1960

Dependence of the solution of a Goursat problem on the characteristic data

Orville Goering
Iowa State University

DEPENDENCE OF THE SOLUTION

OF A GOURSAT PROBLEM ON THE CHARACTERISTIC DATA

by

Orville Goering

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1960

TABLE OF CONTENTS

I. INTRODUCTION

Associated with the partial differential equation

$$(1) \quad L(U) = U_{xy} + A(x, y)\, U_x + B(x, y)\, U_y + C(x, y)\, U = D(x, y)$$

are two classical problems referred to as the Goursat problem and the Cauchy problem. The Goursat problem for equation (1) consists of finding a function $U(x, y)$ which satisfies (1) and the given conditions

$$(2) \qquad U(x, 0) = f(x), \quad U(0, y) = g(y), \quad f(0) = g(0).$$

The Cauchy problem for equation (1) consists of obtaining a function which satisfies equation (1) and the conditions

$$(3) \qquad U(x(t), y(t)) = \alpha(t), \quad U_x \frac{dx}{dt} - U_y \frac{dy}{dt} = \beta(t),$$

where $x = x(t)$, $y = y(t)$ are parametric equations of some curve which is nowhere tangent to a coordinate line. Various conditions of regularity must be imposed on the functions A, B, C, D, f, g, α, β to insure the existence and uniqueness of solutions to these problems.

The differential equation

$$(4) \quad M(V) = V_{st} - [A(s, t)\, V]_s - [B(s, t)\, V]_t + C(s, t)\, V = 0$$

is called the adjoint of equation (1) provided that A_x and B_y are continuous in the region considered. A solution of (4) which satisfies the additional conditions

$$(5) \qquad V_t - A\,V = 0 \quad \text{when } s = x$$

$$V_s - B\,V = 0 \quad \text{when } t = y$$

$$V(x, y) = 1,$$

is called a Riemann (sometimes Riemann-Green) function for equation (1).

Solutions of the Goursat and Cauchy problems can be expressed in terms of the Riemann function by Riemann's method as is shown in Chapter III. This method was introduced by Riemann (7) for two special cases of (1) which arise in the theory of gas dynamics. The method, in the form in which it is given here, appeared in one of Darboux's Volumes (4). E. T. Copson (2) gives a list of the Riemann functions for which explicit formulae have been obtained by one method or another. No general method for obtaining these functions is known.

The question of dependence of the solution of the Goursat problem on the functions f and g has been considered by C. Corduneanu (3). In that paper the following example is given: Consider the problem

$$(6) \qquad U_{xy} - p(x)\,U_x = 0,$$

$$(7) \quad U(x, 0) = n^{-\frac{1}{3}} \sin nx, \; U(0, y) = 0 \quad (n \text{ an integer}).$$

The solution is easily obtained in the form

$$(8) \qquad U_n(x, y) = n^{\frac{2}{3}} \int_0^x \cos nt \exp yp(t)\, dt.$$

At the point $(2\pi, 1)$,

$$(9) \qquad U_n(2\pi, 1) = n^{\frac{2}{3}} \int_0^{2\pi} \exp p(t) \cos nt\, dt = n^{\frac{2}{3}} \pi a_n$$

where a_n is the nth Fourier coefficient of $\exp p(t)$. It is known that there exist positive continuous bounded functions whose Fourier coefficients a_n are such that $a_n \neq O(n^{-\frac{1}{2}})$. (See Hobson (5)). Let $q(t)$ denote such a function and let $p(t) = \log q(t)$. With $p(t)$ chosen in this manner $\lim_{n \to \infty} U_n(2\pi, 1)$ does not exist. Thus, a sequence of the prescribed functions on the x-axis which converges uniformly to zero does not necessarily give rise to a convergent solution function. Hence, in this sense, the solution does not depend continuously on the functions f and g. In the same paper (3), sufficient conditions for continuity in this sense are given. The results are the same as those of Theorems 4.1 and 4.2 in Chapter IV.

The possibility that some of the functions A, B, C, D may not possess partial derivatives must also be considered. The existence and uniqueness of the solutions of both the Goursat and Cauchy problems require only the continuity of these functions. A difficulty appears to arise in considering the Riemann function since equation (4) may not make sense unless A_s and B_t are assumed to exist. This difficulty can be resolved and the Riemann method can be retained by replacing equation (4) and conditions (5) by the integral equation

$$(10)\quad V(s, t) = -1 + \rho(s) + \sigma(t) - \int_s^x \left\{ \int_t^y C(\xi, \eta)\, d\eta \right\} d\xi + \int_t^y \left\{ A(x, \eta) V(x, \eta) - A(s, \eta) V(s, \eta) \right\} d\eta + \int_s^x \left\{ B(\xi, y) V(\xi, y) - B(\xi, t) V(\xi, t) \right\} d\xi .$$

where

$$(11)\quad \sigma(t) = \exp \int_y^t A(x, \tau)\, d\tau, \quad \rho(s) = \exp \int_x^s B(\tau, y)\, d\tau .$$

If A and B possess the partial derivatives required in (4), it is easy to show that a function V(s, t) satisfies (10) if and only if it satisfies (4) and (5). A. Wintner (9) has shown that the solutions of the Goursat and Cauchy problems can be expressed by Riemann's method using the solution of (10) for the Riemann function even in the case when A and B are assumed only continuous.

The purpose of this thesis is to characterize the dependence of the solutions of the Goursat and Cauchy problems on the given functions (f, g, α and β) by using Riemann's method. It is shown that under certain restrictions on A and B, the solution of (10) belongs to a certain Banach space. The dependence of the solutions of the Goursat and Cauchy problems on the given functions is then determined by utilizing the properties of this Banach space.

II. DEFINITIONS AND NOTATION

The concept of total variation of a function plays an important role in this thesis. Some of the results needed later are given in this chapter.

Definition 2.1. Let f be defined on $[a, b]$. Let $P_n = [x_0, x_1, \cdots, x_n]$ be a partition of $[a, b]$ and set $\Delta f_k = f(x_k) - f(x_{k-1})$, $k = 1, 2, \cdots, n$. If there exists a positive number M such that

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

for all partitions P_n then f is said to be of bounded variation on $[a, b]$.

Definition 2.2. Let f be of bounded variation on $[a, b]$, and let $\sum(P_n)$ denote the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P_n of $[a, b]$. The number

$$V_a^b(f) = \sup\left\{ \sum (P_n) \mid P_n \in P \right\},$$

where P is the collection of all partitions of $[a, b]$, is called the total variation of f on $[a, b]$.

For proofs of the following theorems see Apostol (1).

Theorem 2.1. Let f be of bounded variation on $[a, b]$. Let $p(x) = V_a^x(f)$. Then

(a) p is monotone non-decreasing,

(b) p-f is monotone non-decreasing.

Theorem 2.2. Let f be defined on $[a, b]$. Then f is of bounded variation if and only if f can be expressed as the difference of two monotone functions.

<u>Theorem 2.3.</u> If f is continuous on $[a, b]$ and f' exists and is bounded in (a, b), then f is of bounded variation on $[a, b]$.

<u>Theorem 2.4.</u> If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$ and

$$|f(x)| \leq |f(a)| + V_a^b(f).$$

When there is no danger of confusion, $V_a^b(f)$ is replaced by $V(f)$ for the sake of simplicity.

<u>Theorem 2.5.</u> If f and g are of bounded variation on $[a, b]$ so are their sum, difference and product. Also we have

$$(12) \quad V(f \pm g) \leq V(f) + V(g) \text{ and } V(fg) \leq A\, V(f) + B\, V(g),$$

where $A \geq |f(x)|$ and $B \geq |g(x)|$ for $x \in [a, b]$.

<u>Theorem 2.6.</u> Let $\{f_n\}$ be a sequence of functions on $[a, b]$ satisfying

(i) $V(f_n)$ exists for each integer n,

(ii) $f_n(a) = 0$ for all n,

(iii) for every $\epsilon > 0$ there exists an integer N such that for all $n, m > N$, $V(f_n - f_m) < \epsilon$.

Then there exists a unique function f of bounded variation on $[a, b]$ such that $\lim_{n \to \infty} V(f_n - f) = 0$.

Proof: Since $f_n(a) - f_m(a) = 0$, we have by Theorem 2.4

$$|f_n - f_m| \leq V(f_n - f_m). \tag{13}$$

Hence, by (iii) $\{f_n\}$ converges uniformly. By Theorem 2.1, each f_n can be written in the form $f_n = p_n - h_n$, where $p_n(x) = V_a^x(f_n)$ and $h_n(x)$ are monotone non-decreasing functions. We have

$$|V_a^x(f_n) - V_a^x(f_m)| \leq V_a^x(f_n - f_m) \leq V(f_n - f_m). \tag{14}$$

Hence, from (iii), the sequence $\{p_n\}$ converges uniformly to some function, say p. Since each p_n is non-decreasing, p has this property. It is also clear that $\{h_n\}$ converges to a non-decreasing function h. Therefore

$$f = \lim_{n \to \infty} f_n = \lim_{n \to \infty} (p_n - h_n) = p - h \tag{15}$$

is the difference of two non-decreasing functions and is thus of bounded variation.

