Generalized polykays and application to obtaining variances and covariances of components of variation

Eldred Eugene Dayhoff

Iowa State University
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Eldred Eugene Dayhoff

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Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
of Science and Technology
Ames, Iowa

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I. INTRODUCTION

As originally conceived, the main purpose of the research leading to this thesis was to obtain the variances and covariances of mean squares and hence of estimates of components of variation in the analysis of variance of balanced samples from completely balanced finite population structures under no assumptions of distribution or independence of factors in the models to be considered. However, due to the extremely tedious algebra involved in working with third and fourth moments of sample observations, much of the effort represented by this thesis was directed toward finding the appropriate methods and functions to be used under the very general conditions herein considered. As a result somewhat limited emphasis is placed on the variances and covariances of estimates of components of variation.

The use of "sampling variables"—an ingenious device used in working with samples from finite populations, first introduced by Cornfield (1944), and later exploited by Kempthorne (1952) and Wilk (1955)—was considered in the beginning of this research but was soon found to be too pedestrian in view of the one-to-one correspondence between the population and sample structures in a pure sampling situation and the very nice property that symmetric sample means (later defined explicitly) are, as Tukey (1950) terms it, "inherited on the average," i.e., the expected value of a symmetric mean of sample observations is the same symmetric mean of
population values. This present thesis deals entirely with symmetric functions which can be expressed in terms of these symmetric means.

The use of sampling variables in experimental situations, with the concept of randomly applying treatments to units is quite relevant as was demonstrated by Kempthorne (1952), Wilk (1955) and later by Kempthorne, Zyskind and others (1961). As contrasted to pure random samples, samples of this nature are referred to as "fractionated samples" by the latter authors. Only pure random sampling is considered in this present work.

Tukey (1950) introduced a certain class of functions of symmetric means, later naming them polykays (1956a), which greatly simplified manipulating with higher moments in simple random sampling. Hooke (1956a, 1956b) generalized these to so-called "bipolykays" for the case of sampling from a two-way structure. Again much of the complexities arising in the algebra involved were eliminated.

In view of this and also because, as it is shown in a later chapter, the polykays and bipolykays arise as the coefficients of quantities appearing as functions of sample size only in sampling from certain balanced population structures, it was decided to approach the problem by use of these functions and appropriate extensions of these. Another compelling reason or motivation for the interest in use of polykays is that it is shown in Chapter 5 that generalized polykays of degree two are equivalent to the functions called $\Sigma$'s (read as cap sigmas) which are the basic quantities
appearing in much of the work reported by Kempthorne, Zyskind and others (1961).

Thus, much of the content of this thesis is devoted to the extension of the polykays and the development of certain useful properties of these. The coefficients of variation are expressible in terms of polykays, as is later shown, and hence the problem of obtaining the variances and covariances of estimates of the components of variation becomes that of obtaining the variances and covariances of certain polykays which is done in the last chapter of the thesis.

Though general formulas are presented, the amount of work involved in dealing with finite populations is laborious and exhausting enough that for the most part examples involving one, two and three factors only are illustrated in detail.
II. LITERATURE REVIEW

Most of the work concerning higher moments of functions of sample observations, specifically moments of mean squares in the analysis of variance, has been done under the assumption of infinite populations with errors normally and independently distributed. Under these assumptions the usual procedure for estimating variance components in balanced cases is to equate observed and expected mean squares in the analysis of variance. In Model II of Eisenhart (1947) where all factors in the linear model are assumed random, these estimates of variance components are essentially maximum likelihood estimates, while properties of this estimating procedure are virtually unknown under other assumptions. Some pertinent work has been reported by Herbach (1957).

Under Model II with normal, independent constituent errors, any mean square, say $S$, based on $f$ degrees of freedom, in the analysis of variance of a balanced multiple classification is distributed as

$$
E(S) \frac{\chi_f^2}{f}
$$

where $\chi_f^2$ is a chi-square variate based upon $f$ degrees of freedom. Also, all mean squares in the analysis are mutually independent and the estimate of any variance component is of the form

$$
\hat{\sigma}^2 = a_1 S_1 + a_2 S_2 + \ldots + a_k S_k
$$
where \( S_i, i=1, \ldots, k \), is a mean square based upon \( f_i \) degrees of freedom.

Hence the variance of \( \hat{\sigma}^2 \) is

\[
V(\hat{\sigma}^2) = 2a_1 \frac{[E(S_1)]^2}{f_1} + 2a_2 \frac{[E(S_2)]^2}{f_2} + \ldots + 2a_k \frac{[E(S_k)]^2}{f_k}
\]

and an estimate of this variance is obtained by replacing \( E(S_i) \) by \( S_i \) in the above equation. [See Satterthwaite (1946)].

