INTRODUCTION

Lamb wave ultrasonic testing has been employed as a practical nondestructive method to detect defects in a thin plate. The quantitative evaluation of the ultrasonic testing has not, however, been established since the scattering process of elastic waves in a plate depends on various testing conditions such as frequencies, wave modes, a plate thickness, defect’s properties and so on. Therefore, numerical simulations for the wave scattering in a plate are necessary to make the ultrasonic method more quantitative.

Lamb wave scattering problems by edges or defects in a plate have been investigated using various methods such as the T-matrix method [1], the boundary element method (BEM) combined with the normal mode technique [2,3] and the hybrid method of the finite element method and the Lamb wave modal expansions [4]. Hirose and Yamano [5] developed the BEM to solve the scattering problems of Lamb waves by a crack and investigated the backscattered far-fields mode by mode. In their BEM formulation, the fundamental solution for a full space was used even for a plate problem. Since the fundamental solution does not satisfy the boundary conditions on the upper and bottom surfaces of a plate, the BEM requires the integrations over the plate surfaces as well as the integrations over the boundaries of defects. Therefore the system of equations to be solved becomes large and the truncation error is induced in the solutions.

In this paper, a boundary element method (BEM) with the Green’s function is presented for an SH wave scattering by a crack in a plate. The Green’s function satisfies the traction free conditions on plate surfaces. Hence, the boundary integral equations are discretized only over the boundaries of defects.

PROBLEM STATEMENT AND BEM FORMULATION

Let $D$ be a two dimensional domain of a homogeneous, isotropic, linearly elastic plate with the thickness $2h$ including a crack $S$ as shown Fig. 1. The wave field is
assumed to be in an antiplane state with the time factor $e^{-i\omega t}$, where $\omega$ is the angular frequency. In the following, the time factor $e^{-i\omega t}$ will be omitted.

The governing equation for the antiplane displacement $u$ is given by

$$\nabla^2 u(\vec{x}) + k_T^2 u(\vec{x}) = 0, \quad \vec{x} \in D$$

where $k_T$ is the wavenumber of the transverse wave. The traction free boundary conditions are given on the crack face $S$ and on the upper and bottom surfaces $B$ of the plate.

$$t(\vec{x}) \equiv \partial u(\vec{x})/\partial n(\vec{x}) = 0, \quad \vec{x} \in S \text{ and } B$$

where $\partial/\partial n$ denotes the normal derivative and the shear modulus $\mu$ is chosen as one for simplicity.

To construct the integral equation, we use the Green’s function, which satisfies the equation of motion and the boundary condition as follows:

$$\nabla^2 G(\vec{x} - \vec{y}) + k_T^2 G(\vec{x} - \vec{y}) = -\delta(\vec{x} - \vec{y}),$$

$$\partial G(\vec{x} - \vec{y})/\partial n(\vec{x}) = 0, \quad \vec{x} \in B \ (x_2 = \pm h).$$

The total wave field $u$ can be represented as a sum of the incident wave $u^{in}$ and the scattered wave $u^{sc}$. Applying the reciprocal theorem to the scattered wave $u^{sc}$ and the Green’s function $G$, we have the integral expressions for the displacement $u^{sc}$ and the traction $t^{sc}$ of the scattered wave.

$$u^{sc}(\vec{x}) = \int_S \frac{\partial G(\vec{x} - \vec{y})}{\partial n(\vec{y})} [u(\vec{y})] dS_y, \quad \vec{x} \in D$$

$$t^{sc}(\vec{x}) = \frac{\partial}{\partial n(\vec{x})} \int_S \frac{\partial G(\vec{x} - \vec{y})}{\partial n(\vec{y})} [u(\vec{y})] dS_y, \quad \vec{x} \in D$$

where $[u]$ is the crack opening displacement. It is noted that the above integral equations involve the integration over the crack face $S$ only, because both $t^{sc}$ and $G$ satisfy the traction free condition on $B$ as seen in eqs.(2) and (4). Taking the limit of $\vec{x} \in D \to \vec{x} \in S$, and substituting eq.(6) into the boundary condition (2), namely,

$$t(\vec{x}) = t^{in}(\vec{x}) + t^{sc}(\vec{x}) = 0, \quad \vec{x} \in S,$$

the boundary integral equation is obtained in the following form:

$$\text{p.f.} \frac{\partial}{\partial n(\vec{x})} \int_S \frac{\partial G(\vec{x} - \vec{y})}{\partial n(\vec{y})} [u(\vec{y})] dS_y = -t^{in}(\vec{x}), \quad \vec{x} \in S$$

where p.f. indicates the finite part of the integral.
If one discretizes eq.(7) using the collocation method with constant elements, for example, the system of equations may be obtained as follows:

$$
\begin{align*}
\left[ \frac{\partial}{\partial n(\bar{x})} \int \frac{\partial G}{\partial n(\bar{y})} \, ds_y \right] \{ [u] \} &= \{ -t^n \}.
\end{align*}
$$