To show that $\lim_{n \to \infty} V(f_n - f) = 0$, let $\epsilon > 0$ be given and choose N so large that $V(f_n - f_m) < \frac{\epsilon}{3}$ whenever n, m > N. Let P_M be a partition of $[a, b]$. Now

$$V(f_n - f) = \sup\{ \sum(P_M) | P_M \in P\}. \tag{16}$$

But,

$$(17) \quad \sum(P_M) = \sum_{k=1}^{M} |f_n(x_k) - f(x_k) - f_n(x_{k-1}) + f(x_{k-1})|$$

$$\leq \sum_{k=1}^{M} |f_n(x_k) - f_s(x_k) - f_n(x_{k-1}) + f_s(x_{k-1})|$$

$$+ \sum_{k=1}^{M} |f_s(x_k) - f(x_k)| + \sum_{k=1}^{M} |f(x_{k-1}) - f_s(x_{k-1})|$$

is true for arbitrary s. Let us choose s so large that $s > N$ and

$$|f_s(x) - f(x)| < \frac{\epsilon}{3M}$$

for all x in $[a, b]$. Then, if in addition $n > N$, we have

$$(18) \quad \sum(P_M) \leq V(f_n - f_s) + 2\sum_{k=1}^{M} \frac{\epsilon}{3M} \leq \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon .$$

Therefore, we have

$$V(f_n - f) = \sup \left\{ \sum(P_M) \,|\, P_M \; \varepsilon \; P \right\} \leq \epsilon$$

since (18) is true for all partitions. It follows that

$$(19) \quad \lim_{n \to \infty} V(f_n - f) = 0.$$

To show that f is unique, we assume g is a function such that

$$\lim_{n \to \infty} V(f_n - g) = 0.$$

Let $\epsilon > 0$ be given. We have the inequality

$$V(f - g) \leq V(f - f_n + f_n - g) \leq V(f_n - f) + V(f_n - g)$$

for every integer n. Let n be chosen so large that

$$V(f_n - f) < \frac{\epsilon}{2} \text{ and } V(f_n - g) < \frac{\epsilon}{2}.$$

Then $V(f - g) < \epsilon$. Therefore, since ϵ is arbitrary, $V(f - g)$ is equal to zero. Thus, $f - g$ is constant and since $f(a) - g(a) = 0$, $f - g = 0$. This completes the proof.

Theorem 2.7. Let $\{f_n\}$ be a sequence of functions of bounded variation on $[a, b]$ and let us denote by $\|f_n\|_1$ the expression

$$\|f_n\|_1 = V(f_n) + |f_n(a)|. \tag{20}$$

If for every $\epsilon > 0$ there exists an integer N such that $\|f_n - f_m\|_1 < \epsilon$ for all n, m > N, then there exists a function f of bounded variation on $[a, b]$ such that

(a) $\|f\|_1$ exists and $\|f\|_1 \leq \sup\{\|f_n\|_1\} < \infty$,

(b) $\lim_{n \to \infty} \|f_n - f\|_1 = 0$,

(c) $\{f_n\}$ converges uniformly to f,

(d) $\|f_n - f\|_1 < 2\epsilon$ for $n > N$.

Proof: We first prove that $\{f_n\}$ converges uniformly and then that the uniform limit f of $\{f_n\}$ satisfies (b), (a) and (d) in that order. Let us consider the sequence $\{g_n\}$ where $g_n(x) = f_n(x) - f_n(a)$. Clearly, $g_n(a) = 0$ for all n and we have

$$V(g_n - g_m) = V(f_n - f_m) \leq \|f_n - f_m\|_1.$$

Therefore, the sequence $\{g_n\}$ satisfies all of the hypotheses of Theorem 2.6. Let $g = \lim_{n \to \infty} g_n$. From the definition (20) it is clear that

$$|f_n(a) - f_m(a)| \leq \| f_n - f_m \|_1$$

and thus the sequence $\{f_n(a)\}$ converges to some number, say A. Thus,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \{g_n(x) + g_n(a)\} = g(x) + A$$

uniformly. Further

$$\lim_{n \to \infty} \| f_n - f \|_1 = \lim_{n \to \infty} \| g_n + f_n(a) - g - A \|_1$$

$$= \lim_{n \to \infty} \{ V(g_n - g) + |f_n(a) - A| \}$$

$$= 0$$

by Theorem 2.6. Results (b) and (c) have now been established.

To show (a), we observe that

$$\| f_n \|_1 \leq \| f_n - f \|_1 + \| f \|_1. \tag{21}$$

Let $\epsilon > 0$ be given and let N be an integer so large that for all $n > N$, $\| f_n - f \|_1 < \epsilon$. Then for all $n > N$, (21) becomes

$$\| f_n \|_1 < \epsilon + \| f \|_1.$$

$\overline{T}(f) = \sup \left\{ T_o^y(f) \mid s \in [o, x] \right\}$. Let us denote by B(R) the space of all functions f defined on R such that $\overline{T}(f)$ and $\overline{S}(f)$ exist. Let

$$(22) \qquad \| f \| = \max \left\{ \overline{S}(f), \overline{T}(f) \right\} + |f(x, y)|.$$

Theorem 2.8. The space B(R) with the usual addition and real scalar multiplication together with $\| f \|$ as norm is a Banach space.

Proof: First we show that B(R) is a linear space. Let f, g $\in$ B(R) and let α, β be real numbers. For fixed t, f and g are of bounded variation in s and therefore by Theorem 2.5 we get

$$S(\alpha f + \beta g) \leq S(\alpha f) + S(\beta g) = |\alpha| S(f) + |\beta| S(g)$$

$$\leq |\alpha| \overline{S}(f) + |\beta| \overline{S}(g).$$

Thus $S(\alpha f + \beta g)$ is a bounded function of t and hence $\overline{S}(\alpha f + \beta g)$ exists. In the same way we can show that $\overline{T}(\alpha f + \beta g)$ exists. Therefore, $\alpha f + \beta g \in B(R)$ and so B(R) is a linear space.

To show that $\| f \|$ has the properties of a norm, we must establish that for arbitrary f, g B(R) and for an arbitrary real number α the following are satisfied:

$$(23) \qquad \begin{aligned} &\| \alpha f \| = |\alpha| \, \| f \|, \\ &\| f \| = 0 \text{ if and only if } f = 0, \\ &\| f + g \| \leq \| f \| + \| g \| . \end{aligned}$$

We have for each t, $S(\alpha f) = |\alpha| S(f)$. Hence $\overline{S}(\alpha f) = |\alpha|\overline{S}(f)$. Similarly we have that $\overline{T}(\alpha f) = |\alpha|\overline{T}(f)$. Therefore it follows that

$$\begin{aligned}\| \alpha f \| &= \max \left\{ \overline{S}(\alpha f), \overline{T}(\alpha f)\right\} + |\alpha f(x, y)| \\ &= \max \left\{ |\alpha|\overline{S}(f), |\alpha|\overline{T}(f)\right\} + |\alpha| \, |f(x, y)| \\ &= |\alpha| \max \left\{\overline{S}(f), \overline{T}(f)\right\} + |\alpha| \, |f(x, y)| \\ &= |\alpha| \, \| f \|.\end{aligned}$$

Thus the first of (23) is established.

Clearly, $\| 0 \| = 0$. Suppose $f \neq 0$. Then either $f(x, y) \neq 0$ in which case $\| f \| \neq 0$ or there exists at least one point (s_1, t_1) other than (x, y) in R such that $f(s_1, t_1) \neq 0$. If $S(f) = 0$ when $t = t_1$, then $f(x, t_1) = f(s_1, t_1)$. But in that case $T(f) \neq 0$ when $s = x$ since $f(x, t_1) \neq f(x, y)$. Thus, if $f \neq 0$ we must have either $|f(x, y)| \neq 0$, $\overline{S}(f) \neq 0$ or $\overline{T}(f) \neq 0$. In each case $\| f \| \neq 0$. Hence, $\| f \| = 0$ if and only if $f = 0$.

We next show that the inequality of (23) is valid. By definition we have

$$\| f + g \| = \max\left\{\overline{S}(f + g), \overline{T}(f + g)\right\} + |f(x, y) + g(x, y)|.$$

We have already established that

$$(24) \qquad \begin{aligned} S(f + g) &\leq S(f) + S(g) \quad \text{for all } t, \\ T(f + g) &\leq T(f) + T(g) \quad \text{for all } s. \end{aligned}$$

Hence we have

$$\overline{S}(f + g) \leq \overline{S}(f) + \overline{S}(g),$$

$$\overline{T}(f + g) \leq \overline{T}(f) + \overline{T}(g).$$

Using these we deduce that

$$(25)\ \max\left\{\overline{S}(f + g),\ \overline{T}(f + g)\right\} \leq \max\left\{\overline{S}(f) + \overline{S}(g),\ \overline{T}(f) + \overline{T}(g)\right\}$$

$$= \tfrac{1}{2}\ [\ \overline{S}(f) + \overline{S}(g) + \overline{T}(f) + \overline{T}(g)$$

$$+ \ |\overline{T}(f) + \overline{T}(g) - \overline{S}(f) - \overline{S}(g)\ |\ ]$$

$$\leq \tfrac{1}{2}\ [\ \overline{T}(f) + \overline{S}(f) + |\overline{T}(f) - \overline{S}(f)|\ ]$$

$$+ \ \tfrac{1}{2}\ [\ \overline{T}(g) + \overline{S}(g) + |\overline{T}(g) - \overline{S}(g)|\ ]$$

$$= \max\left\{\overline{S}(f),\ \overline{T}(f)\right\} + \max\left\{\overline{S}(g),\ \overline{T}(g)\right\}.$$

It is also clear that

$$(26) \qquad |f(x,\ y) + g(x,\ y)| \leq |f(x,\ y)| + |g(x,\ y)|.$$

Combining (25) and (26) we are led to

$$(27) \qquad \max\left\{\overline{S}(f + g),\ \overline{T}(f + g)\right\} + |f(x,\ y) + g(x,\ y)|$$

$$\leq \max\left\{\overline{S}(f),\ \overline{T}(f)\right\} + |f(x,\ y)| + \max\left\{\overline{S}(g),\ \overline{T}(g)\right\}$$

$$+ \ |g(x,\ y)|,$$

which is the same as

$$\| f + g \| \leq \| f \| + \| g \|.$$

To complete the proof of this theorem we must show that B(R) is complete in this norm. Let $\{f_n\}$ be a sequence of functions in B(R) with the property that for every $\epsilon > 0$ there exists an integer N such that $\| f_n - f_m \| < \epsilon$ for all $n, m > N$. It is necessary to establish the existence of a unique function $f \in B(R)$ which has the property that $\lim_{n \to \infty} \| f_n - f \| = 0$. This is done by showing that $\{f_n\}$ is uniformly convergent and that f, the limit function, is in fact the one and only function having the property that $\lim_{n \to \infty} \| f_n - f \| = 0$.

