1959

Fixed point theorem for a contraction mapping in a regular developable space

Jerold Chase Mathews
Iowa State University

FIXED POINT THEOREM FOR A CONTRACTION MAPPING IN A REGULAR DEVELOPABLE SPACE

by

Jerold Chase Mathews

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1959

TABLE OF CONTENTS

I. INTRODUCTION

Kolmogorov and Fomin in (4) prove the well-known theorem that in a complete metric space each contraction mapping has one and only one fixed point. This theorem, like several other widely known fixed point theorems, gives rise to techniques which find application in many fields of mathematics. For example, the contraction mapping theorem makes possible brief and elegant proofs of existence and uniqueness theorems in both of the fields differential equations and integral equations. The proof of the contraction mapping theorem itself, as given by Kolmogorov and Fomin, depends heavily upon the assumption that the space on which and into which the mapping is defined be both metric and complete. Roughly, the existence of a metric is needed to characterize the contraction mapping, and the completeness is necessary to obtain the fixed point of the mapping.

The primary purpose of this dissertation is to establish that the concept of a contraction mapping can be realized in certain non-metric spaces in a quite natural manner and, further, to establish a theorem which insures that these contraction mappings possess essentially the same fixed point property as those in a metric space. As a sort of adjunct to the above, it will be established that in the class of spaces to be considered here, a method of completion exists which is similar to the classical method of completing a metric space. The class of spaces for which the results obtained in this dissertation are true is a subclass of the class of all developable topological spaces. The members of this subclass are not, in general, metrizable.

The ensuing discussion divides itself quite naturally into three parts. Chapter II is devoted primarily to proving a fixed point theorem for the generalized contraction mapping. Chapter III consists mainly of a detailed description of a method for completing a developable space. Chapter IV is a collection of examples of developable topological spaces which are not metrizable.

The notation and unstated definitions used in this dissertation are those used in Kelley (3). A few of the notations or conventions which are used frequently are either new or slightly different from the usual and therefore these will be listed below.

1. The set of positive integers is denoted by I^+.
2. The set of non-negative integers is denoted by I.
3. Sequences will always be written in the form $\{x_i\}$ and it will always be understood that the index i ranges over I^+.

Other definitions and notational conventions will be introduced as they become necessary.

II. THE CONTRACTION MAPPING THEOREM

If f is a mapping of a metric space into itself it is easy to formulate conditions on f which make precise the concept of a mapping which tends to contract or to shrink the space. If the metric is denoted by d, then one may require, for example, that for each pair (x,y) of points of the space the inequality $d(f(x),f(y)) < d(x,y)$ is true. A slightly more stringent condition is to require the existence of a real number α , $0 < \alpha < 1$, such that for each pair (x,y) of points of the space the inequality $d(f(x),f(y)) \leq \alpha d(x,y)$ is true. The latter condition is used by Kolmogorov and Fomin in (4) to define what they call a contraction mapping. In case the topological space under consideration is not metrizable it is not immediately clear how the intuitively realizable concept of a contraction mapping can be formulated if, indeed, it can be formulated at all. However, a little reflection will reveal that one can probably dispense with the condition that a metric function exist and require, instead, that the space have a topology which is sufficiently metric-like to insure that the concept of contraction can be characterized.

The class of topological spaces with metric-like topologies occupies a significant position in topology. This is hardly surprising since topology itself emerged from a process of abstracting and generalizing the properties of such familar metric spaces as the real line or higher dimensional Euclidean spaces. In particular, the concern in this dissertation will be with the so-called developable topological spaces. The definition that will be given below was essentially given by Bing in (1). The basic concepts which characterize what are now called developable topological spaces have been given several statements, not all of which, however, are precisely equivalent, and have been known at least since 1919 when Chittenden and Pitcher in (2) used the idea of a developable topological space in connection with an investigation of sufficient conditions for the metrizability of a space.

Definition 1. A topological space (X,T) will be called a developable topological space if and only if there exists a sequence $\{G_i\}$ of open covers for X such that the following two conditions are satisfied: (1) For each i in I^+ it is true that each element of G_{i+1} is a subset of at least one element of G_i. (2) For each p in X and each open set U which contains p it is true that there exists an N in I^+ such that for each n in I^+, $n \geq N$, any member of G_n which contains p is a subset of U. In order to facilitate subsequent notation the following conventions will be adopted: (1) For each j in I^+ let I_j be an index set for G_j and let U_j^α denote the element of G_j corresponding to α in I_j. (2) Let the sequence $\{G_i\}$ be denoted by G and be called a development for (X,T).

If one recalls the definition used by Kolmogorov and

Fomin for a contraction mapping in a metric space, it is easy to verify that a nearly equivalent definition can be given which asserts that a mapping is a contraction mapping if and only if the image of each spherical neighborhood is contained in a spherical neighborhood of smaller radius. It is this condition that will be used in the present context of a developable topological space where the notion of a metric is not available but where the notion of a spherical neighborhood is still present.

<u>Definition 2</u>. Let (X,T) be a developable topological space with a development G; a mapping $f:X\to X$ will be called a (p,G)-contraction mapping if and only if (1) there exists a p in I^+ such that for each k in I^+ and each α_k in I_k it is true that there exists an α_{k+p} in I_{k+p} such that $f(U_k^{\alpha_k}) \subset U_{k+p}^{\alpha_{k+p}}$, and (2) there exist an s in I^+, an α in I_s, and an x_o in X such that x_o and $f(x_o)$ are both in U_s^{α}.

