)$. We remark that*

(i) All the results stated in this section remain true for an arbitrary (not stationary) initial distribution of the Markov chain η.

(ii) The proofs of our results given in Section 3.1 rest on the exploiting of a regenerative (renewal) structure associated with η, i.e. on the use of random times τ_n which are introduced below in Section 3.1.1. It is then not hard to verify that all the results stated in this section, with the only exception of the generalized law of iterated logarithm given in part (a)-(ii) of Theorem 3.0.6, remain true for a class of regenerative (in the sense of [6]) Markov chains η whose stationary distribution are supported on general Borel subsets of $\mathcal{S}^{d-1}$ rather than on a finite set $U \subset \mathcal{S}^{d-1}$. For instance, the following strong version of the classical Doeblin's conditions is sufficient for our purposes:

- *There exist a constant $c_r > 1$ and a probability measure ψ on $(\mathcal{S}^{d-1}, \mathcal{T}_d)$ such that*

$$c_r^{-1}\psi(A) < H(x, A) < c_r\psi(A) \qquad \forall x \in \mathcal{S},\ A \in \mathcal{T}_d. \tag{3.4}$$

A regenerative (renewal) structure for Markov chains which satisfies Doeblin's condition is described in [6]. Due to the fact that under the assumption (3.4)*, the kernel $H(x, A)$ is dominated uniformly from above and below by a probability measure ψ, such Markov chains share two key features with finite-state Markov chains. Namely, 1) the exponential bound stated in* (3.6) *holds for the renewal times which are defined in [6]; and 2) $c_r^{-1} < P_x(A)/P_y(A) < c_r$ for any non-null event $A \in \mathcal{T}_d$ and almost every states $x, y \in \mathcal{S}^{d-1}$ (with respect to the stationary law). Here P_x stands for the law of the Markov chain η starting from the initial state $x \in \mathcal{S}^{d-1}$. Once these two crucial properties are verified, our proofs (except only the proof of part (a)-(ii) of Theorem 3.0.6) work nearly verbatim* *for directionally reinforced random walks governed by a Markov chain η which satisfies condition* (3.4).

3.1 Proofs

This section is devoted to the proof of the results stated in the beginning of this chapter. Some preliminary observations are stated in Section 3.1.1 below. The proof of Theorem 3.0.2 is contained in Section 3.1.2. Theorems 3.0.4 and 3.0.6 are proved in Section 3.1.3 and Section 3.1.4, respectively. Finally, the proof of Theorem 3.0.7 is given in Section 3.1.5.

3.1.1 Preliminaries

Our approach relies on the use of a renewal structure which is induced on the paths of the random walk by the cycles of the underlying Markov chain η. To define the renewal structure, set $\tau_0 = 0$ and let

$$\tau_{i+1} = \inf\big\{ j > \tau_i : \eta_j = u_1 \big\}, \qquad i \geq 0.$$

Thus, for $i \geq 1$, τ_i are steps when the Markov chain η visits the distinguished state u_1. Correspondingly, s_{τ_i} are successive times when the random walk chooses u_1 as the direction of its motion. Recall N_t (see a few lines preceding (1.3)). Denote by $c(t)$ the number of times that the walker chooses direction u_1 before time $t > 0$. That is,

$$c(t) := \sup\big\{ i \geq 0 : s_{\tau_i} \leq t \big\} = \sum_{j=1}^{N_t} \mathbf{1}_{\{\eta_j = u_1\}},$$

where $\mathbf{1}_A$ stands for the indicator function of an event A. Notice that N_t is the unique mapping from $\mathbb{R}_+$ to $\mathbb{Z}_+$ which has the following property:

$$s_{N_t} \leq t < s_{N_t+1} \qquad \text{and} \qquad \tau_{c(t)} \leq N_t < \tau_{c(t)+1}.$$

For $i \geq 0$, let $\xi_i = \sum_{j=\tau_i+1}^{\tau_{i+1}} T_j \eta_j$. Then

$$X_t = \xi_0 + \sum_{i=1}^{c(t)-1} \xi_i + \sum_{j=\tau_{c(t)}+1}^{N_t} T_j \eta_j + \big(t - s_{N_t} \big) \cdot \eta_{N_t}. \qquad (3.5)$$

The strong Markov property implies that the pairs $\big(\xi_i, \tau_{i+1} - \tau_i\big)_{i\in\mathbb{N}}$ form an i.i.d. sequence which is independent of (ξ_0, τ_1). Furthermore, since η is an irreducible finite-state Markov chain, there exist positive constants $K_1, K_2 > 0$ such that the inequality

$$P(\tau_{i+1} - \tau_i > t) \le K_1 e^{-K_2 t} \tag{3.6}$$

holds uniformly for all reals $t \ge 0$ and all integers $i \ge 0$.