In Model I of Eisenhart, the fixed model, the mean squares in the analysis of variance are for the most part distributed as non-central chi-squares and the variance of such a mean square, say \( M_1 \), based upon \( f_1 \) degrees of freedom, is

\[
V(M_1) = \frac{2(f_1 + 2\lambda)[E(M_1)]^2}{f_1^2}
\]

where \( \lambda \) is a non-centrality parameter. An estimate of a variance component in this case is a linear function of such non-central chi-squares but in general the problem of obtaining estimates of the variance of these component estimates is extremely difficult.

Working under slightly more general conditions Hammersley (1949) obtained the sampling variances and covariances of the estimates of variance components for a one-way classification under the assumptions that the populations involved were infinite but with arbitrary density functions.

Other work which may be of interest to the reader is that by Searle (1956, 1958, 1961), Mahamunulu (1963), and that summarized in a paper
by Crump (1951). Crump (1951) summarized work on the sampling variances of least squares estimates of components of variance in an unbalanced (non-orthogonal) one-way classification and the large sample variance of the maximum likelihood estimates of these quantities. Searle (1956, 1958, 1961) obtains the same results reported by Crump by use of matrix notation and extends these to the sampling variances of estimates of components of covariance when two variables are considered. Matrix methods were also used to obtain the large sample variance-covariance matrix of the maximum likelihood estimate of the components of variance and covariance. Mahamunulu (1963) extended the results of Searle to the case of 3-way hierachical classifications. As with Searle's work, the completely random model, that is, Eisenhart's Model II, was assumed.

If we consider models in which the populations involved are finite and make no assumptions about the errors we find much work has been done concerning second moments, specifically the expected mean squares, but very little on the higher moments, particularly the variances of mean squares. Examples of the work done on second moments are found in recent theses at Iowa State University [Wilk (1953, 1955), Zyskind (1958), Throckmorton (1961), and White (1963)], in Wilk and Kempthorne's work [Wilk and Kempthorne (1955, 1956, 1957)] and work by Cornfield and Tukey (1956). A summary of most of the research done at Iowa State may be found in the Wright Air Development Center Report by Kempthorne, Zyskind and others (1961).
In working with higher moments in sampling from finite populations, Tukey (1950, 1951, 1956a) introduced "polykays," polynomial symmetric functions which are extensions of Fisher's k-statistics [Fisher (1928)]. Using these polykays, Tukey (1956b) attempted to derive variances of variance component estimates in certain "balanced" models without the usual assumptions of infinite populations and normality of distributions, but maintained the customary and often very unrealistic independence assumptions. He presents results for two balanced single classifications, a two-way crossed classification without interaction, a balanced incomplete block and a latin square. A general definition of balance for an analysis of variance is given, and the application of the technique used in the specific examples to "balanced" situations is indicated. In later papers, Tukey (1957a, 1957b) deals with an unbalanced single classification and finds third moments of variance component estimates in a balanced single classification.

Tukey's results, not withstanding the assumptions of independence, are fairly nice and manageable but yet limited in many respects. For example, consider his balanced single classification model:

\[ x_{ij} = \mu + \eta_i + \omega_{ij} ; i = 1, 2, \ldots, c; j = 1, 2, \ldots, r \]

\{\eta_i\} sampled from \(n, k_1, k_{11}, \ldots\)

\{\omega_{ij}\} sampled from \(N, K_1, K_{11}, \ldots\)
sampling independent, order randomized. Here the k's and K's denote popu-
lation polykays. Though the variances and covariances of the estimates of
variance components are given in terms of fourth degree polykays, the
meaning of the polykays K is not clear since the components $w_{ij}$ depend upon
the value of two subscripts i and j. Also the estimation of these polykays
is not discussed.

Hooke (1956a, 1956b) extended, from sets to matrices, the family
of symmetric polynomials, or polykays, of Tukey. His polynomials were
named "bipolykays" and are symmetric in the sense that they are invariant
under permutations of rows and/or columns of the matrix. Hooke proves
many of the properties of bipolykays corresponding to those of the simple
polykays, e.g., the property of inheritance on the average, and develops
multiplicative and random pairing formulas as did Tukey. Then Hooke
applies these results to (a) find expressions for sampling moments of
functions of the elements of a matrix which is a "bi-sample" from a larger
matrix (b) find expressions for sampling moments of functions, specifically
estimates of variance components associated with the analysis of variance
of a two-way table with systematic interactions, in terms of bipolykays
and (c) find unbiased estimators for the variance and covariances of esti-
mated variance components in a two-way table without interaction.