The conditions given in the foregoing definition are sufficient to prove that a (p,G)-contraction mapping is continuous on X.

<u>Lemma 1</u>. Let (X,T) be a developable topological space with a development G and let $f:X\to X$ be a (p,G)-contraction mapping; then f is continuous on X.

Proof: It is sufficient to show that for all H in T it is true that $f^{-1}(H)$ is in T. Thus let H in T be given and let x be in $f^{-1}(H)$. It is clear that there exists an i in

I^+ and an α in I_i such that f(x) is in U_i^α and U_i^α is a subset of H. Since U_i^α is an open set containing f(x) it follows that there exists an N in I^+ such that for each n in I^+, $n \geq N$, it is true that any member of G_n which contains f(x) is a subset of U_i^α. Further, it follows that there exists a β in I_t, where t = max(1,N-p), such that x is in U_t^β. Since f is a (p,G)-contraction mapping it is true that $f(U_t^\beta)$ is a subset of U_i^α. As noted before, U_i^α is a subset of H. Hence f is continuous on X.

The next lemma is a well known result and it together with its proof may be found in most of the texts on either the theory of functions of a real variable or on general topology. Hence only the statement of this result will be given here.

Lemma 2. Let (X,T) be a topological space and $f:X \to X$ a mapping which is continuous on X; then if $\{x_n\}$ is a sequence in X which converges to x, it is true that the sequence $\{f(x_n)\}$ converges to f(x).

These two lemmas will be used in the proof of the main theorem of this section. In order to complete the preliminaries to this theorem, three more definitions will be needed. The second of these definitions gives the structure required of a sequence in order that it have properties resembling those of a Cauchy sequence in a metric space. It will become clear, however, that the quite natural definition given above of a contraction-like mapping leads one, in a space without a metric, to consider what in certain instances are somewhat unusual Cauchy-like sequences.

Definition 3. Let (X,T) be a developable topological space with a development G; a sequence $\{U_i^{\alpha_i}\}$, where α_i is in I_i, will be called a G-chain if and only if for each k in I^+ it is true that $U_k^{\alpha_k} \cap U_{k+1}^{\alpha_{k+1}} \neq \emptyset$.

Definition 4. Let (X,T) be a developable topological space with a development G; a sequence $\{x_i\}$ in X will be called a G-Cauchy sequence in X if and only if there exist an N in I^+, a G-chain $\{U_i^{\alpha_i}\}$, and a subsequence $\{m(i)\}$ of the sequence $\{i\}$ such that for all p in I^+ it is true that x_{N+p} and x_{N+p+1} are both in $U_{m(p)}^{\alpha_{m(p)}}$.

Definition 5. Let (X,T) be a developable topological space with a development G; (X,T) will be called G-complete if and only if every G-Cauchy sequence in X converges.

It is pertinent to note that in the instances where one may choose from several possibilities the development for a particular topological space, the particular choice made can be crucial in determining the characteristics of the class of G-Cauchy sequences. Of course, in general one may not choose the development - it is just given. The following example may serve to illustrate these considerations.

Example 1. Let (R,T) be the topological space consisting of the set R of all real numbers and the usual topology T. Let

$S(x,r)$ denote the spherical neighborhood of radius r with center at x and for each j in I^+ let G_j denote the set $\{S(x,1/j) : x \text{ in } R\}$. It is clear that the sequence $\{G_j\}$ is a development for (R,T). Now consider the sequence $\{x_n\}$, where for each n in I^+ $x_n = 1 + 1/2 + 1/3 + \cdot \cdot \cdot + 1/n$; it is easily verified that this sequence is a G-Cauchy sequence in R and, further, manifestly does not converge. If, on the other hand, G_j had been defined as the set $\{S(x,1/2^j) : x \text{ in } R\}$, the same sequence is now not a G-Cauchy sequence.

Before leaving these matters and proceeding to the main theorem of this section, several other comments seem pertinent. The definition of a Cauchy-like sequence given in Definition 4 is clearly designed with the characteristics of a (p,G)-contraction mapping in mind. It will be seen, however, in Chapter IV that the definitions given in this dissertation are essentially comparable to the usual definitions when the space under consideration is a metric space. Finally, it will be seen shortly that the property of being G-complete is an essential property for the topological space under consideration. Hence it would be desirable to prove that at least some developable topological spaces can be homeomorphically embedded in a G-complete space. This problem is considered at length in Chapter III.

Theorem 1. Let (X,T) be a developable topological space with a development G and assume that (X,T) is G-complete; then if $f:X\to X$ is a (p,G)-contraction mapping it is true that f has a fixed point; that is, there exists an x^* in X such that $f(x^*) = x^*$.

Proof: Since f is a (p,G)-contraction mapping there exist an s in I^+, an α_s in I_s, and an x_o in X such that both x_o and $f(x_o)$ are in $U_s^{\alpha_s}$. It will be shown that the sequence $\{f^{(j-1)}(x_o)\}$, where $f^{(o)}(x_o) = x_o$, $f^{(1)}(x_o) = f(x_o)$, $f^{(2)}(x_o) = f(f(x_o))$, etc., is a G-Cauchy sequence. The following statement is clearly true: For each m in I it is true that $f^{(m)}(x_o)$ and $f^{(m+1)}(x_o)$ are both in $U_{s+mp}^{\alpha_{s+mp}}$, and, further, $f(U_{s+mp}^{\alpha_{s+mp}}) \subset U_{s+(m+1)p}^{\alpha_{s+(m+1)p}}$. It follows directly that $\{f^{(j-1)}(x_o)\}$ is a G-Cauchy sequence. Since (X,T) is G-complete there exists an x^* in X such that $\{f^{(j-1)}(x_o)\}$ converges to x^*. In order to complete the proof it must be shown that $f(x^*) = x^*$. Using Lemmas 1 and 2 the following statements are clear:

$$\lim_n f(f^{(n)}(x_o)) = f(\lim_n f^{(n)}(x_o)) = f(x^*), \text{ and}$$

$$\lim_n f(f^{(n)}(x_o)) = \lim_n f^{(n+1)}(x_o) = x^*.$$

Hence, $f(x^*) = x^*$.