We next list some direct consequences of the law of large numbers that will be frequently exploited in the subsequent proofs. Let $v(n)$ be the number of times that the Markov chain η visits u_1 during its first n steps. Thus, while $c(t)$ is the number of visits of η to u_1 up to time $t > 0$ on the clock of the random walk, $v(n)$ is the number of occurrences of u_1 among first n directions of the random walk. In particular, $v(N_t) = c(t)$. Taking into account (3.6), the law of large numbers and the renewal theorem imply that

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \lim_{n\to\infty} \frac{n}{v(n)} = E(\tau_2 - \tau_1) = \pi_1^{-1}, \qquad \text{a. s.,}$$

and, letting $\Lambda_k := \sum_{i=\tau_k+1}^{\tau_{k+1}} \eta_i$,

$$\mu \;=\; \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \eta_i}{n} = \lim_{n\to\infty} \frac{\sum_{k=0}^{v(n)} \Lambda_k}{n} = \pi_1 \cdot E(\Lambda_1), \qquad \text{a. s.}$$

Since η and $(T_k)_{k\in\mathbb{N}}$ are independent, it follows that

$$E(\xi_1) = E(T_1) \cdot E(\Lambda_1) = \pi_1^{-1} \mu \cdot E(T_1). \tag{3.7}$$

Finally, $\frac{c(t)}{t} = \frac{v(N_t)}{t} = \frac{v(N_t)}{N_t} \cdot \frac{N_t}{t}$ yields

$$\lim_{t\to\infty} \frac{c(t)}{t} = \frac{\pi_1}{E(T_1)}, \qquad \text{a. s.} \tag{3.8}$$

We now turn to the proofs of our main results.

3.1.2 Proof of Theorem 3.0.2

Part (a) of Theorem 3.0.2. Recall (3.6) and observe that the moment condition of the theorem along with the independence of the Markov chain η and $(T_k)_{k\in\mathbb{N}}$ of each other, implies that

$$\begin{aligned} E(\|\xi_1\|^p) \;&\le\; E\big[\big(s_{\tau_2} - s_{\tau_1}\big)^p\big] = \sum_{n=1}^{\infty} P(\tau_2 - \tau_1 = n) \cdot E\Big[\Big(\sum_{k=1}^{n} T_k\Big)^p\Big] \\ &\le\; K_1 \sum_{n=1}^{\infty} e^{-K_2(n-1)} n^p E(T_1^p) < \infty, \end{aligned} \tag{3.9}$$

where we used Minkowski's inequality and (3.6). It follows that $\|\xi_k\| = \mathrm{o}\big(k^{1/p}\big)$. Indeed, for any $\varepsilon > 0$, Chebyshev's inequality implies that

$$\sum_{k=1}^{\infty} P\big(\|\xi_k\| > k^{\frac{1}{p}}\varepsilon\big) = \sum_{k=1}^{\infty} P\big(\|\xi_k\|^p > \varepsilon^p k\big) \le \varepsilon^{-p} E\big(\|\xi_1\|^p\big) < \infty,$$

and hence $P\big(\|\xi_k\| > k^{\frac{1}{p}}\varepsilon \text{ i.o.}\big) = 0$ by the Borel-Cantelli lemma.

For now we will make a simplifying assumption (to be removed later on) that $\mu = 0$. By virtue of (3.8), the Marcinkiewicz-Zigmund law of large numbers implies that

$$\lim_{t\to\infty} \frac{\sum_{i=0}^{c(t)-1} \xi_i}{t^{1/p}} = 0, \qquad \text{a. s.}$$

Furthermore, by (3.5), $\big\|X_t - \sum_{i=0}^{c(t)-1} \xi_i\big\| \le r_{c(t)}$, where

$$r_k := \sum_{i=\tau_k+1}^{\tau_{k+1}} T_i. \tag{3.10}$$

An argument similar to the one which we used to estimate the order of $\|\xi_n\|$, shows that with probability one $r_n = o(n^{1/p})$. Then (3.8) implies that

$$r_{c(t)} = o(t^{1/p}). \tag{3.11}$$

This completes the proof of part (a) of Theorem 3.0.2 for the particular case $\mu = 0$.

