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On highly regular digraphs

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DEDICATION

I would like to dedicate this thesis to my wife, Sevim, without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance during the writing of this work.

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ABSTRACT

We explore directed strongly regular graphs (DSRGs) and their connections to association schemes and finite incidence structures. More specifically, we study flags and antiflags of finite incidence structures to provide explicit constructions of DSRGs. By using this connection between the finite incidence structures and digraphs, we verify the existence and non-existence of $1\frac{1}{2}$-designs with certain parameters by the existence and non-existence of corresponding digraphs, and vice versa. We also classify DSRGs of given parameters according to isomorphism classes. Particularly, we examine the actions of automorphism groups to provide explicit examples of isomorphism classes and connection to association schemes. We provide infinite families of vertex-transitive DSRGs in connection to non-commutative association schemes. These graphs are obtained from tactical configurations and coset graphs.

CHAPTER 1. GENERAL OVERVIEW AND INTRODUCTION

In this chapter, we provide a brief history of the theory of association schemes and related combinatorial objects. Then we briefly explain our motivations and give an overview of the thesis, outlining the main results.

The origins of theory of association schemes lie in the work of Bose and Shimamoto [5] in their study of experimental designs. Association schemes have an important impact on combinatorics because of their close connections with other combinatorial objects such as codes, designs, and distance regular graphs. For instance, Delsarte, in his thesis, used subsets of association schemes as a powerful tool to contribute to the field of coding theory and design theory [11].

A d-class association scheme $Y = (X, \{R_i\}_{0\le i\le d})$ of order $v = |X|$ may be considered as a decomposition of a complete (di)graph $K_v = (X, X \times X)$ of v vertices into regular digraphs, $\Gamma_i = (X, R_i)$, so that $R_1, R_2, \cdots R_d$ form a partition of $X \times X$ together with $R_0 = \{(x, x) : x \in X\}$ and satisfy certain regularity conditions. If the association scheme is symmetric, that is, all relations R_i are symmetric (binary) relations, then the (non-trivial) relation graphs $\Gamma_i = (X, R_i)$, $i = 1, 2, \cdots, d$, are undirected simple regular graphs.

Strongly regular graphs are a family of graphs which has close connections to codes and association schemes. A strongly regular graph with parameters (v, k, λ, μ) is an undirected regular graph G with v vertices satisfying the properties that the number of common neighbors of vertices x and y is k, if $x = y$, λ if x and y are adjacent, and μ, if x and y are non-adjacent distinct vertices. Particularly, a strongly regular graph and its complement forms a symmetric

2-class association scheme. An interesting group theoretical connection is every vertex transitive permutation group of rank 3 gives rise to a pair of strongly regular graphs.

The concept of directed strongly regular graphs was introduced in 1988 by A. M. Duval [13] as a directed version of strongly regular graphs. A directed strongly regular graph (DSRG) is a loopless directed graph, D, with parameters (v, k, t, λ, μ), if D satisfies the following conditions: (i) every vertex has in-degree and out-degree k; (ii) every vertex x has t out-neighbors, all of which are also in-neighbors of x; and (iii) the number of directed paths of length two from a vertex x to another vertex y is λ, if there is an edge from x to y, and is μ if there is no edge from x to y. Among the DSRGs, ones with $t = k$ are strongly regular graphs. Also a DSRG with $t = 0$ is a pure digraph known as a doubly regular tournament.

The sources for directed strongly regular graphs (with $0 < t < k$) are also rich and diverse as reported by many researchers [7, 8, 13, 14, 17, 16, 22, 24, 25, 28, 27]. Klin et al. showed coherent algebra of a mixed directed strongly regular graph is a non-commutative algebra of rank at least 6 [28]. They have also provided examples of these graphs arising from dihedral groups and flag algebras of *BIBD* with $\lambda = 1$. We will also provide examples of DSRGs obtained from tactical configurations in connection to non-commutative association schemes. DSRGs obtained from semidirect product of cyclic groups, Cayley graphs, were investigated by Duval [14]. Godsil et al. and Jørgensen independently provided excellent tools to investigate the non-existence of DSRGs [19, 26]. Fiedler et al. provided a complete list of vertex transitive DSRGs with $v \leq 20$ by the aid of computer.

The organization of this dissertation is as follows. In Chapter 2, we provide basic facts and definition on theory of finite incidence structures and investigate some of the well-studied structures. We also provide a review of theory of association schemes and an introduction to Delsarte's work on error-correcting codes in Chapter 2. In Chapter 3, we will provide existence and non-existence results on DSRGs obtained from flags (or anti-flags) of certain finite incidence structures called $1\frac{1}{2}$-designs. We will also discuss construction of non-isomorphic DSRGs and will investigate connections to association schemes. In Chapter 4, we will focus on vertex transitive DSRGs and will provide infinite families of vertex transitive DSRGs in connection to coset graphs and tactical configurations.

CHAPTER 2. PREMINILARIES

In this chapter, we will recall some basic definitions and facts about the finite incidence structures, association schemes and codes.

2.1 Finite Incidence Structures

Definition 2.1.1 *An incidence structure is a triple $\mathcal{S} = (\mathcal{P};\mathcal{B};\mathcal{I})$ where*

(i) $\mathcal{P}$ is a finite set, the elements called points,

(ii) $\mathcal{B}$ is a finite set, the elements called blocks (or lines),

(iii) $\mathcal{I}$ is an incidence relation between $\mathcal{P}$ and $\mathcal{B}$, i.e., $\mathcal{I}$ is a subset of $\mathcal{P} \times \mathcal{B}$.

The elements of $\mathcal{I}$ are called flags. If $(p, B) \in \mathcal{I}$, then we say point p and block B are incident. Let

$$(p) := \{B \in \mathcal{B} : (p, B) \in \mathcal{I}\} \text{ for } p \in \mathcal{P}$$

$$(B) := \{p \in \mathcal{P} : (p, B) \in \mathcal{I}\} \text{ for } B \in \mathcal{B},$$

and let $r_p = |(p)|$ and $k_B = |(B)|$, the cardinality of the two sets (p) and (B). If $r_p = r$ and $k_B = k$ are constant, then the incidence structure, $\mathcal{S} = (\mathcal{P};\mathcal{B};\mathcal{I})$, is called a tactical configuration.

Theorem 2.1.2 *In a tactical configuration, $\mathcal{S} = (\mathcal{P};\mathcal{B};\mathcal{I})$ with $|\mathcal{P}| = v$ and $|\mathcal{B}| = b$, $vr = bk$.*

Definition 2.1.3 *Let $\mathcal{S} = (\mathcal{P}; \mathcal{B}; \mathcal{I})$ be a finite incidence structure where $\mathcal{P} = \{p_1, \cdots, p_v\}$ and $\mathcal{B} = \{B_1, \cdots, B_v\}$. The incidence matrix of $\mathcal{S}$ is the $v \times b$ $\{0,1\}$-matrix N defined by*

$$N_{i,j} = \begin{cases} 1, & \text{if } p_i \in B_j; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.1.4 *Let $\mathcal{S} = (\mathcal{P}; \mathcal{B}; \mathcal{I})$ be a finite incidence structure and N be the incidence matrix of $\mathcal{S}$. The finite incidence structure having incidence matrix N^t (the transpose of N) is called the dual of $\mathcal{S}$.*

Definition 2.1.5 *Let $\mathcal{S} = (\mathcal{P}; \mathcal{B}; \mathcal{I})$ and $\mathcal{S}' = (\mathcal{P}'; \mathcal{B}'; \mathcal{I}')$ be two finite incidence structures. $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, if there exists a bijection $f : \mathcal{P} \to \mathcal{P}'$, such that $\{f((B)) : B \in \mathcal{B}\} = \{(B') : B' \in \mathcal{B}'\}$.*

Now, we will give definitions and examples of the most-studied tactical configurations.

Definition 2.1.6 *Let v, k, λ be positive integers such that $v > k \geq 2$. A (v, k, λ)-balanced incomplete block design (which we abbreviate to (v, k, λ)-*BBID*) is a tactical configuration, such that each distinct point is contained in exactly λ blocks.*

Example 2.1.7 *A* $(7, 3, 1)$-BIBD.

$$\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6\}.$$

$$\mathcal{B} = \{123, 145, 016, 246, 025, 034, 356\}.$$

This BIBD *has a nice diagrammatic representation which is known as the Fano plane.*

Theorem 2.1.8 *Let I denote the identity matrix and J denote all-ones matrix. Let N be a $v \times b$ $\{0,1\}$-matrix and $2 \leq k < v$. Then, N is the incidence matrix of a (v, k, λ)-*BBID *if and only if $NN^t = \lambda J + (r - \lambda)I$.*

An *automorphism* of a *BIBD* is an isomorphism from this *BIBD* to itself. For instance, the map $f(x) = 2x(mod 7)$ is an automorphism of the $(7, 3, 1)$-design described in Example 2.1.7.

Definition 2.1.9 *A* BIBD *with $b = v$ (or equivalently $r = k$) is called a symmetric* BIBD*.*

Definition 2.1.10 *An* $(n^2+n+1, n+1, 1)$- BIBD *with* $n \geq 2$ *is called a projective plane of order* n*.*

Definition 2.1.11 *An* $(n^2, n, 1)$- BIBD *with* $n \geq 2$ *is called an affine plane of order* n*.*

Example 2.1.12 *An affine plane of order* 3 *(*$(9,3,1)$-BIBD*).*

$$\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

$$\mathcal{B} = \{123, 456, 789, 147, 258, 369, 159, 267, 348, 168, 249, 357\}.$$

Definition 2.1.13 *A partial geometry pg*(κ, ρ, τ) *is a tactical configuration with the following properties:*

(i) Every line is incident with κ *points (*$\kappa \geq 2$*), and every point is incident with* ρ *lines (*$\rho \geq 2$*).*

(ii) Any two points are incident with at most one line.

(iii) If a point p *and a line* L *are not incident, then there exist exactly* τ *(*$\tau \geq 1$*) lines that are incident with* p *and incident with* L*.*

Example 2.1.14 *Let* q *be a prime and* $\overline{AP}^{l}(q)$ *denote the finite incidence structure obtained from an affine plane of order* q *by considering all* q^2 *points and taking the lines of* $l \leq q$ *parallel classes of the plane. Then,* $\overline{AP}^{l}(q)$ *satisfies the following properties:*

(i) Every point is incident with l *lines,*

(ii) Every line is incident with q *points,*

(iii) Any two points are incident with at most one line,

(iv) If p *and* L *are a non-incident point-line pair, there are exactly* $l-1$ *lines containing* p *which meet* L*.*

Thus, $\overline{AP}^{l}(q)$ *is a pg*$(q, l, l-1)$ *and will be called a degenerated affine plane in what follows.*

Example 2.1.15 *The dual of affine plane of order* 2 *is a* $pg(3,2,2)$

$$\mathcal{P} = \{1,2,3,4,5,6\}$$

$$\mathcal{B} = \{123, 156, 345, 246\}.$$

Definition 2.1.16 *A transversal design* $TD(n,k)$ *of order* n*, block size* k *is a triple* $(\mathcal{P}, \mathcal{G}, \mathcal{B})$*, such that*

(i) $\mathcal{P}$ *is a set of* kn *elements called points,*

(ii) $\mathcal{G}$ *is a partition of* $\mathcal{P}$ *into* k *subsets, each of size* n *(the "groups"),*

(iii) $\mathcal{B}$ *is a collection of* k*-subsets of* $\mathcal{P}$ *(the "blocks"),*

(iv) Any group and any block contain exactly one common point, and

(v) Every pair of elements from distinct groups is contained in exactly one block.

2.2 Difference Sets

We now introduce difference sets as they will be used as an important construction method for symmetric *BIBD*s.

Definition 2.2.1 *Let* $(G,+)$ *be a finite group of order* v *in which the identity element is denoted by* "0"*. Let* k *and* λ *be positive integers, such that* $2 \le k < v$*. A* (v,k,λ)*-difference set in* $(G,+)$ *is a subset* $D \subseteq G$*, such that the* $k(k-1)$ *possible differences between members of* D *comprise all elements of* $G \setminus \{0\}$ *exactly* λ *times.*

Example 2.2.2 $D = \{1,2,4\}$ *is a* $(7,3,1)$*-difference set in* $\mathbb{Z}_7$*.*

Example 2.2.3 *Let* G *be the additive group of finite field* $GF(q)$*, where* $q = 4n-1$*, and let* $D = \{a^2 : \ a \in G \setminus \{0\}\}$*. Then,* D *is a* $(4n-1, 2n-1, n-1)$*-difference set, known as a Hadamard difference set.*

For any $g \in G$, define $D + g = \{x + g : x \in D\}$. Any set $D + g$ is called a translate of D. Then, define $\mathrm{Dev}(D)$ to be the collection of all v translates of D. $\mathrm{Dev}(D)$ is called the development of D.

Theorem 2.2.4 *Let D be a (v, k, λ)-difference set in $(G, +)$. Then, $(G, Dev(D))$ is a symmetric (v, k, λ)*-BIBD.

Definition 2.2.5 *Let D be a (v, k, λ)-difference set in $(G, +)$. For an integer, m, define $mD = \{mx : x \in G\}$. Then, m is called a multiplier of D if $mD = D + g$ for some $g \in G$.*

Theorem 2.2.6 (Multiplier Theorem) *Suppose there exists (v, k, λ)-difference set D in an abelian group $(G, +)$. Suppose also that the following conditions are satisfied:*

(i) p is prime.

(ii) gcd(p,v)=1.

(iii) $k - \lambda \equiv 0 (mod\ p)$.

(iv) $p > \lambda$.

Then, p is a multiplier of D.

Example 2.2.7 *$p = 2$ is a multiplier of the $(7, 3, 1)$-difference set $D = \{1, 2, 4\}$ in $\mathbb{Z}_7$.*

Lemma 2.2.8 *Suppose m is a multiplier of a (v, k, λ)-difference set D in an abelian group $(G, +)$. Define $\phi : G \to G$ by the rule $\phi(x) = mx$. Then, ϕ belongs to the automorphism group of $(G, Dev(D))$.*

The group ring is an essential tool to study many combinatorial objects. For a group G, the group ring $\mathbb{Z}G$ is the set of formal sums, $\sum_{g \in G} c_g g$, where $c_g \in \mathbb{Z}$. Then, $\mathbb{Z}G$ is a ring with sum

$$\sum_{g \in G} c_g g + \sum_{g \in G} d_g g = \sum_{g \in G} (c_g + d_g) g$$

and product

$$\left(\sum_{g \in G} c_g g \right) \left(\sum_{h \in G} d_h h \right) = \sum_{g,h \in G} (c_g d_h) gh.$$

The element $\underline{S} = \sum_{s \in S} s$ for a nonempty set $S \subseteq G$ is called a *simple quantity.*

Theorem 2.2.9 *Let D be a subset of an abelian group $(G, +)$ and e be the identity element of G. Then, D is a (v, k, λ)-difference set in the group $(G, +)$ if and only if $\underline{D}\underline{D}^{-1} = \lambda \underline{G} + (k - \lambda) \underline{e}$.*

2.3 $1\frac{1}{2}$-Designs

In this section, we will discuss certain tactical configurations called $1\frac{1}{2}$-designs (also known as partial geometric designs).

Definition 2.3.1 *Let $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a tactical configuration with parameters (v, b, k, r). For every point $x \in P$ and every block $B \in \mathcal{B}$, let $s(x, B)$ be the number of flags $(y, C) \in \mathcal{I}$ such that $y \in B \setminus \{x\}$, $C \ni x$ and $C \neq B$. A $1\frac{1}{2}$-design with parameters $(v, b, k, r; \alpha, \beta)$ is a tactical configuration $\mathcal{T}$, such that*

$$s(x, B) = \begin{cases} \alpha, & x \ni B; \\ \beta, & x \notin B. \end{cases}$$

If $\alpha > 0, 3 \leq k \leq v - 3, 3 \leq r \leq b - 3$, then we call $1\frac{1}{2}$-design proper.

Proposition 2.3.2 *[30] Let $\mathcal{T}$ be a $1\frac{1}{2}$-design with parameters $(v, b, k, r; \alpha, \beta)$ and $n = r + k + \beta - \alpha - 1$. Then, the following holds*

(i) $(v - k)\alpha + k\beta = k(k - 1)(r - 1)$.

(ii) $v = \frac{k(kr-n)}{\alpha}$.

(iii) $b = \frac{r(kr-n)}{\alpha}$.

(iv) $k + r \leq n + \alpha + 1 \leq kr$.

Example 2.3.3 *We have the following examples of $1\frac{1}{2}$-designs:*

(i) A (v, k, λ)-BIBD.

(ii) A transversal design.

(iii) The dual of transversal designs and BIBDs.

(iv) Partial geometries.

Lemma 2.3.4 *[30] The parameters of a proper $1\frac{1}{2}$-design $\mathcal{T}$ satisfy*

$$\alpha \geq k(r - n)$$

with equality, if and only if $\mathcal{T}$ is a BIBD *with $\lambda = r - n$.*

Theorem 2.3.5 *[30] Let $\mathcal{T}$ be an incidence structure with incidence matrix A. $\mathcal{T}$ is a proper $1\frac{1}{2}$-design with parameters $(v, b, k, r; \alpha, \beta)$ if and only if $AA^tA = nA + \alpha J$, where $n = r + k + \beta - \alpha - 1$.*

Corollary 2.3.6 *Let $\mathcal{T}$ be a proper $1\frac{1}{2}$-design with parameters $(v, b, k, r; \alpha, \beta)$ and $v \times b$ incidence matrix A. Then, $B = J_{v\times b} - A$ is an incidence matrix of a $1\frac{1}{2}$-design.*

Lemma 2.3.7 *[30] If A is an incidence matrix of a proper $1\frac{1}{2}$-design, then $N = AA^t$ satisfies*

$$N^2 = nN + \alpha rJ.$$

N has eigenvalues kr, n, 0 with multiplicities 1,σ, $v - 1 - \sigma$ respectively, where

$$\sigma = r(v - k)/n.$$

Notes 2.3.8 *For those who are interested in a more comprehensive view of finite incidence structures and related combinatorial objects, we recommend Design Theory [4], Combinatorial Design: Construction and Analysis [38], and Finite Geometries[12].*

$1\frac{1}{2}$-designs are also known as partial geometric designs by Bose, Shrikhande, and Singhi [5].

Research Problem : *An interesting research problem is classifying all $1\frac{1}{2}$-designs, whose automorphism group has a subgroup G, acting on blocks and points sharply transitively. In this case, the points will be the elements of G and the blocks will be the translates of a single block. For example, any difference set will provide an example of this classification.*

2.4 Association Schemes

In algebraic combinatorics, association schemes provide a unified approach to many topics, for example, combinatorial designs and coding theory. We will use association schemes as a tool to investigate graphs and their automorphism groups.

Definition 2.4.1 *A graph is an ordered pair $\Gamma = (V, E)$, comprising a set V of vertices together with a set E of edges, which are 2-element subsets of V.*

Definition 2.4.2 *A (simple) graph is called a complete graph, if its edge set consists of all two-element subsets of its vertex set.*

Definition 2.4.3 *Let d denote a nonnegative integer. By a commutative d-class association scheme, we mean a non-empty finite set X together with a sequence $R_0, R_1, \ldots, R_d$ of non-empty subsets of the Cartesian product $X \times X$ satisfying the properties:*

(i) $R_0 = \{(x, x) : x \in X\}$,

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ *(disjoint union),*

(iii) For all integers i $(0 \leq i \leq d)$, there exists $i' \in \{0, 1, \cdots, d\}$, such that $R_i^t = R_{i'}$, where $R_i^t = \{(y, x) : (x, y) \in R_i\}$,

(iv) For all integers h, i, j $(0 \leq h, i, j \leq d)$, and for all $x, y \in X$, such that $(x, y) \in R_h$, the number $p_{ij}^h(x, y) := |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant depends only on h, i, j, and not on x or y,

(v) $p_{ij}^h = p_{ji}^h$ $(0 \leq h, i, j \leq d)$.

We denote this association scheme by $Y = (X, \{R_i\}_{0 \leq i \leq d})$. An association scheme is said to be symmetric if $R_{i'} = R_i$ for all $i \in \{0, 1, \ldots, d\}$.

Example 2.4.4 *Let G denote a finite group. Let $C_0 = 1, C_1, C_2, ..., C_d$ denote the conjugacy classes of G, and define $R_i := \{(x, y) : x, y \in G, x^{-1}y \in C_i\}$ for $i \in \{0, 1, \ldots, d\}$. Then, $Y = (G, \{R_i\}_{0 \leq i \leq d})$ is an association scheme.*

An association scheme can be visualized as a complete digraph with labeled edges. The graph has X as its vertex set, and the edge joining vertices x and y is labeled i, if $(x, y) \in R_i$. Each edge has a unique label. The number of triangles with a fixed base labeled h having the other edges labeled i and j is a constant p_{ij}^h, depending only on h, i, j, but not on the choice of the base. In this way, given an association scheme $Y = (X, \{R_i\}_{0 \leq i \leq d})$, each relation R_i gives rise to a graph $\Gamma_i = (X, R_i)$. This graph will be referred to as the ith-relation graph associated with Y in what follows.