The next theorem will establish that under certain more restrictive conditions the set of all fixed points for a (p,G)-contraction mapping is "widely dispersed" over X; that is, the fixed point guaranteed by Theorem 1 is locally unique.

Theorem 2. Let (X,T) be a developable topological space with a development G and assume that (X,T) is G-complete. If $f:X\rightarrow X$ is a (p,G)-contraction mapping and (X,T) is a T_o-space, then for each k in I^+ and α in I_k it is true that the set of fixed points for f has at most one point in common with U_k^α.

Proof: Assume there are two distinct fixed points x_1 and x_2 of f and suppose, further, that there exist a k in I^+ and an α in I_k such that both x_1 and x_2 are in U_k^α. Since f is a (p,G)-contraction mapping and (X,T) is a T_o-space, the contradiction is immediately clear.

Corollary. If in addition to the hypotheses of Theorem 2 it is assumed that G_1 consists of one and only one element (which, then, must be X), then f has one and only one fixed point.

Proof: Direct application of Theorem 2.

It is now clear that Kolmogorov and Fomin's fixed point theorem is susceptible of generalization to a class of spaces which, in general, are not metrizable provided that these spaces possess the completeness property. In the next chapter it will be established that a certain class of developable topological spaces can be completed.

III. THE COMPLETION OF A DEVELOPABLE SPACE

If one recalls from the preceding section the definitions of a G-Cauchy sequence and a G-chain, it is evident that these two concepts are quite closely related; that is, in each G-chain there can be found a G-Cauchy sequence and with each G-Cauchy sequence there can be associated a containing G-chain. Thus it is seen that one may impose conditions on, say, the class of all G-chains and then see these conditions reflected in the behavior of the class of all G-Cauchy sequences. The device of imposing conditions on the most basic structures available rather than on the structures of most immediate interest is not an uncommon means to attain a desired result and it will, in fact, be employed shortly. In the present case it seems that placing conditions on the open sets constituting the development is more desirable than stipulating conditions to be satisfied by the class of all G-Cauchy sequences.

In choosing conditions on the development for the space which, if satisfied, would insure that the space have a completion (the precise meaning of the phrase "have a completion" is given in Definition 8), the following considerations seem to offer some direction: (1) In a metric space each Cauchy sequence is bounded. (2) If one refers back to Example 1, pages 7 and 8, it is suggestive that of the two sets of radii used to construct the two developments, the set which made possible an unbounded Cauchy-like sequence was associated with a divergent infinite series while, on the other hand, the set which made this same sequence no longer Cauchy-like was associated with a convergent infinite series.

(3) In a metric space, a condition equivalent to the more usual condition for completeness is that the intersection of every nested sequence of closed spheres whose radii shrink to zero consists of exactly one point. With these considerations in mind one is led to the following definitions.

Definition 6. Let (X,T) be a developable topological space with a development G; a G-chain $\{U_i^{\alpha_i}\}$ will be called convergent if and only if there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$, where $\{m(i)\}$ is a subsequence of the sequence $\{i\}$ and $\beta_{m(i)}$ is in $I_{m(i)}$, such that (1) for each j in I^+ it is true that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that $U_k^{\alpha_k} \subset U_{m(j)}^{\beta_{m(j)}}$, and (2) for each i in I^+ it is true that $\overline{U}_{m(i+1)}^{\beta_{m(i+1)}} \subset U_{m(i)}^{\beta_{m(i)}}$. It will be understood in the remainder of this dissertation that the notations m(i) or n(i), when used to index a sequence, denote the entries of a subsequence of the sequence $\{i\}$ of positive integers. Further, it will always be understood that a symbol of the form U_j^{α} denotes the member of G_j corresponding to α in I_j.

Definition 7. Let (X,T) be a developable topological space with a development G; G will be called a convergent development if and only if the following conditions are satisfied: (1) Every G-chain is convergent. (2) For each nested sequence $\{U^{\alpha m(i)}_{m(i)}\}$ it is true that the set $\bigcap\{\overline{U}^{\alpha m(i)}_{m(i)} : i \text{ in } I^+\}$ has at most one member.

Definition 8. If (X,T) is a developable topological space with a convergent development it will be said that (X,T) has a completion if and only if (X,T) can be homeomorphically embedded in a developable topological space (W,U) with a convergent development H where (W,U) is H-complete, in such a fashion that the image in (W,U) of (X,T) is everywhere dense.

The principal result to be established in this section is that each developable topological space with a convergent development has a completion. Instead of proceeding with this directly, however, several interesting and illuminating results will be established which arise from the assumption of a convergent development for a space. Some of these results will find application in the later parts of this section.

Lemma 3. Let (X,T) be a developable topological space with a convergent development G; then (X,T) is a T_1-space.