We now turn to the general case of arbitrary finite $\mu \in \mathbb{R}^d$. Let

$$\tilde\eta_i = \eta_i - \mu \qquad \text{and} \qquad \widetilde{X}_t = \sum_{i=0}^{N_t} T_i \tilde\eta_i + (t - s_{N_t})\tilde\eta_{N_t}. \tag{3.12}$$

Then $\widetilde{X}_t$ is a directionally reinforced random walk associated with $(T_n)_{n\in\mathbb{N}}$ and $\tilde\eta = (\tilde\eta_n)_{n\in\mathbb{N}}$. Since $E(\tilde\eta_i) = 0$, we have $\big\|\widetilde{X}_t\big\| = o\big(t^{1/p}\big)$. To complete the proof of part (a) of the theorem, observe that $X_t - \widetilde{X}_t = \mu \cdot \sum_{i=1}^{N_t} T_i + \mu \cdot (t - s_{N_t}) = \mu t.$ □

Part (b) of Theorem 3.0.2. Recall (3.7). Let

$$\bar\xi_k := \xi_k - E(\xi_1) = \xi_k - E(T_1)\pi_1^{-1}\mu$$

and

$$\Delta_k := s_{\tau_{k+1}-1} - s_{\tau_k - 1} - E(T_1)\pi_1^{-1}.$$

Let $\gamma_k = (\bar{\xi}_k, \Delta_k) \in \mathbb{R}^{d+1}$. Then $(\gamma_k)_{k\geq 1}$ is an i.i.d. sequence with $E(\gamma_1) = 0 \in \mathbb{R}^{d+1}$. Define

$$\Gamma(t) = \sum_{1\leq k\leq t} \gamma_k.$$

By virtue of Theorem 1 2. 1 in [15], there is a Brownian motion $\big(B(t)\big)_{t\geq 0}$ in $\mathbb{R}^{d+1}$ such that

$$\sup_{0\leq t\leq T} \big\|\Gamma(t) - B(t)\big\| = o\big(T^{1/p}\big).$$

Then Theorem 2. 3. 6 in [15] implies that there exists a Brownian motion $\big(W(t)\big)_{t\geq 0}$, such that

$$\sup_{0\leq t\leq T} \Big\| \sum_{k=0}^{c(t)-1} \xi_k - t\mu - W(t) \Big\| = o\big(T^{1/p}\big).$$

Recall r_k from (3.10). Since

$$\sup_{0\leq t\leq T} \Big\| X_t - \sum_{k=0}^{c(t)-1} \xi_k \Big\| \leq \sup_{0\leq t\leq T} r_{c(t)},$$

it suffices to show that

$$\sup_{0\leq t\leq T} r_{c(t)} = o(T^{1/p}). \tag{3.13}$$

Notice that

$$\sup_{0\leq t\leq T} r_{c(t)} = \sup_{0\leq k\leq c(T)} r_k.$$

Therefore, by virtue of (3.8), it suffices to show that

$$\lim_{n\to\infty} n^{-1/p} \cdot \sup_{0\leq k\leq n} r_k = 0, \qquad \text{a. s.} \tag{3.14}$$

Toward this end, let

$$g(n) = \max\big\{k \leq n : r_k \geq r_i \text{ for all } 1 \leq i \leq n\big\}, \qquad n \in \mathbb{N}.$$

Thus $g(n) \leq n$ and $\sup_{0\leq k\leq n} r_k = r_{g(n)}$. Furthermore, since r_k are i.i.d. random variables, $\lim_{n\to\infty} g(n) = \infty$ with probability one. Therefore, $r_n = o(n^{1/p})$ yields (3.14). The proof of Theorem 3.0.2 is completed. $\square$

3.1.3 Proof of Theorem 3.0.4

Part (a) of Theorem 3.0.4. By (3.9), $E\big(\|\xi_1\|^2)\big) < \infty$ under the conditions of the theorem. Assume first that $\mu = 0$. Then the invariance principle for i.i.d. sequences implies that

$$\frac{\sum_{k=1}^{[nt]} \xi_k}{\sqrt{n}} \Rightarrow W(t), \qquad t \in [0,1],$$

where $W(t)$ is a d-dimensional Brownian motion. It follows then from (3.8) and Theorem 14.4 in [12, p. 152] that

$$\frac{\sum_{k=0}^{c(nt)-1} \xi_k}{\sqrt{n}} \Rightarrow \sqrt{b} \cdot W(t), \qquad t \in [0,1], \tag{3.15}$$

where $b = \frac{\pi_1}{\cdot E(T_1)}$. Under the moment condition of Theorem 3.0.4 we have the following counterpart of (3.13):

$$\sup_{0 \le t \le T} r_{c(t)} = o(T^{1/2}).$$

Since $\big\|X_{nt} - \sum_{k=0}^{c(nt)-1} \xi_k\big\|$ is bounded above by $r_{c(nt)}$, it follows that

$$n^{-1/2} \cdot \Big\|X_{nt} - \sum_{k=0}^{c(nt)-1} \xi_k\Big\| \Rightarrow 0,$$

which implies the desired convergence of $n^{-1/2} \cdot X_{nt}$ when $\mu = 0$. To prove the general case of arbitrary $\mu \in \mathbb{R}^d$ one can apply the result with $\mu = 0$ to the Markov chain $\tilde{\eta}_n$ and the random walk $\tilde{X}_t$ that were introduced in (3.12). The proof of part (a) of the theorem is completed. □