Using Theorem 2.4 twice gives us for all n, m

$$(28) \quad |f_n(s, t) - f_m(s, t)| \leq \overline{S}(f_n - f_m) + |f_n(x, t) - f_m(x, t)|$$

$$|f_n(x, t) - f_m(x, t)| \leq \overline{T}(f_n - f_m) + |f_n(x, y) - f_m(x, y)|.$$

Combining these two inequalities we have

$$(29) \quad |f_n(s, t) - f_m(s, t)| \leq \overline{S}(f_n - f_m) + \overline{T}(f_n - f_m)$$

$$+ |f_n(x, y) - f_m(x, y)| \leq 2 \| f_n - f_m \| .$$

This inequality assures us of the uniform convergence of the sequence $\{f_n\}$ to a limit function f. It is evident that $\| f_n \|$ is a bounded sequence. Let $\sup \{\| f_n \|\} = M$. Let $t = t_1$ be fixed. Then by Theorem 2.4, we have

$$(30) \quad S(f_n - f_m) + |f_n(x, t_1) - f_m(x, t_1)|$$

$$\leq S(f_n - f_m) + \overline{T}(f_n - f_m) + |f_n(x, y) - f_m(x, y)|$$

$$\leq 2\| f_n - f_m \| .$$

Each function $f_n(s, t_1)$ is of bounded variation in s. The left side of inequality (30) is the quantity $\| f_n(s, t_1) - f_m(s, t_1)\|_1$ introduced in Theorem 2.7. Inequality (30) shows that the sequence $\{f_n(s, t_1)\}$ satisfies the hypotheses of Theorem 2.7. Therefore, $f(s, t_1)$ is of bounded variation and

$$S(f) + |f(x, t_1)| \leq \sup \left\{ S(f_n) + |f_n(x, t_1)| \;\middle|\; n\varepsilon I \right\} < \infty$$

follows from (a) of Theorem 2.7. But we also have

$$S(f_n) + |f_n(x, t_1)| \leq 2 \| f_n \| \leq 2 M$$

for all t_1. Therefore, we have

$$S(f) + |f(x, t_1)| \leq 2 M,$$

and it follows that $\overline{S}(f)$ exists. Similarly we can show that $\overline{T}(f)$ exists and thus $f \in B(R)$.

It remains to be shown that $\lim_{n \to \infty} \| f - f_n \| = 0$ and that f is unique. From inequality (30) we conclude that if $n, m > N$ then

$$S(f_n - f_m) + |f_n(x, t_1) - f_m(x, t_1)| < 2 \epsilon .$$

Using Theorem 2.7 (d), we have

$$S(f_n - f) + |f_n(x, t_1) - f(x, t_1)| < 2(2\epsilon) = 4\epsilon.$$

Therefore it is clear that $\overline{S}(f_n - f) \leq 4\epsilon$. Similarly, we can establish that $\overline{T}(f_n - f) \leq 4\epsilon$. In addition, since for all $n, m > N$ it is true that

$$|f_n(x, y) - f_m(x, y)| \leq \| f_n - f_m \| < \epsilon,$$

it follows that $|f_n(x, y) - f(x, y)| < \epsilon$ for all $n > N$. Therefore we have

$$\max\left\{\overline{S}(f_n - f), \overline{T}(f_n - f)\right\} + |f_n(x, y) - f(x, y)| \leq 5\epsilon, \tag{31}$$

and $\| f_n - f \| \leq 5\epsilon$. It follows that $\lim_{n \to \infty} \| f - f_n \| = 0$.

To show f is unique, let $g \in B(R)$ be a function such that $\lim_{n \to \infty} \| f_n - g \| = 0$. We have for all n

$$\| f - g \| \leq \| f - f_n \| + \| f_n - g \|. \tag{32}$$

Let $\epsilon > 0$ be given. Then n can be chosen so large that $\| f - f_n \| < \frac{\epsilon}{2}$ and $\| f_n - g \| < \frac{\epsilon}{2}$. Thus for arbitrary $\epsilon > 0$ we have $\| f - g \| < \epsilon$. Therefore $\| f - g \| = 0$ and from the second of (23) we can conclude that $f - g = 0$ and, therefore, $f = g$. This completes the proof.

The next theorems are of considerable importance in the next chapter.

<u>Theorem 2.9.</u> If $f(x)$ is of bounded variation on $[a, b]$,

and if

$$F(x) = \int_a^x f(t)\, dt, \ a \leq x \leq b,$$

then

(33) $$V_a^x(F) = \int_a^x |f(t)|\, dt.$$

A proof of this theorem can be found in Widder (9, p. 20 ff).

Theorem 2.10. Let $f(s, t)$ be defined on R, let $\overline{S}(f)$ exist and let $T(f)$ exist for all s. If we define $F(s, t)$ by

$$F(s, t) = \int_t^y f(s, \tau)\, d\tau,$$

then $S(F)$ exists and $S(F) \leq \overline{S}(f)\, y$.

Proof: By definition we have

$$S(F) = \sup \left\{ \textstyle\sum(P_n) | P_n \in P \right\}.$$

Using the definition of $\sum(P_n)$, we get for an arbitrary partition P_n

(34) $$\begin{aligned} \textstyle\sum(P_n) &= \sum_{k=1}^n \left| \int_t^y f(s_k, \tau) d\tau - \int_t^y f(s_{k-1}, \tau) d\tau \right| \\ &\leq \sum_{k=1}^n \int_t^y |f(s_k, \tau) - f(s_{k-1}, \tau)|\, d\tau \\ &= \int_t^y \sum_{k=1}^n |f(s_k, \tau) - f(s_{k-1}, \tau)|\, d\tau \\ &\leq \int_t^y \overline{S}(f) d\tau \leq \overline{S}(f)\, y. \end{aligned}$$

Hence it follows that $S(F) \leq \overline{S}(f)\, y$.

III. BASIC EXISTENCE THEOREMS

A function U is called a solution of the Goursat problem in some open region containing the origin if at every point in the region U_x, U_y and U_{xy} exist and are continuous, and equations (1) and (2) are satisfied. In showing that a unique solution exists, the procedure involves a suitable restriction of the functions A, B, C, D, f, g, the representation of the problem as an equivalent integral equation and then showing that from some specified class of functions there is one and only one function satisfying the integral equation. A fixed point theorem is used to facilitate the procedure.

Let B be a metric space and T a mapping of B into itself. The mapping T is called a contraction mapping if there exists a real number α such that $0 < \alpha < 1$ and for every x_1, $x_2 \varepsilon B$, $\rho(Tx_1, Tx_2) \leq \alpha\, \rho(x_1, x_2)$ where ρ is the metric of B. The theorem which is used in the existence theorems which follow is given now. A proof can be found in Kolmogorov and Fomin (6, p. 50).

<u>Theorem 3.1.</u> Let B be a complete metric space and let T be a continuous mapping of B into itself. If there exists a positive integer n such that T^n is a contraction mapping then there exists one and only one $x \varepsilon B$ such that $x = Tx$.

<u>Theorem 3.2.</u> Let R be the region $R = \left\{(x, y);\ 0 \leq x \leq a,\ 0 \leq y \leq \beta\right\}$ and let A(x, y), B(x, y), C(x, y), D(x, y), f'(x),

$g'(y)$ be continuous in an open region $H \supset R$. Then there exists one and only one continuous function on R which satisfies the partial differential equation (1) and the conditions (2) in R.

Proof: It is necessary to change the form of the Goursat problem so that it can be formulated in the manner described earlier. If, $U(x, y)$ is any solution of the Goursat problem for (1), define $W(x, y)$ by

$$(35) \qquad W(x, y) = U(x, y) - f(x) - g(y) + f(0).$$

It is clear that

$$(36) \quad W(0, y) = U(0, y) - f(0) - g(y) + f(0) = g(y) - g(y) = 0,$$

$$W(x, 0) = U(x, 0) - f(x) - g(0) + f(0) = 0,$$

since $f(0) = g(0)$. In addition $W(x, y)$ must satisfy

$$\begin{aligned}(37) \quad W_{xy} + AW_x + BW_y + CW &= U_{xy} + AU_x + BU_y + CU \\ &\quad - Af' - Bg' - Cf - Cg + Cf(0) \\ &= D - Af' - Bg' - Cf - Cg + Cf(0).\end{aligned}$$

Furthermore, it is clear that if W is a solution of the Goursat problem

$$(38) \quad L(W) = W_{xy} + AW_x + BW_y + CW = D - Af' - Bg' - Cf - Cg + Cf(0)$$

$$W(x, 0) = W(0, y) = 0,$$

then $U(x, y) = W(x, y) + f(x) + g(y) - f(0)$ is a solution of the Goursat problem for (1). Therefore, it will suffice to show that under the hypotheses of this theorem there exists a unique solution of (38).