Proof: In order to establish that (X,T) is a T_1-space it is sufficient to show that for each x in X it is true that the singleton set $\{x\}$ is closed. Thus let x in X be given; it is immediately clear from Definition 1 that there exists a nested sequence $\{U_{n(i)}^{\alpha_{n(i)}}\}$ such that for each i in I^+ it is true that x is in $U_{n(i)}^{\alpha_{n(i)}}$. Using Definition 7 it follows that $\{x\} = \bigcap \{\overline{U}_{n(i)}^{\alpha_{n(i)}} : i \text{ in } I^+\}$.

Lemma 4. Let (X,T) be a developable topological space with a convergent development G; then (X,T) is regular.

Proof: In order to establish that (X,T) is regular it is sufficient to show that for each element p in X and each open set V containing p it is true that there exists an open set W containing p such that $\overline{W} \subset V$. Thus let p in X and V in T, p in V, be given; then, using Definition 1, it follows that there exists a G-chain $\{U_i^{\alpha_i}\}$ such that each entry of this sequence contains p. From Definition 6 it follows that there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ such that for each i in I^+ it is true that p is in $U_{m(i)}^{\beta_{m(i)}}$ and $\overline{U}_{m(i+1)}^{\beta_{m(i+1)}} \subset U_{m(i)}^{\beta_{m(i)}}$. Using Definition 1 again, it follows that there exists a k in I^+ such that for each j in I^+, $j \geq k$, it is true that $U_{m(j)}^{\beta_{m(j)}} \subset V$. It now follows that $U_{m(k+1)}^{\beta_{m(k+1)}}$ contains p, and $\overline{U}_{m(k+1)}^{\beta_{m(k+1)}} \subset V$.

It is well known that a topological space which is both regular and T_1 is a Hausdorff space. Hence a developable topological space with a convergent development is a Hausdorff space since by Lemmas 3 and 4 it is both regular and T_1.

Lemma 5. Let (X,T) be a developable topological space with a convergent development G; then if $\{x_i\}$ is a G-Cauchy sequence in X it is true that this sequence has at most one limit point.

Proof: Let $\{x_i\}$ be a G-Cauchy sequence in X; then, since G is a convergent development, there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ such that for each i in I^+ it is true that the sequence $\{x_j\}$ is eventually in $U_{m(i)}^{\beta_{m(i)}}$. Now suppose a and b are in X, $a \neq b$, and are limit points of $\{x_i\}$. It is evident that for each i in I^+ it is true that both a and b are in $\overline{U}_{m(i)}^{\beta_{m(i)}}$. However, by virtue of condition (2) of Definition 7, this leads directly to a contradiction.

Lemma 6. Let (X,T) be a developable topological space with a convergent development G; then if $\{x_i\}$ is a convergent sequence in X it is true that there exists a subsequence of $\{x_i\}$ such that this subsequence is a G-Cauchy sequence in X.

Proof: Let $\{x_i\}$ be a convergent sequence in X and suppose it converges to p; then there exist an α_1 in I_1 and an $N(1)$ in I^+ such that p is in $U_1^{\alpha_1}$ and for each k in I^+, $k \geqslant N(1)$, it is true that x_k is in $U_1^{\alpha_1}$. Next, there exist an $n(1)$ in I^+, $n(1)\ \ 1$, an $\alpha_{n(1)}$ in $I_{n(1)}$, and an $N(2)$ in I^+, $N(2) > N(1)$, such that p is in $U_{n(1)}^{\alpha_{n(1)}}$, $U_{n(1)}^{\alpha_{n(1)}} \subset U_1^{\alpha_1}$, and for each k in I^+, $k \geqslant N(2)$, it is true that x_k is in $U_{n(1)}^{\alpha_{n(1)}}$. This process may be continued indefinitely, and it is easily verified that the sequence $\{x_{N(i)}\}$ thus formed is a G-Cauchy sequence.

Lemma 7. Let (X,T) be a developable topological space with a convergent development G; then if $\{x_i\}$ is a G-Cauchy sequence and x is a limit point for this sequence it is true that $\{x_i\}$ converges to x.

Proof: Let $\{x_i\}$ be a G-Cauchy sequence in X and let x be a limit point of $\{x_i\}$; then, by virtue of Definition 4, there exists a G-chain $\{U_i^{\alpha_i}\}$ such that for each n in I^+ it is true that $\{x_i\}$ is eventually in $\bigcup\{U_k^{\alpha_k} : k \text{ in } I^+, k \geq n\}$. Next, since G is a convergent development, it follows that there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ such that for each i in I^+ it is true that the entries of the G-chain $\{U_j^{\alpha_j}\}$ are eventually subsets of $U_{m(i)}^{\beta_{m(i)}}$ and $\overline{U}_{m(i+1)}^{\beta_{m(i+1)}} \subset U_{m(i)}^{\beta_{m(i)}}$. Thus it is true that for each i in I^+ the sequence $\{x_i\}$ is eventually in $U_{m(i)}^{\beta_{m(i)}}$ and, further, x is in $U_{m(i)}^{\beta_{m(i)}}$. It now follows that $\{x_i\}$ converges to x.

The foregoing lemmas establish fairly clearly that there is some degree of similarity between the G-Cauchy sequences in a developable topological space with a convergent development G and the Cauchy sequences in a metric space. Of course, this is not unexpected in view of the structures common to both kinds of spaces. It hardly needs to be pointed out that an arbitrary metric space may be regarded as a developable topological space with a convergent development. Attention will now be directed to the problem of finding a completion for an arbitrarily given developable topological space with a convergent development. The method that will be used here is similar to the classical method for completing a metric space. Thus the initial step is to define an equivalence relation on the class of all G-Cauchy sequences.