Example 2.4.5 *Let Γ be a complete graph on $V = \{1, 2, 3, 4\}$.*

$$R_0 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}.$$

$$R_1 = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,1),(1,4)\}.$$

$$R_2 = \{(1,3),(3,1),(2,4),(4,2)\}.$$

Then, $Y = (V, \{R_i\}_{0\leq i\leq 2})$ *is a 2-class (symmetric) association scheme. The first relation graph* Γ_1 *is a rectangle, a strongly regular graph with parameters* $(4,2,0,1)$.

2.4.1 Bose-Mesner algebra

Let $Y = (X, \{R_i\}_{0\leq i\leq d})$ be an association scheme. For each integer i $(0 \leq i \leq d)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } (x,y) \in R_i; \\ 0, & \text{otherwise.} \end{cases}$$

We refer to A_i as the i-th adjacency matrix of Y.

Lemma 2.4.6 *Let* $Y = (X, \{R_i\}_{0\leq i\leq d})$ *be an association scheme with adjacency matrices* $A_0, A_1, \cdots, A_d$. *Then,*

(i) $A_0 = I$ *(identity matrix),*

(ii) $A_0 + A_1 + \cdots + A_d = J$ *(all-ones matrix),*

(iii) $A_i^t = A_{i'}$ *for some* $i' \in \{0, 1, \cdots, d\}$,

(iv) $A_iA_j = \sum_{h=0}^{d} p_{ij}^h A_h$,

(v) $A_iA_j = A_jA_i$.

Let $Y = (X, \{R_i\}_{0\leq i\leq d})$ be an association scheme. The adjacency matrices, $A_0, A_1, \cdots, A_d$, form a basis for a commutative subalgebra, which will be denoted by M, of $\mathrm{Mat}_X(\mathbb{C})$. We refer to M as the Bose-Mesner algebra of Y.

Given a symmetric 2-class association scheme $Y = (X, \{R_0, R_1, R_2\})$, the relation graphs $\Gamma_1 = (X, R_1)$ and $\Gamma_2 = (X, R_2)$ are strongly regular graphs and they are complement to each other. In other words, all symmetric 2-class association schemes arise from strongly regular graphs.

Definition 2.4.7 *The adjacency matrix for a graph with v vertices is a $v \times v$ matrix, whose xy entry is 1 if vertex x and vertex y are adjacent, and 0 if they are not.*

Example 2.4.8 *A strongly regular graph with its adjacency matrix A gives rise to the 2-class association scheme with adjacency matrices, $A_0 = I$, $A_1 = A$ and $A_2 = J - I - A$.*

Theorem 2.4.9 *Let $Y = (X, \{R_i\}_{0 \le i \le d})$ be a d-class association scheme, with Bose-Mesner algebra M. Then, there exists a basis, $E_0, E_1, \cdots, E_d$, for M, such that*

(i) $E_0 = \frac{1}{|X|}J$,

(ii) $I = E_0 + E_1 + \cdots + E_d$,

(iii) $E_iE_j = \delta_{i,j}E_i \ (0 \le i, j \le d)$.

E_i's are primitive idempotents of the algebra M.

Lemma 2.4.10 *Let $Y = (X, \{R_i\}_{0 \le i \le d})$ denote a scheme with adjacency matrices, $A_0, A_1, \cdots, A_d$, and primitive idempotents, $E_0, E_1, ..., E_d$. Then, there exists scalars $p_j(i), q_i(j)$ $(0 \le i, j \le d)$, such that*

$$A_i = \sum_{j=0}^{d} p_i(j)E_j.$$

$$E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_i(j)A_j.$$

Lemma 2.4.11 *Let $Y = (X, \{R_i\}_{0 \le i \le d})$ denote a scheme, P denote a $d+1$ by $d+1$ matrix, whose ij entry is $p_j(i)$ and Q denote a $d+1$ by $d+1$ matrix, whose ij entry is $q_j(i)$. Then,*

(i) P *and* $\frac{1}{|X|}Q$ *are inverses,*

(ii) $\sum_{h=0}^{d} p_h(i)q_j(h) = \delta_{ij}|X|$,

(iii) $\sum_{h=0}^{d} q_h(i)p_j(h) = \delta_{ij}|X|$.

2.4.2 Permutation groups and association schemes

The set of all permutations of a set V is denoted by $\mathrm{Sym}(V)$. A permutation group on V is a subgroup of $\mathrm{Sym}(V)$. The image of an element $v \in V$ under a permutation $g \in \mathrm{Sym}(V)$ will be denoted by v^g.

Definition 2.4.12 *Let G be a group. A group action of G on a set V is a binary operator:*

$$\circ : G \times V \to V,$$

$$(g, x) \longmapsto x^g$$

that satisfies the following two axioms:

(i) For all $g, h \in G$, for each $x \in V$ $x^{gh} = (x^g)^h$

(ii) For all $x \in V$, $x^e = x$.

The set V is called a G-set. The group G is said to act on V.

Definition 2.4.13 *Let G be a group. The orbit of a point x in V is the set of elements of V to which x can be moved by the elements of G. The orbit of x is denoted by x^G, where $x^G = \{x^g : g \in G\}$.*

Definition 2.4.14 *A permutation group G on V is transitive, if given any two points, x and y from V there is an element $g \in G$ such that $x^g = y$.*

Definition 2.4.15 *Let G be a permutation group on V. For any $x \in V$, the stabilizer, G_x, is the set of all permutations $g \in G$, such that $x^g = x$.*

Lemma 2.4.16 *Let G be a permutation group acting on V and let x be a point in V. Then,*

$$|G_x||x^G| = |G|.$$

Lemma 2.4.17 *Let G be a permutation group on V. Then the number of orbits of G on V is equal to the average number of points fixed by an element of G.*

Definition 2.4.18 *Let G be a transitive permutation group on V. An orbital of G is an orbit of G on the set $V \times V$ under the action $(x,y)^\rho = (x^\rho, y^\rho)$ for $x, y \in V$ and $\rho \in G$.*

Definition 2.4.19 *The rank of the transitive permutation group G is the number of orbitals.*

Lemma 2.4.20 *Let G be a group acting transitively on V, and let x be a point of V. Then, there is a one-to-one correspondence between orbitals of G and the orbits of G_x on V.*

Let $R_0, R_1, \ldots, R_d$ be the orbitals of G on V, where $R_0 = \{(z,z) : \; z \in V\}$. If we denote the set $\{y \in V : \; (x,y) \in R_i\}$ by $R_i(x)$. Then, $R_0(x) = \{x\}, R_1(x), \ldots, R_d(x)$ are the orbits of G_x on V. It holds that $R_i(x)^\rho = R_i(x^\rho)$ for all $\rho \in G$, $0 \leq i \leq d$. For any orbital R, $R' = \{(y,x) : \; (x,y) \in R\}$ is also an orbital. The orbital R' is referred to as the *paired orbital* of R. We say the orbital R is *self-paired* if $R = R'$. The *orbital graph* associated with an orbital R is the graph with vertex set V and edge set R; i.e., there is an edge from x to y for each $(x,y) \in R$. If R is self-paired, then its orbital graph (V, R) can be regarded as an undirected graph. The orbital graphs $(V, R_1), (V, R_2), \ldots, (V, R_d)$ give a decomposition of the complete on $v = |V|$ vertices.

In this case, the set V, together with the set of orbitals, forms an association scheme of class d, called an *orbital scheme* of the transitive permutation group G on V.

Example 2.4.21 *Let G be the following subgroup of $Sym(V)$ where $V = \{1,2,3,4,5\}$*

$$G = \{(1), (2,3)(4,5), (1,2)(3,4), (1,2,4,5,3),$$

$$(1,3,5,4,2), (1,3)(2,5), (1,4)(3,5), (1,4,3,2,5), (1,5,2,3,4), (1,5)(2,4)\}.$$

G is isomorphic to dihedral group of order 10. Clearly, G acts transitively on V. The orbitals of G are

$$R_0 = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}.$$

$$R_1 = \{(1,2), (2,4), (4,5), (5,3), (3,1), (1,3), (3,5), (5,4), (4,2), (2,1)\}.$$

$$R_2 = \{(1,4), (2,5), (4,3), (5,1), (3,2), (1,5), (3,4), (5,2), (4,1), (2,3)\}.$$

Thus, $Y = (G, \{R_i\}_{0 \leq i \leq 2})$ is an association scheme.

2.5 Error-Correcting Codes

Definition 2.5.1 *The Hamming space $H(n,q)$ is the set $H = F^n = \{(x_1, \cdots, x_n) : x_i \in F\}$ where F is a fixed alphabet of size q. The elements of H are words of length n which are n-tuples of letters taken from the alphabet F of size q.*

Definition 2.5.2 *The Hamming distance d (or Hamming metric d) on H is defined by*

$$d : H \times H \to \mathbb{Z}^+$$

$$(x,y) \longmapsto d(x,y) = |\{i : x_i \neq y_i\}|$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.

Definition 2.5.3 *A code of length n over F is a subset C of $H(n,q)$ which contains at least two words. The elements of the code are codewords.*

Definition 2.5.4 *For $e \in \mathbb{Z}^+$, the code C is an e-error-correcting if given any word $w \in H(n,q)$, there exists at most one codeword $c \in C$ such that $d(w,c) \leq e$.*

Definition 2.5.5 *Given a code C, let $d = min\{d(x,y) : x, y \in C\}$. Then d is called the minimum distance of the code C.*

Definition 2.5.6 *A binary (n, M, d) error-correcting code is a set of M vectors 0's and 1's of length n, such that the (Hamming) distance between any two codewords is at least d.*

Proposition 2.5.7 *Let C be a binary (n, M, d) error-correcting code. Then C can correct $[\frac{d-1}{2}]$ or fewer errors.*

Example 2.5.8 *The Hamming* $(7, 16, 3)$ *code:*

0000000	*1111111*
1101000	*0010111*
0110100	*1001011*
0011010	*1100101*
0001101	*1110010*
1000110	*0111001*
0100011	*1011100*
1010001	*0101110*

The code words $1101000, 0110100, 0011010, 0001101, 1000110, 0100011, 1010001$ form the rows of incidence matrix of $(7, 3, 1)$-*BIBD*, which is a projective plane of order 2.

Definition 2.5.9 *An e-error-correcting code is called perfect, if every binary vector of length n is within Hamming distance e of some codeword.*

In a (n, M, d) error-correcting code, we expect to have n small for fast transmission, M large for efficiency and d large to correct many errors. Mathematicians use association schemes as a tool to answer how large M can be for a given n and d.

Hamming scheme is an important example for coding theory. In this scheme, $X = Q_n$, the set binary vectors of length n and two vectors are in the i-th association relation if they are distance i apart. The intersection numbers are given by

$$p_{i,j}^k = \begin{cases} \binom{k}{\frac{i-j+k}{2}}\binom{n-k}{\frac{i-j+k}{2}} & \text{if } i+j-k \text{ is even;} \\ 0 & \text{if } i+j-k \text{ is odd.} \end{cases}$$

Also the i-th valency $v_i = \binom{n}{i}$. This association scheme is denoted by $H(n, 2)$, the n-class binary Hamming scheme. In this scheme, the $\mathbf{uv}$-th entry of the k-th adjacency matrix $(A_k)_{u,v} = 1$, if and only if $\text{dist}(\mathbf{u}, \mathbf{v}) = k$.

Example 2.5.10 *Let* $n = 2$*, and label the rows and columns of the adjacency matrices by the* *binary vectors* 00, 01, 10, 11. *Then adjacency matrices of* $H(2,2)$,

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 2.5.11

$$\sum_{\mathbf{u} \in Q_n} (-1)^{\mathbf{u} \odot \mathbf{v}} = 2^n \delta_{\mathbf{0},\mathbf{v}} \quad ,$$

where $\mathbf{u} \odot \mathbf{v}$ *denotes the real scalar product.*

Lemma 2.5.12 *Let* $wt(\mathbf{u})$ *denote the number of nonzero components of the vector* $\mathbf{u}$. *The primitive idempotent* E_k, *is the matrix whose* $\mathbf{uv}$*-th entry is*

$$\frac{1}{2^n} \sum_{wt(\mathbf{w})=k} (-1)^{(\mathbf{u}+\mathbf{v}) \odot w}, \qquad k = 0, 1 \cdots, n.$$

Example 2.5.13 *For* $n = 2$, *apply the above lemma:*

$$E_0 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, E_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, E_2 = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Definition 2.5.14 *For any positive integer* n, k*-th* Krawtchouk polynomial *is defined by*

$$K_k(x;n) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}, \qquad k = 0, 1, 2, \cdots$$

where x *is an indeterminate. The first few* Krawtchouk polynomials *are*

$$K_0(x;n) = 1.$$

$$K_1(x;n) = n - 2x.$$

$$K_2(x;n) = \binom{n}{2} - 2nx + x^2.$$

Lemma 2.5.15 *If $wt(\mathbf{u}) = i$,*

$$\sum_{wt(\mathbf{v})=i} (-1)^{\mathbf{u}\odot\mathbf{v}} = K_k(i;n).$$

Theorem 2.5.16 *The eigenvalues of the Hamming scheme are $p_k(i) = q_k(i) = K_k(i;n)$ for $i, k = 0, \cdots, n$.*

Example 2.5.17 *For $n = 2$, we have*

$$P = Q = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

2.5.1 Linear programming bound

The definition of an error-correcting code saying that a code is a subset of $X = Q_n$ in the association scheme $H(n, |Q|)$. More generally, define a code Y in any association scheme $Z = (X, \{R_i\}_{0\leq i\leq n})$ to be a non-empty subset of points X. Elements of Y are called codewords.

The distance distribution of the code Y is defined to be the $(n+1)$-tupe of rational numbers $(B_0, B_1, \cdots, B_n)$, where

$$B_i = \frac{1}{|Y|}|R_i \cap (Y \times Y)|$$

is the average number of codewords which are i-th associates of given a codeword.

Theorem 2.5.18 *[10] The distance distribution of any code, Y, satisfies*

$$B'_k = \frac{1}{|Y|}\sum_{i=0}^{n} B_i q_k(i) \geq 0,$$

for $k = 0, \cdots, n$.

It is said that a code Y in an association scheme has a minimum distance d, if no codeword is an i-th associate of any other codeword for $0 < i < d$. The distance distribution of Y must satisfy

$$B_1 = B_2 = \cdots = B_{d-1} = 0.$$

The problem of determining the largest code of minimum distance d is related to the following problem:

Linear Programing Problem I: Choose $B_d, B_{d+1}, \cdots, B_n$ so as to maximize

$$g = \sum_{i=d}^{n} B_i \quad ,$$

subject to inequalities

$$B_i \geq 0, \quad i = d, \cdots, n \quad ,$$
$$q_k(0) + \sum_{i=d}^{n} B_i q_k(i) \geq 0, \quad k = 1, \cdots, n.$$

An $(n+1)$-tuple $B = (B_0, B_1, \cdots, B_n)$ with $B_1 = B_2 = \cdots = B_{d-1} = 0$ is called a feasible solution to *Linear Programing Problem I*, if it satisfies the above inequalities. A feasible solution is called optimal, if g is maximized.

If a code Y with minimum distance d exists in an association scheme, its distance distribution $B = (B_0, B_1, \cdots, B_n)$ is a feasible solution to *Linear Programing Problem I.* Hence,

$$|Y| \leq g_{max} + 1 \quad .$$

Linear Programing Problem II: Choose the real variables $\beta_1, \cdots, \beta_2$ so as to minimize:

$$\gamma = \sum_{k=1}^{n} \gamma_k q_k(0)$$

subject to the in equalities

$$\beta_k \geq 0, \quad k = 1, \cdots, n \quad ,$$
$$1 + \sum_{k=1}^{n} \beta_k q_k(i) \leq 0, \quad i = d, \cdots, n.$$

An $(n+1)$-tuple $\beta = (\beta_0, \beta_1, \cdots, \beta_n)$ with $\beta_0 = 1$ is called a feasible solution to *Linear Programming Problem II*, if it satisfies the above inequalities. A feasible solution is called optimal, if γ is minimized.

Theorem 2.5.19 *[35]*

(i) *If B is a feasible solution to* Problem I *and β is a feasible solution to* Problem II, *then* $g \leq \gamma$.

(ii) *Optimal solutions exist both to* Problems, *and the optimal values g and γ are equal.*

(iii) *If B is an optimal solution to* Problem I *and β is an optimal solution to* Problem II, *then*

$$\beta_k \left(q_k(0) + \sum_{i=d}^{n} B_i q_k(i) \right) = 0, \qquad k = 1, \cdots, n.$$

$$B_i \left(\sum_{k=0}^{n} \beta_k q_k(i) \right) = 0, \qquad i = d, \cdots, n.$$

(iv) *Conversely, if a pair of feasible solutions B, β satisfy*

$$\beta_k \left(q_k(0) + \sum_{i=d}^{n} B_i q_k(i) \right) = 0, \qquad k = 1, \cdots, n.$$

$$B_i \left(\sum_{k=0}^{n} \beta_k q_k(i) \right) = 0, \qquad i = d, \cdots, n.$$

then they are optimal solutions.

Theorem 2.5.20 *[10] Suppose a polynomial $\beta(x)$ of degree at most n can be found with the following properties. Let*

$$\beta(x) = \sum_{k=0}^{n} \beta_k q_k(x).$$

Then, $\beta(x)$ should satisfy

$$\beta_0 = 1$$

$$\beta_k \geq 0 \quad \textit{for} \quad k = 1, \cdots, n$$

$$\beta(i) \leq 0 \quad \textit{for} \quad i = d, \cdots, n.$$

Then, if Y is any code with minimum distance d,

$$|Y| \leq \beta(0).$$

For theoretical purposes, the above theorem is easier to use and a very powerful technique. The following theorems are applications of this technique.

Theorem 2.5.21 *[33] If C is an (n, M, d) error-correcting code with $n < 2d$, then*

$$M \leq \frac{2d}{2d - n}.$$

Theorem 2.5.22 *[23, 11] If C is an $(V, M, 2\delta)$ error-correcting code in which for every code word $\mathbf{u}$, $wt(\mathbf{u}) = n$, then*

$$M \leq \frac{\delta V}{\delta V - n(V - n)},$$

provided $n(V - n) < \delta V$.

Theorem 2.5.23 *[36, 11] If C is an (n, M, d) error-correcting code, then*

$$M \leq 2^{n-d+1}.$$

Notes 2.5.24 *For more information on theory of association schemes we recommend the book Algebraic Combinatorics I: Association Schemes [3] and on theory of error-correcting codes we recommend the books Orthogonal Arrays [21] and Introduction to Error-Correcting Codes [32].*

Most of the results on error-correcting codes in this chapter can be found in Delsarte's thesis [11] and the paper by Sloane [37]. Delsarte pointed out that certain linear combinations of parameters of a subset of a association scheme must be nonnegative. By this result we can state, the problem of finding bounds on the size of subsets of an association scheme having certain properties, is a linear programming problem. The linear programming bound technique has been applied to combinatorial problems, including codes, orthogonal arrays, t-designs, families of lines with a prescribed number of angles between them, bilinear, and alternating forms over a finite field of order q.

CHAPTER 3. DIRECTED STRONGLY REGULAR GRAPHS ARISING FROM FINITE INCIDENCE STRUCTURES

In this section, we will introduce a directed version of strongly regular graphs and discuss related combinatorial structures.

3.1 Directed Strongly Regular Graphs

Directed strongly regular graphs were introduced by Duval in 1988 as a generalization of the notion of strongly regular graphs [13].

Definition 3.1.1 *A loopless directed graph* Γ *with* v *vertices is called directed strongly regular graph (DSRG) with parameters* (v, k, t, λ, μ)*, if* Γ *satisfies the following conditions:*

(i) Every vertex has in-degree and out-degree k*.*

(ii) Every vertex x *has* t *out-neighbors, all of which are also in-neighbors of* x*.*

(iii) The number of directed paths of length two from a vertex x *to another vertex* y *is* λ *if there is an edge from* x *to* y*, and is* μ *if there is no edge from* x *to* y*.*

Another definition of a DSRG, in terms of its adjacency matrix, is often conveniently used. Let Γ be a directed graph with v vertices. Let A denote the adjacency matrix of Γ. Then, Γ is a DSRG with parameters (v, k, t, λ, μ), if A satisfies

$$JA = AJ = kJ. \tag{3.1}$$

$$A^2 = tI + \lambda A + \mu(J - I - A). \tag{3.2}$$

Example 3.1.2 *Let* Γ *be a digraph with the following adjacency matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $A^2 + A = J$, *so that* Γ *is an DSRG with parameters* $(6, 2, 1, 0, 1)$.

Lemma 3.1.3 *[13] If* Γ *is a DSRG with parameters* (v, k, t, λ, μ) *and adjacency matrix* A *then the complementary graph* $\overline{\Gamma}$ *is a DSRG with parameters* $(v, k', t', \lambda', \mu')$ *with adjacency matrix* $\overline{A} = J - I - A$, *where*

- $k' = (v - 2k) + (k - 1)$.
- $t' = (v - 2k) + (t - 1)$.
- $\lambda' = (v - 2k) + (\mu - 2)$.
- $\mu' = (v - 2k) + \lambda$.

Theorem 3.1.4 (Duval's Main Theorem) *[13] If* Γ *is a DSRG with parameters* (v, k, t, λ, μ) *and adjacency matrix* A, *and* Γ *is neither a strongly regular graph* $(t = k)$ *nor a complete graph* $(A = J - I)$, *then* A *is equivalent to a Hadamard matrix* $(A + A^t = J - I,\ AA^t = (\mu - 1)J + \mu I)$ *or for some positive integer* d,

- $k(k + (\mu - \lambda)) = t + (v - 1)\mu$.
- $(\mu - \lambda)^2 + 4(t - \mu) = d^2$.
- $2k - (\mu - \lambda)(v - 1) \equiv 0 \ (mod\ d)$.
- $\dfrac{2k - (\mu - \lambda)(v - 1)}{d} \equiv v - 1 \ (mod\ 2)$.
- $\left|\dfrac{2k - (\mu - \lambda)(v - 1)}{d}\right| \leq v - 1$.