Some related work has been done at Cornell University by Robson
(1960) and Rao (1961). Robson (1960) considers the one-way classification,
\[ x_{ij} = \mu + a_i + e_{ij}, \]
where the $a_i$ and $e_{ij}$ are from infinite but arbitrary populations. Rao (1961) considers the same model but allows the subpopulations $j = 1, \ldots, N_i$, to be finite. In both cases the first four cumulants of $x_{ij}$ are worked out and the general formulas presented. The research done at Cornell is part of a more general scheme of non-parametric estimation, and as Robson indicates

"Variance component estimation, a technique of increasing practical importance in such fields as plant and animal breeding, has been the topic most intensively studied in terms of the polykay system. The polykay approach to variance component estimation is non-parametric in the sense that no assumptions are made concerning the function form of the underlying distribution functions, other than the assumption of the existence of moments. In the case of finite populations, of course, the required moments always exist. Since the problem is viewed nonparametrically the variance components estimates actually represent only a small fraction of the information contained in the sample, and while it is certainly desirable to estimate moments of the sampling distribution of the variance component estimates [Hooke (1956a), Robson (1957), Tukey (1956b, 1957a)] it would seem even more desirable to direct this computing effort toward a further description of the population itself. The polykay approach and the modern computing machinery now make practicable the estimation of higher cumulant components and, therefore, the extraction of additional information from the sample."
In his earlier work in randomization theory, Wilk (1953) derived the variances and covariances of the mean squares in the generalized randomized block design with no assumptions of normality, infinite populations or independence, but under the assumption of no treatment effects. His approach involved sampling or indicator variables, the algebra being quite tedious, and the resulting equations were rather complex.

Other results on simple random sampling not especially related to this present work but of possible interest to the reader have been given by Irwin and Kendall (1944), Wishart (1952), and Abdel-Aty (1954). Combinatorial methods for obtaining products of simple polykays have been given by Dwyer and Tracy (1962) and an extension of simple polykays to multivariate polykays has been given by Robson (1957).
III. REVIEW OF POLYKAYS, Σ'S AND RELATED CONCEPTS

Because this thesis deals mostly with polykays and their extension and discusses these with reference to population structures, Σ's and certain other concepts associated with the analysis of variance, it is necessary to review the pertinent elements of these concepts as developed by previous authors.

A. Polykays

Fisher (1928) introduced a family of statistics \( k_1, k_2, \ldots, k_p \), which are symmetric functions of observations in simple random sampling and are such that the mean value of \( k_p \) is the \( p \)-th cumulant,

\[
E_k_p = K_p.
\]

Kendall and Stuart (1958) give an explicit definition of the \( k \)-statistics in terms of the sample observations. Fisher (1928) also proved that the \( k \) statistics were inherited on the average from finite populations, i.e., the expected value of the functions of sample values is the same function of population values.

Tukey (1950) introduced multiple subscript \( k \)'s which also are inherited on the average. These statistics, named polykays by Tukey (1956a), had the property that for an infinite population

\[
E_{k_{pqr}} = K_p K_q K_r,
\]

where the \( K \)'s are cumulants.
These polykays were defined in terms of polynomial symmetric functions of \( n \) numbers \( x_1, x_2, \ldots, x_n \), specifically, symmetric means. Tukey denoted the symmetric means by angle brackets such as \(<2>\), \(<34>\), etc., and defined the symmetric means as the means of products of powers of different \( x \)'s so that, for example,

\[
<ab> = \frac{\sum_{i \neq j} a_i b_j}{n(n-1)},
\]

where the sum is over the \( n(n-1) \) pairs \((i,j)\) with \( i \neq j \). The formal definition of the polykays given by Tukey involved a symbolic multiplication, he called \( O \)-multiplication, and certain generating functions. This definition would require an extensive development of concepts and will be omitted here; instead a simpler alternate definition proposed by Hooke (1956a) will be given.

As a specific example of the definitions of the polykays the mean and variance of the sample mean, denoted by \( k_1 \) and \( k_2 \) respectively, are defined as

\[
k_1 = <1>
\]

and

\[
k_2 = <2> - <11>.
\]

Hooke (1956a, 1956b) used a "secondary" notation in dealing with simple polykays and the extension of these, the so-called bipolykays, which are defined later. For example, the secondary notation for \(<ab\ldots d>\) is \( <p_1 p_2 \ldots p_a, q_1 q_2 \ldots q_b, r_1 r_2 \ldots r_d> \) where \( p_i, i=1, \ldots, a \) denotes
the individual $x_i, q_i, i = 1, \ldots, b$ denotes the individual $x_j$, etc. The entries $a, b, \ldots, d$ in the angle brackets are said to form a partition of the integer $m = a + b + \ldots + d$, where $m$ is the degree of the symmetric mean. Thus in the secondary notation the comma separates the parts of the partitions.