Next let us consider the integro-differential system

$$(39)\quad W = - \int_0^y \int_0^x [A(s, t)p + B(s, t)q + C(s, t)W - F(s, t)]\, dsdt,$$

$$p = - \int_0^y [A(x, t)p + B(x, t)q + C(x, t)W - F(x, t)]\, dt,$$

$$q = - \int_0^x [A(s, y)p + B(x, y)q + C(s, y)W - F(s, y)]\, ds,$$

where $F = D - Af' - Bg' - Cf - Cg + Cf(0)$. If it is true that there exist continuous functions $W(x, y)$, $p(x, y)$, and $q(x, y)$ which satisfy the three equations (39), then $W(x, y)$ is also a solution of the Goursat problem (38). For, if such functions exist, using the continuity hypothesis of the coefficients, the integrand in each of these equations is continuous. Thus, differentiation of the first of equations (39) yields $W_x = p$, $W_y = q$ and

$$(40)\quad W_{xy} = - A(x, y)\, W_x - B(x, y)\, W_y - C(x, y)\, W - F(x, y).$$

In addition, it is easy to see that $W(0, y) = W(x, 0) = 0$. Hence a continuous solution of (39) is a solution of the Goursat problem (38). Conversely, suppose $W(x, y)$ is a solution of (38). Then, since this implies the existence of the indicated partial derivatives, the integration of both sides of the equation

(41) $W_{xy} = - A(x, y)\, W_x - B(x, y)\, W_y - C(x, y)\, W - F(x, y)$

can be carried out to give the equations (39). Thus, $W(x, y)$ is a solution of (38) if and only if it is a solution of (39). It suffices, therefore, to prove that the system (39) has a unique solution.

Consider the Banach space $C(R)$ which consists of all continuous functions on $R = \{(s, t) \mid 0 \leq s \leq x,\ 0 \leq t \leq y\}$ with the usual addition and real scalar multiplication and norm given by

$$\| f \| = \sup \{ |f(s, t)| \mid (s, t) \in R \}.$$

The cartesian product space

$$C_3(R) = C(R) \times C(R) \times C(R)$$

is a complete metric space in the following manner. Let $(f_1, g_1, h_1), (f_2, g_2, h_2) \in C_3(R)$. Define addition and scalar multiplication by

(42) $$\alpha(f_1, g_1, h_1) + \beta(f_2, g_2, h_2)$$

$$= (\alpha f_1 + \beta f_2, \alpha g_1 + \beta g_2, \alpha h_1 + \beta h_2).$$

For a metric, use

$$(43) \qquad \rho[\,(f_1, g_1, h_1), (f_2, g_2, h_2)]$$

$$= \| f_2 - f_1 \| + \| g_2 - g_1 \| + \| h_2 - h_1 \|.$$

Completeness of $C_3(R)$ follows from the completeness of $C(R)$.

To complete the proof, it is convenient to express equations (39) in the abbreviated form

$$(44) \quad W = H_1(W, p, q),\ p = H_2(W, p, q),\ q = H_3(W, p, q),$$

or still more concisely,

$$(45) \quad (W, p, q) = [\, H_1(W, p, q), H_2(W, p, q), H_3(W, p, q)\,].$$

If $(f, g, h) \in C_3(R)$, then $H_1(f, g, h) \in C(R)$, $H_2(f, g, h) \in C(R)$, and $H_3(f, g, h) \in C(R)$ since each represents an integral of continuous functions. Hence, the right sides of equations (39) represent a mapping of $C_3(R)$ into itself. Let H denote this mapping. Then we can write the equations (39) in the abbreviated form

$$(46) \qquad (W, p, q) = H[\,(W, p, q)\,].$$

It remains to be shown that H is continuous and that there exists an integer n such that H^n is a contraction mapping. Under these conditions, Theorem 3.1 assures us of the existence of a unique fixed point of H, that is, there exist continuous functions W, p, q such that

$$(W, p, q) = H[\,(W, p, q)\,].$$

Hence, we conclude that there exists one and only one solution of (38).

To show that H is continuous let (f_1, g_1, h_1) and (f_2, g_2, h_2) belong to $C_3(R)$ and suppose that $\rho[(f_1, g_1, h_1), (f_2, g_2, h_2)] < \delta$. Let x, y be fixed and let α, β be two variables such that $0 \leq \alpha \leq x$, $0 \leq \beta \leq y$. Then it follows that

$$(47) \quad \Big| - \int_0^\beta\int_0^\alpha [A(s, t)g_1 + B(s, t)h_1 + C(s, t)f_1 - F(s, t)]dsdt$$
$$+ \int_0^\beta\int_0^\alpha [A(s, t)g_2 + B(s, t)h_2 + C(s, t)f_2 - F(s, t)]dsdt\Big|$$
$$\leq M \int_0^\beta\int_0^\alpha \Big\{|g_1 - g_2| + |h_1 - h_2| + |f_1 - f_2|\Big\} dsdt$$
$$\leq M\, \alpha\beta(\| g_1 - g_2\| + \| h_1 - h_2 \| + \| f_1 - f_2 \|)$$
$$\leq M\, \alpha\beta\, \rho\, [(f_1, g_1, h_1), (f_2, g_2, h_2)] < Mxy\delta ,$$

where $M = \max \Big\{\| A \|, \| B \|, \| C \|\Big\}$. In addition we have

$$(48) \quad \Big| - \int_0^\beta [A(\alpha, t)g_1 + B(\alpha, t)h_1 + C(\alpha, t)f_1 - F(\alpha, t)]\, dt$$
$$+ \int_0^\beta [A(\alpha, t)g_2 + B(\alpha, t)h_2 + C(\alpha, t)f_2 - F(\alpha, t)]\, dt\Big|$$
$$\leq M \int_0^\beta \Big\{\| g_1 - g_2 \| + \| h_1 - h_2 \| + \| f_1 - f_2 \|\Big\}\, dt$$
$$\leq M\, y\,\delta$$

for all α such that $0 \leq \alpha \leq x$. In the same way we have

$$(49)\quad \Big| - \int_0^{\alpha} [A(s, y)g_1 + B(s, y)h_1 + C(s, y)f_1 - F(s, y)]\, ds$$

$$+ \int_0^{\alpha} [A(s, y)g_2 + B(s, y)h_2 + C(s, y)f_2 - F(s, y)]\, ds \Big|$$

$$\leq M\, x\delta$$

for all β such that $0 \leq \beta \leq y$.

The inequality (47) can be written as

$$(50)\qquad |H_1(f_1, g_1, h_1) - H_1(f_2, g_2, h_2)| < Mxy\delta\ .$$

The left side of (50) is a function of α, β. Taking the supremum over all $(\alpha, \beta) \varepsilon R$ gives us

$$(51)\qquad \| H_1(f_1, g_1, h_1) - H_1(f_2, g_2, h_2)\| \leq Mxy\delta\ .$$

In just the same way we have established by (48) and (49) that

$$(52)\qquad \| H_2(f_1, g_1, h_1) - H_2(f_2, g_2, h_2)\| \leq My\delta,$$

$$\| H_3(f_1, g_1, h_1) - H_3(f_2, g_2, h_2)\| \leq Mx\delta\ .$$

The inequalities (51) and (52) give us

$$(53)\quad \rho\,[(H_1(f_1, g_1, h_1), H_2(f_1, g_1, h_1), H_3(f_1, g_1, h_1)),$$

$$(H_1(f_2, g_2, h_2), H_2(f_2, g_2, h_2), H_3(f_2, g_2, h_2))]$$

$$< M\,\delta\,(xy + y + x),$$

which can be written

$$(54)\quad \rho\,[H[(f_1, g_1, h_1)], H[(f_2, g_2, h_2)]] < M\,\delta(xy + y + x).$$

Thus, the continuity of H has been established for it is now clear that for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that $\rho[(f_1, g_1, h_1), (f_2, g_2, h_2)] < \delta$ implies $\rho[H[(f_1, g_1, h_1)], H[(f_2, g_2, h_2)]] < \epsilon$ since δ may be chosen as

$$\delta = \frac{\epsilon}{M(xy + x + y)}.$$

To show that there exists an integer n such that H^n is a contraction mapping, it is convenient to characterize H^n in terms of successive approximations. Let (f, g, h) be an arbitrary element in $C_3(R)$ and define

$$(f_1, g_1, h_1) = H\,[\,(f, g, h)\,] \tag{54}$$

and inductively for $n \geq 2$

$$(f_n, g_n, h_n) = H\,[\,(f_{n-1}, g_{n-1}, h_{n-1})\,]. \tag{55}$$

It is evident that

$$(f_n, g_n, h_n) = H^n(f, g, h). \tag{56}$$

But, we have

$$(f_n, g_n, h_n) = H\,[\,(f_{n-1}, g_{n-1}, h_{n-1})\,], \tag{57}$$

which gives us

$$(58) \qquad \begin{aligned} f_n &= H_1(f_{n-1}, g_{n-1}, h_{n-1}), \\ g_n &= H_2(f_{n-1}, g_{n-1}, h_{n-1}), \\ h_n &= H_3(f_{n-1}, g_{n-1}, h_{n-1}). \end{aligned}$$

To show that H^n is a contraction mapping for some integer n, let (f, g, h) and (f', g', h') be arbitrary elements in $C_3(R)$ and let us consider

$$(59) \qquad \rho\ [\, H^n\, [(f, g, h)\,],\ H^n\, [(f', g', h')]\,]$$
$$= \rho\ [(f_n, g_n, h_n), (f'_n, g'_n, h'_n)],$$

where (f'_n, g'_n, h'_n) is defined analogously to (f_n, g_n, h_n). But by definition we have

$$(60) \qquad \rho\ [\ (f_n, g_n, h_n), (f'_n, g'_n, h'_n)\ ]$$
$$= \| f_n - f'_n \| + \| g_n - g'_n \| + \| h_n - h'_n \|.$$