Definition 9. Let (X,T) be a developable topological space with a convergent development G. For each pair $(\{x_i\}, \{y_i\})$ of G-Cauchy sequences in X, $\{x_i\} \backsim \{y_i\}$ if and only if for each n in I^+ it is true that there exist an α in I_n and an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that x_k and y_k are both in U_n^α. In order to shorten the notation the following convention will be adopted: If $\{x_i\}$ is a sequence, then this sequence will frequently be denoted by x when no confusion seems possible.

Before establishing that the binary relation defined in Definition 9 is an equivalence relation for the class of all G-Cauchy sequences, let it be agreed that in the remainder of this section the topological space (X,T) is tacitly understood to be given and that it will always be developable with a convergent development G. Further, any structures, or notation designating these structures, will be used, once having been introduced, consistently without further explicit mention in the remainder of this section.

Lemma 8. The binary relation $\backsim$ defined in Definition 9 is an equivalence relation for the class S of all G-Cauchy sequences in X.

Proof: In order to establish that $\backsim$ is an equivalence relation for S it is sufficient to show that $\backsim$ has the following three properties: (1) For each x in S it is true that $x \backsim x$. (2) For each pair (x,y) of elements of S such that $x \backsim y$, it is true that $y \backsim x$. (3) For each triplet (x,y,z) of elements of S such that $x \backsim y$ and $y \backsim z$, it is true that $x \backsim z$. The properties (1) and (2) given above are immediate consequences of Definitions 4 and 9 together with the fact that G is a convergent development. In order to verify the property (3) let $\{x_i\}$, $\{y_i\}$, and $\{z_i\}$ be in S and assume that $x \backsim y$ and $y \backsim z$. Since $x \backsim y$ it is true that for each j in I^+ there exist an α_{2j-1} in I_{2j-1} and an N_{2j-1} in I^+ such that for each k in I^+, $k \geqslant N_{2j-1}$, it is true that x_k and y_k are in $U_{2j-1}^{\alpha_{2j-1}}$. Further, since $y \backsim z$ it is true that for each j in I^+ there exist an α_{2j} in I_{2j} and an N_{2j} in I^+ such that for each k in I^+, $k \geqslant N_{2j}$, it is true that y_k and z_k are in $U_{2j}^{\alpha_{2j}}$. It is clear that the sequence $\{U_k^{\alpha_k}\}$ is a G-chain. Since G is a convergent development there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ such that for each j in I^+ it is true that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that $U_k^{\alpha_k} \subset U_{m(j)}^{\beta_{m(j)}}$. Now, using the properties of the development G, the sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ can be considered as a subsequence of a sequence $\{U_i^{\gamma_i}\}$ formed as follows: For each i in I^+ such that there exists a j in I^+ such that $i = m(j)$, define $U_i^{\gamma_i}$ to be $U_{m(j)}^{\beta_{m(j)}}$. For each j in I^+ define the set R_j to be $\{i : i \text{ is in } I^+, m(j) < i < m(j+1)\}$; then for each k in R_j it is true that there exists a γ_k in I_k such that

$$U_{m(j+1)}^{\beta_{m(j+1)}} \subset U_{m(j+1)+1}^{\gamma_{m(j+1)+1}} \subset \ldots \subset U_{m(j)-1}^{\gamma_{m(j)-1}} .$$

It is now evident that $x \backsim z$.

Since $\backsim$ is an equivalence relation for S, it follows that S may be written as the union of a collection of disjoint equivalence classes of S. Using the axiom of choice it is possible to form a set W of G-Cauchy sequences where one and only one element is taken from each equivalence class of S. This procedure is given more formally in the next paragraph.

The equivalence relation $\backsim$ induces a partitioning of S; namely, there exists a collection H of non-null subsets of S such that (1) $\bigcup\{A : A \text{ is in } H\} = S$, (2) for each pair (A,B) of elements of H it is true either A = B or A and B are disjoint, and (3) for each A in H and each pair (x,y) of elements of A it is true that $x \backsim y$. The axiom of choice asserts that there exists a mapping $\Phi : H \rightarrow S$ such that Φ is a biunique mapping and for each A in H it is true that $\Phi(A)$ is in A. Now let W be defined to be the set $\Phi(H)$. The set W with a suitable topology will eventually be seen to be a completion of (X,T) in the sense of Definition 8. The next concern, then, will be to define a topology for W in such a manner that the resulting space has the desired structure.

Definition 10. For each x in S the symbol C_x will be used to denote the set $\{L$: L is a nested sequence of the form $\{U_{m(i)}^{\alpha m(i)}\}$ and for each j in I^+ it is true that x is eventually in $U_{m(j)}^{\alpha m(j)}\}$.

Definition 11. Let M denote the collection of all subsets of W. The mapping Ψ:T→M is defined as follows: for each B in T it is true that x is in Ψ(B) if and only if x is in W and for each C in C_x it is true that the entries of C are eventually subsets of B. The set Ψ(T) will be denoted by U and for each B in T, the set Ψ(B) will be denoted by $\hat{B}$.

An important property of the mapping Ψ is given in the following lemma; namely, Ψ is biunique.

Lemma 9. The mapping Ψ:T→U defined in Definition 11 is biunique.

Proof: In order to establish that Ψ:T→U is biunique it is sufficient to show that for each B in U it is true that if A and A' are in T, and $\Psi(A) = \Psi(A') = B$, then A = A'. Thus let B in U be given and suppose A and A' are in T and $\Psi(A) = \Psi(A') = B$. Let p be in A and consider the element y in W which, when considered as a G-Cauchy sequence in X, is equivalent to the G-Cauchy sequence $\{x_i\}$, where for each i in I^+ it is true that $x_i = p$. It is clear that y must eventually be in both A and A' and, consequently, p must be in A'. Hence $A \subset A'$. Similarly, $A' \subset A$. Hence A = A'.