Part (b) of Theorem 3.0.4. Suppose first that $\mu = 0$. For $x \in \text{Span}(U) \subset \mathbb{R}^d$ and $i \in \mathbb{N}$ define

$$\xi_{i,x} := \xi_i \cdot x. \tag{3.16}$$

Then, in view of (3.8), the law of iterated logarithm for i.i.d. sequences implies that there exists a constant $K(x) \in (0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{\sum_{i=0}^{c(t)-1} \xi_{i,x}}{\sqrt{t \ln \ln t}} = K(x), \qquad \text{a. s.}$$

By (3.7) and (3.11)

$$\lim_{t\to\infty} \frac{\big|X_t \cdot x - \sum_{i=0}^{c(t)-1} \xi_{i,x}\big|}{\sqrt{t \ln \ln t}} = 0, \qquad \text{a. s.}$$

Thus

$$\limsup_{t\to\infty} \frac{X_t \cdot x}{\sqrt{t \ln\ln t}} = K(x), \qquad \text{a. s.},$$

in the case $\mu = 0$. To obtain the general case with an arbitrary $\mu \in \mathbb{R}^d$, apply this result to the random walk $\widetilde{X}_t$ defined in (3.12) and recall that $X_t - \widetilde{X}_t = \mu t$. The proof of part (b) of Theorem 3.0.4 is completed. □

3.1.4 Proof of Theorem 3.0.6

Part (a)-(i) and part (b) of Theorem 3.0.6. Let $\overline{\mathbb{R}}_0^d := [-\infty, \infty]^d \backslash \{0\}$, where 0 stands for the zero vector in $\mathbb{R}^d$, and equip $\overline{\mathbb{R}}_0^d$ with the topology inherited from $\mathbb{R}^d$. Recall (see for instance [9, 41]) that a random vector $\xi \in \mathbb{R}^d$ is said to be regularly varying with index $\alpha > 0$ if there exists a function $a : \mathbb{R}_+ \to \mathbb{R}$, regularly varying with index $1/\alpha$, and a Radon measure ν_ξ on $\overline{\mathbb{R}}_0^d$ such that

$$nP\big(a_n^{-1}\xi \in \cdot\big) \overset{v}{\Rightarrow} \nu_\xi(\cdot), \qquad \text{as } n \to \infty, \tag{3.17}$$

where $\overset{v}{\Rightarrow}$ denotes the vague convergence of measures. We will denote by $\mathcal{R}_{d,\alpha,a}$ the set of all random d-vectors regularly varying with index α, associated with a given function $a \in \mathcal{R}_{1/\alpha}$ by (3.17). The measure ν is referred to as the *measure of regular variation* associated with ξ. We will also use the following equivalent definition of the regular variation for random vectors (see, for instance, [9, 41]). Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$ with respect to the norm $\|\cdot\|$. Then $\xi \in \mathcal{R}_{d,\alpha,a}$ if and only if there exists a finite Borel measure $\mathfrak{S}_\xi$ on S^{d-1} such that for all $t > 0$,

$$nP\big(\|\xi\| > ta_n;\, \xi/\|\xi\| \in \cdot\big) \overset{v}{\Rightarrow} t^{-\alpha}\mathfrak{S}_\xi(\cdot), \qquad \text{as } n \to \infty, \tag{3.18}$$

where $\overset{v}{\Rightarrow}$ denotes the vague convergence of measures on S^{d-1}. The following well-known result is the key to the proof of the next lemma: if $\xi, \eta \in \mathcal{R}_{d,\alpha,a}$ and ξ, η are independent of each other, then $\nu_{\xi_1+\eta} = \nu_\xi + \nu_\eta$ and $\mathfrak{S}_{\xi+\eta} = \mathfrak{S}_\xi + \mathfrak{S}_\eta$. We have:

Lemma 3.1.1. *Let Assumption 3.0.5 hold. For $t \geq 0$, let a_t be defined as in (3.3). Then*

(a) $\sum_{\tau_1+1}^{\tau_2} T_i \in \mathcal{R}_{1,\alpha,a}$.

(b) $\xi_1 \in \mathcal{R}_{d,\alpha,a}$.

Proof of Lemma 3.1.1. It is not hard to see that the claim of part (a) can be formally deduced from that of part (b). Thus we will focus on proving the more general claim (b).