It is only necessary to show that the last expression becomes less than $\rho\ [\ (f, g, h), (f', g', h')\ ]$ for sufficiently large n. This is accomplished by showing inductively that

$$(61) \qquad \begin{aligned} |f_n - f'_n| &\leq \frac{3\ ML\ K^{n-1}(\alpha + \beta)^{n+1}}{(n+1)!}, \\ |g_n - g'_n| &\leq \frac{3\ ML\ K^{n-1}(\alpha + \beta)^{n}}{n!}, \\ |h_n - h'_n| &\leq \frac{3\ ML\ K^{n-1}(\alpha + \beta)^{n}}{n!}, \end{aligned}$$

where $M = \max \left\{ \| A \|, \| B \|, \| C \| \right\}$, $L = \max \left\{ \| f - f' \|, \| g - g' \|, \| h - h' \| \right\}$ and $K = M(2 + x + y)$. From these inequalities it follows that

$$(62) \qquad \| f_n - f_n' \| \leq \frac{3\ ML\ K^{n-1}(x + y)^{n+1}}{(n + 1)!} ,$$

$$\| g_n - g_n' \| \leq \frac{3\ ML\ K^{n-1}(x + y)^{n}}{n!} ,$$

$$\| h_n - h_n' \| \leq \frac{3\ ML\ K^{n-1}(x + y)^{n}}{n!} ,$$

from which it is easily seen that the right side of equation (60) becomes less than $\rho\ [\ (f, g, h), (f', g', h')\ ]$ for sufficiently large n. We now use equations (58) and the definition of H_1, H_2, and H_3 to establish (61). We have

$$(63) \quad |f_n - f_n'| = \left| \int_0^\beta dt \int_0^\alpha [\ A(g_{n-1} - g_{n-1}') + B(h_{n-1} - h_{n-1}') + C(f_{n-1} - f_{n-1}')\ ]\ ds \right| .$$

With n = 1 we get

$$(64) \quad |f_1 - f_1'| \leq M \int_0^\beta dt \int_0^\alpha \left\{ |g - g'| + |h - h'| + |f - f'| \right\} ds$$

$$\leq 3\ LM\alpha\beta \leq \frac{3\ LM\ (\alpha + \beta)^2}{2} .$$

Therefore, the first of the inequalities (61) is true for n = 1. In much the same way we have

$$(65)\quad |g_n - g_n'| = \left| \int_0^\beta \left\{ A(g_{n-1} - g_{n-1}') + B(h_{n-1} - h_{n-1}') + C(f_{n-1} - f_{n-1}') \right\} dt \right|$$

$$|g_1 - g_1'| \le M \int_0^\beta \left\{ |f - f'| + |g - g'| + |h - h'| \right\} dt$$

$$\le 3\, ML\beta \le 3\, ML(\alpha + \beta).$$

In the same way we can show that

$$(66)\qquad |h_1 - h_1'| \le 3\, ML(\alpha + \beta).$$

Hence, each of the inequalities (61) holds true for $n = 1$. Let us assume they hold true for $n = p - 1$. Then from (63) we get

$$(67)\quad |f_p - f_p'| = \left| \int_0^\beta dt \int_0^\alpha \left[A(g_{p-1} - g_{p-1}') + B(h_{p-1} - h_{p-1}') + C(f_{p-1} - f_{p-1}') \right] ds \right|$$

$$\le M \int_0^\beta dt \int_0^\alpha \left\{ |g_{p-1} - g_{p-1}'| + |h_{p-1} - h_{p-1}'| + |f_{p-1} - f_{p-1}'| \right\} ds$$

$$\le M \int_0^\beta dt \int_0^\alpha \left\{ \frac{6MLK^{p-2}(s + t)^{p-1}}{(p - 1)!} + \frac{3MLK^{p-2}(s + t)^p}{p!} \right\} ds$$

$$\le \frac{3M^2LK^{p-2}}{(p - 1)!} \int_0^\beta \left\{ \frac{2(\alpha + t)^p}{p} + \frac{(\alpha + t)^{p+1}}{p(p + 1)} - \frac{\alpha^p}{p} - \frac{\alpha^{p+1}}{p(p + 1)} \right\} dt$$

$$\le \frac{3M^2LK^{p-2}}{p!} \int_0^\beta \left\{ 2(\alpha + t)^p + \frac{(\alpha + t)^{p+1}}{(p + 1)} \right\} dt$$

$$\le \frac{3M^2LK^{p-2}}{p!} \int_0^\beta (\alpha + t)^p \left\{ 2 + \frac{(\alpha + t)}{(p + 1)} \right\} dt$$

$$\leq \frac{3M^2LK^{p-2}(2 + x + y)}{p!} \left[\frac{(\alpha + \beta)^{p+1}}{(p + 1)} \right]$$

$$\leq \frac{3MLK^{p-1}}{(p + 1)!} (\alpha + \beta)^{p + 1}.$$

From (64) we get

$$(68) \quad |g_p - g'_p| = \left| \int_0^\beta \left\{ A(g_{p-1} - g'_{p-1}) + B(h_{p-1} - h'_{p-1}) + C(f_{p-1} - f'_{p-1}) \right\} dt \right|$$

$$\leq M \int_0^\beta \left\{ \frac{6MLK^{p-2}(\alpha + t)^{p-1}}{(p - 1)!} + \frac{3MLK^{p-2}(\alpha + t)^p}{p!} \right\} dt$$

$$\leq \frac{3M^2LK^{p-2}}{(p - 1)!} \int_0^\beta (\alpha + t)^{p-1} \left[2 + \frac{\alpha + t}{p} \right] dt$$

$$\leq \frac{3M^2LK^{p-2}(2 + x + y)}{(p - 1)!} \int_0^\beta (\alpha + t)^{p-1} dt$$

$$\leq \frac{3MLK^{p-1}}{p!} (\alpha + \beta)^p.$$

In the same way we can show that

$$(69) \qquad |h_p - h'_p| \leq \frac{3MLK^{p-1}}{p!} (\alpha + \beta)^p.$$

This establishes the validity of the inequalities (61) which completes the proof of the theorem.

An important corollary of this theorem is the fact that under suitable hypotheses there exists a unique continuous function satisfying the adjoint equation (4) and the conditions (5).

<u>Corollary 3.1.</u> Let A, B, C, As, Bt be continuous on R. Then there exists one and only one function $V(s, t) \in C(R)$ which satisfies

$$(70) \quad M(V) = V_{st} - [\,A(s, t)V\,]_s - [\,B(s, t)V\,]_t + C(s, t)V = 0$$

and the conditions

$$(71) \qquad V_t - A(s, t)V = 0 \quad \text{when } s = x,$$

$$V_s - B(s, t)V = 0 \quad \text{when } t = y,$$

$$V(x, y) = 1.$$

Proof: If we change independent variables by setting

$$(72) \qquad s = x - \xi\,, \quad t = y - \eta\,,$$

equations (70) and (71) become

$$(73) \qquad V_{\xi\eta} + (AV)_\xi + (BV)_\eta + CV = 0$$

$$V(0, \eta) = \exp\left[\int_y^{\eta+y} A(x, \tau)\,d\tau\right]$$

$$V(\xi, 0) = \exp\left[\int_x^{\xi+x} B(\tau, y)\,d\tau\right].$$

The conclusion of this corollary now follows from the theorem.

The existence and uniqueness of a solution of the Cauchy problem for (1) can be established in much the same way as for the Goursat problem. Since the procedure and conditions on the coefficients are so similar to the above it is omitted here. Also, having established the existence of the Riemann function (solution of (70) satisfying (71)) we can prove existence and uniqueness by considering Riemann's method for the solution of the Cauchy problem.

Theorem 3.2. (Riemann's method). Let A_s, B_t, C, D ε C(R) and let $f'(x)$ and $g'(y)$ be continuous. Then the solution of the Goursat problem for equation (1) can be expressed in terms of the Riemann function V(s, t; x, y) by

$$(74)\quad U(x, y) = f(x) + g(y) - f(0) + \int_0^y\int_0^x V(s, t; x, y)\Big\{D(s,t) - A(s, t)\, f'(s) - B(s, t)\, g'(t) - C(s, t)\,[\, f(s) + g(t) - f(0)\,]\Big\}\, ds dt.$$

Proof: Let W be the unique solution of the Goursat problem (38). The identity

$$(75)\quad V\, L(W) - W\, M(V) = \frac{1}{2}\,[\, V W_s - W V_s + 2 B V W\,]_t + \frac{1}{2}\,[\, V W_t - W V_t + 2 A V W\,]_s$$

can be established by differentiation. Let us set

$$P = V W_s - W V_s + 2 B V W,\quad Q = V W_t - W V_t + 2 A V W.$$

Then we have

$$(76)\quad \iint_R [\, V L(W) - W M(V)\,]\, dA = \frac{1}{2}\iint_R (P_t + Q_s)\, dA.$$

The use of Green's theorem gives us

$$(77) \qquad \iint_R (P_t + Q_s)\, dA = \oint - P\, ds + Q\, dt,$$

in which the latter integral is taken around the boundary of R which consists of the lines $s = 0$, $s = x$, $t = 0$, $t = y$. Recalling that on $t = 0$, $W = 0$, $W_s = 0$ and on $s = 0$, $W = 0$, $W_t = 0$ (see (38)) and also using the facts that on $t = y$, $V_s = B\,V$ and on $s = x$, $V_t = A\,V$, (see (71)) we can simplify the line integral to

$$(78) \quad \oint - P\, ds + Q\, dt = \int_0^x V(W_s + B\,W)|_{t=y}\, ds + \int_0^y V(W_t + A\,W)|_{s=x}\, dt$$

$$= \int_0^x V\,W_s|_{t=y}\, ds + \int_0^x B\,V\,W|_{t=y}\, ds + \int_0^y V\,W_t|_{s=x}\, dt + \int_0^y A\,V\,W|_{s=x}\, dt.$$