Lemma 10. The collection U is a topology for W.

Proof: It is clear that $\bigcup\{A : A \text{ is in } U\} = W$ since W is in U. In addition, in order to establish that U is a topology for W it must be shown that the union of each subfamily of U is an element of U and the intersection of each finite subfamily of U is a member of U. These conditions are obviously satisfied by virtue of Lemma 9 and the fact that T is a topology for X.

The necessary definitions and lemmas have now been given in order to state and prove two important theorems which establish, respectively, that the topological space (W,U) is a developable topological space and that, in fact, it possesses a convergent development.

Theorem 3. (W,U) is a developable topological space.

Proof: The proof consists of an explicit construction of a development $\hat{G}$ for (W,U). For each i in I^+ let $\hat{G}_i = \{\Psi(U_i^\alpha) : \alpha \text{ is in } I_i\}$. It is evident that the sequence $\{\hat{G}_i\}$ is a development for (W,U). This follows directly from the structure of the topology U and its relationship to the topology T through the mapping Ψ.

The construction of the development $\hat{G}$ for (W,U) in the foregoing theorem makes it clear that for each i in I^+ the set $\hat{G}_i$ is indexed by I_i. For the sake of a consistent notation, when reference is made to the development $\hat{G}$, for each i in I^+ the set I_i will be denoted by $\hat{I}_i$.

Theorem 4. The development $\hat{G}$ for (W,U) is a convergent development.

Proof: It must be established that conditions (1) and (2) of Definition 7 are satisfied. Thus, let $\{\hat{U}_i^{\alpha_i}\}$ be a $\hat{G}$-chain and consider the G-chain $\{U_i^{\alpha_i}\}$, where, as was notationally agreed previously, $U_i^{\alpha_i} = \Psi^{-1}(\hat{U}_i^{\alpha_i})$. Since G is a convergent development for (X,T) it is true that there exists a nested sequence $\{U_{m(i)}^{\beta_{m(i)}}\}$ such that (1) for each j in I^+ it is true that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that $U_k^{\alpha_k} \subset U_{m(j)}^{\beta_{m(j)}}$, and (2) for each i in I^+ it is true that $\overline{U}_{m(i+1)}^{\beta_{m(i+1)}} \subset U_{m(i)}^{\beta_{m(i)}}$. Now consider the sequence $\{\hat{U}_{m(i)}^{\beta_{m(i)}}\}$. It is clear that this sequence is a nested sequence and, further, it follows that for each j in I^+ it is true that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that $\hat{U}_k^{\alpha_k} \subset \hat{U}_{m(j)}^{\beta_{m(j)}}$. Finally, it follows from the fact that Ψ is biunique that for each i in I^+ it is true that $\overline{\hat{U}}_{m(i+1)}^{\beta_{m(i+1)}} \subset \hat{U}_{m(i)}^{\beta_{m(i)}}$. Hence, condition (1) of Definition 7 is satisfied. Next, let $\{\hat{U}_{m(i)}^{\alpha_{m(i)}}\}$ be a nested sequence. If condition (2) of Definition 7 is to be satisfied then it must be true that the set $\bigcap\{\overline{\hat{U}}_{m(i)}^{\alpha_{m(i)}} : i \text{ in } I^+\}$ have at most one element. Thus, let it be assumed that there exist an x and a y in W, $x \neq y$, such that for each i in I^+ it is true that x and y are members of $\overline{\hat{U}}_{m(i)}^{\alpha_{m(i)}}$. Now, by an easy construction, it can be shown that there exists a $\hat{G}$-chain $\{\hat{U}_i^{\beta_i}\}$ such that for each j in I^+ it is true that $\hat{U}_{m(j)}^{\beta_{m(j)}} = \hat{U}_{m(j)}^{\alpha_{m(j)}}$. Since condition (1) of Definition 7 has already been established, it follows that there exists a nested sequence $\{\hat{U}^{\gamma_{n(i)}}_{n(i)}\}$ such that (1) for each j in I^+ it is true that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that $\hat{U}^{\beta_k}_k \subset \hat{U}^{\gamma_{n(j)}}_{n(j)}$, and (2) for each j in I^+ it is true that $\overline{\hat{U}}^{\gamma_{n(j+1)}}_{n(j+1)} \subset U^{\gamma_{n(j)}}_{n(j)}$. It follows easily that for each j in I^+, x and y are members of $\hat{U}^{\gamma_{n(j)}}_{n(j)}$. Now consider the nested sequence $\{\Psi^{-1}(\hat{U}^{\gamma_{n(j)}}_{n(j)})\}$; for each j in I^+ it is true that the sequences x and y are eventually in $U^{\gamma_{n(j)}}_{n(j)}$. Hence, by virtue of Definition 9 it follows that $x \backsim y$. But this is impossible. Thus condition (2) of Definition 7 is satisfied.

The next theorem will establish that (W,U) is $\hat{G}$-complete. The proof of this result is somewhat lengthy and, consequently, will be divided between a lemma and the main theorem. One may again note that here, as in the foregoing, the sequence of results is quite analogous to that of the classical method of completing a metric space.

Lemma 11. The set $K = \{x : x$ is in W, there exists a p in X such that x converges to $p\}$ is everywhere dense in (W,U).