Proof. The eigenvalues of J are v with multiplicity 1 corresponding to eigenvector $\mathbf{j}$, where $\mathbf{j}$ is all-ones vector, and 0 with multiplicity $v-1$. Since $AJ = JA = kJ$, $\mathbf{j}$ is also an eigenvector of A, corresponding to eigenvalue k. Let the other eigenvalues of A be θ_i $(i = 1, \cdots, v-1)$. By (3.2)

$$k^2 + (\mu - \lambda)k - (t - \mu) = \mu v \tag{3.3}$$

$$\theta_i^2 + (\mu - \lambda)\theta_i - (t - \mu) = 0 \tag{3.4}$$

so,

$$\theta_i = \frac{1}{2}\left(-(\mu - \lambda) \pm \sqrt{(\mu - \lambda)^2 + 4(t - \mu)}\right) \tag{3.5}$$

and (3.3) proves $k(k + (\mu - \lambda)) = t + (v - 1)\mu$. Let m_1 and m_2 be multiplicities θ_1 and θ_2 respectively. Then,

$$m_1 + m_2 = v - 1 \tag{3.6}$$

$$0 = \text{trace } (A) = k + m_1\theta_1 + m_2\theta_2 \tag{3.7}$$

so,

$$0 = k - \frac{1}{2}(v - 1)(\mu - \lambda) + (m_1 - m_2)\frac{1}{2}\sqrt{(\mu - \lambda)^2 + 4(t - \mu)}. \tag{3.8}$$

First, suppose $(\mu - \lambda)^2 + 4(t - \mu) \neq d^2$ for any positive integer. Then, $k = \frac{1}{2}(v - 1)(\mu - \lambda)$. So, $\mu - \lambda$ is 1 or 2 as $0 < k \leq v - 1$.

Case 1: $\mu - \lambda = 2$, $k = v - 1$. Because $k = v - 1$, we have $A = J - I$, which contradicts our hypothesis.

Case 2: $\mu - \lambda = 1$, $k = \frac{1}{2}(v - 1)$. Substituting into (3.3) gives

$$k(k + 1 - 2\mu) = t.$$

So, $t = k + 1 - 2\mu = 0$ as $0 \leq t < k$. Because $t = 0$, there are no undirected edges and $k = \frac{1}{2}(v - 1)$ implies that

$$A + A^t = J - I.$$

Also, we have

$$A^2 + A = \mu(J - I).$$

Thus,

$$AA^t = A(J - I - A) = kJ - A - A^2 = kJ - \mu(J - I),$$

$$AA^t = (k - \mu)J + \mu I = (\mu - 1)J + \mu I.$$

So, A is equivalent to a Hadamard matrix of order 4μ. Therefore, we may assume $(\mu - \lambda)^2 + 4(t - \mu) = d^2$ for some positive integer d. So,

$$\theta_1 = \frac{1}{2}(-(\mu - \lambda) + d), \quad \theta_2 = \frac{1}{2}(-(\mu - \lambda) - d).$$

We also have

$$m_1 = \frac{k + \theta_1(v - 1)}{\theta_1 - \theta_2}, \quad m_2 = \frac{k + \theta_2(v - 1)}{\theta_1 - \theta_2}.$$

Thus,

$$m_1 - m_2 = \frac{2k - (\mu - \lambda)(v - 1)}{d}.$$

As m_1 and m_2 are eigenvalue multiplicities, they must be integers. This will occur if and only if $m_1 + m_2$ and $m_1 - m_2$ are integers and have the same parity. Therefore, d divides $2k - (\mu - \lambda)(v - 1)$ and $\dfrac{2k - (\mu - \lambda)(v - 1)}{d} \equiv v - 1 \pmod 2$. We also require that m_1 and m_2 be nonnegative. $m_1 + m_2 > 0$ occurs if and only if $|m_1 - m_2| \leq m_1 + m_2 = v - 1$ i.e. $\left|\dfrac{2k - (\mu - \lambda)(v - 1)}{d}\right| \leq v - 1$. □

Theorem 3.1.5 *[13] If Γ is a DSRG with parameters (v, k, t, λ, μ), then*

$$0 \leq \lambda < t < k,$$

$$0 < \mu \leq t < k.$$

Theorem 3.1.6 *[13] If Γ is a DSRG with parameters (v, k, t, λ, μ), then*

$$-2(k - t - 1) \leq \mu - t \leq 2(k - t).$$

Theorem 3.1.7 *[13] If Γ is a DSRG with parameters (v, k, t, λ, μ), then v is not a prime.*

Theorem 3.1.8 *[13] Let A be the adjacency matrix of a DSRG with parameters (v, k, t, λ, μ) and J be the $m \times m$ all-ones matrix. Then, the Kronecker matrix product $A \otimes J$ of A and J is the adjacency matrix of a DSRG with the parameters $(v', k', t', \lambda', \mu')$ if and only if $t = \mu$. In this case, $(v', k', t', \lambda', \mu') = (mv, mk, mt, m\lambda, m\mu)$.*

Theorem 3.1.9 *[13] Let A be the adjacency matrix of a DSRG with parameters (v, k, t, λ, μ) and J be an $m \times m$ all-ones matrix. Then, $A \otimes J \oplus (I \otimes (J - I))$ is the adjacency matrix of a DSRG with parameters $(v', k', t', \lambda', \mu')$ if and only if $t = \lambda - 1$. In this case, $(v', k', t', \lambda', \mu') = (mv, m(k+1) - 1, m(t+1) - 1, m(t+1) - 2, m\mu)$.*

We now explicitly construct DSRGs using antiflags and flags of $1\frac{1}{2}$-designs.

Theorem 3.1.10 *Let $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a tactical configuration. Let $\Gamma = \Gamma(\mathcal{T})$ be the directed graph defined on the set*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : p \notin B\}$$

with adjacency defined by

$$(p, B) \to (q, C) \text{ if and only if } p \in C.$$

Then, Γ is directed strongly regular if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof. First, we assume $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(v^*, b^*, k^*, r^*; \alpha, \beta)$ and show $\Gamma = \Gamma(\mathcal{T})$ is a DSRG with parameters

$$v = b^*(v^* - k^*), \quad k = r^*(v^* - k^*), \quad t = \mu = k^*r^* - \alpha, \quad \lambda = k^*r^* - k^* - r^* + 1 - \beta.$$

Parameter k is obtained immediately from the definition of Γ, and it follows that Γ is a regular (mixed) digraph.

Given a vertex $(p, B) \in V(\Gamma)$, the parameter t is given by

$$t = |\{(q, C) \in V(\Gamma) : \ q \in B, \ C \ni p\}| = \sum_{p \in C \in \mathcal{B}} (|B| - |C \cap B|) = k^*r^* - \alpha.$$

For the given two distinct vertices (p, B) and (q, C), such that $p \notin C$, μ counts the number of vertices $(r, D) \in V(\Gamma)$ such that $r \in C$ and $D \ni p$. So,

$$\mu = \sum_{p \in D \in \mathcal{B}} (|C| - |C \cap D|) = k^*r^* - \alpha = t.$$

For the given two vertices $(p, B), (q, C)$ with $p \in C$, λ counts the number of vertices (r, D), such that $r \in C$ and $D \ni p$. So,

$$\lambda = \sum_{p \in D \in \mathcal{B}} (|C| - |C \cap D|) = k^*r^* - (\beta + k^* + r^* - 1).$$

It follows Γ is directed strongly regular.

For the converse, suppose Γ is directed strongly regular. The parameter t exists subject to the fact that for any $(p, B) \in V(\Gamma)$, the numbers $\tau_1(p, B) = \sum_{p \in C \in \mathcal{B} \setminus \{B\}} |C \cap B|$ and $\tau_2(p, B) = \sum_{q \in B} |(p) \cap (q)|$ are equal to a constant τ, and τ does not depend on the choice of vertex (p, B). Such a τ exists if and only if the number $\alpha(p, B)$ is constant α for all antiflags (p, B) in the tactical configuration.

Similarly, parameter λ exists if and only if the number $\sum_{p \in D \in \mathcal{B}} |C \cap D|$ is constant over all flags (p, C). This is equivalent to the condition that the number

$$\alpha(p, C) = |\{(r, D) : \ r \in C \setminus \{p\}, \ D \ni p, \ r \in D, D \neq C\}|$$

is constant for all flags (p, C). Hence, the tactical configuration must be a $1\frac{1}{2}$-design. This completes the proof. □

Example 3.1.11 *Let P be a ql-element set with a partition $\mathcal{P}$ of P into l parts of size q. Let $\mathcal{P} = \{S_1, S_2, \ldots, S_l\}$. Let*

$$\mathcal{B} = \{B \subset P : \ |B \cap S_i| = 1 \textit{ for all } i = 1, 2, \ldots, l\}.$$

Then, the incidence structure, $\mathcal{T} = (P, \mathcal{B}, \in)$, becomes a $1\frac{1}{2}$-design with parameters

$$v^* = ql, b^* = q^l, k^* = l, r^* = q^{l-1}; \alpha = (l-1)q^{l-2}, \ \beta = (l-1)(q^{l-2} - 1).$$

Therefore, the graph $D = D(\mathcal{T})$ is a DSRG with parameters

$$\begin{aligned} v &= lq^l(q-1), \\ k &= lq^{l-1}(q-1), \\ t &= \mu = q^{l-2}(lq - l + 1), \\ \lambda &= q^{l-2}(l-1)(q-1). \end{aligned}$$

Example 3.1.12 *Let P be the set of $2n$ vertices, and $\mathcal{B}$ the set of n^2 edges of the complete bipartite graph $K_{n,n}$. Then, the incidence structure $\mathcal{T} = (P, \mathcal{B}, \in)$ is a $1\frac{1}{2}$-design with parameters*

$$v^* = 2n, b^* = n^2, k^* = 2, r^* = n; \alpha = 1, \beta = 0.$$

Therefore, the graph $\Gamma(\mathcal{T})$ is a DSRG with parameters

$$(2n^2(n-1), 2n(n-1), 2n-1, n-1, 2n-1).$$

Example 3.1.13 *Let $\mathcal{T}$ be a degenerated affine plane of order q. Then $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(v^*, b^*, k^*, r^*; \alpha, \beta) = (q^2, ql, q, l; l-1, 0)$. $\Gamma(\mathcal{T})$ is a DSRG with parameters*

$$(lq^2(q-1),\ lq(q-1), lq-l+1, (l-1)(q-1), lq-l+1).$$

Theorem 3.1.14 *Let $\mathcal{T} = (\mathcal{P}, \mathcal{B}, \in)$ be a tactical configuration. Let $\Gamma = \Gamma(\mathcal{T})$ be the directed graph defined by*

$$V(\Gamma) = \{(p, B) \in P \times \mathcal{B} :\ p \in B\}$$

and adjacency by

$$(p, B) \to (q, C) \textit{ if and only if } (p, B) \neq (q, C) \textit{ and } p \in C.$$

Then, Γ is a DSRG if and only if $\mathcal{T}$ is a $1\frac{1}{2}$-design.

Proof. If $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters $(v^*, b^*, k^*, r^*; \alpha, \beta)$, then it is shown that Γ is a DSRG with parameters

$$(v^*r^*, \quad r^*k^* - 1, \quad k^* + r^* + \beta - 2, \quad k^* + r^* + \beta - 3, \quad \alpha).$$

The argument is similar to the proof of Theorem 3.1.10.

Given a vertex $(p, B) \in V(\Gamma)$,

$$t = |\{(q, C) \in V(\Gamma) :\ q \in B,\ C \ni p,\ (q, C) \neq (p, B)\}|$$

$$= |\{(q, C) \in P \times \mathcal{B} :\ q \in B,\ C = B,\ q \neq p\}|$$

$$+\ |\{(q, C) \in P \times \mathcal{B} :\ q = p,\ C \ni p,\ C \neq B\}|$$

$$+|\{(q, C) \in P \times \mathcal{B} :\ q \in B \setminus \{p\},\ C \ni p,\ C \ni q,\ C \neq B\}|$$

$$= \quad (k^* - 1) + (r^* - 1) + \beta.$$

Given adjacent vertices (p, B) and (q, C) with $p \in C$,

$$\lambda = |(r, D) \in V(\Gamma) :\ r \in C,\ D \ni p,\ (r, D) \neq (p, B),\ (r, D) \neq (q, C)\}|$$

$$= |\{(r,D) \in P \times \mathcal{B} : r \in D = C,\ r \neq q\}|$$
$$+ |\{(r,D) \in P \times \mathcal{B} :\ r = p \in D,\ D \neq B,\ D \neq C\}|$$
$$+ |\{(r,D) \in P \times \mathcal{B} :\ r \in C \setminus \{p\},\ D \supseteq \{p,r\}, D \neq C\}|$$
$$= (k^* - 1) + (r^* - 2) + \beta$$

Given two distinct vertices (p,B) and (q,C), such that $p \notin C$,

$$\mu = |\{(r,D) \in V(\Gamma) :\ r \in C,\ D \ni p,\ (r,D) \neq (p,B),\ (r,D) \neq (q,C)\}|$$

$$= |\{(r,D) \in P \times \mathcal{B} :\ r \in C,\ D \ni p,\ D \ni r,\ (r,D) \neq (q,C)\}|,$$

since $p \in D$, $(r,D) \neq (q,C)$; and so, μ equals $\alpha(p,C)$ for $p \notin C$ and $\mu = \alpha$. Hence, it follows Γ is directed strongly regular.

For the converse, suppose the regular directed graph Γ is a DSRG with parameters

$$(b^*k^*, \quad r^*k^* - 1, \quad t, \quad \lambda, \quad \mu).$$

Then, in the above counting of the parameters t, λ, and μ, the parameters cannot be constants unless the numbers $\alpha(p,B)$, $\alpha(p,D)$, and $\alpha(p,C)$, respectively, are constants. Thus the $1\frac{1}{2}$-design 'regularity condition' is necessary for Γ to be directed strongly regular. This completes the proof. □

Example 3.1.15 *Let $\mathcal{T}$ be a $(\overline{v}, \overline{k}, \overline{\lambda})$-BIBD. Then, $\mathcal{T}$ is a $1\frac{1}{2}$-design with parameters* $(v^*, b^*, k^*, r^*; \alpha, \beta) = (\overline{v}, \frac{\overline{\lambda}\overline{v}(\overline{v}-1)}{\overline{k}-1}, \overline{k}, \frac{\overline{\lambda}(\overline{v}-1)}{\overline{k}-1}; \overline{\lambda k}, (\overline{\lambda}-1)(\overline{k}-1))$. *$\Gamma(\mathcal{T})$ is a DSRG with parameters*

$$(\overline{v}r^*, \quad r^*k^* - 1, \quad k^* + r^* + \beta - 2, \quad k^* + r^* + \beta - 3, \quad \alpha).$$

The above characterization theorems may be used to show the non-existence of $1\frac{1}{2}$-designs with given parameter sets. We provide an example. Suppose there exists a $1\frac{1}{2}$-design with parameters $(8, 16, 5, 10; 25, 35)$. Then, there is a DSRG with parameters $(48, 30, 25, 15, 25)$. However, it is known that there is no DSRG $(48m, 30m, 25m, 15m, 25m)$ for any positive integer m by Jørgensen [26]. So, although the parameter set $(v^*, b^*, k^*, r^*; \alpha, \beta) = (8, 16, 5, 10; 25, 35)$ satisfies all the necessary conditions, there is no $1\frac{1}{2}$-design with these parameters. More generally, we will introduce the Jorgensen's non-existence theorem to provide a non-existence result for $1\frac{1}{2}$-designs.

Theorem 3.1.16 *[26] Suppose* (v, k, t, λ, μ) *are the parameters of a DSRG with* $t < k$ *whose adjacency matrix has rank at most 4. Then either the adjacency matrix has rank 3 and* $(v, k, t, \lambda, \mu) = (6s, 2s, s, 0, s)$*, or* $(v, k, t, \lambda, \mu) = (8s, 4s, 3s, s, 3s)$*, or else the adjacency matrix has rank 4 and then* $(v, k, t, \lambda, \mu) = (6s, 3s, 2s, s, 2s)$ *or* $(v, k, t, \lambda, \mu) = (12s, 3s, s, 0, s)$*.*

Theorem 3.1.17 *Suppose there exists a* $1\frac{1}{2}$*-design with parameters* $(v^*, b^*, k^*, r^*; \alpha, \beta)$ *such that* $r^*(v^* - k^*) = 3n$ *where* $n = k^* + r^* - 1 + \beta - \alpha$*. Then either*

$$\frac{b^*}{r^*} = 2 \text{ and } k^*r^* = 2(k^* + r^* - 1 + \beta) - \alpha$$

or

$$\frac{b^*}{r^*} = 4 \text{ and } k^*r^* = k^* + r^* - 1 + \beta.$$

Proof. Assume there exists a $1\frac{1}{2}$-design with parameters $(v^*, b^*, k^*, r^*; \alpha, \beta)$, such that $r^*(v^* - k^*) = 3n$. Then, there exists a DSRG with parameters $v = b^*(v^* - k^*)$, $k = r^*(v^* - k^*)$, $t = \mu = k^*r^* - \alpha$, and $\lambda = k^*r^* - (k^* + r^* - 1 + \beta)$. Let $d = \mu - \lambda = n$ and A be an adjacency matrix of the DSRG. Then, the eigenvalues of A are k, d, and 0 and their multiplicities are 1, $\frac{k}{d}$ and $v - 1 - \frac{k}{d}$ respectively. By our assumption $\frac{k}{d} = 3$. Hence, the rank of A is 4. The rest follows from Theorem 3 of Jørgensen [26]. □

Remark 3.1.18 *By the above theorem, there is no* $1\frac{1}{2}$*-design with parameters* $(9, 9, 3, 3; 1, 2)$ *and* $(16, 16, 6, 6; 6, 15)$*.*

Theorem 3.1.19 *Suppose there exists a* $1\frac{1}{2}$*-design with parameters* $(v^*, b^*, k^*, r^*; \alpha, \beta)$*, such that* $r^*(v^* - k^*) = 2n$ *where* $n = k^* + r^* - 1 + \beta - \alpha$*. Then, either*

$$\frac{b^*}{r^*} = 2 \text{ and } k^*r^* = \frac{3(k^* + r^* - 1 + \beta) - \alpha}{2}$$

or

$$\frac{b^*}{r^*} = 3 \text{ and } k^*r^* = k^* + r^* - 1 + \beta.$$

Remark 3.1.20 *By the above theorem, there is no* $1\frac{1}{2}$*-design with parameters* $(9, 9, 6, 6; 18, 16)$ *and* $(12, 12, 8, 8; 32, 33)$*.*

Notes 3.1.21 *A compact proof of equivalence of $1\frac{1}{2}$-designs and DSRGs which uses incidence matrices of the designs, can be found in "Directed strongly regular graphs from $1\frac{1}{2}$-designs" [9].*

Research Problem : *Let $\mathcal{T}$ be a tactical configuration. Define an adjacency, which is different from the ones defined in the above theorems, between anti-flags and flags of $\mathcal{T}$ which yields DSRGs. Does $\mathcal{T}$ must be a $1\frac{1}{2}$-design?*

Research Problem : *Let Γ be a DSRG obtained from a $1\frac{1}{2}$-design $\mathcal{T}$. Is there any relationship between $Aut(\Gamma)$ and $Aut(\mathcal{T})$?*

3.2 Non-isomorphic Directed Strongly Regular Graphs Obtained from Certain Tactical Configurations

Identifying co-spectral graphs is a key concept to understand which properties cannot be distinguished by eigenvalues or the parameters of the graphs. Here, we will use tactical configurations to construct non-isomorphic DSRGs.

Let r and q be positive integers, such that $q - 1 = ab$ for some positive integers a and b. We assume all these integers are greater than 1. We consider the case of $a = 1$ and the case of $b = 1$ separately later. Let $P = \{1, 2, \ldots, n\}$ be an n-element set with $n = rq$. Let $\{G_1, G_2, \ldots, G_r\}$ be a partition of P into r subsets (called 'groups') of size q. For each $j = 1, 2, \ldots, r$, let $(G_j, \mathcal{P}_j)$ be a tactical configuration with parameters (q, q, a, a). That is, the block set $\mathcal{P}_j$ consists of q blocks, such that each block is an a-element subset of G_j, and every point of G_j appears in exactly a blocks. Let the blocks in $\mathcal{P}_j$ be labeled by $P_{j1}, P_{j2}, \ldots, P_{jq}$. It is clear that $P_{ig} \cap P_{jh} = \emptyset$ if $i \neq j$, since groups are disjoint. Let $\{B_1, B_2, \ldots, B_q\}$ be a family of ra-element subsets of P defined in such a way that

(i) every B_i contains exactly one block from every $\mathcal{P}_j$, and

(ii) each block in each $\mathcal{P}_j$ is contained in B_i for exactly one i.