Of the last four integrals the first and third can be integrated by parts and combined with the other two to give

$$(79) \quad \oint - P\, ds + Q\, dt = V(x, y)\,W(x, y) - \int_0^x W(V_s - B\,V)|_{t=y}\, ds + V(x, y)\,W(x, y) - \int_0^y W(V_t - AV)|_{s=x}\, dt$$

$$= 2\,W(x, y),$$

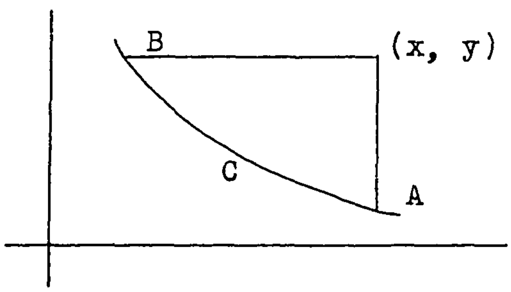
since $V(x, y) = 1$ and $V_s - B\,V = 0$ when $t = y$ and $V_t - AV = 0$ when $s = x$. Also, we have $M(V) = 0$ and $L(W) = D - A\,f'(s) -$

$- B\, g'(t) - C\, [\, f(s) + g(t) - f\,(0)\,]$. Therefore (76) becomes

$$(80) \qquad W(x, y) = \iint V \left\{ D - Af'(s) - Bg'(t) - C[\, f(s) + g(t) - f(0)\,] \right\} dA.$$

But from (35) $U(x, y) = W(x, y) + f(x) + g(y) - f(0)$, and (74) follows.

The solution of the Cauchy problem for (1) can also be obtained by this method. Let C denote the curve in (3) given by $x(t)$ and $y(t)$. It is convenient to let x be the parameter and have C be given by $y(x)$. Let $\bar{\bar{R}}$ be the region bounded by C and two coordinate lines as in the diagram.



Again let us consider the identity (75) in the form

$$(81) \qquad V\, L\,(W) - W\, M\,(V) = \tfrac{1}{2}\,(P_t + Q_s).$$

Integration over P leads us to

$$(82) \quad U(x, y) = \alpha(x) + \int_B^A V(s, y(s);\, x, y)\, [\, \beta(s) - \alpha'(s)\,]ds + \iint_{\bar{\bar{R}}} V(s, t;\, x, y)\, [\, D(s, t) - A(s, t)\, \alpha'(s) - C(s, t)\, \alpha(s)\,]\, dA.$$

Thus, $U(x, y)$ is given explicitly in terms of the coefficients of (1), the Riemann function and the data given in (3).

The existence of the solution of the Goursat problem was demonstrated under the assumption of continuity of the coefficients. The use of Riemann's method, however, requires that A_s and B_t exist and are continuous. The question arises as to whether the solution of the Goursat and Cauchy problems can be expressed by equations (74) and (82) respectively even under the weaker hypotheses. This has been answered affirmatively in the paper by Wintner (9).

The integral equation

$$(83)\quad V(s, t) = -1 + \rho(s) + \sigma(t) - \int_s^x \left\{ \int_t^y C(\xi, \eta) V(\xi, \eta)\, d\eta \right\} d\xi$$

$$+ \int_t^y [\, A(x, \eta)\, V(x, \eta) - A(s, \eta)\, V(s, \eta)\,]\, d\eta$$

$$+ \int_s^x [\, B(\xi, y)\, V(\xi, y) - B(\xi, t)\, V(\xi, t)\,]\, d\xi$$

in which

$$\rho(s) = \exp \int_x^s B(\tau, y)\, d\tau\,, \quad \sigma(t) = \exp \int_y^t A(x, \tau)\, d\tau$$

can be obtained by assuming $V(s, t)$ satisfies the adjoint equation and formally integrating. By differentiating (83) it is easy to see that if A, B, C, A_s, B_t are continuous then $V(s, t)$ must satisfy the adjoint equation (4) and the equations (5). Therefore, under these assumptions, $V(s, t)$

satisfies (83) if and only if it satisfies equations (4) and (5). For simplicity, the right side of equation (83) will be denoted by $G(V)$.

The following theorem will be stated without proof since the proof is much like that of Theorem 3.2.

<u>Theorem 3.3.</u> Let A, B, $C \varepsilon C(R)$. Then there exists a unique $V \varepsilon C(R)$ such that $V = G(V)$. That is, the integral equation (83) has one and only one continuous solution.

To show that the solution of (83) in the case that A, B, C are only continuous still serves as a Riemann function, it must be shown that the solution of (83) depends continuously on the coefficients A and B. It is well known that for arbitrary $A \varepsilon C(R)$ there exist sequences $\{A_n\}$ in $C(R)$ such that A_n is differentiable for each n and $\lim_{n \to \infty} \| A_n - A \| = 0$. But an examination of (74), (82) and (83) makes it clear that if $A_n \to A$, $V_n \to V$ then $U_n \to U$ where V_n and U_n are the solutions corresponding to A_n. Hence, the solution of the Goursat and Cauchy problems in the case that A and B are assumed continuous can be expressed by equations (74) and (82) respectively, the function V being the unique solution of (83). (See Wintner (9)).

The theorems which follow establish the fact that under suitable hypothesis the integral equation (83) has a unique solution in $B(R)$.

<u>Theorem 3.4.</u> Let $\overline{S}(A)$ and $\overline{T}(B)$ exist and let A, B, C

be continuous on R. Then $G(V)$ is a continuous mapping on $B(R)$ into itself.

Proof: Let $V \varepsilon B(R)$. $G(V)$ is defined by

$$(84)\ G(V) = -1 + \rho(s) + \sigma(t) - \int_s^x \left\{ \int_t^y C(\xi, \eta)\ V(\xi, \eta)\ d\eta \right\} d\xi$$
$$+ \int_t^y \left\{ A(x, \eta)\ V(x, \eta) - A(s, \eta)\ V(s, \eta) \right\} d\eta$$
$$+ \int_s^x \left\{ B(\xi,\ y)\ V(\xi,\ y) - B(\xi,\ t)\ V(\xi,\ t) \right\} d\xi,$$

where $\sigma(t)$ and $\rho(s)$ are given in (83). With t fixed it is clear that $S(G(V))$ exists since each term of (84) has a bounded derivative with respect to s on R except the term

$$\int_t^y \left\{ A(x, \eta)\ V(x, \eta) - A(s, \eta)\ V(s, \eta) \right\} d\eta$$

and this integral is of bounded variation in s as a consequence of Theorem 2.10. Using Theorems 2.5, 2.9, and 2.10 we get

$$(85)\ S(G(V)) \leq S(\rho) + \int_0^x \left| \int_t^y C(\xi,\ \eta)\ V(\xi,\ \eta)\ d\eta \right| d\xi$$
$$+ \int_t^y \overline{S}(AV) d\eta + \int_0^x |B(\xi,\ y)V(\xi,\ y) - B(\xi,\ t)V(\xi,\ t)| d\xi$$
$$\leq S(\rho) + \int_0^x \int_t^y |C(\xi,\ \eta)V(\xi, \eta)| d\eta d\xi + \overline{S}(AV)y$$
$$+ \int_0^x \overline{T}(BV) d\xi$$
$$\leq S(\rho) + M \int_0^x \int_t^y |V(\xi, \eta)| d\eta d\xi + \overline{S}(AV)y + \overline{T}(BV)x$$
$$\leq S(\rho) + 2\ M\ \|\ V\ \|\ xy + \overline{S}(AV)y + \overline{T}(BV)x,$$

where $M = \sup \left\{ C(s, t) \mid (s, t) \in R \right\}$. Therefore $\overline{S}(G(V)$ exists. Similarly we can show that $\overline{T}(G(V))$ exists. Hence, $G(V) \in B(R)$. Therefore, G is a mapping on B(R) into itself.

To show continuity, suppose $V_1, V_2 \in B(R)$ and $\| V_2 - V_1 \| < \delta$. Then we have

$$
\begin{aligned}
(86)\quad & \| G(V_2) - G(V_1) \| \\
& = \| - \int_s^x \left\{ \int_t^y C(\xi, \eta) \left[V_2(\xi, \eta) - V_1(\xi, \eta) \right] d\eta \right\} d\xi \\
& + \int_t^y A(x, \eta) \left[V_2(x, \eta) - V_1(x, \eta) \right] d\eta \\
& - \int_t^y A(s, \eta) \left[V_2(s, \eta) - V_1(s, \eta) \right] d\eta \\
& + \int_s^x B(\xi, y) \left[V_2(\xi, y) - V_1(\xi, y) \right] d\xi \\
& - \int_s^x B(\xi, t) \left[V_2(\xi, t) - V_1(\xi, t) \right] d\xi \|.
\end{aligned}
$$

The expression on the right is equal zero when $s = x$ and $t = y$. Therefore,

$$
\| G(V_2) - G(V_1) \| = \max \left\{ \overline{S}(G(V_2) - G(V_1)), \overline{T}(G(V_2) - G(V_1)) \right\}.
$$