(8)

The crack opening displacement $[u]$ can be obtained by solving the above equation. In this paper, however, we do not follow the collocation procedure. The reason is that the integral kernel in eq.(7) has hypersingularity, which makes the numerical implementation difficulty.

Instead of the collocation method, the Galerkin method is applied to regularize eq.(7). We approximate the crack opening displacement $[u]$ as

$$
[u(\bar{y})] = \sum_m [u]_m \psi^m(\bar{y})
$$

(9)

where $[u]_m$ denote the coefficients to be determined and $\psi^m$ are appropriate shape functions, which satisfy the condition $\psi^m = 0$ at the crack edge points $\partial S$. Multiplying eq.(7) by a shape function $\psi^n(\bar{x})$ and integrating over the crack face $S$, we then have

$$
\sum_m \int_S \psi^n(\bar{x}) \frac{\partial}{\partial n(\bar{x})} \int_S \frac{\partial G(\bar{x} - \bar{y})}{\partial n(\bar{y})} \psi^m(\bar{y}) \, ds_y \, ds_x \times [u]_m = - \int_S \psi^n(\bar{x}) t^n(\bar{x}) \, ds_x.
$$

(10)

The above equation constitutes the system of equations with the unknown coefficients $[u]_m$, given by

$$
A_{nm} [u]_m = b_n
$$

(11)

where $A_{nm}$ is the integral part of the L.H.S. in eq.(10) and $b_n = - \int_S \psi^n(\bar{x}) t^n(\bar{x}) \, ds_x$.

The components $A_{nm}$ are regularized using the Stokes theorem as follows[6]:

$$
A_{nm} = - \int_S \epsilon_{npq} n_p(\bar{x}) \frac{\partial \psi^n(\bar{x})}{\partial x_q} \int_S \epsilon_{hij} n_i(\bar{y}) \frac{\partial \psi^m(\bar{y})}{\partial y_j} G(\bar{x} - \bar{y}) \, ds_y \, ds_x
$$

$$
+ k_T^2 \int_S \psi^n(\bar{x}) n_i(\bar{x}) \int_S \psi^m(\bar{y}) n_i(\bar{y}) G(\bar{x} - \bar{y}) \, ds_y \, ds_x.
$$

(12)

where $n_p$ is the component of the normal vector and $\epsilon_{npq}$ stands for the 2-D permutation tensor. As seen in the above equation, the singularity in the integral kernels is reduced to the logarithmic one, which is very tractable in the numerical implementation.

GREEN’S FUNCTION

In this section, the Green’s function is derived in the Fourier integral form. The Green’s function $G(\bar{x}, \bar{y})$ for a plate can be divided into two parts as follows:

$$
G(\bar{x}, \bar{y}) = U(\bar{x}, \bar{y}) + P(\bar{x}, \bar{y}).
$$

(13)

Here, $U(\bar{x}, \bar{y})$ is the fundamental solution in an infinite domain, which is given by

$$
U(\bar{x}, \bar{y}) = \frac{i}{4} H_0^{(1)}(k_T |\bar{x} - \bar{y}|).
$$

(14)
where $H_0^{(1)}$ denotes the zero order Hankel function of the first kind. The singularity due to a point force is included in the solution $U$. The fundamental solution $U(\bar{x}, \bar{y})$ can also be expressed in the Fourier integral form:

$$U(\bar{x}, \bar{y}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-R|x_2-y_2|} \frac{e^{ik(x_1-y_1)}}{R} dk$$  \hspace{1cm} (15)$$

where $R = \sqrt{k^2 - k^2}$ and the sign of $R$ is chosen so that $Re(R) > 0$. $P(\bar{x}, \bar{y})$ in eq.(13) is the regular function representing the reflected waves from the upper and lower surfaces of the plate. Since the reflected waves are composed of waves traveling upward and downward, the regular part $P(\bar{x}, \bar{y})$ may be written as