For each $g \in G_h$, let $\{X_{g1}, X_{g2}, \ldots, X_{gb}\}$ be a partition of $G_h \setminus \{g\}$ with $|X_{gl}| = a$ for all $l \in \{1, 2, \ldots, b\}$. That is, $G_j \setminus \{g\} = X_{g1} \cup X_{g2} \cup \cdots \cup X_{gb}$ and $|X_{gl}| = a$ for every l. Then, we have the following tactical configuration.

Lemma 3.2.1 *For each point $g \in G_h$ and each $l \in \{1, 2, \ldots, b\}$, if we define*

$$B_{gl,j} = X_{gl} \cup (B_j \setminus P_{hj}) \ \ \textit{for } j = 1, 2, \ldots, q,$$

$$\mathcal{B}_g = \{B_{gl,j} : \ 1 \le j \le q, \ 1 \le l \le b\},$$

and

$$\mathcal{B} \ = \ \bigcup_{g=1}^{rq} \mathcal{B}_g \ = \ \{B_{gl,j} : \ 1 \le g \le rq, \ 1 \le l \le b, \ 1 \le j \le q\},$$

then the pair $(P, \mathcal{B})$ forms a tactical configuration with parameters

$$(rq, \ rq^2 b, \ ra, \ rq(q-1)).$$

Proof. From the definition, $\overline{v}$ and $\overline{k}$ are clear and $\overline{b} = |P||\mathcal{B}_i| = rq \cdot qb$. For $\overline{r}$, given a point $g \in G_h$, we must determine the size of the set $\{B' \in \mathcal{B} : \ g \in B'\}$. We claim that $\overline{r}$ is the sum of $q(q-1)$ and $(r-1)qab$. The first summand $q(q-1)$ comes from the fact that g is a member of an X_{il} for each $i \in G_h \setminus \{g\}$, and each X_{il} is contained in q blocks in $\mathcal{B}_i$. The second summand $(r-1)q \cdot ab$ is the number of blocks B', such that $g \in B' \in \mathcal{B} \setminus (\bigcup_{i \in G_h} \mathcal{B}_i)$, since g belongs to B_j for a different j's (because g belongs to a blocks of $(G_h, \mathcal{P}_h)$), and each B_j is contained in b blocks of $\mathcal{B}_i$ for each of $(r-1)q$ points $i \in P \setminus G_h$. This completes the proof. □

We now use this tactical configuration to construct a directed strongly regular graph as follows.

Theorem 3.2.2 *Let $\mathcal{T}$ be the above tactical configuration $(P, \mathcal{B})$. Let $D = D(\mathcal{T})$ be the directed graph defined on the vertex set*

$$V(D) = \{(g, B) : \ B \in \mathcal{B}_g, \ g \in P\},$$

with adjacency between vertices $(g, B), (g', B') \in V(D)$ defined by $(g, B) \to (g', B')$ if and only if $g \in B'$. Then D is a DSRG with parameters (v, k, t, λ, μ) equals to

$$(rq^2(q-1)/a, \ rq(q-1), \ r(q-1)a + a, \ q(a-1) + (r-1)(q-1)a, \ r(q-1)a + a).$$

Proof. It is clear that $v = |\mathcal{B}| = rq^2 b = rq^2(q-1)/a$. Parameter k is the size of the set $\{(g', B') \in V(D) : \ g \in B'\}$ for a given vertex $(g, B) \in V(D)$, and equals $b = rq(q-1)$. To compute t, let $(g,B) \in V(D)$ with $g \in G_h$ and let $B = B_{gl,j} = X_{gl} \cup (B_j \setminus P_{hj})$ for some l and j. Then, $t = |\{(g',B') \in V(D) : \; g' \in B, \; g \in B'\}|$. We see for each $g' \in X_{gl} \subset B$, there are q blocks in $\mathcal{B}_{g'}$, all of which contain g. On the other hand, for each $g' \in B_j \setminus P_{hj} \subset B$, there are $ab = q-1$ blocks in $\mathcal{B}_{g'}$ containing g. Together, we have $t = qa + (q-1)(r-1)a$ as desired, since $|X_{gl}| = a$ and $|B_j \setminus P_{hj}| = (r-1)a$.

Let (g,B) and (g',B') be two adjacent vertices with $g \in B'$. Suppose $g' \in G_f$ and $B' = B_{g'l,j} = X_{g'l} \cup (B_j \setminus P_{fj})$. To show $\lambda = |\{(g^*,B^*) \in V(D) : \; g^* \in B', \; B^* \ni g\}|$ is constant, we consider two cases:

Case 1. Suppose $g \in G_f$, that is, $g \in X_{g'l}$. Then, (i) for each element, say g^*, of $X_{g'l} \setminus \{g\}$, there are q blocks of $\mathcal{B}_{g^*}$ containing g; while (ii) for each element $g^* \in B_j \setminus P_{fj}$, there are ab blocks of $\mathcal{B}_{g^*}$ containing g. Therefore, $\lambda = (a-1)q + (r-1)a(q-1)$ in this case.

Case 2. If $g \notin G_f$, then g must be an element of $B_j \setminus P_{fj}$. Suppose $g \in P_{hj} \subset G_h$. Then (i) for each choice of $g^* \in X_{g'l}$ there are $ab = q-1$ blocks possessing g (so available for B^*) in $\mathcal{B}_{g^*}$; (ii) for each choice of $g^* \in P_{hj} \setminus \{g\}$, there are q blocks possessing g in $\mathcal{B}_{g^*}$; and (iii) for each element g^* of the remaining $(r-2)a$ elements in B', there are $(q-1)$ blocks available for B^* in $\mathcal{B}_{g^*}$. Hence, together, we have $\lambda = a(q-1) + (a-1)q + (r-2)a(q-1)$ as well.

Hence, λ has a constant value $(a-1)q + (r-1)a(q-1)$.

For μ, let $(g,B) \nrightarrow (g',B')$, (so $g \notin B'$). Let g belong to G_h for some h. Then, by the similar counting argument, we can verify that the number of vertices (g^*,B^*) such that $(g,B) \rightarrow (g^*,B^*) \rightarrow (g',B')$ (or equivalently the number of choices for g^* and B^* such that $g^* \in B'$ and $g \in B^*$) is $aq + (r-1)a(q-1)$, whether g and g' belong to the same group G_h for some h or not, as a vertices in B' can be paired with q blocks, while the remainder can be paired with $ab = (q-1)$ blocks. This completes the proof. $\square$

Corollary 3.2.3 *Let r, q and a be positive integers such that $a|(q-1)$ as before. Then there exist directed strongly regular graphs with parameters*

$$(mrq^2(q-1)/a, \; mrq(q-1), \; m(rqa-ra+a), \; m\{q(a-1)+(r-1)(q-1)a\}, \; m(rqa-ra+a))$$

for all positive integer m.

Example 3.2.4 *To illustrate the above construction, we consider the case when* $r = 2, q = 5$, *and* $a = b = 2$. *This will give us DSRG-*$(100, 40, 18, 13, 18)$.

Let $P = \{0, 1, \ldots, 9\}$, $G_1 = \{1, 2, 3, 4, 5\}$, $G_2 = P \setminus G_1$. $\mathcal{P}_1 = \{12, 23, 34, 45, 15\}$ *and* $\mathcal{P}_2 = \{67, 78, 89, 90, 60\}$. *Then, one example of tactical configuration that will produce a DSRG-*$(100, 40, 18, 13, 18)$ *may be described as in the following table. In this table entries* 23, 45, *and* 2367 *represent the sets* $\{2, 3\}$, $\{4, 5\}$ *and* $\{2, 3, 6, 7\}$, *respectively.*

Table 3.1 *The blocks* $\mathcal{B}_i$ *for each point* i.

i	X_{i1}, X_{i2}	$B_{i1,1}$	$B_{i1,2}$	$B_{i1,3}$	$B_{i1,4}$	$B_{i1,5}$	$B_{i2,1}$	$B_{i2,2}$	$B_{i2,3}$	$B_{i2,4}$	$B_{i2,5}$
1	23, 45	2367	2378	2389	2390	2360	4567	4578	4589	4590	4560
2	13, 45	1367	1378	1389	1390	1360	4567	4578	4589	4590	4560
3	12, 45	1267	1278	1289	1290	1260	4567	4578	4589	4590	4560
4	12, 35	1267	1278	1289	1290	1260	3567	3578	3589	3590	3560
5	12, 34	1267	1278	1289	1290	1260	3467	3478	3489	3490	3460
6	78, 90	7812	7823	7834	7845	7815	9012	9023	9034	9045	9015
7	89, 60	8912	8923	8934	8945	8915	6012	6023	6034	6045	6015
8	79, 60	7912	7923	7934	7945	7915	6012	6023	6034	6045	6015
9	67, 80	6712	6723	6734	6745	6715	8012	8023	8034	8045	8015
0	67, 89	6712	6723	6734	6745	6715	8912	8923	8934	8945	8915

3.3 DSRG-$(r(1+a)^2,\ r(1+a)a,\ ra^2+a,\ ra^2-1,\ ra^2+a)$

In this section, we consider the particular case of the above construction for the case when $b = 1$. Let r and q be positive integers greater than 1, and $P = \{1, 2, \ldots, rq\}$ a set of rq elements. Let $\{G_1, G_2, \ldots, G_r\}$ be a partition of P into r groups of size q. For each $j = 1, 2, \ldots, r$, let $\mathcal{P}_j$ be the family of all $(q-1)$-element subsets of G_j. Let $B_1, B_2, \ldots, B_q$ be $r(q-1)$-element subsets of P defined as follows:

(1) Select one set from each family to have $B_1 = \bigcup_{j=1}^{r} P_{j1}$, where $P_{j1} \in \mathcal{P}_j$ for $j = 1, 2, \ldots, r$.

(2) For B_2, select one set from each $\mathcal{P}_j \setminus \{P_{j1}\}$, for $j = 1, 2, \ldots, r$, so that $B_2 = \bigcup_{j=1}^{r} P_{j2}$.

(3) Continue this process to obtain

$$B_i = P_{1i} \cup P_{2i} \cup \cdots \cup P_{ri} \quad \text{where } P_{ji} \in \mathcal{P}_j \setminus \{P_{j1}, P_{j2}, \ldots, P_{j(i-1)}\}$$

for $i = 3, 4, \ldots, q$.

Then, for each point $g \in G_h$, define

$$B_{g,j} = (G_h \setminus \{g\}) \cup (B_j \setminus P_{hj}) \ \text{ for } j = 1, 2, \ldots, q$$

and $\mathcal{B}_g = \{B_{g,1}, B_{g,2}, \ldots, B_{g,q}\}$. Then, with

$$\mathcal{B} = \bigcup_{g \in P} \mathcal{B}_g = \{B_{g,j} : \ 1 \le g \le rq, \ 1 \le j \le q\},$$

the pair $(P, \mathcal{B})$ becomes a tactical configuration with parameters

$$(v, b, k, r) = (rq, \ rq^2, \ r(q-1), \ rq(q-1)).$$

Theorem 3.3.1 *Let $\mathcal{T}$ be the above tactical configuration $(P, \mathcal{B})$. Let $D = D(\mathcal{T})$ be the directed graph defined on the vertex set,*

$$V(D) = \{(g, B) : \ B \in \mathcal{B}_g, \ g \in P\},$$

with adjacency between vertices (g, B) and (g', B') defined by $(g, B) \to (g', B')$ if and only if $g \in B'$. Then, D is a directed strongly regular graph with parameters (v, k, t, λ, μ) equal to

$$(rq^2, \ rq(q-1), \ (q-1)(rq-r+1), \ r(q-1)^2 - 1, \ (q-1)(rq-r+1)).$$

Proof. It is clear that $v = rq^2$, $k = q(q-1) + (r-1)q(q-1)$, and $t = q(q-1) + (r-1)(q-1)^2$.

In order to show that λ is constant, consider vertices (g, B) and (g', B') with $(g, B) \to (g', B')$ (and so $g \in B'$). We consider two cases.

Case 1. Suppose both g and g' belong to the same group, say G_j for some j. Then, the number of vertices (g^*, B^*) such that $B^* \ni g$ and $g^* \in B'$ may be counted as follows. (i) Since $G_j \setminus \{g'\} \subset B'$, with any of $q-2$ choices for g^* from $G_j \setminus \{g, g'\}$, all q blocks in $\mathcal{B}_{g^*}$ provide the legitimate pairs (g^*, B^*) as every block in $\mathcal{B}_{g^*}$ has g in it. (ii) Since $|B' \cap (P \setminus G_j)| = (r-1)(q-1)$, so there are $(r-1)(q-1)$ possible points available for $g^* \in B' \cap (P \setminus G_j)$. For each point g^* of these possible points, there are $q-1$ blocks possessing g in $\mathcal{B}_{g^*}$. Hence, we must have $\lambda = q(q-2) + (r-1)(q-1)(q-1) = r(q-1)^2 - 1$.

Case 2. Suppose $g \in G_j$ for some j and $g' \notin G_j$. Then the number of ways to pick suitable (g^*, B^*) may be counted as follows: (i) With each of $q-2$ possible $g^* \in (B' \setminus \{g\}) \cap G_j$, there are q blocks possessing g in $\mathcal{B}_{g^*}$; and thus, we can have $q(q-2)$ such vertices (g^*, B^*). (ii) With any g^* of $r(q-1) - (q-1)$ points in $B' \setminus G_j$, there are $q-1$ blocks in $\mathcal{B}_{g^*}$ for B'. Hence we also have $\lambda = q(q-2) + (r-1)(q-1)^2$ as desired. Thus, we see that λ is a constant.

For μ, suppose $(g, B) \nrightarrow (g', B')$, (so $g \notin B'$). Let $g \in G_j$ for some j.

Case 1. Suppose $g = g'$ and $B \neq B'$. Then vertices (g^*, B^*) such that $(g, B) \rightarrow (g^*, B^*) \rightarrow (g', B')$ may be counted as follows. For each $g^* \in B'$, the number of blocks B' in $\mathcal{B}_{g^*}$ that can be paired with g^* is q blocks if $g^* \in B' \cap (G_j \setminus \{g\})$, while is $q-1$ blocks if $g^* \in B' \cap (P \setminus G_j)$. Since there are $q-1$ choices for g^* in the former and $(r-1)(q-1)$ choices for the latter, we must have $\mu = q(q-1) + (r-1)(q-1)^2$.

Case 2. If $g \neq g'$, then g' must be in $P \setminus G_j$ since neither g nor g' may be in B'. This means B' should be a block that contains all $q-1$ elements of $G_j \setminus \{g\}$. For any of $G_j \setminus \{g\}$ as g^*, there are q blocks that contain g in $\mathcal{B}_{g^*}$. (This gives us $q(q-1)$ desired vertices (g^*, B^*).) For each of $(r-1)(q-1)$ possible points in $B' \setminus G_j$, there are $(q-1)$ blocks containing g. Hence, we have $q(q-1) + (r-1)(q-1)^2$ for μ in this case as well.

This completes the proof. □

Example 3.3.2 *Let* $r = 2, q = 3$, $P = \{1, 2, 3, 4, 5, 6\}$, $G_1 = \{1, 2, 3\}$ *and* $G_2 = \{4, 5, 6\}$. *With the tactical configuration described in Table 3. 2, we have a DSRG-*$(18, 12, 10, 7, 10)$. *This graph is shown to be non-isomorphic to its orientation reversing conjugate. We know that these are the two non-isomorphic graphs with the parameters* $(18, 12, 10, 7, 10)$ *and there are no more.*

Table 3.2 $\mathcal{T}-(6,18,4,12)$.

i	$B_{i,j}, j=1,2,3$
1	$2356, 2346, 2345$
2	$1356, 1346, 1345$
3	$1256, 1246, 1245$
4	$2356, 1356, 1256$
5	$2346, 1346, 1246$
6	$2345, 1345, 1245$

Table 3.3 $\mathcal{T}-(6,12,3,6)$.

i	$B_{i,j}, j=1,2$
1	$235, 246$
2	$135, 146$
3	$415, 426$
4	$315, 326$
5	$613, 624$
6	$513, 524$

Example 3.3.3 *Let* $r=3, q=2$, $P=\{1,2,3,4,5,6\}$, $G_1=\{1,2\}$ $G_2=\{3,4\}$ *and* $G_3=\{5,6\}$. *With the tactical configuration described in Table 3.3 above, we have a DSRG-*$(12,6,4,2,4)$. *It is easy to see that there are* $2^6=64$ *different tactical configurations available for the given combinations of* $r=3$ *and* $q=2$. *These 64 tactical configurations yield seven non-isomorphic graphs. (Their adjacency matrices are given below.) It is easy to verify that the orientation reversing conjugates, whose adjacency matrices are the transpose of the seven adjacency matrices, are all non-isomorphic. Therefore, our construction provides us 14 distinct graphs with parameters* $(12,6,4,2,4)$. *The table showing the description of the automorphism groups of these graphs and the size of the isomorphism classes are followed by the adjacency matrices.*

Tables 3.4 *The adjacency matrices of the graphs with parameters* $(12,6,4,2,4)$ *constructed in Theorem 3.3.1.*

$$N_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad N_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$N_3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad N_4 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$N_5 = \begin{pmatrix} 0&0&1&1&1&0&1&0&1&0&1&0 \\ 0&0&1&1&1&0&1&0&1&0&1&0 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&0&1&0&0&0&1&1&0&1&0&1 \\ 1&0&1&0&0&0&1&1&0&1&0&1 \\ 0&1&0&1&1&1&0&0&1&0&1&0 \\ 0&1&0&1&1&1&0&0&1&0&1&0 \\ 1&0&1&0&1&0&1&0&0&0&1&1 \\ 1&0&1&0&1&0&1&0&0&0&1&1 \\ 0&1&0&1&0&1&0&1&1&1&0&0 \\ 0&1&0&1&0&1&0&1&1&1&0&0 \end{pmatrix} \quad N_6 = \begin{pmatrix} 0&0&1&1&1&0&1&0&1&0&1&0 \\ 0&0&1&1&1&0&1&0&1&0&1&0 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&0&1&0&0&0&1&1&0&1&1&0 \\ 1&0&1&0&0&0&1&1&0&1&1&0 \\ 0&1&0&1&1&1&0&0&1&0&0&1 \\ 0&1&0&1&1&1&0&0&1&0&0&1 \\ 0&1&1&0&0&1&1&0&0&0&1&1 \\ 0&1&1&0&0&1&1&0&0&0&1&1 \\ 1&0&0&1&1&0&0&1&1&1&0&0 \\ 1&0&0&1&1&0&0&1&1&1&0&0 \end{pmatrix}$$

$$N_7 = \begin{pmatrix} 0&0&1&1&1&0&1&0&1&0&1&0 \\ 0&0&1&1&1&0&1&0&1&0&1&0 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&1&0&0&0&1&0&1&0&1&0&1 \\ 1&0&1&0&0&0&1&1&0&1&0&1 \\ 1&0&1&0&0&0&1&1&0&1&0&1 \\ 0&1&0&1&1&1&0&0&1&0&1&0 \\ 0&1&0&1&1&1&0&0&1&0&1&0 \\ 0&1&1&0&1&0&1&0&0&0&1&1 \\ 0&1&1&0&1&0&1&0&0&0&1&1 \\ 1&0&0&1&0&1&0&1&1&1&0&0 \\ 1&0&0&1&0&1&0&1&1&1&0&0 \end{pmatrix}$$

Graph	N_1	N_2	N_3	N_4	N_5	N_6	N_7
Automorphism Group	D_{12}	D_8	$C_2 \times C_2$	$C_2 \times C_2$	D_{12}	S_4	C_2
Size of Isomorphism Class	4	6	12	12	4	2	24

In the above example, we produced 14 DSRGs with parameters $(12, 6, 4, 2, 4)$. However, Jørgensen has shown that there exist exactly twenty non-isomorphic graphs with parameters $(12, 5, 3, 2, 2)$, which are the complementary graphs of DSRGs with parameters $(12, 6, 4, 2, 4)$. Therefore, there are six graphs not obtained from the above construction.

3.4 DSRG-$(r(1+b)^2b,\ r(1+b)b,\ rb+1,\ rb-b,\ rb+1)$

Let r and q be positive integers greater than 1, such that $r \le q^{r-3}$, and let P be a set of rq elements. Let $\mathcal{P} = \{G_1, G_2, \ldots, G_r\}$ be a partition of P into r groups of size q. Let

$$\mathcal{B} = \{B \subset P :\ |B \cap G_i| = 1 \text{ for all } i = 1, 2, \ldots, r\}.$$

Then, $\mathcal{B}$ consists of q^r subsets (which will be called 'blocks') of P of size r. For each $i \in P$, let

$$\mathcal{B}_i = \{B \in \mathcal{B} : i \in B\}.$$

Then $|\mathcal{B}_i| = q^{r-1}$. Let $\mathcal{B}_i$ be partitioned into q^{r-2} parts, each of which consists of q blocks, such that no two blocks in the same part share any other common point besides i. To be precise, let $\mathcal{B}_{i,1}, \mathcal{B}_{i,2}, \ldots, \mathcal{B}_{i,w}$, where $w = q^{r-2}$, denote the parts of the partition of $\mathcal{B}_i$, so that

$$\mathcal{B}_i = \bigcup_{j=1}^{w} \mathcal{B}_{i,j},$$

where (i) $\mathcal{B}_{i,j} \cap \mathcal{B}_{i,h} = \emptyset$, for any distinct $j, h \in \{1, 2, \ldots, w\}$; (ii) $|\mathcal{B}_{i,j}| = q$, for every $j \in \{1, 2, \ldots, w\}$; and (iii) $B \cap C = \{i\}$ for any $B, C \in \mathcal{B}_{i,j}$ for each $j \in \{1, 2, \ldots, w\}$.