Again, using Theorems 2.5, 2.9, 2.10 we get

$$
\begin{aligned}
(87)\quad & S(G(V_2) - G(V_1)) \\
& \leq \int_0^x \left| \int_t^y C(\xi, \eta) \left[V_2(\xi, \eta) - V_1(\xi, \eta) \right] d\eta \right| d\xi +
\end{aligned}
$$

$$+ \int_t^y S(A(s,\eta)\,[\,V_2(s,\eta) - V_1(s,\eta)\,])\,d\eta$$

$$+ \int_0^x |\,B(\xi, y)\,[\,V_2(\xi, y) - V_1(\xi, y)\,]$$

$$- B(\xi, t)\,[\,V_2(\xi, t) - V_1(\xi, t)\,]\,|\,d\xi$$

$$\leq 2\,M\,\|\,V_2 - V_1\,\|\,xy + \overline{S}(A(V_2 - V_1))y + \overline{T}(B(V_2 - V_1))x$$

$$\leq 2\,M\,\|\,V_2 - V_1\,\|\,xy + M\|\,V_2 - V_1\| + 2\overline{S}(A)\|\,V_2 - V_1\|\,y$$

$$+ M\,\|\,V_2 - V_1\,\|\,x + 2\,\overline{T}(B)\,\|\,V_2 - V_1\,\|\,x$$

$$< [\,2\,Mxy + My + 2\,\overline{S}(A)y + Mx + 2\,\overline{T}(B)x\,]\,\delta,$$

where $M = \max\left\{\sup |A|,\ \sup |B|,\ \sup |C|\right\}$. (We have made use of the fact that since $S(fg) \leq \sup |f|\ S(g) + \sup |g|\ S(f)$, $\overline{S}(fg) \leq \sup |f|\ \overline{S}(g) + \sup |g|\ \overline{S}(f)$. Also, we have used the fact that $|f(s, t)| \leq 2\ \|\,f\,\|$. See (29).). In the same way, we can show that

$$(88)\quad T(G(V_2) - G(V_1)) < [\,2\,Mxy + Mx + My + 2\,\overline{S}(A)y + 2\,\overline{T}(B)x\,]\,\delta\,.$$

Hence, it follows that

$$(89)\quad \|\,G(V_2) - G(V_1)\,\| < [\,2\,Mxy + Mx + My + 2\,\overline{S}(A)\,y + 2\,\overline{T}(B)x\,]\,\delta\,.$$

The continuity of G follows.

<u>Theorem 3.5.</u> Let $\overline{S}(A)$ and $\overline{T}(B)$ exist and let A, B, C be

continuous. Then there exists an integer n such that G^n is a contraction mapping.

Proof: Let V_1 and $V_2 \in B(R)$. We define $V_1^1 = G(V_1)$, $V_2^1 = G(V_2)$ and inductively $V_1^n = G(V_1^{n-1})$ and $V_2^n = G(V_2^{n-1})$ for $n \geq 2$. It is clear that $V_1^n = G^n(V_1)$ and $V_2^n = G^n(V_2)$. Thus we have

$$(90) \qquad \| V_1^n - V_2^n \| = \| G^n(V_1) - G^n(V_2) \|.$$

Let us define θ^n by $\theta^o = V_1 - V_2$ and for $n \geq 1$, $\theta^n = V_1^n - V_2^n$. From (83) we get

$$(91) \quad \theta^n = - \int_s^x \left\{ \int_t^y C(\xi, \eta)\, \theta^{n-1}(\xi, \eta)\, d\eta \right\} d\xi$$

$$+ \int_t^y \left[A(x, \eta)\theta^{n-1}(x, \eta) - A(s, \eta)\theta^{n-1}(s, \eta) \right] d\eta$$

$$+ \int_s^x \left[B(\xi, y)\theta^{n-1}(\xi, y) - B(\xi, t)\theta^{n-1}(\xi, t) \right] d\xi.$$

Note that $\theta^n = 0$ when $s = x$ or $t = y$ for each n. Therefore, $\| \theta^n \| = \max \left\{ \overline{S}(\theta^n), \overline{T}(\theta^n) \right\}$. We show that there exists an integer n such that $\| \theta^n \| < \| \theta^o \|$. This is accomplished by showing by induction that

$$(92) \qquad S_s^x (\theta^n) \leq \frac{M^n L^{n-1} K}{(n-1)!} \left[(x - s) + (y - t) \right]^{n-1},$$

$$T_t^y (\theta^n) \leq \frac{M^n L^{n-1} K}{(n-1)!} \left[(x - s) + (y - t) \right]^{n-1},$$

in which $M = \max \left\{ \sup |A|, \sup |B|, \sup |C|. \overline{S}(A), \overline{T}(B) \right\}$.

$L = x + y + 4$, $K = V_0(xy + 2y + 2x)$ and $V_0 = \| V_1 - V_2 \|$. From (92) we conclude

$$\text{(93)} \qquad \overline{S}(\theta^n) \leq \frac{M^n L^{n-1} K(x+y)^{n-1}}{(n-1)!},$$

$$\overline{T}(\theta^n) \leq \frac{M^n L^{n-1} K(x+y)^{n-1}}{(n-1)!},$$

from which it follows immediately that $\| \theta^n \| < \| \theta^0 \|$ if n is large enough.

To establish (92) we make use of the following inequalities, each of which is easily verified with the help of the theorems in Chapter II:

$$\text{(94)} \qquad S_s^x(A\theta^n) \leq 2\ M\ S_s^x(\theta^n)$$

$$T_t^y(B\theta^n) \leq 2\ M\ T_t^y(\theta^n).$$

The first of (94) can be verified by using Theorem 2.5. We have

$$\text{(95)} \quad S_s^x(A\theta^n) \leq (\sup |A|)\ S_s^x(\theta^n) + (\sup |\theta^n|)\ S_s^x(A)$$

$$\leq M\ S_s^x(\theta^n) + M(\sup |\theta^n|).$$

The expression $\sup |\theta^n|$ means supremum over all $\xi \in [\ s, x\ ]$. But by Theorem 2.4 we have

$$\text{(96)} \qquad |\theta^n(\xi, \eta)| \leq |\theta^n(s, \eta)| + S_\xi^x(\theta^n)$$

$$\leq S_s^x(\theta^n),$$

of which the right side does not depend on η. Hence the first inequality of (94) follows from (95). Similarly, the validity of the second inequality of (94) can be shown.

From equation (91), making use of (94), we get

$$(97)\quad S_s^x(\theta^n) \leq \int_s^x \left| \int_t^y C(\xi, \eta)\theta^{n-1}(\xi, \eta)d\eta \right| d\xi$$

$$+ \int_t^y S_s^x \left[A(x, \eta)\theta^{n-1}(x, \eta) - A(s, \eta)\theta^{n-1}(s, \eta)\right] d\eta$$

$$+ \int_s^x |B(\xi, y)\theta^{n-1}(\xi, y) - B(\xi, t)\theta^{n-1}(\xi, t)| d\xi.$$

But $|B(\xi, y)\theta^{n-1}(\xi, y) - B(\xi, t)\theta^{n-1}(\xi, t)| \leq T_t^y(B\theta^{n-1})$ holds true and therefore from (97) we obtain

$$(98)\quad S_s^x(\theta^n) \leq \int_s^x \left\{ \int_t^y | C(\xi, \eta) | S^x(\theta^{n-1}) d\eta \right\} d\xi$$

$$+ \int_t^y S_s^x(A\theta^{n-1})d\eta + \int_s^x T_t^y(B\theta^{n-1}) d\xi$$

$$\leq M \int_s^x \left\{ \int_t^y S^x(\theta^{n-1}) d\eta \right\} d\xi$$

$$+ 2 M \int_t^y S_s^x(\theta^{n-1})d\eta + 2 M \int_s^x T_t^y(\theta^{n-1}) d\xi .$$

Similarly, we get

$$(99)\quad T_t^y(\theta^n) \leq M \int_s^x \left\{ \int_t^y T_t^y(\theta^{n-1})d\eta \right\} d\xi + 2 M \int_t^y S_s^x(\theta^{n-1}) d\eta$$

$$+ 2 M \int_s^x T_t^y(\theta^{n-1}) d\xi .$$

If we set n = 1 in (98) we get

$$(100)\quad S_s^x(\theta^1) \le M \int_s^x \int_t^y V_o \, d\eta \, d\xi + 2\,M \int_t^y V_o \, d\eta + 2\,M \int_s^x V_o \, d\xi$$

$$= V_o\, M\, [\,(x - s)(y - t) + 2(y - t) + 2(x - s)\,]$$

$$\le V_o\, M\,(xy + 2y + 2x) = M\,K.$$

Hence, the first of inequalities (92) holds for n = 1. In the same way we can show that the second inequality holds for n = 1. Let us assume the inequalities are correct for $S_s^x(\theta^{n-1})$ and for $T_t^y(\theta^{n-1})$. Then we obtain from (98)

$$(101)\quad S_s^x(\theta^n) \le M \int_s^x \left\{ \int_t^y \frac{M^{n-1}L^{n-1}K}{(n-2)!} \,[\,(x - \xi) + (y - \eta)\,]^{n-2} d\eta \right\} d\xi$$

$$+ 2\,M \int_s^x \frac{M^{n-1}L^{n-2}K}{(n-2)!} \,[\,(x - \xi) + (y - t)\,]^{n-2} d\xi$$

$$+ 2\,M \int_t^y \frac{M^{n-1}L^{n-2}K}{(n-2)!} \,[\,(x - s) + (y - \eta)\,]^{n-2} d\eta.$$

Evaluating the integrals gives us

$$(102)\quad S_s^x(\theta^n) \le \frac{M^nL^{n-2}K}{n!} \left\{ -\,(x - s)^n - (y - t)^n + [(x-s) + (y-t)\,]^n \right\}$$

$$+ \frac{2\,M^nL^{n-2}K}{(n-1)!} \left\{ -\,(y - t)^{n-1} + [(x-s) + (y-t)]^{n-1} \right\}$$

$$+ \frac{2\,M^nL^{n-2}K}{(n-1)!} \left\{ -\,(x - s)^{n-1} + [(x-s) + (y-t)]^{n-1} \right\}.$$

This can be simplified to

$$(103) \quad S_s^x(\theta^n) \leq \frac{M^n L^{n-2} K}{(n-1)!} [\,(x-s) + (y-t)\,]^{n-1} \left\{ \frac{(x-s) + (y-t)}{n} + 4 \right\}$$

$$\leq \frac{M^n L^{n-2} K}{(n-1)!} [\,(x-s) + (y-t)\,]^{n-1} [\,x + y + 4\,]$$

$$\leq \frac{M^n L^{n-1} K}{(n-1)!} [\,(x-s) + (y-t)\,]^{n-1}.$$

In the same way, the second inequality of (92) can be shown to hold true under the induction hypothesis.