First, observe that (3.18) implies that $T_1 u \in \mathcal{R}_{d,\alpha,a}$ for any $u \in U$. Let

$$H(u,v) = P(\eta_{n+1} = v | \eta_n = u), \qquad u, v \in U,$$

be the transition matrix of the Markov chain η. Further, define a sub-Markovian kernel Θ by setting

$$\Theta(u,v) = H(u,v) \cdot \mathbf{1}_{\{v \neq u_1\}}, \qquad u, v \in U.$$

Fix any $t > 0$ and a Borel set $B \subset S^{d-1}$, and let

$$A_n = \big\{ \|\xi_1\| > t a_n;\ \xi_1/\|\xi_1\| \in B \big\}, \qquad n \in \mathbb{N}.$$

Then,

$$P(\xi_1 \in A_n) = \sum_{k=1}^{\infty} P(\tau_2 - \tau_1 = k) P(T_1 u_1 + T_2 \eta_2 + \ldots + T_k \eta_k \in A_n | \tau_2 - \tau_1 = k\big) =$$

$$\sum_{k=1}^{\infty} \sum_{v_2 \neq u_1} \cdots \sum_{v_k \neq u_1} \Theta(u_1, v_1) \cdots \Theta(v_{k-1}, v_k) H(v_k, u_1) P\big(T_1 u_1 + T_2 v_2 + \ldots + T_k v_k \in A_n\big),$$

where we assume that the sums $\sum_{v_2 \neq u_1} \cdots \sum_{v_k \neq u_1}$ are empty if $k = 1$. Let

$$J_n(v_2, \ldots, v_k) = T_1 u_1 + T_2 v_2 + \ldots + T_k v_k.$$

Notice that for any $k \in \mathbb{N}$ and fixed set of vectors $v_2, \ldots, v_k \in U$, we have

$$n \cdot P\big(J_n(v_2, \ldots, v_k) \in A_n\big) \leq n \cdot P\big(\|J_n(v_2, \ldots, v_k)\| \geq t a_n\big) \leq n P\Big(\sum_{j=1}^{k} T_j \geq t a_n\Big)$$
$$\leq nk P\big(T_1 \geq t a_n / k\big) \leq C t^{-\alpha} k^{1+\alpha}$$

for some $C > 0$. Furthermore,

$$\lim_{n \to \infty} n \cdot P\big(J_n(v_2, \ldots, v_k) \in A_n\big) = t^{-\alpha} \Big(\mathfrak{S}_{T_1 u_1}(B) + \sum_{j=2}^{k} \mathfrak{S}_{T_1 v_j}(B) \Big).$$

Observe that the spectral radius of the matrix Θ is strictly less than one and that $\mathfrak{S}_{T_1 v_j}(B)$ is uniformly bounded from above by $\max_{v \in U} \mathfrak{S}_{T_1 v}\big(S^{d-1}\big)$. Therefore, the dominated convergence theorem implies that the following limit exists and the identity holds:

$$\lim_{n \to \infty} n \cdot P(\xi_1 \in A_n)$$
$$= \sum_{k=1}^{\infty} t^{-\alpha} \sum_{v_2 \neq u_1} \cdots \sum_{v_k \neq u_1} \Theta(u_1, v_1) \cdots \Theta(v_{k-1}, v_k) H(v_k, u_1) \Big(\mathfrak{S}_{T_1 u_1}(B) + \sum_{j=2}^{k} \mathfrak{S}_{T_1 v_j}(B) \Big).$$

Since the spectral radius of Θ is strictly less than one, Fubini's theorem implies that the right-hand side of the above identity defines a measure on S^{d-1}. The proof of the lemma is therefore completed. □

We are now in a position to complete the proof of the limit results stated in parts (a) and (b) of Theorem 3.0.6. Suppose first that $\mu = 0$. It follows from Lemma 3.1.1 and the stable limit theorem for i.i.d. sequences (see, for instance, Section 1.6 in [14, p. 75]) that

$$\frac{\sum_{k=1}^{[nt]} \xi_k}{a_n} \Rightarrow S_\alpha(t), \qquad t \in [0,1], \tag{3.19}$$

where $S_\alpha(t)$ is a homogeneous vector-valued process in $D(\mathbb{R}^d)$ with independent increments and $S_\alpha(1)$ distributed according to a stable law of index α. Then (similarly to (3.15)), asymptotic equivalence (3.8) along with the suitable modification of Theorem 14.4 in [12, p. 152] implies

$$\frac{\sum_{k=0}^{c([nt])-1} \xi_k}{a_n} \Rightarrow b^{1/\alpha} \cdot S_\alpha(t), \qquad t \in [0,1],$$

where $b = \frac{\pi_1}{\cdot E(T_1)}$. In particular, using $t = 1$,

$$\frac{\sum_{k=0}^{c(n)-1} \xi_k}{a_n} \Rightarrow b^{1/\alpha} \cdot S_\alpha(1), \tag{3.20}$$

Recall r_k from (3.10). Since

$$\Big\| X_t - \sum_{k=0}^{c(t)-1} \xi_k \Big\| \le r_{c(t)},$$

an application of the renewal theorem shows that

$$\frac{X_n}{n} \Rightarrow \mathcal{L}_\alpha \qquad \text{and hence} \qquad \frac{X_{\lfloor t \rfloor}}{t} \Rightarrow \mathcal{L}_\alpha.$$