Two partitions are said to be equivalent (not distinct) if they are identical, except possibly for the order of parts and the order of symbols within a part. A partition $\delta$ is a subpartition of a partition $\gamma$ if $\delta$ can be made equivalent to $\gamma$ by simply inserting one or more commas. Tukey and Hooke both used parentheses to denote a polykay as contrasted to the angle bracket which denotes a symmetric mean. Hooke's definition of the polykays is as follows:

Definition 3.1: The polykays of degree $m$ are defined implicitly by the equations

\[ \langle \alpha \rangle = (\alpha) + \sum (\beta), \]

where there is one equation for each symmetric mean $\langle \alpha \rangle$ of degree $m$, and where the summation is over all distinct subpartitions $\beta$ of $\alpha$.

As an example of the above consider the case $m = 3$. The symmetric means in the "primary notation" of Tukey are $<111>$, $<21>$, and $<3>$, or expressed in the "secondary notation $<p,q,r>$, $<pq,r>$ and $<pqr>$, respectively. The polykays are then defined by the equations

\[ <p,q,r> = (p,q,r), \]
\[ <pq,r> = (pq,r) + (p,q,r), \]
\(<pqr> = (pqr) + (pq,r) + (pr,q) + (p,qr) + (p,q,r)\),

which may be manipulated to yield

\((p,q,r) = <p,q,r>\),

\((pq,r) = <pq,r> - <p,q,r>\),

\((pqr) = <pqr> - <pq,r> - <pr,q> - <p,qr> + 2<p,q,r>\),

or in the primary notation,

\(k_{111} = (111) = <111>\),

\(k_{12} = (21) = <21> - <111>\),

\(k_{3} = (3) = <3> - 3<21> + 2<111>\).

B. Generalized Symmetric Means and Bipolykays

In developing the bipolykays Hooke (1956a) considers the problem of sampling from a population matrix

\(\|X_{ij}\|; I = 1, 2, \ldots, R; J = 1, 2, \ldots, C\),

the sample being denoted by

\(\|x_{ij}\|; i = 1, 2, \ldots, r; j = 1, 2, \ldots, c\),

and selected by taking a sample of \(r\) of the \(R\) rows and another sample of \(c\) of the \(C\) column and forming the matrix whose elements are at the intersections of these selected rows and columns.

Hooke defined generalized symmetric means to be averages of monomial functions over the matrix, i.e., a g.s.m. (abbreviation for generalized symmetric mean) is a polynomial

\[\frac{1}{M} \left( \sum_{\overset{a \neq \text{pq}}{x \in \text{pq}}} a_{pq} \ldots x_{st} \right)\]
where the summation is over all subsequent subscripts, with the restriction that all subscripts represented by different letters must remain different throughout the sum and M is the number of terms in the summation. A g.s.m. is specified by the exponents along with information which tells which ones correspond to elements that lie in the same row and which correspond to elements that lie in the same column.

Because of this, Hooke used a convenient matrix notation for g.s.m.'s illustrated by the following example:

\[
\begin{bmatrix}
a & b & c \\
c & d & e \\
e & f & g
\end{bmatrix} = \frac{1}{rc(r-1)(c-1)(c-2)} \sum_{ij} x_{ij}^a x_{ij}^b x_{ij}^c .
\]

The author here has used different number of primes to denote different letters. The zeros are usually replaced by dashes, and these dashes are also used to extend every matrix of entries to at least two rows and two columns to avoid confusion with simple symmetric means. Thus, for example,

\[
\begin{bmatrix}
1 & - \\
- & -
\end{bmatrix} = \frac{1}{rc} \sum x_{ij}
\]

and

\[
\begin{bmatrix}
3 & - \\
2 & -
\end{bmatrix} = \frac{1}{rc (r-1)} \sum_{ij} x_{ij}^3 x_{ij}^2 .
\]