Therefore, the inequalities (93) have been shown to hold true and the proof is complete.

IV. DEPENDENCE THEOREMS

In this chapter some theorems concerning the dependence on the prescribed functions of the solutions of the Goursat and Cauchy problems are given.

We consider again the Goursat problem

$$(104)\quad U_{nxy} + A(x, y)U_{nx} + B(x, y)U_{ny} + C(x, y)U_n = D(x, y),$$

$$U_n(x, 0) = f_n(x),\ U(0, y) = g(y),\ f_n(0) = g(0),$$

in which A, B, C, D, f_n' and g' are continuous functions. It has been shown by the example in Chapter I that a uniformly convergent sequence of differentiable functions $\{f_n\}$ does not necessarily give rise to the convergence of $U_n(x, y)$. As has been shown in Chapter III the solution of (104) can be written in the form

$$(105)\quad U_n(x, y) = f_n(x) + g(y) - g(0) + \int_0^x \int_0^y V(s, t; x, y) \Big\{ D(s, t) - A(s, t)f_n'(s) - B(s, t)g'(t) - C(s, t) \left[f_n(s) + g(t) - g(0) \right] \Big\} \, dsdt.$$

An examination of this equation enables us to prove the next theorem.

<u>Theorem 4.1.</u> In equations (104) let A, B, C, D, f_n', g' be continuous functions and let $\{f_n(s)\}$ converge uniformly

to $f(s)$ and let $\{f_n'(s)\}$ converge uniformly to $f'(s)$. Let $U(x, y)$ be the solution of the Goursat problem given by (104) with $f_n(x)$ replaced by $f(x)$. Then $\lim_{n \to \infty} U_n(x, y) = U(x, y)$.

Proof: Let δ be an arbitrary positive real number and choose an integer N so large that for all $n > N$, $|f_n(s) - f(s)| < \delta$ and $|f_n'(s) - f'(s)| < \delta$ for all $s \in [0, x]$. Then from (105) we get

$$(106) \quad |U_n(x, y) - U(x, y)| \le |f_n(x) - f(x)| + \int_0^y \int_0^x |V(s, t; x, y)| \{|A(s, t)|\ |f'(s) - f_n'(s)| + |C(s, t)|\ |f(s) - f_n(s)|\}\, dsdt$$

$$\le \delta + 2\, V_0 Mxy\,\delta = (1 + 2\, MV_0 xy)\delta ,$$

where $M = \max\ \{\sup |A|, \sup |B|, \sup |C|\}$ and $V_0 = \sup |V(s, t)|$. Since δ can be chosen arbitrarily near zero it follows that $|U_n(x, y) - U(x, y)| \to 0$ as $n \to \infty$. This completes the proof.

The example given in Chapter I of non-dependence was, of course, not of this type. In that example $f_n(x) = n^{-\frac{1}{3}} \sin nx$ so $f_n'(x) = n^{\frac{2}{3}} \cos nx$ tends to infinity as $n \to \infty$. The next theorem states that the convergence of $\{f_n'(x)\}$ is not a necessary condition for the convergence of $U_n(x, y)$ if A, B are further restricted.

Theorem 4.2. In equations (104) let A, B, C, D, f_n', g' be continuous functions and let $\overline{S}(A)$ and $\overline{T}(B)$ exist. Let $\{f_n(s)\}$ converge uniformly to $f(s)$ for $s \in [0, x]$. Let $\bar{U}(x, y)$ be the solution of (104) with $f_n(x)$ replaced by $f(x)$. Then $\lim_{n \to \infty} U_n(x, y) = U(x, y)$.

Proof: As in the previous theorem, let $\delta > 0$ be arbitrary and choose N so large that $|f_n(s) - f(s)| < \delta$, for all $s \in [0, x]$. Then we get from (105)

$$(107)\quad \begin{aligned} |U_n(x, y) - U(x, y)| &\le |f_n(x) - f(x)| \\ &+ \left|\int_0^y \int_0^x V(s, t; x, y)\right. \\ &\Big\{A(s, t)\,[\,f'(s) - f_n'(s)\,] \\ &+ C(s, t)\,[\,f(s) - f_n(s)\,]\Big\}\,dsdt\Big| \\ &\le \delta + \int_0^y \left|\int_0^x V(s, t; x, y)\right. \\ &A(s, t)\,[\,f'(s) - f_n'(s)|\,ds\Big|dt \\ &+ M\delta xy. \end{aligned}$$

The inner integral can be written as a Stieltjes integral and integrated by parts to obtain

$$(108)\quad \begin{aligned} &\int_0^x V(s, t; x, y)\, A(s, t)\,[\,f'(s) - f_n'(s)\,]\,ds \\ &= \int_0^x V(s, t; x, y)\, A(s, t)\, d_s\,[\,f(s) - f_n(s)\,] \\ &= V(x, t; x, y)\, A(x, t)\,[\,f(x) - f_n(x)\,] \end{aligned}$$

$$- V(0, t; x, y)\, A(0, t)\, [\, f(0) - f_n(0)\,]$$

$$- \int_0^x [\, f(s) - f_n(s)\,]\, d_s\, [\, V(s, t; x, y)\, A(s, t)]$$

$$= V(x, t)\, A(x, t)\, [\, f(x) - f_n(x)\,]$$

$$- \int_0^x [\, f(s) - f_n(s)\,]\, d_s\, [\, V(s, t; x, y)\, A(s, t)].$$

If we substitute this into (107) we get

$$(109) \quad |U_n(x, y) - U(x, y)| \leq \delta + Mxy\delta$$

$$+ \int_0^y \Big\{ M\, V_o \delta + \Big| \int_0^x [\, f(s) - f_n(s)\,]\, ds$$

$$[\, V(s, t; x, y)\, A(s, t)\,]\, \Big| \Big\}\, dt$$

$$\leq \delta + Mxy\delta + M\, V_o y \delta + \int_0^y \delta\, \overline{S}(V(s, t; x, y)$$

$$A(s, t))\, dt$$

$$\leq \delta + Mxy\delta + M\, V_o y \delta + \overline{S}(V\, A) y \delta$$

$$= |\, 1 + Mxy + M\, V_o y + \overline{S}(V\, A) y\, |\, \delta .$$

It follows that $\lim\limits_{n \to \infty} |\, U_n(x, y) - U(x, y)\, | = 0$. This completes the proof.

To examine the behavior of the solution of the Cauchy problem let us examine the equation

$$(110) \quad U_{n_{xy}} + A\, U_{n_x} + B\, U_{n_y} + C\, U_n = D$$

$$U_n(x,\ y(x)) = \alpha_n(x)$$

$$U_{n_x}(x,\ y(x)) - U_{n_y}(x,\ y(x))\, y'(x) = \beta(x).$$

Under suitable hypothesis the solution of these equations is given by (82). The following theorem can be proved in just the same way as Theorem 4.1.

Theorem 4.3. Let A, B, C, β, α_n, α_n' be continuous and let $\{\alpha_n\}$ converge uniformly to α and let $\{\alpha_n'\}$ converge uniformly to α'. Let U(x, y) be the solution of (110) when α_n is replaced by α. Then $\lim_{n \to \infty} | U_n(x, y) - U(x, y) | = 0$.

V. BIBLIOGRAPHY

1. Apostol, T. M. Mathematical analysis. Reading, Mass., Addison-Wesley Publishing Co., Inc., 1957.
2. Copson, E. T. On the Riemann-Green function. Arch. Rational Mech. Anal. 1: 324 - 348., 1958.
3. Corduneanu, C. Depedenţa soluţiilor ecuaţiilor hiperbolice de coeficienţi şi de datele pe caracteristice. Acad. R. P. Romine. Bul. Sti. Sect. Sti. Mat. Fiz. 2: 313 - 317., 1955.
4. Darboux, G. Leçons sur la théorie des surfaces. Volume II. Paris, Gauthier-Villars., 1915.
5. Hobson, E. W. The theory of functions of a real variable and the theory of Fourier series. Volume II. 2nd ed., Cambridge, Cambridge University Press., 1926.
6. Kolmogorov, A. N. and Fomin, S. V. Elements of the theory of functions and functional analysis. Volume I. Rochester, N. Y., Graylock Press., 1957.
7. Riemann, B. Uber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Abh. Koniglichen Gesell. Wiss. Gottingen. 8: 1860. Reprinted in Collected works of Bernard Riemann, 156 - 175. New York, N. Y., Dover Press., 1953.
8. Widder, David Vernon. The Laplace transform. Princeton, Princeton University Press., 1941.

9. Wintner, A. On the conditions of validity of Riemann's method of integration. Quart. Appl. Math. 15: 94 - 98., 1957.

VI. ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. C. G. Maple for instruction and encouragement given during the preparation of this thesis.