Given any injective map $\pi : \{1, 2, \ldots, rq\} \to \{1, 2, \ldots, w\}$, if $g \in G_h$, let $\mathcal{C}_g^\pi$ denote the collection of all blocks in the $\mathcal{B}_{i,\pi(i)}$ for all $i \in G_h \setminus \{g\}$, and let $\mathcal{B}^\pi$ be the union of $\mathcal{C}_g^\pi$ over all points in P. That is, for each given injection π, define

$$\mathcal{B}^\pi = \bigcup_{g \in P} \mathcal{C}_g^\pi \quad \text{where} \quad \mathcal{C}_g^\pi = \bigcup_{i \in G_h \setminus \{g\}} \mathcal{B}_{i,\pi(i)} \ \text{ for } g \in G_h.$$

Then $\mathcal{T}^\pi = (P, \mathcal{B}^\pi)$ becomes a tactical configuration with parameters

$$(v, b, k, r) = (rq,\ rq^2(q-1),\ r,\ rq(q-1)).$$

We obtain a directed strongly regular graph from this tactical configuration as follows.

Theorem 3.4.1 *Let $D = D(\mathcal{T}^\pi)$ be the directed graph defined on the vertex set*

$$V(D) = \{(g, B): \ B \in \mathcal{C}_g^\pi, \ g \in P\}$$

with adjacency defined by $(g, B) \to (g', B')$ if and only if $g \in B'$. Then, D is a DSRG with parameters (v, k, t, λ, μ) equal to

$$(rq^2(q-1), \ rq(q-1), \ rq - r + 1, \ rq - r - q + 1, \ rq - r + 1).$$

Proof. Since for each $g \in P$, $|\mathcal{C}_g^\pi| = q(q-1)$, we have

$$v = |V(D)| = \sum_{g \in P} |\mathcal{C}_g^\pi| = rq \cdot q(q-1).$$

A vertex (g', B') is an out-neighbor of (g, B), if B' contains g. There are q blocks containing g in $\mathcal{B}_{g,\pi(g)}$. Every block $B' \in \mathcal{B}_{g,\pi(g)}$ can be paired with any point besides the r points of B' to become a neighbor of (g, B). Hence, we have $k = q \cdot (rq - 1)$.

To count the (in and out)-neighbors of a vertex (g, B), we need to count the vertices (g', B'), such that $g' \in B$ and $B' \ni g$. If g belongs to G_j for some j and if g' belongs to $B \cap G_j$, then g' can be paired with any block containing g to become both (in and out)-neighbors of (g, B). Any of the remaining $r-1$ points belonging to B (except g') can be paired with any $q-1$ blocks containing g (excluding the block containing both g and g'); thus, we have $t = q + (r-1)(q-1)$.

Given $(g, B) \to (g', B')$, (and so $g \in B'$), the parameter λ counts the vertices $(g^*, B^*) \in V(D)$, such that $B^* \ni g$ and $g^* \in B'$. There are $q - 1$ choices for B^* (except for the block B') and $r - 1$ choices for g^* in B' excluding g; and thus, $\lambda = (r-1)(q-1)$.

For μ, let $(g, B) \nrightarrow (g', B')$, (and so $g \notin B'$). If g belongs to G_j for some j and $B' \cap G_j = \{g^*\}$, then any block B^* containing g can be paired with g^* to form a path of length two from $(g, B) \to (g^*, B^*) \to (g', B')$. Every other point in B' can be paired with any $q - 1$ blocks containing g (excluding the block containing both g and itself). Hence, we have $\mu = q + (r-1)(q-1)$. This completes the proof. □

3.5 DSRG-$(ns, \ ls + s - 1, \ ld + s - 1, \ ld + s - 2, \ ld + d)$ and

DSRG-$(ns,\ ls,\ ld,\ ld-d,\ ld)$ **with** $d(n-1)=ls$

Let n, d, l and s be positive integers such that $d(n-1) = ls$, or equivalently, $n = 1 + \frac{ls}{d}$. Let $P = \{1, 2, \ldots, n\}$. For each $i \in P$, suppose there exists a tactical configuration $\mathcal{P}_i = (P\backslash\{i\}, \mathcal{B}_i)$ with parameters $(\overline{v}, \overline{b}, \overline{k}, \overline{r}) = (n-1, s, l, d)$. We define the tactical configuration $\mathcal{T} = (P, \mathcal{B})$ with $\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{B}_i$ by collecting the blocks of all configurations, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. Then $\mathcal{T} = (P, \mathcal{B})$ has parameters $(\overline{v}, \overline{b}, \overline{k}, \overline{r}) = (n, ns, l, ls)$. Using this configuration, we now construct two DSRGs on the set

$$V = \{(g, B) :\ B \in \mathcal{B}_g,\ g \in P\}.$$

Theorem 3.5.1 *Let $\mathcal{T} = (P, \mathcal{B})$ be the above tactical configuration $\mathcal{T} - (n, ns, l, ls)$, where $n = 1 + \frac{ls}{d}$. Let $D_1 = D_1(\mathcal{T})$ be the directed graph with its vertex set,*

$$V = \{(g, B) :\ B \in \mathcal{B}_g,\ g \in P\},$$

and adjacency defined by

$$(g, B) \to (g', B') \ \ \text{if and only if} \ \ g \in B'.$$

Then, D_1 is a DSRG with parameters

$$(v, k, t, \lambda, \mu) = (ns,\ ls,\ ld,\ (l-1)d,\ ld)\,.$$

Proof. It is clear that $v = \sum_{g \in P} |\mathcal{B}_g| = ns$. A vertex (g', B') is an out-neighbor of (g, B), g' can be any point different from g, and B' can be any member of $\mathcal{B}_{g'}$ containing g. Since there are d blocks in $\mathcal{B}_{g'}$ containing g, $k = (n-1)d$. A vertex (g', B') is an (in and out)-neighbor of (g, B), g' should be one of $l = |B|$ points while B' must be any one of d blocks containing g and belonging to $\mathcal{B}_{g'}$. Hence, $t = ld$.

Given $(g, B) \to (g', B')$, (and so $g \in B'$), the number of vertices $(g^*, B^*) \in V(D)$, such that $g^* \in B'$, $B^* \in \mathcal{B}_{g^*}$, and $B^* \ni g$, is $(l-1)d$ since there are $l-1$ choices for g^* in $B' \setminus \{g\}$ and for any g^*, there are d blocks in $\mathcal{B}_{g^*}$ that contain g. Thus, $\lambda = (l-1)d$.

For μ, let $(g, B) \nrightarrow (g', B')$, (and so $g \notin B'$). For any point g^* in B', there are d blocks in $\mathcal{B}_{g^*}$ that contain g. Hence, we have $\mu = ld$. This completes the proof. $\square$

Theorem 3.5.2 *Let $\mathcal{T} = (P, \mathcal{B})$ be the same tactical configuration as in the above theorem. Let $D_2 = D_2(\mathcal{T})$ be the directed graph with its vertex set,*

$$V = \{(g, B) : \; B \in \mathcal{B}_g, \; g \in P\},$$

and adjacency defined by

$$(g, B) \to (g', B') \quad \text{if and only if either } g \in B' \text{ or } g = g' \text{ and } B \neq B'.$$

Then D_2 is a DSRG with the parameters,

$$(v, k, t, \lambda, \mu) = (ns, \; ls + s - 1, \; ld + s - 1, \; ld + s - 2, \; (l + 1)d).$$

Corollary 3.5.3 *Let $\mathcal{T} = (P, \mathcal{B})$ be the tactical configuration, and let $D_1 = D_1(\mathcal{T})$ and $D_2 = D_2(\mathcal{T})$ as in the above theorems. In the constructions for D_1 and D_2, if we take the multi-set consisting of m copies of the vertex set V as its vertex set, we can obtain the DSRGs with parameters*

$$(v, k, t, \lambda, \mu) = (m(ns), \; mls, \; mld, \; m(l - 1)d, \; mld)$$

and

$$(m(ns), \; m(ls + s) - 1, \; m(ld + s) - 1, \; m(ld + s) - 2, \; m(l + 1)d),$$

respectively.

3.6 DSRG-$(ls^2 + s, \; ls + s - 1, \; l + s - 1, \; l + s - 2, \; l + 1)$ and DSRG-$(ls^2 + s, \; ls, \; l, \; l - 1, \; l)$

Let l and s be positive integers. Consider the $(ls + 1)$-element set $P = \{1, 2, \ldots, ls + 1\}$. For each $i \in P$, let $\mathcal{B}_i = \{B_{i1}, B_{i2}, \ldots, B_{is}\}$ be a partition of $P \setminus \{i\}$ into s parts (blocks) of equal size l. Let

$$\mathcal{B} = \bigcup_{i=1}^{ls+1} \mathcal{B}_i = \{B_{ig} : \; 1 \leq g \leq s, \; 1 \leq i \leq ls + 1\}.$$

Then, the pair $(P, \mathcal{B})$ forms a tactical configuration, $\mathcal{T} - (ls + 1, s(ls + 1), l, ls)$. We construct directed strongly regular graphs on the set,

$$V = \{(i, B) : \; B \in \mathcal{B}_i, \; i \in P\},$$

in two ways.

Theorem 3.6.1 *Let* $(P,\mathcal{B})$ *be* $\mathcal{T}-(ls+1, s(ls+1), l, ls)$. *Let* $D_1 = D_1(\mathcal{T})$ *be the directed graph with its vertex set*

$$V = \{(i, B_{ig}) \in P \times \mathcal{B} : 1 \le i \le ls+1, 1 \le g \le s\}$$

and adjacency defined by

$$(i, B_{ig}) \to (j, B_{jh}) \text{ if and only if } i \in B_{jh}.$$

Then, D_1 *is a DSRG with parameters*

$$(v,k,t,\lambda,\mu) = (ls^2+s,\ ls,\ l,\ l-1,\ l).$$

Theorem 3.6.2 *Let* $(P,\mathcal{B})$ *be* $\mathcal{T}-(ls+1, ls^2+s, l, ls)$. *Let* $D_2 = D_2(\mathcal{T})$ *be the directed graph with its vertex set,*

$$V = \{(i, B_{ig}) \in P \times \mathcal{B} : 1 \le i \le ls+1, 1 \le g \le s\}$$

and adjacency defined by

$$(i, B_{ig}) \to (j, B_{jh}) \text{ if and only if either } i \in B_{jh} \text{ or } i = j \text{ and } B_{ig} \ne B_{jh}.$$

Then, D_2 *is a DSRG with the parameters*

$$(v,k,t,\lambda,\mu) = (ls^2+s,\ ls+s-1,\ l+s-1,\ l+s-2,\ l+1).$$

Corollary 3.6.3 *Let* $(P,\mathcal{B})$ *be the tactical configuration* $\mathcal{T}-(ls+1, ls^2+s, l, ls)$ *as above. Let* $D_1 = D_1(\mathcal{T})$ *and* $D_2 = D_2(\mathcal{T})$. *In the constructions for* D_1 *and* D_2, *if we take the multi-set consisting of* m *copies of the vertex set* V *as its vertex set, we can obtain the directed strongly regular graph with parameters,*

$$(v,k,t,\lambda,\mu) = (m(ls^2+s),\ mls,\ ml,\ m(l-1),\ ml)$$

and

$$(m(ls^2+s),\ m(ls+s)-1,\ m(l+s)-1,\ m(l+s)-2,\ m(l+1)),$$

respectively.

In the above constructions, different tactical configurations coming from different partitions of P may produce non-isomorphic graphs with the same parameters as before. For example, for $l = s = 2$, we obtain 13 different DSRGs with the same parameter set $(v, k, t, \lambda, \mu) = (10, 4, 2, 1, 2)$. To illustrate the above claim and to show the connections to other combinatorial structures, we will describe them in detail in the remainder of the current section.

3.6.1 Isomorphism classes of DSRG-$(10, 4, 2, 1, 2)$

When $l = s = 2$, the number of ways to form tactical configurations with parameters $(v, b, k, r) = (5, 10, 2, 4)$ is 243. Let $\mathbf{F}$ be the set of these tactical configurations. Each tactical configuration, $\mathcal{T} = (P, \mathcal{B}) \in \mathbf{F}$, gives rise to a DSRG $D(\mathcal{T})$ with its vertex set $V(\mathcal{T}) = \{(i, B_{ij}) : i \in P,\ B_{ij} \in \mathcal{B}\}$ by Theorem 3.6.1. Consider the action of S_5 on $\mathbf{F}$ under the rule that $\mathcal{T}_1^\sigma = \mathcal{T}_2$ if and only if $V(\mathcal{T}_1)^\sigma = V(\mathcal{T}_2)$ where

$$V(\mathcal{T})^\sigma = \{(i^\sigma, (B_{ij})^\sigma) : i \in P,\ B_{ij} \in \mathcal{B}\}$$

with natural action on B_{ij}; i.e., $(B_{ij})^\sigma = \{x^\sigma, y^\sigma\}$ if $B_{ij} = \{x, y\}$. Under this action $\mathbf{F}$ is partitioned into seven orbits. The tactical configurations belong to the same orbit produce isomorphic DSRGs. Let $\mathcal{T}_1$, $\mathcal{T}_2$, ..., $\mathcal{T}_7$ denote the representatives of the orbits. The block sets of these representatives are provided in Table 3.5.

Table 3.5 The block sets of the representatives of seven orbits.

i	$\mathcal{B}(\mathcal{T}_1)$	$\mathcal{B}(\mathcal{T}_2)$	$\mathcal{B}(\mathcal{T}_3)$	$\mathcal{B}(\mathcal{T}_4)$	$\mathcal{B}(\mathcal{T}_5)$	$\mathcal{B}(\mathcal{T}_6)$	$\mathcal{B}(\mathcal{T}_7)$
1	$23, 45$	$23, 45$	$23, 45$	$23, 45$	$23, 45$	$23, 45$	$23, 45$
2	$13, 45$	$13, 45$	$13, 45$	$14, 35$	$13, 45$	$13, 45$	$13, 45$
3	$12, 45$	$14, 25$	$12, 45$	$15, 24$	$14, 25$	$14, 25$	$14, 25$
4	$12, 35$	$12, 35$	$12, 35$	$13, 25$	$12, 35$	$12, 35$	$13, 25$
5	$12, 34$	$12, 34$	$13, 24$	$12, 34$	$14, 23$	$13, 24$	$14, 23$

Table 3.5 shows the group structure of each stabilizer of $\mathcal{T}_i$, $i = 1, 2, \ldots, 7$ and its generators. The last row of the table indicates the size of the orbit represented by the corresponding

tactical configuration.

Table 3.6 Stabilizers and the size of orbits for the action of S_5 on **F**.

$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$
D_8 $(1524), (15)(24)$	$C_2 \times C_2$ $(12)(45), (15)(24)$	C_2 $(23)(45)$	$C_5 \rtimes C_4$ $(15234), (1345)$	C_2 $(15)(23)$	C_2 $(15)(34)$	D_{10} $(12435), (12)(45)$
15	30	60	6	60	60	12

Let $D(\mathcal{T}_i)$, $i = 1, 2, \ldots, 7$ be the directed strongly regular graphs with parameters $(10, 4, 2, 1, 2)$ obtained from the seven orbit representatives given in Table 3.5 by Theorem 3.6.1. Then, it is shown that the orientation-reversing conjugates of $D(\mathcal{T}_i)$ for $i = 1, 2, \ldots, 6$ are non-isomorphic to any of the seven. The graph $D(\mathcal{T}_7)$ is isomorphic to its orientation-reversing conjugate. Therefore, together with their conjugates, our construction produces 13 directed strongly regular graphs for the given parameter set. However, Jørgensen has shown that there are 16 graphs for the given parameter set [25].

The adjacency matrices for seven graphs, $D(\mathcal{T}_1)$, $D(\mathcal{T}_2), \ldots,$ $D(\mathcal{T}_7)$ are as follows. (The rows of the matrices are indexed by the vertices of corresponding graphs.)

$D(\mathcal{T}_1)$

$$
\begin{array}{r}
(1,23)\\(2,13)\\(3,12)\\(3,45)\\(4,35)\\(5,34)\\(1,45)\\(4,12)\\(2,45)\\(5,12)
\end{array}
\left(\begin{array}{ccc|ccc|cccc}
0&1&1&0&0&0&0&1&0&1\\
1&0&1&0&0&0&0&1&0&1\\
1&1&0&0&1&1&0&0&0&0\\
\hline
1&1&0&0&1&1&0&0&0&0\\
0&0&0&1&0&1&1&0&1&0\\
0&0&0&1&1&0&1&0&1&0\\
\hline
0&1&1&0&0&0&0&1&0&1\\
0&0&0&1&0&1&1&0&1&0\\
1&0&1&0&0&0&0&1&0&1\\
0&0&0&1&1&0&1&0&1&0
\end{array}\right)
$$

$D(\mathcal{T}_2)$

$$
\begin{array}{r}
(1,45)\\(4,12)\\(2,45)\\(5,12)\\(1,23)\\(2,13)\\(3,25)\\(5,34)\\(4,35)\\(3,14)
\end{array}
\left(\begin{array}{cccc|cccccc}
0&1&0&1&0&1&0&0&0&1\\
1&0&1&0&0&0&0&1&0&1\\
0&1&0&1&1&0&1&0&0&0\\
1&0&1&0&0&0&1&0&1&0\\
\hline
0&1&0&1&0&1&0&0&0&1\\
0&1&0&1&1&0&1&0&0&0\\
0&0&0&0&1&1&0&1&1&0\\
1&0&1&0&0&0&1&0&1&0\\
1&0&1&0&0&0&0&1&0&1\\
0&0&0&0&1&1&0&1&1&0
\end{array}\right)
$$

$D(\mathcal{T}_3)$

$$
\begin{array}{r}
(1,23)\\(2,13)\\(3,12)\\(1,45)\\(4,12)\\(2,45)\\(5,24)\\(4,35)\\(3,45)\\(5,13)
\end{array}
\left(\begin{array}{ccc|ccccccc}
0&1&1&0&1&0&0&0&0&1\\
1&0&1&0&1&0&1&0&0&0\\
1&1&0&0&0&0&0&1&0&1\\
\hline
0&1&1&0&1&0&0&0&0&1\\
0&0&0&1&0&1&1&0&1&0\\
1&0&1&0&1&0&1&0&0&0\\
0&0&0&1&0&1&0&1&1&0\\
0&0&0&1&0&1&1&0&1&0\\
1&1&0&0&0&0&0&1&0&1\\
0&0&0&1&0&1&0&1&1&0
\end{array}\right)
$$

$D(\mathcal{T}_4)$

$$
\begin{array}{r}
(1,23)\\(2,14)\\(4,25)\\(5,34)\\(3,15)\\(1,45)\\(4,13)\\(3,24)\\(2,35)\\(5,12)
\end{array}
\left(\begin{array}{ccccc|ccccc}
0&1&0&0&1&0&1&0&0&1\\
1&0&1&0&0&0&0&1&0&1\\
0&1&0&1&0&1&0&1&0&0\\
0&0&1&0&1&1&0&0&1&0\\
1&0&0&1&0&0&1&0&1&0\\
\hline
0&1&0&0&1&0&1&0&0&1\\
0&1&0&1&0&1&0&1&0&0\\
1&0&0&1&0&0&1&0&1&0\\
1&0&1&0&0&0&0&1&0&1\\
0&0&1&0&1&1&0&0&1&0
\end{array}\right)
$$

$D(\mathcal{T}_5)$

$D(\mathcal{T}_6)$

$$\begin{array}{c}(1,23)\\(2,13)\\(3,25)\\(5,23)\\(2,45)\\(4,12)\\(1,45)\\(5,14)\\(4,35)\\(3,14)\end{array}\begin{pmatrix}0&1&0&0&0&1&0&1&0&1\\1&0&1&1&0&1&0&0&0&0\\1&1&0&1&0&0&0&0&1&0\\0&0&1&0&1&0&1&0&1&0\\1&0&1&1&0&1&0&0&0&0\\0&0&0&0&1&0&1&1&0&1\\0&1&0&0&0&1&0&1&0&1\\0&0&1&0&1&0&1&0&1&0\\0&0&0&0&1&0&1&1&0&1\\1&1&0&1&0&0&0&0&1&0\end{pmatrix}\qquad\begin{array}{c}(1,23)\\(2,13)\\(3,25)\\(5,13)\\(1,45)\\(4,12)\\(2,45)\\(5,24)\\(4,35)\\(3,14)\end{array}\begin{pmatrix}0&1&0&1&0&1&0&0&0&1\\1&0&1&0&0&1&0&1&0&0\\1&1&0&1&0&0&0&0&1&0\\0&0&1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&0&0&1\\0&0&0&0&1&0&1&1&0&1\\1&0&1&0&0&1&0&1&0&0\\0&0&1&0&1&0&1&0&1&0\\0&0&0&0&1&0&1&1&0&1\\1&1&0&1&0&0&0&0&1&0\end{pmatrix}$$

$D(\mathcal{T}_7)$

$$\begin{array}{c}(1,23)\\(2,13)\\(3,25)\\(5,23)\\(2,45)\\(4,25)\\(5,14)\\(1,45)\\(4,13)\\(3,14)\end{array}\begin{pmatrix}0&1&0&0&0&0&1&0&1&1\\1&0&1&1&0&1&0&0&0&0\\1&1&0&1&0&0&0&0&1&0\\0&0&1&0&1&1&0&1&0&0\\1&0&1&1&0&1&0&0&0&0\\0&0&0&0&1&0&1&1&0&1\\0&0&1&0&1&1&0&1&0&0\\0&1&0&0&0&0&1&0&1&1\\0&0&0&0&1&0&1&1&0&1\\1&1&0&1&0&0&0&0&1&0\end{pmatrix}$$

We now describe the three graphs that are not produced by our construction. Their adjacency matrices are given by J_8 and J_9 below, and the transpose of J_8 gives for the third. The graph of J_9 is self-transpose and has the trivial automorphism group. The automorphism groups for the graphs of J_8 and its transpose are isomorphic to C_2.

$$J_8 = \begin{pmatrix} 0&1&1&1&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&1&0&0\\ 1&1&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&1&1&0&1&1\\ 1&1&0&0&1&0&0&1&0&0\\ 1&0&1&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&1&0&1&1\\ 0&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&1&1&0 \end{pmatrix} \qquad J_9 = \begin{pmatrix} 0&1&1&1&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&1&0&0\\ 1&1&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&1&0&1&1\\ 0&0&0&0&0&1&1&0&1&1\\ 1&1&0&0&1&0&0&1&0&0\\ 1&0&1&1&0&0&0&1&0&0\\ 0&0&0&0&0&1&1&0&1&1\\ 0&1&1&1&0&0&0&0&0&1\\ 0&0&0&0&1&1&1&0&1&0 \end{pmatrix}$$

For each graph $D(\mathcal{T}_i)$, $i = 1, 2, \ldots, 7$, we can see that the full automorphism group of $D(\mathcal{T}_i)$ is determined by the stabilizer of $\mathcal{T}_i$ under the action of the symmetric group S_5 on **F**. Hence from the knowledge of the orbits or the stabilizers of the permutation action of S_5 on **F**, we can obtain the number of distinct graphs produced by our construction. For instance, as we have seen in Table 3.6 we have the following 6 different tactical configurations all which produce graph $D(\mathcal{T}_4)$.