Two g.s.m.'s are identical if the matrix of entries of one can be obtained from that of the other by permuting rows and/or columns. The distinct g.s.m.'s of degree 2, for example, are
The general term, $x_{pq} \ldots x_{st}$ of the generalized symmetric mean contains $m$ factors, $a_{pq}$ of which are equal to $x_{pq}$, etc. To each of these factors is assigned a different symbol, and the resulting set of symbols may be partitioned in two ways, once by rows and once by columns. The secondary notation then, analogous for that of the simple polykay, for the g.s.m. will be an ordered pair of partitions $\alpha$ and $\beta$ denoted by $<\alpha/\beta>$, each partition being on the same set of letters. The different parts of $\alpha$ will correspond to factors having particular row subscripts and similarly the parts of $\beta$ are determined by column subscripts. For example, the secondary notation for

$$\left[ \begin{array}{c} 3 \\ - \\ - \end{array} \right] = \frac{1}{rc(r-1)(c-1)} \sum_{ij} x_{ij}^3 x_{i'j'}^3$$

is

$$<\text{pqr}, s/\text{pqr}, s>.$$ 

In order to define the bipolykay Hooke introduced a "dot-multiplication" defined as follows:
Definition 3.2:

\[ <\alpha> \cdot <\beta> = \begin{cases} <\alpha/\beta> & \text{if } \alpha \text{ and } \beta \text{ consist of the same symbols} \\ 0 & \text{otherwise} \end{cases} \]

By extending this non-commutative multiplication by distributivity, dot-products of linear combinations of symmetric means can be handled, and hence the following definition of a bipolykay is developed.

Definition 3.3: The bipolykay \((\alpha/\beta)\), where \(\alpha\) and \(\beta\) are partitions of the same set of symbols is

\[ (\alpha/\beta) = (\alpha) \cdot (\beta) \]

it being understood that \((\alpha)\) and \((\beta)\) are expressed as linear combinations of symmetric means before dot-multiplication.

As an example consider the bipolykay of degree three

\[
\begin{pmatrix}
1 & 1 & - \\
- & - & 1
\end{pmatrix}
= (pq,r/p,q,r)
\]

\[
= [(pq,r)-<p,q,r>] \cdot <p,q,r>
\]

\[
= <pq,r/p,q,r> - <p,q,r/p,q,r>
\]

\[
= \begin{bmatrix}
1 & 1 & - \\
- & - & 1
\end{bmatrix} - \begin{bmatrix}
1 & - & - \\
- & 1 & -
\end{bmatrix}
\]

Hooke proved the property of inheritance on the average for bipolykays and developed other properties of the bipolykays analogous to those of the simple polykays given by Tukey.
One concept developed by Tukey (1950) and used by both Tukey and Hooke is that of random pairing. Random pairing refers to taking two samples, \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\), the order within each having been independently randomized, and adding the two to obtain a new sample \((z_1, \ldots, z_n)\) where \(z_i = x_i + y_i\). This technique seems appropriate of course when dealing with additive models in which the observation is the sum of two or more independent components, each being obtained from its respective population by the appropriate random pairing. Because of the restrictive assumption of independently sampled components the idea of random pairing is not exploited in this present work and only the polykays of the yields or observations will be considered, just as was done by Hooke (1956b) in considering moments in bisampling.

C. Population Structures, Admissible Means, Σ's

Most of the following discussion and definitions are abstracted from the work done by Zyskind (1958, 1962) and Throckmorton (1961).

Throughout this thesis only balanced samples (defined later) from a balanced population of individuals with a specified structure will be considered. Also since only pure sampling is considered (no random association of treatments with units) there is a one-to-one correspondence between the sample structure and the population structure. In all cases finite population structures are initially considered, and fixed, mixed and random situations arise as particular cases of the general formulation.
In discussing population or sample structures we must first define a few concepts which are pertinent. A given response or yield in the linear models usually employed in statistics is said to depend entirely on a finite number of entities everyone of which is indicated by a corresponding subscript in the notation for the response, where the range of these subscripts is over the possible levels of the entity in question. An entity represented by subscript \( J \) say, in a given response is said to be hierarchical, or nested, within another set of entities \( S^* \) whose representative subscripts belong to the set \( S \) of subscripts of the response, if the unique identification of each level of \( J \) requires also the specification of the levels of each subscript of \( S^* \). For example, if we are examining the scores of a group of students assigned numbers 1, 2, 3, ... etc., within each of five sections, say, a typical response or score in this case may be represented by the symbol \( Y_{\text{II}} \) where \( I = 1, \ldots, 5 \), denotes the particular section the student whose grade is being examined came from and the subscript \( J \), where \( J = 1, \ldots, n_i \), say, denotes the particular number of the student. To specify a particular student's score precisely we must first specify which section the student is in. We say that the students are "nested" within sections.