$$P(\bar{x}, \bar{y}) = \int_{-\infty}^{\infty} \left( \tilde{P}^+(\bar{y})e^{-R x_2} + \tilde{P}^-(\bar{y})e^{R x_2} \right) e^{ikx_1} dk$$  \hspace{1cm} (16)$$

where $\tilde{P}^+$ and $\tilde{P}^-$ are the coefficients determined from the boundary condition (4), i.e.,

$$\left( \frac{\partial U(\bar{x}, \bar{y})}{\partial x_2} + \frac{\partial P(\bar{x}, \bar{y})}{\partial x_2} \right) \bigg|_{x_2=\pm h} = 0.$$  \hspace{1cm} (17)$$

Substituting eqs.(15) and (16) into eq.(17), then we have

$$\tilde{P}^+ + \tilde{P}^- = \frac{e^{-2Rh}}{4\pi R(1 - e^{-2Rh})}(e^{Ry_2} + e^{-Ry_2})e^{-iky_1}, \hspace{1cm} (18)$$

$$\tilde{P}^+ - \tilde{P}^- = \frac{-e^{-2Rh}}{4\pi R(1 + e^{-2Rh})}(e^{Ry_2} - e^{-Ry_2})e^{-iky_1}. \hspace{1cm} (19)$$

Consequently, the regular function $P$ can be obtained in the following form:

$$P(\bar{x}, \bar{y}) = \frac{1}{8\pi} \int_{-\infty}^{\infty} \left[ \frac{e^{-2Rh}}{R(1 - e^{-2Rh})}(e^{Ry_2} + e^{-Ry_2})(e^{Ry_2} + e^{-Ry_2}) \right. \left. + \frac{e^{-2Rh}}{R(1 + e^{-2Rh})}(e^{Ry_2} - e^{-Ry_2})(e^{Ry_2} - e^{-Ry_2}) \right] e^{ik(x_1-y_1)} dk. \hspace{1cm} (20)$$

The first and second terms of the integrand in eq.(20) show the symmetric and antisymmetric modes, respectively.

**NUMERICAL PROCEDURE**

In the numerical calculation, the crack face is divided into piecewise linear elements and the crack opening displacements are also approximated by piecewise linear interpolation functions. Using these approximations, the normal vectors involved in eq.(12) are constant and the derivatives of the shape functions like $\partial \psi^n / \partial x_q$ are independent of the coordinates. Thus, the integral terms on the R.H.S. of eq.(12) are reduced to $\int_S \int_S G(\bar{x} - \bar{y})ds_yds_x$ and $\int_S \psi^n(\bar{x}) \int_S \psi^n(\bar{y})G(\bar{x} - \bar{y})ds_yds_x$.

As seen in eq.(13), the Green's function is divided into the fundamental solution $U$ and the regular term $P$. Using the explicit expression for $U$ as given in eq.(14), the double integrals of $\int_S \int_S Uds_yds_x$ and $\int_S \psi^n(\bar{x}) \int_S \psi^n Uds_yds_x$ can be evaluated analytically and numerically.

In the numerical implementation, the most difficulty exists in evaluating the regular part $P(\bar{x}, \bar{y})$ of the Green's function. From eq.(20), $P(\bar{x}, \bar{y})$ is composed of the
following improper integral with respect to the wavenumber $k$:

$$P_e(\vec{x}, \vec{y}) = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{e^{-2Rh}}{R(1 \pm e^{-2Rh})} e^{\pm R x_2 + R y_2} e^{ik(x_1 - y_1)} \, dk.$$  

Figure 2. Integral path in the complex $k$-plane.

It can be seen that the spatial variables $\vec{x}$ and $\vec{y}$ appear only in the exponential functions. Therefore the double integrations of the exponential functions over the crack face $S$ can be done analytically. For example, consider the integrals

$$\int_{S_n} \int_{S_m} P_e(\vec{x}, \vec{y}) \, ds_y \, ds_x,$$

where $S_n$ is the $n$-th line element with the nodal points $\vec{x}_n$ and $\vec{x}_{n+1}$, and $S_m$ is the $m$-th line element with the nodal points $\vec{y}_m$ and $\vec{y}_{m+1}$. Changing the order of the integrations with respect to the spatial variables and the wavenumber, we have

$$\int_{S_n} \int_{S_m} P_e(\vec{x}, \vec{y}) \, ds_y \, ds_x = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{e^{-2Rh}}{R(1 \pm e^{-2Rh})}$$

$$\times \left( e^{\pm R x_2 + i k x_1} (e^{e^{R y_2 + i k y_1} - e^{R y_2 + i k y_1}}) \right)$$

$$\left( e^{e^{R x_2 + i k x_1} + e^{R y_2 + i k y_1}} (e^{R y_2 + i k y_1} - e^{R y_2 + i k y_1}) \right) \, dk.$$  