Since $\|X_{\lfloor t \rfloor} - X_t\| \le 1$, the proof of part (a)-(i) of Theorem 3.0.6 is completed. □

Part (a)-(ii) of Theorem 3.0.6. For $V \in U$ let $c_v(t)$ be the number of occurrences of v in the set $\{\eta_1, \eta_2, \ldots, \eta_{N_t}\}$. That is,

$$c_v(t) = \sum_{k=1}^{N_t} \mathbf{1}_{\{\eta_k = v\}}, \qquad n \in \mathbb{N},\, i \in \mathcal{D}.$$

Notice that $c_{u_1}(t)=c(t)$, where $c(t)$ is introduced in Section 3.1.1. Similarly to (3.8) we have

$$\lim_{t\to\infty}\frac{c_v(t)}{t}=\frac{\pi_v}{E(T_1)},\qquad \text{a. s.,} \tag{3.21}$$

where π_v is the mass that the stationary distribution of the Markov chain η puts on v.

Define $\tau_v(0)=0$ and $\tau_v(j)=\inf\{k>\tau_v(j-1):\eta_k=v\}$ for $j\in\mathbb{N}$. For $v\in U$ and $t\geq 0$, let

$$\widetilde{B}_v(t)=\sum_{i=0}^{c_v(t)-1}T_{\tau_v(i)}-c_v(t)\cdot E(T_1).$$

Then, the law of iterated logarithm for heavy-tailed i.i.d. sequences (see Theorems 1.6.6 and 3.9.1 in [14]) combined with (3.8) yields

$$\limsup_{t\to\infty}\frac{\widetilde{B}_v(t)}{a_t\cdot(\ln t)^{1/\alpha+\varepsilon}}=\begin{cases}0 & \text{if } \varepsilon>0,\\ \infty & \text{if } \varepsilon<0\end{cases}\qquad \text{a. s.} \tag{3.22}$$

For $v\in U$, let

$$B_v(t)=\sum_{i=0}^{c_v(t)-1}\big(T_{\tau_v(i)}-E(T_1)\big)+(t-s_{N_t})\mathbf{1}_{\{\eta_{N_t}=v\}}.$$

Then, (1.3) implies that

$$X_t=\sum_{v\in U}vB_v(t)+\Big(\sum_{v\in U}v\cdot c_v(t)\cdot E(T_1)-\mu\cdot t\Big).$$

Taking into account (3.21), a standard inversion argument allows one to deduce from the law of iterated logarithm for $\tau_v(n)$ that

$$\limsup_{t\to\infty}\frac{\Big\|\sum_{v\in U}v\cdot c_v(t)\cdot E(T_1)-\mu\cdot t\Big\|}{\sqrt{t\ln\ln t}}<\infty,\qquad \text{a. s.} \tag{3.23}$$

Since $a_t\in\mathcal{R}_\alpha$ with $\alpha\in(1,2)$,

$$\lim_{t\to\infty}\frac{\sqrt{t\ln\ln t}}{a_t\cdot(\ln t)^{1/\alpha+\varepsilon}}=0.$$

Thus (3.22) along with (3.23) yields part (a)-(ii) of Theorem 3.0.6, provided that we are able to show that for any $u,v\in U$ and all $\delta\in(1/(2\alpha),1/\alpha)$,

$$P\Big(\big(G_{n,v}\cap E_{n,v}\big)\text{ and }\big(G_{n,u}\cap E_{n,u}\big)\text{ i. o.}\Big)=0, \tag{3.24}$$

where the events $G_{n,v}$ and $E_{n,v}$ are defined for $n \in \mathbb{N}$ and $v \in U$ as follows. For $n \in \mathbb{N}$ let $\gamma_n = 2n \cdot \max_{v \in V} \pi_v$. Let

$$E_{n,v} = \Big\{ \max_{1 \le m \le \gamma_n} \Big| \sum_{i=0}^{m-1} T_{\tau_v(i)} - m \cdot E(T_1) \Big| > a_n \cdot (\ln n)^\delta \Big\} \quad \text{and} \quad G_{n,v} = \{c_v(n) > \gamma_n\}.$$

We then have:

$$\begin{aligned} &P\Big(\big(G_{n,v} \cap E_{n,v}\big) \text{ and } \big(G_{n,u} \cap E_{n,u}\big) \Big) \\ &\quad \le P\big(E_{n,v} \bigcap E_{n,u}\big) + P\big(c_v(n) > \gamma_n\big) + P\big(c_u(n) > \gamma_n\big) \\ &\quad = P(E_{n,v}) \cdot P(E_{n,u}) + P\big(c_v(n) > \gamma_n\big) + P\big(c_u(n) > \gamma_n\big). \end{aligned}$$