Table 3.7 The block sets of 6 tactical configurations that are isomorphic to $\mathcal{T}_4$.
(Top row indicates the isomorphism $\sigma \in S_5$ to the first tactical configuration.)

(1)	(23), (45)	(14), (35)	(15), (24)	(13), (25)	(12), (34)
23 45	23 45	25 34	25 34	24 35	24 35
14 35	15 34	14 35	13 45	15 34	13 45
15 24	14 25	12 45	15 24	12 45	14 25
13 25	12 35	15 23	12 35	13 25	15 23
12 34	13 24	13 24	14 23	14 23	12 34

Therefore, graph $D(\mathcal{T}_4)$ is isomorphic to the graphs obtained from the following vertex sets.

$V = V_1$ (1)	V_2 $(23), (45)$	V_3 $(14), (35)$	V_4 $(15), (24)$	V_5 $(13), (25)$	V_6 $(12), (34)$
$(1, 23), (1, 45)$	$(1, 23), (1, 45)$	$(1, 25), (1, 34)$	$(1, 25), (1, 34)$	$(1, 24), (1, 35)$	$(1, 24), (1, 35)$
$(2, 14), (2, 35)$	$(2, 15), (2, 34)$	$(2, 14), (2, 35)$	$(2, 13), (2, 45)$	$(2, 15), (2, 34)$	$(2, 13), (2, 45)$
$(3, 15), (3, 24)$	$(3, 14), (3, 25)$	$(3, 12), (3, 45)$	$(3, 15), (3, 24)$	$(3, 12), (3, 45)$	$(3, 14), (3, 25)$
$(4, 13), (4, 25)$	$(4, 12), (4, 35)$	$(4, 15), (4, 23)$	$(4, 12), (4, 35)$	$(4, 13), (4, 25)$	$(4, 15), (4, 23)$
$(5, 12), (5, 34)$	$(5, 13), (5, 24)$	$(5, 13), (5, 24)$	$(5, 14), (5, 23)$	$(5, 14), (5, 23)$	$(5, 12), (5, 34)$

3.6.2 Association schemes and an SRG arising from a DSRG-$(10, 4, 2, 1, 2)$.

Let A be the adjacency matrix of $D(\mathcal{T}_4)$[1] and $\bar{A}$ be the matrix given by

$$\bar{A}_{ij} = \begin{cases} 1 & \text{either } A_{ij} = 1 \text{ or } A_{ji} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\bar{A} = \begin{array}{c} (1,23) \\ (2,14) \\ (4,25) \\ (5,34) \\ (3,15) \\ (1,45) \\ (4,13) \\ (3,24) \\ (2,35) \\ (5,12) \end{array} \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{array} \right)$$

[1] This graph was constructed in [13, Sec. 5] and [28].

Let G be the graph whose adjacency matrix is $\bar{A}$. Then, G is the strongly regular graph with parameters $(v, k, \lambda, \mu) = (10, 6, 3, 4)$, which is known as the Johnson graph $J(5, 2)$. We note that $J(5, 2)$ is also obtained from the Jørgensen's graph, J_9, by 'symmetrizing' the matrix J_9. In fact, this is the only strongly regular graph that can be obtained from any of the directed strongly regular graphs with parameters $(10, 4, 2, 1, 2)$ through the symmetrization process.

Among the DSRGs with parameters $(10, 4, 2, 1, 2)$, $D(\mathcal{T}_4)$ has the largest automorphism group. It is the only one that has a vertex transitive automorphism group. The automorphism group, $H = \mathrm{Aut}(D(\mathcal{T}_4))$, is isomorphic to the group $C_5 \rtimes C_4$ of order 20. From the transitive permutation group, H, on the vertex set of $D(\mathcal{T}_4)$, we obtain a 5-class association scheme. Let $\mathcal{X}(H, V(\mathcal{T}_4))$ denote this association scheme. Then its association relation table is given by the matrix on the left below.

Tables 3.8 Relation matrices of $\mathcal{X}(H, V(\mathcal{T}_4))$ and its 2-class symmetric fusion scheme.

$$\left(\begin{array}{ccccc|ccccc}
0 & 3 & 2 & 2 & 3 & 5 & 1 & 4 & 4 & 1 \\
3 & 0 & 3 & 2 & 2 & 4 & 4 & 1 & 5 & 1 \\
2 & 3 & 0 & 3 & 2 & 1 & 5 & 1 & 4 & 4 \\
2 & 2 & 3 & 0 & 3 & 1 & 4 & 4 & 1 & 5 \\
3 & 2 & 2 & 3 & 0 & 4 & 1 & 5 & 1 & 4 \\
\hline
5 & 1 & 4 & 4 & 1 & 0 & 3 & 2 & 2 & 3 \\
4 & 1 & 5 & 1 & 4 & 3 & 0 & 3 & 2 & 2 \\
1 & 4 & 4 & 1 & 5 & 2 & 3 & 0 & 3 & 2 \\
1 & 5 & 1 & 4 & 4 & 2 & 2 & 3 & 0 & 3 \\
4 & 4 & 1 & 5 & 1 & 3 & 2 & 2 & 3 & 0
\end{array}\right)
\qquad
\left(\begin{array}{ccccc|ccccc}
0 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\
2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\
2 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 1 \\
\hline
2 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 \\
1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 2 \\
1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 0
\end{array}\right)$$

It is observed that $\mathcal{X}(H, V(\mathcal{T}_4))$ is isomorphic to a 5-class non-commutative association scheme. This scheme has two fusion schemes of class 3. They are $K_2 \times K_5$, the direct product of two trivial schemes of order 2 and 5, and $C_5 \wr K_2$, the wreath product of the scheme coming from a pentagon and the trivial scheme of order 2. The scheme $\mathcal{X}(H, V(\mathcal{T}_4))$ also has three symmetric fusion schemes of class 2, $K_2 \wr K_5, K_5 \wr K_2$ and the Johnson scheme $J(5,2)$. The Johnson scheme $J(5,2)$ is obtained from $\mathcal{X}(H, V(\mathcal{T}_4))$ by fusing the relations R_1, R_3, and R_4, and fusing R_2 and R_5 together as easily observed from the above relation tables.

The edge set of $D(\mathcal{T}_4)$ coincides with $R_1 \cup R_3$. The orientation-reversing conjugate of $D(\mathcal{T}_4)$ is the graph with edge set $R_1 \cup R_4$. The edge set of the Johnson graph $J(5,2)$ is $R_1 \cup R_3 \cup R_4$, while its complement, the Petersen graph has edge set $R_2 \cup R_5$. As we have mentioned earlier, although both graphs $D(\mathcal{T}_4)$ and J_9 give rise to Johnson graph $J(5,2)$ via the symmetrization process, $D(\mathcal{T}_4)$ is the graph which yields $\mathcal{X}(H, V(\mathcal{T}_4))$.

Notes 3.6.4 *By using tactical configurations we constructed not only non-isomorphic DSRGs but also we constructed DSRGs previously unknown. For more information, we recommend the website "Parameters of directed strongly regular graphs" by Brouwer and Hobart [7].*

Research Problem : *The adjacency algebra of a graph is the matrix algebra generated by its adjacency matrix. The adjacency algebra of a graph reflects some of the graph-theoretical properties. For instance, an undirected graph is strongly regular if and only if its adjacency algebra contains J and has rank 3. For DSRGs, the adjacency algebra is a proper subalgebra of the coherent algebra generated by the adjacency matrix of the given DSRG. Klin et al. [16] studied DSRGs arising from coherent configuration and showed that the rank of the minimal coherent algebra, which includes the adjacency matrix is greater than or equal to 6. They have given examples of rank 6 and 7 graphs obtained from BIBDs. It is an interesting and open problem to classify all rank 6 and 7 graphs.*

Research Problem : *It is interesting to find a lower bound on the number of non-isomorphic DSRGs for each construction method presented in this chapter.*

3.7 Directed Strongly Regular Graphs Arising from Block Matrices

In this section, we will describe construction methods which use block matrices and give connections to other combinatorial objects. Since all theorems can be checked by evaluating the square of the adjacency matrix, the proofs will be omitted.

Construction 3.7.1 *[13]*

The first construction uses quadratic residue matrices to construct $DSRG(n, k, t, \lambda, \mu)$ *with parameters* $(2q, q-1, \frac{1}{2}(q-1), \frac{1}{2}(q-1)-1, \frac{1}{2}(q-1))$, *where* $q = 4m+1$ *and is a prime power. The adjacency matrices of such DSRGs will take the form*

$$A = \begin{pmatrix} Q & C_1 \\ C_2 & Q \end{pmatrix}.$$

C_1 *and* C_2 *are* σ_1 *and* σ_2 *circulant matrices, respectively, where a* σ *circulant matrix* C *satisfies* $C_{ij} = C_{i-k, j-\sigma k}$. *This means that each row, or each column, is equal to the previous row (column) shifted* σ *entries to the right (down).* Q *is a quadratic residue matrix of order* q, *indexed by the elements of GF(q), the Galois Field of order* q. *When* R *is the set of quadratic residues of GF(q), the nonzero elements* $x \in GF(q)$, *such that* $x = y^2$ *for some* $y \in GF(q)$, *and* N *is the set of quadratic non-residues of GF(q), all other nonzero elements of GF(q),* Q *is defined by*

$$Q_{ij} = \begin{cases} 1 & \text{if } i-j \in R \\ 0 & \text{if } i-j \in N \end{cases}.$$

This construction method produces a DSRG if and only if

- $\sigma_1\sigma_2 = 1 \in GF(q)$.
- $\sigma_1, \sigma_2 \in N$.
- *The partition of* $GF(q)^*$ *into the two sets, each of* $2m$ *elements,*

  $$S = \{x \in GF(q)^* : (C_2)_{0,x} = 1\} \quad \text{and} \quad T = \{x \in GF(q)^* : (C_2)_{0,x} = 0\},$$

  described by the first row satisfies the following "difference partition" property: Each of the $4m$ *elements of* $GF(q)^*$ *occurs exactly* m *times in the* $4m^2$ *differences* $s - t$ *where* $s \in S$ *and* $t \in T$.

An example of an adjacency matrix for the $DSRG(10, 4, 2, 1, 2)$, *using the preceding construction* *with* $\sigma_1 = 2$, $\sigma_2 = 3$, $R = \{1,4\}$ *and* $N = \{2,3\}$, *is*

$$A = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

Construction 3.7.2 *[19] Let* $B_1, B_2, \cdots B_q$ *be (0,1)-matrices satisfying the following conditions.*

- *There is a constant* c, *such that each* B_i *has a constant row sum* c.
- B_i *has* 0*'s on the diagonal, for all* i.
- $\sum_{i=1}^{q} B_i = d(J_n - I_n)$ *for some integer* d.

Then, $A = \begin{bmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_k \end{bmatrix} [I \;\; I \;\; \ldots \;\; I]$ *is the adjacency matrix of a directed strongly regular graph with parameters* $v = nq$, $k = cq$, $t = cd$, $\lambda = cd - d$, $\mu = cd$.

One family of such a set of B_i's can be constructed from regular tournaments.

Definition 3.7.3 *A tournament is a directed graph* Γ, *such that for any* $x, y \in V(\Gamma)$, *exactly one of* $x \to y$ *or* $y \to x$ *holds. A tournament* Γ *is said to be regular if every vertex in* $V(\Gamma)$ *has* *the same out-degree. Thus, a regular tournament has $n = 2k+1$, if n and k denote the number of vertices and the valency of the graph, respectively.*

The adjacency matrix A of a tournament Γ, satisfies the equation $A + A^T = J - I$. If Γ is a regular tournament with valency k, then $JA = AJ = kJ$. So, we have the following corollary.

Corollary 3.7.4 *If A is an adjacency matrix of a regular tournament with valency k, then*

$$M = \begin{pmatrix} A & A \\ A^T & A^T \end{pmatrix}$$

is the adjacency matrix of a directed strongly regular graph with parameters $(4k+2, 2k, k, k-1, k)$.

Remark 3.7.5 *The above parameter set can be also constructed from one of the Godsil's construction by setting $n = 2k+1$, $d = 1$, and $c = k$ (See T12 at [7]).*

An example of an adjacency matrix for the DSRG$(6, 2, 1, 0, 1)$, using the preceding construction is

$$A = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

In the search of small vertex transitive DSRGs Klin et al.[17] observed the following.

$$M = \begin{pmatrix} A & (PA) \\ (PA)^T & A \end{pmatrix}$$

is the adjacency matrix of a DSRG, where A is adjacency matrix of a regular tournament and P is a permutation matrix of order 2. The matrix, M, is the adjacency matrix of a certain class of Cayley graphs arising from the dihedral groups.

Construction 3.7.6 *[19] Let* Γ *be a DSRG with parameters* $(v; k; t; \lambda; \mu)$ *such that* $\lambda = \mu$ *and* $v = 4k - 4\mu$. *Suppose there exists a* $c \times c$ $(1; -1)$ *matrix* H *with* 1*'s on the diagonal such that* $HJ = JH = dJ$ *and* $H^2 = cI$. *Then, there exists a DSRG with parameters*

$$\overline{v} = vc,$$

$$\overline{k} = c(2k - 2\mu) + d(2\mu - k),$$

$$\overline{t} = c(k + t - 2\mu) + d(2\mu - k),$$

$$\overline{\lambda} = \overline{\mu} = c(k - \mu) + d(2\mu - k).$$

Note that H must be a regular Hadamard matrix with a constant diagonal. This implies that $c = 4s^2$ for some integer s, and $d = 2s$.

Definition 3.7.7 *An* (m, r)*-team tournament is a digraph obtained from the complement* $\overline{m \circ K_r}$ *of* m *copies of the complete graph* K_r *by giving an orientation in such a way that every undirected edge* $\{x, y\}$ *is assigned with either* $x \to y$ *or* $x \leftarrow y$*, but not both.*

We note that an (m, r)-team tournament has m maximal independent sets of size r, and the edges are directed links between the vertices of distinct maximal independent sets.

For example, we can construct $(m, 2)$-team tournaments from doubly regular tournaments of order $m - 1$. Let A be an adjacency matrix of a doubly regular tournament T of order $m - 1 = 2k + 1 = 4\lambda + 3$. Then,

$$D(T) = \left(\begin{array}{c|ccc|c|ccc} 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \hline 0 & & & & 1 & & & \\ \vdots & & A & & \vdots & & A^T & \\ 0 & & & & 1 & & & \\ \hline 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ \hline 1 & & & & 0 & & & \\ \vdots & & A^T & & \vdots & & A & \\ 1 & & & & 0 & & & \end{array}\right)$$

is an adjacency matrix of a doubly regular $(m, 2)$-team tournament.

Construction 3.7.8 *[1] Let* $D = D(T)$ *be an* $(m, 2)$*-team tournament described above with* $m = 2k + 2 = 4\lambda + 4$*, then*

$$M = M(D) = \begin{pmatrix} D & D^T + I \\ D + I & D^T \end{pmatrix}$$

is an adjacency matrix of a DSRG with parameters $(4m, 2m-1, m, m-1, m-1) = (16\lambda + 16, 8\lambda + 7, 4\lambda + 4, 4\lambda + 3, 4\lambda + 3)$.

We can extend the above construction to make use of any regular tournament instead of only doubly regular tournaments.

Construction 3.7.9 *[1] Let* A *be the adjacency matrix of a regular tournament* T *of order* h *and*

$$D = D(T) = \begin{pmatrix} 0 & \mathbf{1}^T & 0 & \mathbf{0}^T \\ \mathbf{0} & A & \mathbf{1} & A^T \\ 0 & \mathbf{0}^T & 0 & \mathbf{1}^T \\ \mathbf{1} & A^T & \mathbf{0} & A \end{pmatrix},$$

where $\mathbf{0}$ *and* $\mathbf{1}$ *denote the* n*-dimensional column vectors for all zeros and all ones, respectively. The matrix* $M(D) = \begin{bmatrix} D & D^T + I \\ D + I & D^T \end{bmatrix}$ *is the adjacency matrix of a DSRG* $(4(h+1), 2h+1, h+1, h, h)$*, where* $h \equiv 1 \bmod 2$.

Construction 3.7.10 *[1] For a positive integer* s*, let* L *be the* $(2s+2) \times (2s+2)$*-matrix equal to* $\Pi + \Pi^2 + \cdots + \Pi^s$*, where*

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Then,

$$M(L) = \begin{pmatrix} L & L^T + I \\ L + I & L^T \end{pmatrix}$$

is an adjacency matrix of a DSRG with parameters $(4(s+1), 2s+1, s+1, s, s)$.

Remark 3.7.11 *The above parameter set can be also constructed by one of Jorgensen's methods. (See T4 at [7])*

Notes 3.7.12 *The technique of using block matrices is a simple tool to construct DSRGs. It also gives us connections to other combinatorial objects such as Hadamard matrices, doubly regular tournaments,* (m, r)*-team tournaments, and circulant matrices.*

Research Problem : *The DSRGs obtained from block matrices can also be constructed by various other methods. Make a systematic study of connections between known construction methods and construction by block matrices.*

CHAPTER 4. VERTEX TRANSITIVE DIRECTED STRONGLY REGULAR GRAPHS

In this chapter, we will investigate some families of vertex transitive DSRGs. We will introduce construction of DSRGs arising from Cayley graphs and Coset graphs and give connections to finite incidence structures, such as symmetric *BIBDs*. All graphs and groups considered in this chapter are finite. Given a group G and a subset $S \subseteq G$, $\langle S \rangle$ denotes the subgroup of G generated by S, and $\underline{S}$ denotes the formal sum of the elements of S, as an element of the group ring $\mathbb{Z}G$. Given a graph Γ, $\mathrm{Aut}(\Gamma)$ (or $Aut(\Gamma)$ at times) denotes the full automorphism group of Γ.

4.1 Directed Strongly Regular Graphs Arising from Cayley Graphs

Definition 4.1.1 *A vertex transitive graph is a graph $\Gamma(V, E)$ such that for any given two vertices x, y in Γ, there exists some $\phi \in Aut(\Gamma)$, such that $\phi(x) = y$.*

Definition 4.1.2 *Let S be a subset of a group G, such that $\langle S \rangle = G$. The Cayley graph of G, denoted $Cay(G, S)$, is the directed graph with elements of G as its vertices, where $g \to h$ if and only if $g^{-1}h \in S$.*

Proposition 4.1.3 *[34] A graph $\Gamma(V, E)$ is a Cayley graph of a group G if and only if $Aut(\Gamma)$ contains a regular subgroup isomorphic to G.*

Lemma 4.1.4 *[14] The number of paths of length 2 from g to h in $Cay(G, S)$ equals the coefficient of $g^{-1}h$ in $\underline{S}^2$.*

Proposition 4.1.5 *[14] A Cayley graph $Cay(G, S)$ is a DSRG with parameters (v, k, t, λ, μ) if* *and only if* $|G| = v$, $|S| = k$, *and*

$$\underline{S}^2 = t\underline{e} + \lambda\underline{S} + \mu(\underline{G} - \underline{e} - \underline{S}).$$

Let n denote an integer with $n \geq 3$. Let D_n denote the dihedral group of order $2n$ and C_n denote the cyclic subgroup of D_n of order n.

Lemma 4.1.6 *[28] Let* $X, Y \subseteq C_n$ *where* n *is odd, satisfy the following conditions*

- $\underline{X} + \underline{X}^{-1} = \underline{C_n} - \underline{e}$.
- $\underline{Y}\underline{Y}^{-1} - \underline{X}\underline{X}^{-1} = \epsilon\underline{C_n}$, $\epsilon \in \{0, 1\}$.