A sample (or population) is said to be balanced with respect to all subscripts used in the representation of an arbitrary sample (population) observation if the sample (population) range of any one of the subscripts is the same for every set of particular values the other subscripts may assume.
One can obtain partial population or sample means by averaging over the entire range of values of particular sets of subscripts. Partial means are frequently denoted by the usual symbol for a response but with omission of subscripts over which the average has been taken. An admissible mean is defined as one in which whenever a nested index appears then all the indices which nest it appear also. The indices of an admissible partial mean which nest no other indices of that mean are said to constitute the set of indices belonging to the rightmost bracket. It is convenient to enclose these indices by parentheses.

As a simple example of these ideas we consider a population response $Y_{IJ}$ where the subscript $J$ is considered nested within the subscript $I$. The admissible means are the $Y, Y_I, Y_{I(J)}$. If the subscript $J$ were considered crossed with the subscript $I$, the admissible means would be $Y, Y_I, Y_J$ and $Y_{IJ}$.

From every partial mean linear combinations of means can be formed which are of special physical and formal significance. Any such linear combination, referred to as a component, is obtained by selecting all those partial means which are yielded by the mean in question when some, all, or none of its rightmost bracket subscripts are omitted in all possible ways, and whenever an odd number of indices is omitted the mean is preceded by a negative sign, whenever an even number is omitted it is preceded by a positive sign. As a consequence of this definition, summing the component
over the range of any subscript of the rightmost bracket of a constituent mean gives zero.

Thus the components corresponding to the admissible means in the nested situation of the previous example are \( Y, (Y_i - Y) \) and \( (Y_{ij} - Y_i) \) respectively while those for the crossed situations are \( Y, (Y_i - Y), (Y_j - Y) \) and \( (Y_{ij} - Y_i - Y_j + Y) \) respectively.

A typical response for a given population structure can be expressed identically as a sum of all its corresponding components—this relation is called the population identity. Thus in the previous cases the identities are

\[
Y_i(\jmath) = Y + (Y_i - Y) + (Y_{i(j)} - Y_i)
\]

and

\[
Y_{ij} = Y + (Y_i - Y) + (Y_j - Y) + (Y_{ij} - Y_i - Y_j + Y)
\]

The corresponding sample identities might be denoted by small case \( y \)'s and subscripts, i.e.,

\[
y_{ij} = y + (y_i - y) + (y_{i(j)} - y_i)
\]

and

\[
y_{ij} = y + (y_i - y) + (y_j - y) + (y_{ij} - y_i - y_j + y)
\]

The sum of squares of values of components of a given type over all the population ranges of the indices used to denote the component divided by the number of degrees of freedom (the number of linearly independent values of the component) of the type is said to be the component of variation.
of the type of component. Particular components of variation are denoted by $\sigma^2$'s with subscripts bracketed into groups corresponding to the subscripts of the types of components to which the $\sigma^2$'s refer. Thus in the previous examples if the subscript $I$ corresponds to factor $A$ and has a range $I = 1, \ldots, A$, and the subscript $J$ corresponds to factor $B$ with a range $J = 1, \ldots, B$, the components of variation corresponding to the admissible means $Y, Y_I^1$ and $Y_I^1(\hat{\eta})$ in the nested situation are

$$\sigma^2_{\emptyset} = \bar{Y}^2,$$

$$\sigma^2_A = \frac{\sum (Y_I^1 - \bar{Y})^2}{A - 1},$$

and

$$\sigma^2_{A(B)} = \frac{\sum (Y_I^1(\hat{\eta}) - Y_I^1)^2}{A(B - 1)}.$$

A special group of linear functions of components of variation referred to as $\Sigma$'s were introduced by Wilk and Kempthorne (1956) and formalized and given general definition for a very wide class of situations by Zyskind (1958). The $\Sigma$'s have certain nice properties and are defined for all balanced structure and appear in relatively simple forms in the expected mean squares in the analysis of variance. The $\Sigma$'s are defined as follows:

Definition 3.4: Consider a particular type of component and all $\sigma^2$'s of the following form:

(i) The set of subscripts of $\sigma^2$ includes the set of subscripts corresponding to the leading term of the component as a subset.
(ii) The excess subscripts belong exclusively to the rightmost bracket of $\sigma^2$.

The linear combination of all such $\sigma^2$'s, where the coefficient of a particular $\sigma^2$ with $k$ excess subscripts is

$$(-1)^k \frac{1}{\text{Product of population ranges of the excess indices}},$$

is defined as the $\Sigma$ corresponding to the type of component under consideration. The subscript notation for the $\Sigma$ is to be the same as for the type of component. (The $\Sigma$ corresponding to the component of variation $\sigma^2_\theta = \gamma^2$ and denoted by $\Sigma_\theta$ is uniquely defined.)