It follows from the large deviation principle for $c_v(n)/n$ that $P\big(c_v(n) > \gamma_n\big) < K_v e^{-n\lambda_v}$ for some $K_v > 0$ and $\lambda_v > 0$. Furthermore, for any $A > 0$ and $k_n = [A^n]$, we have $P(E_{k_n,v}) \le Cn^{-\beta}$ for some constants $\beta > 1/2$ and $C > 0$ (see [14, p. 177]; here we exploit the constraint $2\alpha\delta > 1$). The Borel-Cantelli lemma implies then that $P\big(E_{k_n,v} \bigcap E_{k_n,u} \text{ i.o.}\big) = 0$. Since for any $n \in \mathbb{N}$ there is a unique $j(n) \in \mathbb{N}$ such that $k_{j(n)} \le n < k_{j(n)+1}$, and $\lim_{k\to\infty} \frac{a_{k+1}(\ln a_{k+1})^\delta}{a_k(\ln a_k)^\delta} = 1$, this yields (3.24). The proof of part (a)-(ii) of Theorem 3.0.6 is therefore completed. □

3.1.5 Proof of Theorem 3.0.7

Define two families of processes, $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ in $D(\mathbb{R})$, by setting

$$B_n(t) = \frac{\sum_{k=1}^{[nt]} \xi_k}{a_n} \quad \text{and} \quad C_n(t) = \frac{s_{\tau_{[nt]}}}{a_n}, \qquad t \in [0,1]. \tag{3.25}$$

Lemma 3.1.1 combined with [9, Theorem 1.1] implies that $(\xi_1, s_{\tau_2} - s_{\tau_1}) \in \mathcal{R}_{d+1,\alpha,a}$, and hence

$$(B_n, C_n) \Rightarrow (S_\alpha, U_\alpha), \tag{3.26}$$

where S_α and U_α are homogeneous process with independent increments in $D\big(\mathbb{R}^d\big)$ and $D(\mathbb{R})$, respectively, such that $S_\alpha(1)$ and $U_\alpha(1)$ have (multivariate in the former case) stable distributions of index α.

Let U_n^{-1} and C_n^{-1} denote the inverse processes of U_n and C_n, respectively. One can define C_n^{-1} explicitly as follows:

$$C_n^{-1}(t) = n^{-1} c(a_n t), \qquad t \in [0,1]. \tag{3.27}$$

Then the same argument as in [24, pp. 380-381] shows that (alternatively, one can use the result of [48]):

$$\big(B_n, C_n^{-1}\big) \Rightarrow (S_\alpha, U_\alpha^{-1})$$

in $D\big(\mathbb{R}^{d+1}\big)$. This along with (3.27) implies (see, for instance, [12, p. 151]) that

$$\frac{\sum_{i=1}^{c(a_n)-1} \xi_i}{a_n} \Rightarrow \mathcal{L}_\alpha,$$

where

$$\mathcal{L}_\alpha := S_\alpha(U_\alpha^{-1})(1). \tag{3.28}$$

Passing to the subsequence $m_n = \lfloor a_n^{-1} \rfloor$ and using basic properties of regularly varying functions, we obtain

$$\frac{\sum_{i=0}^{c(n)-1} \xi_i}{n} \Rightarrow \mathcal{L}_\alpha. \tag{3.29}$$

To conclude the proof of the theorem one can use verbatim the argument along the lines following (3.20) in the concluding paragraph of the above proof of part (a)-(i) of Theorem 3.0.6. Namely, taking into account the inequality

$$\Big\|X_t - \sum_{k=0}^{c(t)-1} \xi_k\Big\| \leq r_{c(t)}$$

and using the renewal theorem which ensures the weak convergence of $r_{c(t)}$ to a proper random variable, (3.29) yields that $\frac{X_t}{t} \Rightarrow \mathcal{L}_\alpha$. The proof of Theorem 3.0.7 is completed. □

CHAPTER 4. Conclusions and future directions

In this thesis, we obtained several almost-sure limit results concerning the asymptotic behavior of the sequence ξ^*, the number of visits to a most visited site upto time n, for a transient ERW. There are many other related interesting questions one could investigate. One direction for future work is to extend our results, particularly answering the following problem. Suppose that (X_n) is a recurrent ERW. Find the right scaling functions $\phi_1(n)$ and $\phi_2(n)$ in order to have the following limits hold:

$$\liminf_{n\to\infty} \frac{\xi_n^*}{\phi_1(n)} = C_1 \in (0,\infty) \quad \text{and} \quad \limsup_{n\to\infty} \frac{\xi_n^*}{\phi_2(n)} = C_2 \in (0,\infty),$$

where C_1 and C_2 are constants.