Let $a \in D_n \backslash C_n$*. Then the Cayley graph Cay*$(G, X \cup aY)$ *is a DSRG with parameters* $(2n, n-1+\epsilon, \frac{n-1}{2}+\epsilon, \frac{n-3}{2}+\epsilon, \frac{n-1}{2}+\epsilon)$*. In particular, if* X *satisfies* $X + X^{-1} = C_n - e$ *and* $Y = Xg$ *or* $X^{-1}g$ *for some* $g \in C_n$*, then Cay*$(G, X \cup aY)$ *is a DSRG with parameters* $(2n, n-1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$.

Lemma 4.1.7 *[16] Let* n *be even,* $c \in C_n$*, where* $c \neq e$ *is an involution and* $X, Y \subseteq C_n$*, such that*

- $\underline{X} + \underline{X}^{-1} = \underline{C_n} - \underline{e} - \underline{c}$.
- $\underline{Y} = \underline{X}$ *or* $\underline{X}^{-1}$.
- $\underline{X}\underline{c} = \underline{X}^{-1}$.

Let $a \in D_n \backslash C_n$*. Then, the Cayley graph Cay*$(G, X \cup aY \cup a)$ *is a DSRG with parameters* $(2n, n-1, \frac{n}{2}, \frac{n-2}{2}, \frac{n-2}{2})$.

4.2 Vertex Transitive Directed Strongly Regular Graphs Obtained from Tactical Configuration

We will construct two infinite families of vertex transitive DSRGs by using certain tactical configurations.

4.2.1 Construction A

Let p be a prime and let η be a primitive element of the finite field $\mathbb{Z}_p$. Let s be a nontrivial factor of $p-1$, and let $H = \langle \eta^s \rangle$ be the multiplicative subgroup of $\mathbb{Z}_p^*$ of index s. Let $l = \frac{p-1}{s}$, the order of H. Let $B_{i,j}$ denote the translate of the coset $\eta^j H \in \mathbb{Z}_p^*/H$ by $i \in \mathbb{Z}_p$; i.e.,

$$B_{i,j} \ = \ i + \eta^j H \ = \ \{i + \eta^{j+hs} \pmod{p} : \ h = 0, 1, \ldots, l-1\}.$$

Let

$$\mathcal{B} = \{B_{i,j} : \ i \in \{0, 1, \ldots, sl\}, \ j \in \{0, 1, \ldots, s-1\}\}.$$

Then, we see that $\mathcal{T} = (\mathbb{Z}_p, \mathcal{B}, \in)$ is a tactical configuration with parameters

$$(sl+1, \ s(sl+1), \ l, \ sl).$$

Proposition 4.2.1 *Let $\mathcal{T} = (\mathbb{Z}_p, \mathcal{B}, \in)$ be the above tactical configuration. Let Γ be the directed graph defined on the set*

$$V(\Gamma) = \{(i, B_{i,j}) : \ i \in \mathbb{Z}_p, \ j \in \{0, 1, \ldots, s-1\}\}$$

with the adjacency by

$$(g, B_{g,h}) \ \rightarrow \ (i, B_{i,j}) \text{ if and only if } \ g \in B_{i,j}.$$

Then Γ is a DSRG with the parameters

$$(v, k, t, \lambda, \mu) = (s(sl+1), \ sl, \ l, \ l-1, \ l).$$

Proof. For each $i \in \mathbb{Z}_p$, $B_{i,j}$ and $B_{i,h}$ are disjoint whenever $j \neq h$, and $\mathbb{Z}_p \setminus \{i\} = \bigcup_{j=0}^{s-1} B_{i,j}$. However, for each $i \neq g$ in $\mathbb{Z}_p$, there is exactly one j such that $g \in B_{i,j}$. Hence, the out-degree k of a vertex $(g, B_{g,h}) \in V(\Gamma)$ is $sl = |\mathbb{Z}_p \setminus \{g\}|$. This is true for every vertex $(g, B_{g,h})$. Hence, Γ is a regular digraph.

Let $\delta(g, B_{g,h})$ denote the number

$$\delta(g, B_{g,h}) = |\{(i, B_{i,j}) \in V(\Gamma) : \ i \in B_{g,h}, \ B_{i,j} \ni g\}|.$$

Then clearly $\delta(g, B_{g,h}) = l$ since there is $l = |B_{g,h}|$ choices for i, and for each choice of such i, exactly one $B_{i,j}$ contains g among all $j = 0, 1, \ldots, s-1$. Also $\delta(g, B_{g,h})$ is independent of the choice of the vertex $(g, B_{g,h})$; and so, $t = l$.

For any given two vertices $(g, B_{g,h})$ and $(i, B_{i,j})$ with $g \notin B_{i,j}$, the number of vertices $(e, B_{e,f}) \in V(\Gamma)$ such that $e \in B_{i,j}$ and $B_{e,f} \ni g$ equals to $l = \delta(g, B_{i,j})$. This implies that parameter μ is a constant and $t = \mu = l$. Similarly, it can be verified that parameter λ is $l-1$. Thus, Γ is a DSRG with parameters $(s(sl+1),\ sl,\ l,\ l-1,\ l)$. □

4.2.2 Vertex-transitive automorphism groups

Let Γ denote the DSRG with parameters $(s(sl+1),\ sl,\ l,\ l-1,\ l)$ defined in Proposition 4.2.1. For the notational simplicity, let v_g^h denote the vertex $(g, B_{g,h}) \in V(\Gamma)$. Let $Aut(\Gamma)$ denote the full automorphism group of Γ. All automorphism groups discussed in what follows are subgroups of $Aut(\Gamma)$.

Lemma 4.2.2 *Let σ and τ be the maps defined by*

$$\begin{array}{rccc} \sigma: & V(\Gamma) & \to & V(\Gamma) \\ & v_i^j & \mapsto & v_{i+1}^j \end{array} \quad \textit{and} \quad \begin{array}{rccc} \tau: & V(\Gamma) & \to & V(\Gamma) \\ & v_i^j & \mapsto & v_{\eta i}^{j+1} \end{array}.$$

Then, σ and τ belong to $Aut(\Gamma)$.

Proof. Clearly, σ and τ are bijections of $V(\Gamma)$ with their inverses

$$\sigma^{-1}:\ v_i^j \mapsto v_{i-1}^j \quad \text{and} \quad \tau^{-1}:\ v_i^j \mapsto v_{\eta^{-1}i}^{j-1}.$$

Also it is clear that both σ and τ preserve the adjacency since $h \in B_{i,j}$ if and only if $h+1 \in B_{i+1,j}$, and $h \in B_{i,j}$ if and only if $\eta h \in B_{\eta i, j+1}$ for any $h, i \in \mathbb{Z}_p$ and $j \in \{0, 1, \ldots, s-1\}$. □

Lemma 4.2.3 *The σ, as a permutation on $V(\Gamma)$, has a cycle decomposition consisting of s cycles each with length $sl+1$. Thus σ has order $sl+1 = p$.*

Proof. It is clear that σ has the cycle decomposition

$$\sigma = (v_0^0\ v_1^0\ \cdots\ v_{sl}^0)(v_0^1\ v_1^1\ \cdots\ v_{sl}^1)\ \cdots\ (v_0^{s-1} v_1^{s-1}\ \cdots\ v_{sl}^{s-1})$$

so that $\langle\sigma\rangle$ is a cyclic group of order $sl+1$ and has s orbits. □

Lemma 4.2.4 *The automorphism τ has a cycle decomposition consisting of s cycles each with length sl, and one cycle of length s. Thus, τ has order $sl = p-1$.*

Proof. Since $\eta^{js}H = H$ for all $j = 0, 1, \dots, l-1$, we can express τ by

$$\tau = (v_0^0\ v_0^1 \cdots v_0^{s-1})(v_1^0\ v_\eta^1 \cdots v_{\eta^{sl-1}}^{sl-1})\ \cdots\ (v_1^{s-1}\ v_\eta^0\ v_{\eta^2}^1 \cdots v_{\eta^{sl-1}}^{sl-2}).$$

□

Lemma 4.2.5 *Let $H = \langle\sigma\rangle$ and $K = \langle\tau\rangle$ be the cyclic groups generated by the automorphisms σ and τ, respectively. The action of K on H by conjugation gives a homomorphism ϕ of K into $Aut(H)$.*

Proof. It suffices to show that H is closed under conjugation by elements of K. For any $0 \le a \le sl$ and $0 \le b \le sl-1$ and every $v_i^j \in V(\Gamma)$, we have

$$\tau^b\sigma^a\tau^{-b}(v_i^j) = \tau^b\sigma^a(v_{\eta^{-b}i}^{j-b}) = \tau^b(v_{a+\eta^{-b}i}^{j-b}) = v_{\eta^b a+i}^j = \sigma^{\eta^b a}(v_i^j),$$

and thus,

$$\tau^b\sigma^a\tau^{-b} = \sigma^{\eta^b a} \in H.$$

It follows that $kHk^{-1} \subseteq H$ for all $k \in K$. Thus, the map $\phi : K \to Aut(H)$ which maps each $k \in K$ to the inner automorphism of H defined by conjugation by k is well-defined. It is clear that $\phi(k_1k_2) = \phi(k_1)\cdot\phi(k_2)$ for any $k_1, k_2 \in K$. □

Lemma 4.2.6 *Let H and K be the subgroups of $Aut(\Gamma)$ as defined above. Then, $H \cap K = \{1\}$.*

Proof. Suppose $\sigma^a = \tau^b$ for some $0 \le a \le sl$ and $0 \le b \le sl-1$. Then for every $v_i^j \in V(\Gamma)$,

$$\sigma^a(v_i^j) = v_{a+i}^j = v_{\eta^b i}^{j+b} = \tau^b(v_i^j)$$

if and only if $j + b \equiv j \pmod{s}$ and $a + i \equiv \eta^b i \pmod{p}$ for every $0 \le i \le sl$ and $0 \le j \le s-1$. This is possible if and only if $a \equiv 0 \pmod{p}$ and $b \equiv 0 \pmod{(p-1)}$. □

Theorem 4.2.7 *Let H and K be the subgroups of $Aut(\Gamma)$, and let ϕ be the homomorphism from K into $Aut(H)$ as defined above. Let $G = H \rtimes_\phi K$. Then $G \cong HK \le Aut(\Gamma)$ and G is a transitive automorphism group of Γ.*

Proof. By Lemmas 4.2.5 and 4.2.6, we see that $HK \leq Aut(\Gamma)$ and $G \cong HK$. We only need to verify that G is a vertex-transitive automorphism group of Γ. Without loss of generality, let v_g^h and v_i^j be two vertices with $h \leq j$, and let $\rho = \sigma^i \tau^{j-h} \sigma^{p-g}$. Then ρ moves v_g^h to v_i^j as follows:

$$\rho\left(v_g^h\right) = \sigma^i \tau^{j-h} \sigma^{p-g}\left(v_g^h\right) = \sigma^i \tau^{j-h}\left(v_0^h\right) = \sigma^i\left(v_0^j\right) = v_i^j.$$

□

Lemma 4.2.8 *Let x denote the vertex $v_0^0 = (0, H)$. Then, the point stabilizer G_x of G on $V(\Gamma)$ is the cyclic subgroup generated by τ^s and $|G_x| = l$.*

Proof. Since any element in G can be expressed as $\sigma^a \tau^b \in HK$ through the isomorphism $G \cong HK$ by Lemma 4.2.7, we have

$$G_x = \{\sigma^a \tau^b \in G : \ \sigma^a \tau^b(v_0^0) = v_0^0\} = \{\sigma^a \tau^b \in G : \ v_a^b = v_0^0\}.$$

This implies that

$$G_x = \{\sigma^a \tau^b \in G : \ a \equiv 0 (\text{mod } p), \ b \equiv 0 (\text{mod } s)\} = \{\tau^b : \ b \equiv 0 (\text{mod } s)\},$$

and thus, $G_x = \langle \tau^s \rangle$. □

Theorem 4.2.9 *Let G be the automorphism group of Γ defined in Theorem 4.2.7. The rank of the transitive permutation group G on $V(\Gamma)$ is $s(s+1)$.*

Proof. First we claim that each non-identity element of G_x fixes exactly s vertices. Since each element $\rho \in G_x \setminus \{1\}$ can be expressed as τ^{bs} for some $1 \leq b \leq l-1$ by Lemma 4.2.8, the number of vertices fixed by $\rho = \tau^{bs}$ is given by

$$|\{v_i^j \in V(\Gamma) : \ v_i^j = \tau^{bs}(v_i^j) = v_{\eta^{bs} i}^{j+bs}\}|$$

Since $\eta^{bs} i \equiv i$ (mod $(sl+1)$) is possible only when $i \equiv 0$ (mod $(sl+1)$) while $\eta^{j+bs} H = \eta^j H$ for all $0 \leq j \leq s-1$, so the number of vertices fixed by ρ is s.

By Burnside's Lemma, we have

$$N = \frac{1}{|G_x|} \sum_{\rho \in G_x} \text{fix}(\rho) = \frac{1}{l}\{s(sl+1) + s(l-1)\} = s(s+1),$$

where fix(ρ) is the number of $v \in V$ fixed by ρ. □

4.2.3 DSRG-$(s(2s+1), 2s, 2, 1, 2)$ and Johnson graph $J(2s+1, 2)$

If we consider the particular case with $l = 2$ in the construction of DSRGs in Proposition 4.2.1. These DSRGs have a close relationship to strongly regular graphs, known as the triangular graphs or Johnson graphs with diameter 2. The *Johnson graph* $J(p, 2)$ for any integer $(p \geq 5)$ is the graph defined as follows. Let P be a p-element set. The vertex set of $J(p, 2)$ is the set of all 2-element subsets of P, and two distinct vertices x and y are adjacent if $x \cap y \neq \emptyset$. Then $J(p, 2)$ is shown to be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (p(p-1)/2, 2(p-2), p-2, 4)$. The strongly regular graphs $J(p, 2)$ are uniquely determined by their parameters for any integer p, except for $p = 8$.

By Proposition 4.2.1, for each prime $p = 2s+1$ with $s \geq 2$, we have the tactical configuration $\mathcal{T} = (\mathbb{Z}_p, \mathcal{B}, \in)$ with parameters $(p,\ \frac{1}{2}(p-1)p,\ 2,\ p-1)$. With $H = \langle \eta^s \rangle = \{\eta^0, \eta^s\}$ and $B_{i,j} = i + \eta^j H = \{i + \eta^j, i + \eta^{s+j}\}$, the block set

$$\mathcal{B} = \{B_{i,j} :\ i \in \mathbb{Z}_p,\ j \in \{0, 1, \ldots, s-1\}\}$$

of $\mathcal{T}$ consists of all 2-element subsets of $\mathbb{Z}_p$.

Theorem 4.2.10 *Given a prime $p = 2s + 1 \geq 5$, let Γ be the DSRG with parameters $(\frac{1}{2}(p-1)p, p-1, 2, 1, 2)$ constructed by Proposition 4.2.1 with $l = 2$. Then the undirected graph $\overline{\Gamma}$ defined on $V(\Gamma)$ in such a way that each vertex x is adjacent to all the vertices in $N^+(y) \cup N^+(z) \setminus \{x\}$ where both y and z are the vertices in $N^+(x) \cap N^-(x)$, is the strongly regular graph with parameters $(\frac{1}{2}(p-1)p, 2(p-2), p-2, 4)$.*

Proof. In this proof, given a vertex $x = (i, B_{i,j})$, we refer i to the 'point' of x and $B_{i,j}$ to the 'block' of x. Also let $N^+(x)$ and $N^-(x)$ denote the set of out-neighbors of x and that of in-neighbors of x, respectively. Then, since $t = 2$ for Γ, we know that every vertex $x \in V(\Gamma)$, there exist exactly two vertices y and z in $N^+(x) \cap N^-(x)$. In this case, the point of x belongs to the blocks of both y and z while the points of y and z are contained in the block of x. Furthermore, the block of every out-neighbor of y (or z, resp.) contains the point of y (or z, resp.); and thus, the block of each out-neighbor of y or z intersects with the block of x. That is, the block of x and the block of each out-neighbor of y or z has exactly one common element unless the out-neighbor is x. Hence the proof follows from the definition of Johnson graph with $p = 2s + 1$. □

4.2.4 Construction B

It is well-known that the set of flags of a 2-design yields a DSRG (see [7] and references given there). In this section, we show that all DSRGs obtained from a family of Hadamard 2-designs have vertex-transitive automorphism groups. First we recall the Hadamard difference sets and 2-designs from which we will derive the DSRGs of our interest. Let $G = \{0, 1, \ldots, \mathbf{v} - 1\}$ be an (additive) abelian group of order $\mathbf{v}$. A $\mathbf{k}$-element subset D of G is called a $(\mathbf{v}, \mathbf{k}, \Lambda)$-*difference set* if the $\mathbf{k}(\mathbf{k} - 1)$ possible differences modulo $\mathbf{v}$ between members of D comprise all non-zero elements of G exactly Λ times. Whenever we have a $(\mathbf{v}, \mathbf{k}, \Lambda)$-difference set D, we have a 2-$(\mathbf{v}, \mathbf{k}, \Lambda)$ design $(G, \mathcal{B}, \in)$ with $\mathcal{B} = \{D + a : \ a \in G\}$ where $D + a$ denotes the translate $\{d + a : \ d \in D\}$ of D. Paley showed that the set of all non-zero squares in GF(q) gives a difference set of size $(q - 1)/2$ for each $q \equiv 3 \pmod 4$ [31]. Namely, let G be the additive group of GF(q) where $q = 4n - 1$, and let $D = \{a^2 : \ a \in G \setminus \{0\}\}$. Then D is a $(4n - 1, 2n - 1, n - 1)$-difference set, known as a Hadamard difference set. In what follows, we work with this difference set for given prime q with $q \equiv 3 \pmod 4$ [4].

Proposition 4.2.11 *Let q be a prime such that $q \equiv 3 \pmod 4$ and $q \geq 7$. Let $D = \{i^2 : \ i = 1, 2, \ldots, (q - 1)/2\} \subset \mathbb{Z}_q$, a difference set of size $\mathbf{k} = \frac{q-1}{2}$. Let $\mathcal{B} = \{g + D : \ g \in \mathbb{Z}_q\}$. Then $(\mathbb{Z}_q, \mathcal{B}, \in)$ is a $2 - (q, \frac{1}{2}(q - 1), \frac{1}{4}(q - 3))$-design. Furthermore, the graph defined on the set of flags*

$$V(\Gamma) = \{v_i^g : \ i \in \mathbb{Z}_q, \ g \in \{1, 2, \ldots, (q - 1)/2\}\}$$

where $v_i^g = (i + g^2, i + D)$, with adjacency defined by

$$v_i^g \to v_j^h \quad \text{if and only if} \quad v_i^g \neq v_j^h \text{ and } i + g^2 \in j + D$$

is a DSRG with parameters

$$\left(\tfrac{1}{2}q(q - 1), \ \tfrac{1}{4}(q - 1)^2 - 1, \ \tfrac{1}{8}(q + 1)(q - 3), \ \tfrac{1}{8}(q + 1)(q - 3) - 1, \ \tfrac{1}{8}(q - 1)(q - 3)\right).$$

Proof. The proof is routine. (cf. [7].) □

In what follows, graph Γ refers to the DSRG given in Proposition 4.2.11 unless otherwise stated.

Lemma 4.2.12 *The permutation $\sigma : V(\Gamma) \to V(\Gamma)$ where $v_i^g \mapsto v_{i+1}^g$ is an automorphism in $Aut(\Gamma)$. As a permutation of vertices, σ has a cycle decomposition into $\frac{1}{2}(q-1)$ cycles each with length q. Thus the order of σ is q.*

Proof. It is obvious that the map σ is an automorphism of Γ. For the cycle structure of σ, we can verify that

$$\sigma = (v_0^1 \ v_1^1 \ \cdots v_{q-1}^1)(v_0^2 \ v_1^2 \ \cdots v_{q-1}^2) \cdots (v_0^{\mathbf{k}} \ v_1^{\mathbf{k}} \ \cdots v_{q-1}^{\mathbf{k}})$$

for $i \in \mathbb{Z}_q$ and $1 \leq g \leq \mathbf{k} = \frac{1}{2}(q-1)$ as desired. □

Lemma 4.2.13 *Let η be a primitive element of $\mathbb{Z}_q$ and let $a = \eta^2 \in D$. The permutation $\tau : V(\Gamma) \to V(\Gamma)$ where $v_i^g \mapsto v_{ai}^{\eta g}$, is an automorphism of Γ. The permutation τ has a cycle decomposition consisting of q cycles each with length $\frac{1}{2}(q-1)$. Thus, τ has order $\mathbf{k} = \frac{1}{2}(q-1)$.*

Proof. It is easy to see that τ is a bijection. Also, τ maps an adjacent pair of vertices to an adjacent pair since $i + q^2 \in j + D$ implies that $ai + ag^2 = ai + (\eta g)^2 \in aj + aD = aj + D$. (Here we note that $aD = \{ad : \ d \in D\} = D$ since D is a subgroup of $\mathbb{Z}_q^*$.) The cycle decomposition of τ can be expressed as follows.

$$\tau = (v_0^g \ v_0^{\eta g} \ \cdots v_0^{\eta^{\mathbf{k}-1} g})(v_1^g \ v_1^{\eta g} \ \cdots v_1^{\eta^{\mathbf{k}-1} g}) \cdots (v_{q-1}^g \ v_{q-1}^{\eta g} \ \cdots v_{q-1}^{\eta^{\mathbf{k}-1} g}).$$

□

Theorem 4.2.14 *The action of $Aut(\Gamma)$ on $V(\Gamma)$ is transitive.*

Proof. Suppose $v_i^g = (i + g^2, i + D)$ and $v_j^h = (j + h^2, j + D)$ are any two vertices of Γ. Then it is easy to verify that the automorphism $\rho = \sigma^j \tau^f \sigma^{q-i}$ maps v_i^g to v_j^h where $f \in \{0, 1, \ldots, \mathbf{k} - 1\}$ such that $a^f = h^2 g^{-2}$. □

Lemma 4.2.15 *Let $N = \langle \sigma \rangle$ and $K = \langle \tau \rangle$. Then N is closed under conjugation by elements of K in $Aut(\Gamma)$. In particular, K acting on N by conjugation is well-defined. The corresponding permutation representation $\varphi : K \to Aut(N)$ is also well-defined.*

Proof. Let $N = \{\sigma^u \mid 0 \leq u \leq q-1\}$ and $K = \{\tau^w \mid 0 \leq w \leq \frac{q-1}{2} - 1\}$. Then

$$\tau^w \sigma^u \tau^{-w}\left(v_i^g\right) = \tau^w \sigma^u \tau^{-w}\left((i+g^2, i+D)\right) = \tau^w \sigma^u \tau^{-w}(v_i^g) = \sigma^{a^w u}(v_i^g);$$

and thus, $kNk^{-1} \subseteq N$ for all $k \in K$, and the rest follows. □

Corollary 4.2.16 *With the above N and K, we have*

(i) $N \cap K = \{1\}$,

(ii) $NK \leq Aut(\Gamma)$, *and*

(iii) $NK \cong N \rtimes_\varphi K$.