Taking the simple nested and crossed situations again, we have in the first case

$$\Sigma_\theta = \sigma^2_\theta - \frac{1}{A} \sigma^2_A,$$

$$\Sigma_A = \sigma^2_A - \frac{1}{B} \sigma^2_A (B),$$

and

$$\Sigma_{A(B)} = \sigma^2_A (B)$$

and in the latter case

$$\Sigma_\theta = \sigma^2_\theta - \frac{1}{A} \sigma^2_A - \frac{1}{B} \sigma^2_B + \frac{1}{AB} \sigma^2_{AB},$$

$$\Sigma_A = \sigma^2_A - \frac{1}{B} \sigma^2_{AB},$$

$$\Sigma_B = \sigma^2_B - \frac{1}{A} \sigma^2_{AB},$$
and

\[ \Sigma_{AB} = \sigma^2_{AB}. \]

In chapter 5 we will see that these \( \Sigma \)'s are equivalent to certain generalized polykays of degree two. The simple way these functions appear in the analysis of variance will be exploited in obtaining the variances and covariances of estimates of components of variation.

An interesting way of illustrating a given population or sample structure is by the use of structural diagrams, introduced by Throckmorton (1961). Briefly, in his representation, it is assumed that all factors are to be considered nested within the mean and that when one factor is nested within another it is denoted by placing the symbol denoting the nested factor beneath the nesting factor and attaching the two by a line. Thus in the case of the nested and crossed examples we have considered previously the structural diagrams would be

![Structural Diagram](attachment:image.png)

respectively. Throckmorton "closes" the diagram by considering an error, either sampling or technical, to be nested within all other factors. Thus he would have denoted the above by
Corresponding to the population diagrams above would be the sample diagrams respectively.

These diagrams are very useful, for example, in determining admissible means and will be used to illustrate the different structures discussed in this thesis.
IV. EXTENSION OF POLYKAYS

As was previously indicated the simple polykays which involve symmetric means with only one subscript are well defined quantities and many properties of these are known. The bipolykays of Hooke (1956a) are defined in the case of bi-sampling, or sampling from a two-way crossed population. For the case where one factor is nested within another factor we find that an explicit definition of the polykays is not given. Polykays for three or more factor structures have not been previously defined.

In this chapter a definition of a generalized symmetric mean from a \(n\)-way crossed structure is given, then by applying certain restrictions on the subscripts the corresponding g.s.m. for an arbitrary structure, which might contain both nested and crossed factors, is obtained. A concept called random cross labeling is then introduced. A general definition of the polykays of the \(n\)-way crossed population is presented, the generalized symmetric means of which are replaced by the corresponding symmetric means for an arbitrary structure to obtain the corresponding generalized polykay for that structure. In addition, certain useful properties of these g.s.m.'s and polykays are given.

A. Generalized Symmetric Means

Throughout this discussion and the remainder of this thesis we shall adopt the convention that population observations or yields in the population identity are to be denoted by capital letters, say \(X\) or \(Y\), each letter
containing a different subscript, I, J, K, ..., for each of the factors in
the population structure. These factors will simply be referred to as factors
A, B, C, ..., and the population range of the subscript will be as follows:
I has a range of A units, J a range of B units, and K a range of C units, etc.
The corresponding sample observations will be denoted by small letters,
the subscripts will be lower case letters and the range of the subscripts
will be: i, a range of a units; j, a range of b units; and k, a range of c
units, etc.

We now proceed with a definition of a generalized symmetric mean of
degree r for an n-way crossed population structure.

Definition 4.1: A generalized symmetric mean of degree r from an
n-way crossed population structure is a symmetric polynomial made up by
averaging the product of an observation to the $a_1$-th power, a different
observation to the $a_2$-th power, and so on, where $\Sigma a_i = r$. The different
observations occurring can be different with respect to any or all of the
subscripts. The averaging is over all possible selections of observations
in the population subject to the requirements of the relationships of the
subscripts.

Thus the g.s.m. is of the form

$$\frac{1}{M} \Sigma^\infty \sum_{i=1}^{s} X_{[1]}^{a_1} X_{[2]}^{a_2} \cdots X_{[s]}^{a_s},$$

where $\Sigma a_i = r$, M is the number of terms in the sum and
The likeness among the subscripts is indicated by use of the symbols $\theta_v^u$ in which $u$ is the order number of a constituent powered X and $v$ denotes the factor or order number of the subscript. The symbols $\theta_v^u$ are required to take one of the forms 0, 1, 2, ... , $s-1$, which will usually be the number of primes inserted to differentiate levels which must be different in the summation, that is, if $X_{[u_1]}^v$ and $X_{[u_2]}^v$ differ with regard to the $v$-th factor then $\theta_v^{u_1}$ and $\theta_v^{u_2}$ must be different.