Proof: In order to establish that K is everywhere dense in (W,U) it is sufficient to show that $\overline{K} = W$; thus, if it can be shown that every member of W is either a member of K or a point of accumulation of K the proof will be complete. Let it be assumed that there exists an r in W such that r is not in K and r is not a point of accumulation of K. It is clear that there exists an $\hat{A}$ in U such that r is in $\hat{A}$ and $\hat{A}$

and K are disjoint. The set $A = \Psi^{-1}(\hat{A})$ is non-null and, hence, there exists a q in X such that q is in A. Now, there exists a y in W such that the sequence y converges to q. It is true that $y \neq r$ since if $y = r$ then r would converge to q and, hence, r would be in K. Since y is in $\hat{A}$ and K, the proof follows immediately.

<u>Theorem 5</u>. The developable topological space (W,U) with a convergent development $\hat{G}$ is $\hat{G}$-complete.

Proof: In order to establish that (W,U) is $\hat{G}$-complete it is sufficient to show that every $\hat{G}$-Cauchy sequence in W has a limit point since, by virtue of Lemma 7, if a $\hat{G}$-Cauchy sequence has a limit point it converges to this limit point. It will be shown first that any $\hat{G}$-Cauchy sequence in K converges. Thus, let $\{x_i\}$ be a $\hat{G}$-Cauchy sequence in K; since for each i in I^+ it is true that x_i is in K, there exists a p_i in X such that the sequence x_i in X converges to p_i. It is easily seen that the sequence $\{p_i\}$ is a G-Cauchy sequence in X. It will be shown that the sequence $\{x_i\}$ in K converges to a point q in W which, considered as a sequence in X, is equivalent to p. Thus, in S there exists one G-Cauchy sequence, let it be denoted by q, which is equivalent to p and which is in W. Let $\hat{A}$ be any U-open set containing q. It must be established that there exists an N in I^+ such that for each k in I^+, $k \geqslant N$, it is true that x_k is in $\hat{A}$. Since $p \backsim q$ it follows that there exists a nested sequence $\{U_{n(i)}^{\beta_{n(i)}}\}$

such that for each j in I^+, both p and q are eventually in $U_{n(j)}^{\beta n(j)}$. Since this nested sequence is an element of C_p, it follows that the entries of the sequence p are eventually in A. Hence, there exists an N in I^+ such that for each k in I^+, $k \geq N$, it is true that p_k is in A; thus it follows that x_k is in A. Hence, the $\hat{G}$-Cauchy sequence $\{x_i\}$ in K converges. The next step in the proof of this theorem is to establish that an arbitrary $\hat{G}$-Cauchy sequence in W converges. Thus, let $\{x_i\}$ be a $\hat{G}$-Cauchy sequence in W; by virtue of Definition 4 there exist an N in I^+, a $\hat{G}$-chain $\{\hat{U}_i^{\alpha i}\}$, and a subsequence $\{m(i)\}$ of $\{i\}$ such that for each p in I^+ it is true that x_{N+p} and x_{N+p+1} are in $\hat{U}_{m(p)}^{\alpha m(p)}$. Since K is everywhere dense in W it is true that there exists a sequence $\{c_i\}$ in K such that for each j in I^+ it is true that c_j is in $\hat{U}_{m(j)}^{\alpha m(j)}$. It is clear that the sequence $\{c_i\}$ is a $\hat{G}$-Cauchy sequence in K and, hence, by the first part of the proof of this theorem it converges to some point k in W. It follows directly that k is a limit point of the $\hat{G}$-Cauchy sequence $\{x_i\}$ in W and, thus, this sequence converges to k.

The objective of the entire sequence of theorems in this section has been, of course, to establish that (X,T) has a completion. This objective is realized in the next theorem.

Theorem 6. The developable topological space (X,T) with a convergent development G has a completion.

Proof: If the criteria for (X,T) to have a completion, as given in Definition 8, are to be satisfied, it will be sufficient, on the basis of the results afforded by Theorems 4 and 5 and Lemma 11, to establish that K with the relative topology is a homeomorphic image of (X,T). Thus, let the mapping $\Gamma: X \to K$ be defined as follows: if x is in X then $\Gamma(x)$ is defined to be that element of W which is equivalent to the G-Cauchy sequence $\{x_i\}$ in X, where for each i in I^+ it is true that $x_i = x$. It follows readily from the fact that (X,T) is a T_1-space that Γ is biunique. In order to establish that Γ is a homeomorphism it is sufficient to show that Γ is a continuous and an open mapping. This follows from the definition of the topology U for W.

IV. EXAMPLES AND REMARKS

This section will be concerned with the presentation of a number of examples in order to establish that the results obtained in this dissertation are valid on a non-null class of non-metrizable topological spaces. Further, the relations existing between the structures introduced in the context of a developable space and those in a metric space, will be briefly examined. There are several possible instances in Chapters II and III where one could raise the question of whether or not one might relax conditions and thus gain generality; for the most part, these questions are not considered in this dissertation.

One example has already been given, namely, Example 1, which shows that if one is given a topological space whose structure admits of several distinct developments, then it is not a matter of indifference as to which development one chooses if certain properties are to obtain. This example also shows that there do exist developable topological spaces which possess convergent developments. It does not show, however, that there exist non-metrizable spaces which possess developments or, more to the point, convergent developments. Examples 2, 3, and 4 will establish that such spaces do exist.