Perhaps, one could address this problem by studying an associated Bessel process built upon the path of ERW on its visits to 0. More precisely, letting $\tau_{0,n}$ be the time of n-th visits to 0, and upon the path of ERW up to time $\tau_{0,n}$, we can construct two branching processes: the first based on the up-crossing (stepping up from site k to site $k+1$, for all $k \geq 0$) on the positive sites and the second based on the down-crossing (stepping down from site k to site $k-1$, for all $k \leq 0$) on the negative sites. The asymptotic properties of the sequence ξ^* can be related to certain properties of these branching processes. Furthermore, one can show that after rescaling these branching processes, we will have two Bessel processes with dependent initial value. Much is known about such processes, and one may be able to translate the properties of their local time back to the properties of the occupation time of ERW.

Another major direction in studying ERW is concerned with its connection to Excited Brownian motions (EBMs). EBMs are defined as solutions to the stochastic differential equation

$$dX_t = dB_t + \phi\big(L_t(X_t)\big)dt$$

for some bounded and measurable function $\phi : \mathbb{R} \to \mathbb{R}$, where $(B_t)_{t\geq 0}$ is a Brownian motion and $L_t(x)$ is the local time process of $(X_t)_{t\geq 0}$, evaluated at point $x \in \mathbb{R}$. Under the assumption, that ϕ is nonnegative continuous and that its integral is strictly larger than one, Norris, Rogers and Williams in [35] showed that the process is transient, and they obtained a law of large numbers under more technical assumptions. Later, Hu and Yor in [25] obtained a central limit theorem under the same assumptions. Very recently, Raimond and Schapira in [37] found the exact recurrence-transience criteria of EBM for the case that ϕ is bounded and measurable. Their results showed an interesting similarity between these criteria and recurrence-transience criteria of ERW. They also gave an explicit form for the speed of EBM, which is unknown in the case of ERW. Soon after that, in [38], they revealed that indeed a natural limiting process for both recurrent and transient ERW is an EBM.

In connection with EBM, one might be able to answer the following problem concerns a central question about ERW: For an ERW in an i.i.d cookie environment, find an explicit formula for the asymptotic speed v in terms of δ and M.

In [49], Zerner gave a formula for the asymptotic speed based on the expectation of difference between consecutive first hitting times. In addition, Basdevant and Singh [7] gave a different implicit formula for the speed, involving the expectation of the limit of a suitable branching process with migration built upon the random walk path. Yet, because of the lack of an estimation on the exit probability of the random walk from a finite box in the former case and the lack of explicit knowledge of the constants involved in the description of the branching process in the latter case, these formulas do not provide any explicit information about the value of v with respect to δ or other parameters of the walk. Since an explicit form is known for the speed of EBM, we believe that an argument based on embedding EBM into ERW will allow us to gain a considerable insight into the question.

Another interesting problem was posed by Noam Berger and Gady Kozma. Suppose we have two ERWs starting at the origin on $\mathbb{Z}$ and moving simultaneously in the same cookie environment. Under what conditions is one walker transient and the other recurrent? Apart from the usual self-interaction of each walker with its own past, there is also a competition between the two walkers. It appears that as a result of the dual interaction, the problem is very challenging. Partial progress on this problem has been made by Noam Berger and Eviatar Procaccia and reported in [11], yet it remains far from being solved. In order to attain some insight into this problem, I propose to study its continuous analog. Namely, suppose that for some bounded and measurable ϕ,

$$dX_t = dB_t^{(1)} + \phi(L_t^X(X_t) + L_t^Y(X_t))dt$$
$$dY_t = dB_t^{(2)} + \phi(L_t^X(Y_t) + L_t^Y(Y_t))dt,$$

where $B_t^{(1)}$ and $B_t^{(2)}$ are two independent Brownian motions and $L_t^X(x)$ and $L_t^Y(x)$ are local times in x at time t for processes X and Y, respectively. Under what conditions is $(X_t, Y_t)_{t\geq 0}$ a well-defined process? And, what can be said about recurrence-transience criteria of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$?

Self-interacting random processes are notoriously difficult to analyze, and even the question on whether the simplest one-reinforced random walk on $\mathbb{Z}^2$ is recurrent or transient is open. ERW and EBM have all the "usual" difficulties of a self-interacting random process: the environment depends on the walk, and in a dynamic way. However, due to the very localized interaction of these processes with their pasts, especially in the case of ERW, researchers have been able to bring many different techniques together, ranging from combinatorial methods (e.g., lace expansion) to methods of stochastic analysis (e.g., the theory of Bessel processes) and probabilistic methods (e.g., a Ray-Knight type theorem, coupling, etc.), to analyze their behavior. A recent discovery of the connection between ERW and EBW [38] opens a new window into understanding both ERW and EBM. Understanding their relationship, and also the additional properties of EBM, will provide more tools, enabling us to further understand the mechanism behind ERW. Likewise, one may be able to use ERW to extend our knowledge regarding EBM and, as a result, other self-interacting diffusion processes.

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