Proof. (i) The subgroups N and K contains only the identity automorphism in common because $\sigma^u(v_0^g) = \tau^w(v_0^g)$ if and only if $u + D = D$ and $u + g^2 = (\eta^w g)^2$ if and only if $u \equiv 0$ and $w \equiv 0$.

(ii) By Lemma 4.2.15, $(n_1k_1)(n_2k_2) = n_1(k_1n_2k_1^{-1})k_1k_2 = (n_1n_3)(k_1k_2) \in NK$ where $n_3 = knk^{-1} \in N$. Every element nk of NK has its inverse $n'k' \in NK$ with $n' = k^{-1}n^{-1}k \in N$ and $k' = k^{-1} \in K$. So $NK \leq Aut(\Gamma)$.

(iii) By Lemma 4.2.15, $N \trianglelefteq NK$, and $NK \cong N \rtimes_\varphi K$. □

Theorem 4.2.17 *Let $G = NK \cong N \rtimes_\varphi K$. Then G acts sharply transitively on $V(\Gamma)$. In particular, the rank of the permutation group G is $\frac{1}{2}q(q-1) = |V(\Gamma)|$.*

Proof. The transitivity of G has been shown in the proof of Theorem 4.2.14. It is easy to verify that $x = (1, D) \in V(\Gamma)$, the point stabilizer $G_x = \{1\}$. Thus the rank is $|V(\Gamma)|$. It is also evident that for every pair of vertices there is exactly one element in NK which moves one vertex to the other. □

4.3 Directed Strongly Regular Graphs Obtained from Coset Graphs

In this section, we introduce the notion of coset graphs which extends the notion of Cayley graphs. We investigate some properties of vertex transitive DSRGs with $t = \mu$.

Definition 4.3.1 *Let H be a subgroup of a group G and let S be a subset of G. Define a digraph on the set $V = \{xH : x \in G\}$ of cosets of H in G such that the vertex yH is adjacent to xH, (i.e., $xH \to yH$) if and only if $x^{-1}y \in HSH$. This digraph is denoted by $\Gamma(G, H, HSH)$. Such a graph is called a coset graph of G (cf. [34]).*

Proposition 4.3.2 *[34] $\Gamma(G, H, HSH)$ is a vertex transitive digraph.*

Proof. For each $g \in G$ and $xH \in V$. We can define $(xH)^g = gxH$.

$$\begin{aligned} xH \to yH &\Leftrightarrow x^{-1}y \in HSH \\ &\Leftrightarrow x^{-1}g^{-1}gy \in HSH \\ &\Leftrightarrow gxH \to gyH \end{aligned}$$

Thus g is an automorphism of $\Gamma(G, H, HSH)$. Since G acts transitively on V, $\Gamma(G, H, HSH)$ is vertex transitive. □

Proposition 4.3.3 *[34] Suppose Γ is a transitive digraph with transitive subgroup G of $Aut(\Gamma)$. Then Γ is isomorphic to $\Gamma(G, H, HSH)$ for a subgroup H and a subset S of G.*

Proof. Let $V(\Gamma) = \{v_0, v_1, \cdots, v_n\}$, $H = G_{v_0}$ and $S = \{g \in G : v_0^g \in N_{v_0}\}$ where N_{v_0} is the set of out-neighbors of v_0. We are going to show $\Gamma \cong \Gamma(G, H, HSH)$. Define a map ϕ:

$$\phi : v_i \longmapsto g_iH$$

where $v_0^{g_i} = v_i$. Note that $v_0^g = v_i$ if and only if $g \in g_iH$. Thus, ϕ is well-defined. We claim that ϕ is an isomorphism between Γ and $\Gamma(G, H, HSH)$.

Suppose that $g \in HSH$, then $g = h_1sh_2$ for some $h_1, h_2 \in H$ and $s \in S$.

$$v_0^g = v_0^{h_1sh_2} = h_1sh_2v_0 = h_1(sv_0) \in N_{v_0}$$

Therefore, $HSH \subseteq \{g \in G : v_0^g \in N_{v_0}\}$. Now suppose that $g \in G$ such that $v_0^g \in N_{v_0}$. That is $v_0^g = v_i = v_0^{s_i}$ for some $s_i \in S$ and $v_0^{s_i^{-1}g} = v_0$. Hence, $s_i^{-1}g \in G_{v_0}$ and $g \in s_iH$. Thus $\{g \in G : v_0^g \in N_{v_0}\} \subseteq HSH$ so that $\{g \in G : v_0^g \in N_{v_0}\} = HSH$. For any $v_i, v_j \in V$:

$$v_i \to v_j \quad \Leftrightarrow \quad v_0 = v_i^{g_i^{-1}} \to v_0^{g_i^{-1}g_j}$$

$$\Leftrightarrow \quad g_i^{-1} g_j \in HSH$$

$$\Leftrightarrow \quad g_i H \to g_j H.$$

$v_i \to v_j$ in Γ if and only if $v_i^{\phi} \to v_j^{\phi}$ in $\Gamma(G, H, HSH)$ and hence ϕ is an isomorphism between Γ and $\Gamma(G, H, HSH)$. □

Lemma 4.3.4 *Let H be a nontrivial subgroup and S be a non-empty subset of a group G. If Cay(G, HSH) is a DSRG with parameters (v, k, t, λ, μ), then the vertex set of Cay(G, HSH) can be partitioned into $r + 1$ independent sets of size $|H|$.*

Proof. Let Cay(G, HSH) be a DSRG with parameters (v, k, t, λ, μ). Let $T = \{e = g_0, g_1, g_2, \cdots, g_r\}$ be a left transversal (i.e., the set of representatives of left cosets of H in G). Then, for each $g \in T$ the coset gH forms an independent set in Cay(G, HSH). To see this, if we assume that gH is not an independent set, then there exists an $x, y \in gH$ such that $x \to y$. This implies $x^{-1}y = (gh)^{-1}(gh') = h^{-1}h' \in HSH$ and so $e \in HSH$. This contradicts to the fact that there are no loops in Cay(G, HSH). □

Lemma 4.3.5 *Let H be a nontrivial subgroup and S be a non-empty subset of a group G. If Cay(G, HSH) is a DSRG with parameters (v, k, t, λ, μ), then $t = \mu$.*

Proof. Let Cay(G, HSH) be a DSRG with parameters (v, k, t, λ, μ). Let $T = \{e = g_0, g_1, g_2, \cdots, g_r\}$ be a left transversal. For $x \in g_i H$ and $y \in g_j H$, suppose $x \to y$ in Γ. Then $x^{-1}y = (g_i h)^{-1}(g_j h') = h^{-1} g_i^{-1} g_j h' \in HSH$; and thus, $g_i^{-1} g_j \in HSH$. Now if we take any $z = g_i h'' \in g_i H$, then $z \to y$ since $z^{-1}y = (g_i h'')^{-1}(g_j h') = h''^{-1} g_i^{-1} g_j h' \in HSH$. Similarly, for any $w \in g_j H$ we will have $x \to w$. Thus the adjacency in Cay(G, HSH) will be determined by the elements of the set T.

Suppose $a, b \in gH$ for $g \in T$. Then, there is no edge between the vertices a and b. There are exactly t many vertices c, such that $a \longleftrightarrow c$ and $b \longleftrightarrow c$. Thus $\mu \geq t$. Since μ cannot be greater than t we have $t = \mu$. □

Lemma 4.3.6 *Let H be a nontrivial subgroup and S be a non-empty subset of a group G. Let A and B be the adjacency matrices of $\Gamma(G, H, HSH)$ and Cay(G, HSH), respectively. Then, $B = A \otimes J_{|H|}$ where $J_{|H|}$ is the $|H| \times |H|$ all-ones matrix.*

Proof. Let $H = \{e, h_1, \cdots, h_s\}$ and let $T = \{e = g_0, g_1, g_2, \cdots, g_r\}$ be a left transversal. First, index the rows and columns of A by $H, g_1H, g_2H, \cdots, g_rH$. if $g_iH \to g_jH$, then for any $x \in g_iH$ and $y \in g_jH$ we have $x^{-1}y \in HSH$. Similarly, if g_iH and g_jH are not adjacent, then for any $x \in g_iH$ and $y \in g_jH$ we have $x^{-1}y \notin HSH$. Now, we can construct B such that its columns and rows are indexed by $g_0, h_1, \cdots, h_s, g_1, g_1h_1, \cdots, g_1h_s, \cdots, g_r, g_rh_1, \cdots, g_rh_s$. That is,

$$B_{g_ih_j, g_mh_n} = \begin{cases} 1 & \text{if } (g_ih_j)^{-1}(g_mh_n) \in HSH; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $B = A \otimes J_{|H|}$. □

Theorem 4.3.7 *Let H be a nontrivial subgroup and S be a non-empty subset of a group G. Then the coset graph $\Gamma(G, H, HSH)$ is a DSRG with parameters $(v, k, t, \lambda, \mu = t)$ if and only if the Cayley graph Cay(G, HSH) is a DSRG with parameters $(|H|v, |H|k, |H|t, |H|\lambda, |H|\mu)$.*

Proof. Assume $\Gamma(G, H, HSH)$ is a DSRG with parameters (v, k, t, λ, μ) and $t = \mu$. Let the adjacency matrix of $\Gamma(G, H, HSH)$ be A. By Lemma 4.3.6, the adjacency matrix B of Cay(G, HSH) is $B = A \otimes J_{|H|}$. $B = A \otimes J_{|H|}$ is an adjacency matrix of a DSRG with parameters $(|H|v, |H|k, |H|t, |H|\lambda, |H|\mu)$.

Suppose Cay(G, HSH) is a DSRG with parameters $(v', k', t', \lambda', \mu')$. By Lemma 4.3.5, we have $t' = \mu'$ and by Lemma 4.3.4, we can construct a digraph whose vertices are left cosets and the adjacency between two vertices are inherited from Cay(G, HSH). The digraph we defined is isomorphic to $\Gamma(G, H, HSH)$; and so, $\Gamma(G, H, HSH)$ is a DSRG with parameters $(v'/|H|, k'/|H|, t'/|H|, \lambda'/|H|, \mu'/|H|)$. □

Corollary 4.3.8 *Let Γ be a vertex transitive DSRG with $t = \mu$. There are no abelian subgroups of Aut(Γ), which is transitive on the vertex set of Γ.*

Proof. By Proposition 4.3.3, there exists a coset graph $\Gamma(G, H, HSH)$ isomorphic to Γ. First, assume that $H = \{e\}$. Then, $\Gamma(G, H, HSH)$ is actually the graph Cay(G, S). This implies Aut(Γ) has a subgroup which is isomorphic to G and is transitive. By Theorem 5 of [25], we know that G must be a non-abelian group. Now, assume that $H \neq \{e\}$. By Theorem 4.3.7, there exist a Cayley graph, Cay(G, HSH), that is also a DSRG. Again by Theorem 5 of [25], G must be non-abelian. □

Corollary 4.3.9 *Let H be a nontrivial subgroup and S be a non-empty subset of G. Let $T = \{e = g_0, g_1, g_2, \cdots, g_r\}$ be a left transversal. Suppose that $\Gamma(G, H, HSH)$ is a DSRG with $t = \mu$. Then, the number of paths of length 2 from g_iH to g_jH is equal to $\frac{1}{|H|}$ times the coefficient of $g_i^{-1}g_j$ in $\underline{K}^2$ where $K = HSH$.*

Proof. By Theorem 4.3.7, there exists a Cayley graph, Cay(G, K), that is also a DSRG. Let g and h belong to cosets g_iH and g_jH, respectively for some g_i and g_j in T. The number of paths of length 2 from g to h is given by the coefficient of $g^{-1}h$ in $\underline{K}^2$. Instead of g and h we can choose their representatives in T. Thus the number of paths of length 2 in $\Gamma(G, H, HSH)$ is equal to $\frac{1}{|H|}$ times the coefficient of $g_i^{-1}g_j$ in $\underline{K}^2$. □

Example 4.3.10 *Let G be the group given by $\langle \rho, \tau : \rho^5 = \tau^4 = e, \tau\rho = \rho^2\tau \rangle$. We know that $G = NK$, where $N = \langle \rho \rangle$ is a normal subgroup of G and $K = \langle \tau \rangle$. Let $H = \langle \tau^2 \rangle$ and $S = \{\rho, \rho^2\tau\}$.*

Claim 4.3.11 *Cay(G, HSH) is a DSRG with parameters $(20, 8, 4, 2, 4)$.*

Proof. $F = HSH = \{\rho, \rho^4, \rho^2\tau, \rho^3\tau, \rho\tau^2, \rho^4\tau^2, \rho^2\tau^3, \rho^3\tau^3\}$. By direct calculation

$$\underline{F}\,\underline{F} + 2\underline{F} = 4\underline{G}.$$

□

Claim 4.3.12 $\Gamma(G, H, HSH)$ *is a DSRG with parameters* $(10, 4, 2, 1, 2)$.

Proof. It follows from Theorem 5.3.6 □

The above digraph has G as its full automorphism group.

Claim 4.3.13 *Let U be a subgroup of G of order 10. Then, $N \subseteq U$.*

Proof. Assume that U is a subgroup of G with $|U| = 10$. Since N is a normal subgroup, UN should be a subgroup of G.

$$|UN| = \frac{|U||N|}{|U \cap N|} = \frac{50}{|U \cap N|} \mid 20.$$

Hence $U \cap N = N$. $\square$

Proposition 4.3.14 (Dedekind Law) *Let L be a subgroup of a group G. If U, V and UV are subgroups of G, $UV \cap L$ is not only a subgroup, but also the product of the subgroups U and $V \cap L$.*

Claim 4.3.15 *There is only one subgroup of G whose order is 10.*

Proof. Assume that U is a subgroup of G with $|U| = 10$. By the Dedekind Law

$$NK \cap U = N(K \cap U).$$

Thus, the subgroup U is the product of the subgroups N and $(K \cap U)$. Since $|U| = 10$, $|K \cap U| = 2$. This implies $K \cap U = \{e, \tau^2\}$. Hence $U = \{e, \rho, \cdots, \rho^4, \tau^2, \rho\tau^2, \cdots, \rho^4\tau^2\}$ $\square$

Claim 4.3.16 *$\Gamma(G, H, HSH)$ is not a Cayley graph.*

Proof. It is enough to show that the subgroup U is not regular on the vertex set of $\Gamma(G, H, HSH)$. Observe that stabilizer of the vertex H in $\Gamma(G, H, HSH)$ is the subgroup H. Thus, the subgroup U is not regular. $\square$

More generally, we can apply the technique above to show the following.

Lemma 4.3.17 *Let $l, s \geq 2$ and $p = ls + 1$ be a prime. Let α be a primitive element of $\mathbb{Z}_p$. Consider the group $G = \langle \rho, \tau : \rho^p = \tau^{p-1} = e, \tau\rho = \rho^\alpha\tau \rangle$ which is isomorphic to $N \rtimes K$ where $N = \langle \rho \rangle$ and $K = \langle \tau \rangle$ are cyclic subgroups. Then there is a unique subgroup of G of order $s(sl + 1)$.*

Proof. Assume there exist a subgroup U of G such that $|U| = s(sl + 1)$. Since N is a normal subgroup, UN should be a subgroup of G.

$$|UN| = \frac{|U||N|}{|U \cap N|} = \frac{s(sl + 1)(sl + 1)}{|U \cap N|} \mid sl(sl + 1).$$

Hence $U \cap N = N$. Now lets apply the Dedekind law.

$$NK \cap U = N(K \cap U).$$

Thus the subgroup U is the product of the subgroups N and $(K \cap U)$. Since $|U| = s(sl + 1)$, $|K \cap U| = s$. This implies $K \cap U$ is the unique subgroup $\langle \tau^l \rangle$. $\square$

Lemma 4.3.18 *Let G be the group in Lemma 4.3.17, $S = \{\rho, \rho^{\alpha}\tau, \cdots, \rho^{\alpha^{s-1}}\tau\}$ and $H = \langle \tau^s \rangle$. Then, $\Gamma(G, H, HSH)$ is a DSRG.*

Proof. Let $X = \{\rho, \rho^{\alpha^s}, \cdots, \rho^{\alpha^{(l-1)s}}\}$. Observe that $HSH = X \cup X^{\alpha}\tau \cup \cdots \cup X^{\alpha^{p-2}}\tau^{p-2}$. Let $\underline{F} = \underline{X} + \underline{X^{\alpha}\tau} + \cdots + \underline{X^{\alpha^{p-2}}\tau^{p-2}}$. Then

$$\underline{F}\underline{F} = l^2\underline{G} - l\underline{F}.$$

Hence, $\Gamma(G, H, HSH)$ is a DSRG with parameters $(s(sl+1), sl, l, l-1, l)$. □

Lemma 4.3.19 *Let Γ be a coset graph constructed in Lemma 4.3.18. If l and s are relatively prime then Γ can be recognized as a Cayley graph.*

Proof. Let $T = \{e, \rho, \cdots, \rho^{sl}, \tau, \rho\tau, \cdots, \rho^{sl}\tau, \cdots, \tau^{s-1}, \rho\tau^{s-1} \cdots, \rho^{sl}\tau^{s-1}\}$. Since $\rho^i\tau^j H = \{\rho^i\tau^j, \rho^i\tau^{j+s}, \cdots, \rho^i\tau^{j+(l-1)s}\}$, for all j satisfying $0 \le j \le s-1$ the group element $\rho^i\tau^j$ is in a distinct left coset. Hence, T is a left transversal. Assume that l and s are relatively prime. Let $U = \{\rho^i\tau^{jl} : i = 0, \cdots, sl \quad j = 0, \cdots, s-1\}$. By Lemma 4.3.17, U is the unique subgroup of order $s(sl+1)$. We will show that U acts transitively on the vertex set of Γ. Without lost of generality, assume $\rho^g\tau^h H$ and $\rho^i\tau^j H$ be two vertices with $h \le j$ and $\rho^g\tau^h, \rho^i\tau^j \in T$. Since l and s are relatively prime, l has an inverse modulo s. Let $u = (h-j)l^{-1}$. Then $\rho^i\tau^{ul}\rho^{sl+1-g} \in U$ moves $\rho^g\tau^h H$ to $\rho^i\tau^j H$. Thus, U acts transitively on the vertex set of Γ.
Now, we will show U acts semi regularly on the vertex set of Γ. Let $\rho^n\tau^{ml} \in U$ and $\rho^i\tau^j \in T$. Assume $\rho^n\tau^{ml}\rho^i\tau^j H = \rho^i\tau^j H$. This implies $\tau^{-j}\rho^{-i}\rho^n\tau^{ml}\rho^i\tau^j \in H$ and $\rho^{(n-i)\alpha^{-j}+i\alpha^{ml-j}}\tau^{ml} \in H$. Since $0 \le m \le s-1$ and $gcd(l, s) = 1$, $\rho^{(n-i)\alpha^{-j}+i\alpha^{ml-j}}\tau^{ml} = e$. Hence, $m = 0$ and $\rho^{n\alpha^{-j}} = e$. Since α is an primitive element, n is equivalent to 0 in modulo $sl+1$. These arguments show, if $\rho^n\tau^{ml}\rho^i\tau^j H = \rho^i\tau^j H$ then $\rho^n\tau^{ml} = e$. This completes the proof. □

Notes 4.3.20 *For more information on transitive graphs we recommend the book titled* Algebraic Graph Theory *[18] and the paper* vertex transitive graphs *[34]. A similar technique used in Construction A and Construction B can be found in [2].*

Research Problem : *Classify all coset graphs in Lemma 4.3.18, which cannot be recognized as a Cayley graph.*

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