Example 2. Let R denote the set of all real numbers, X the set $\{(x,y) : x \text{ and } y \text{ are in } R\}$, and J the set $\{(x,y) : x \text{ and } y \text{ are in } R,\ y = 0\}$. Let J have imposed on it the usual topology U; then for each H in U define H^* to be the set $\{(x,y) : (x,y)$ is in X, x is in H$\}$. Let the collection $\{H^* :$ H is in U$\}$ be denoted by T and note that T is a topology for X. The topological space (X,T) is not metrizable since it is not a Hausdorff space. However, (X,T) is developable since if G is any development for (J,U), a development G^* for (X,T) is obtainable from G in the same manner as T was obtained from U. If G is a convergent development it is not true, however, that G^* is a convergent development since, although G^* possesses some of the properties of a convergent development, the intersection of a nested sequence contains more than one point. Finally, the space (X,T) is regular and normal.

The next example will be that of a non-metrizable Hausdorff space possessing a development but not, however, possessing a convergent development.

Example 3. Let R denote the set of all real numbers, R^+ the set of all positive real numbers, X the set $\{(x,y) :$ x and y are in R, $y \geq 0\}$, and J the set $\{(x,y) :$ x and y are in R, $y = 0\}$. Let ρ denote the usual metric for Euclidean two-space, restricted to X. A construction procedure will now be given which will form a family B of subsets of X: (1) For each p in X-J and r in R^+ let $B_r(p)$ denote the set $\{q :$ q is in X, $\rho(p,q) < r\}$. (2) For each p in J and r in R^+ let $B_r(p)$ denote the set $\{q :$ q is in X-J, $\rho(p,q) < r\}$

$\cup\{p\}$. The family $B = \{B_r(p) : p$ is in X and r is in $R^+\}$ is a basis for some topology for X. In order to establish this assertion it is sufficient to note that for each pair (S,T) of non-disjoint members of B and for each p in their intersection it is true that there exists a V in B containing p such that $V \subset S \cap T$. If T denotes the family of all unions of subfamilies of B, it follows that T is a topology for X and B is a basis for T. The pair (X,T) is clearly a Hausdorff space. It is not metrizable since it is not regular. In order to establish this last assertion, consider the point p = (0,0) and the set $J - \{p\}$, which is closed. Manifestly, one cannot find disjoint open sets P and Q such that p is in P and $J - \{p\} \subset Q$. It follows from the fact that (X,T) is not regular, that (X,T) does not possess a convergent development. The space (X,T) does, however, possess a development. Thus, for each i in I^+, let G_i be union of the collections E and F, where

$$E = \{B_{2^{-i}}(p) : p \text{ is in } X\text{-}J \text{ and } B_{2^{-i}}(p) \cap J = \emptyset\}, \text{ and}$$

$$F = \{B_{2^{-i}}(p) : p \text{ is in } J\}.$$

It is not difficult to verify that the sequence $\{G_i\}$ is a development for (X,T).

The next example will be that of a non-metrizable Hausdorff space possessing a convergent development.

Example 4. Let (X,T) denote the same space as was given in Example 3 excepting only the following modification: For each p in J and r in R^+ let $B_r(p)$ denote the set $\{q : q \text{ is in } X, \rho(q,t) < r, \text{ where if } p = (x,0),\ t = (x,r)\} \cup \{p\}$. As before, it is easy to verify that B is a basis for some topology T for X. Further, it can be verified that (X,T) is a Hausdorff space and is regular. However, (X,T) is not normal. In order to establish this last assertion, consider the sets $P = \{q : q \text{ is in } J, q \text{ is rational}\}$ and $Q = \{q : q \text{ is in } J, q \text{ is irrational}\}$. P and Q are disjoint closed sets but there do not exist disjoint open sets E and F containing P and Q, respectively. The same construction of a development that was given in Example 3 is sufficient to yield a convergent development for (X,T). Since (X,T) is not normal it is not metrizable.

It is clear that an arbitrary metric space possessses a convergent development. Lemma 6 suggests that the classical procedure used to complete a metric space would yield the same space as formed by the procedure given in Chapter III and conversely. Finally, it is interesting to note a relationship which exists between the constant of contraction, denoted by α on page 3, of Kolmogorov and Fomin's contraction mappings and the constant of contraction p in a (p,G)-contraction mapping. In the space given in Example 1 suppose a convergent development is chosen and the associated convergent series is $\sum_{i=1}^{\infty} r_i$. If f is an α-contraction mapping it follows that spheres of radius r_i are mapped into spheres of radius αr_i. If f is also a (p,G) contraction mapping, spheres of radius r_i are mapped into spheres of radius r_{i+p}. Thus it is seen, albeit somewhat heuristically, that for all i in I^+, α should be approximately the ratio r_{i+p}/r_i. If the sequence $\{r_i\}$ is geometric and decreasing this condition can be satisfied.

V. BIBLIOGRAPHY

1. Bing, R. H. Metrization of topological spaces. Canadian J. Math. 3: 175-186. 1951.

2. Chittenden, E. W. and Pitcher, A. O. On the theory of developments of an abstract class in relation to the calcul fonctionnel. Trans. Amer. Math. Soc. 20: 213-233. 1919.

3. Kelley, John L. General topology. New York, N. Y. D. Van Nostrand Co., Inc. 1955.

4. Kolmogorov, A. N. and Fomin, S. V. Elements of the theory of functions and functional analysis. Vol. 1. Rochester, N. Y. Graylock Press. 1957.

VI. ACKNOWLEDGMENT

The author wishes to take this opportunity to express his sincere appreciation to Dr. Henry P. Thielman for both his characteristic enthusiasm and his helpful suggestions during the preparation of this dissertation.