The improper integration on the wavenumber is carried out numerically as shown in Fig. 2. Let $k_A$ be an arbitrary constant with $k_A > k_T$. The paths of the integrations over $(\pm k_A, \pm \infty)$ are changed so that the R.H.S. in eq.(22) is transformed to the integral

$$\int_0^\infty F(k(s)) e^{-s} \, ds,$$

which can be evaluated numerically by use of the Laguerre integration formula. The integrand in eq.(22) has singular poles at

$$k = \pm \sqrt{k_T^2 - \{m\pi/(2h)\}^2} \quad (m = 0, 1, \ldots),$$

corresponding to the plate wave modes. Therefore, the integration range of $(-k_A, k_A)$ is divided by the poles and the integral over each interval can be evaluated using the residues and the semi-analytical and numerical integrations based on the subtraction method.

**NUMERICAL EXAMPLES**

To confirm the accuracy of the numerical method, the crack opening displacements are obtained using the boundary element models with the different element numbers $N = 100, 200, 300, \text{ and } 500$. Figure 3 (a) and (b) show the real and
Figure 3. (a) Real and (b) imaginary parts of the crack opening displacements obtained using the boundary element models with various element numbers $N$. The crack is a horizontal crack subjected to an SH plate wave of the $A_0$ mode with the wavenumber $hk_T = 5$.

Figure 4. Real parts of the crack opening displacements of the horizontal cracks located at various depths $d/h$, subjected to the incident wave of the $A_0$ mode with the wavenumber $hk_T = 5$.

imaginary parts, respectively, of the crack opening displacements of the horizontal crack of the length $2h$ located at the center of the plate with the thickness $2h$. The crack is subjected to an SH plate wave of the $A_0$ mode with the wavenumber $hk_T = 5$. As the number of the elements increases, the convergence of the solution is obtained. In the sequel, the number of the elements is chosen as $N = 300$.

Figure 4 shows the real parts of the crack opening displacements of the horizontal cracks located at various depths $d/h$. The incident wave is an $A_0$ mode wave with the wavenumber $hk_T = 5$, which has the displacement and traction distributions across the plate thickness as shown in Fig. 5(a). In Fig. 4, large crack opening displacements are found for $d/h \leq 0.5$, where the incident traction shows large values. From eqs.(5) and (6), the scattered amplitudes are approximately proportional to the crack opening displacements. Therefore the ultrasonic testing with the $A_0$ mode incident wave may be sensitive to cracks located at the center part of the plate.

Figure 6 shows the same figure, but for the incident wave of the $S_1$ mode. In this
Figure 5. Displacement and traction distributions across the plate thickness of the SH incident waves of (a) the $A_0$ mode and (b) the $S_1$ mode.

Figure 6. The same as Fig. 4, but for the incident wave of the $S_1$ mode.

case, the incident traction shows the maximum value at $d/h = \pm 0.5$ as shown in Fig. 5(b). Corresponding to the traction distributions, large opening displacements are generated when the crack is located around $d/h \approx 0.6$.

CONCLUSIONS

The scattering problem of the SH incident wave by a crack in a plate was solved using the BEM with the Green’s function, which was expressed in the Fourier integral form. In the Galerkin method, the double integrations of the Green’s function over the crack face were required. The spatial integrations were analytically carried out and the Fourier integral with respect to the wavenumber was evaluated numerically. The crack opening displacements were obtained for the horizontal cracks located at various depths in a plate. A good correlation between the crack opening displacements and the traction distribution of the incident wave was found.

The crack opening displacements obtained here may be substituted into eqs.(5) and (6) to obtain the scattered wave fields at any point. Particularly, the scattered far fields will be calculated to discuss the mode conversion of the plate waves, which may be important for the quantitative Lamb wave ultrasonic testing.
ACKNOWLEDGEMENTS

This work is supported by the Grant-in-Aid (B) 07555636 from the Ministry of Education, Science and Culture, Japan.

REFERENCES