Let $w_v$ denote the number of different $\theta_v^u$'s. Then the number of possible terms in the sum with regard to the $v$-th subscript is

$$N_v (N_v - 1) \ldots (N_v - w_v + 1),$$

where $N_v$ is the population range of the $v$-th subscript.

The g.s.m. is symmetric in the sense that it is invariant under the permutation of any of the subscripts.

Since in the crossed situation, the constituent X's are allowed to be alike or different with respect to a given subscript, say $v$, $\theta_v^1$, $\theta_v^2$, ... , $\theta_v^s$, can be related in any manner ranging from being all alike to being all different. The only restriction in the summation is that the different $\theta_v^u$ must remain unequal. The X's are thus said to be restricted in the sum.

In an arbitrary structure we may classify the factors A, B, C, ... into two general groups, those that are nested in some of the other factors and
those that are not nested in other factors. Denote a g.s.m. from the n-way crossed structure by \( h(X^*) \). For each g.s.m. of the crossed population we define a g.s.m. for an arbitrary structure as follows.

Definition 4.2: For each generalized symmetric mean \( h(X^*) \) from the n-way crossed population structure there exists a corresponding generalized symmetric mean, say \( h(X) \), for any balanced population structure which is defined as that symmetric mean which obtains when the following additional conditions are required of the subscripts of \( h(X^*) \):

(i) If the subscript corresponds to a factor which is not nested in any other factors, the \( \theta^u_v \)'s with respect to this subscript satisfy the same conditions as those of \( h(X^*) \)

(ii) If the \( v \)-th factor say, is nested in the \( v^* \)-th factor, and

\[
\frac{u_1}{\theta^u_v} \neq \frac{u_2}{\theta^u_{v^*}}
\]

in \( h(X^*) \), we must have

\[
\frac{u_1}{\theta^u_v} \neq \frac{u_2}{\theta^u_v}
\]

in \( h(X) \). In this sense subscripts which are nested by unequal subscripts are free to take values over their whole population range. (In connection with this, we use the rule that a nested factor has different subscripts for every combination of levels of all nesting factors.) If \( v \) is nested by \( v^* \), \( v^{**} \), etc., the above condition holds for each of the \( v^* \), \( v^{**} \), ... independently.
In condition (ii) above we say the nested subscript is free. In general there may be several groups of the \( X \)'s of \( h(X^*) \) in which all the subscripts in one group are alike with respect to the nesting subscripts but different from another group of like subscripts with regard to at least one of the nesting subscripts. The nested subscripts in this case are said to be **group-wise free** in the sum of the \( g.s.m. \) \( h(X) \), though they may be restricted within each group as in \( h(X^*) \).

This definitional process gives all the possible \( g.s.m. \)'s of an arbitrary structure although these could have been defined without reference to the \( g.s.m. \)'s of the completely crossed structure. The approach used has the advantage of providing a more systematic way of enumerating \( g.s.m. \)'s of an arbitrary structure because we have a relatively simple system of enumerating \( g.s.m. \)'s of the crossed population. This approach is also useful in defining the generalized polykays of an arbitrary structure, as we shall see.

It is clear from these definitions that because more conditions are required of the subscripts of a \( g.s.m. \) from a structure involving nesting of factors than of the \( g.s.m. \)'s from a completely crossed structure, there will be more distinct \( g.s.m. \)'s in the completely crossed structure than in any other balanced structure involving the same number of factors and the same number of levels of each factor. The correspondence between the \( g.s.m. \)'s of each structure is, in general, many-to-one.
In a later section, another important relationship between the g.s.m.'s of the completely crossed structure and those of other balanced structures will be discussed.

As examples of generalized symmetric means let us first look at a four-factor completely crossed structure with the factors A, B, C, D represented by the subscripts I, J, K, L in the typical response $Y_{IJKL}$, where I has a range of A units; J, a range of B units; K, a range of C units; and L, a range of D units. Three typical generalized symmetric means, one each of degrees two, three and four are given below.

\[(4.1a) \quad \frac{\sum_{IJKL} Y_{IJKL} - Y_{IJK'KL}}{ABCD(A-1)(B-1)} \]
\[(4.1b) \quad \frac{\sum_{IJKL} Y^2_{IJKL} - Y_{IJK'KL}}{ABCD(C-1)(D-1)} \]
\[(4.1c) \quad \frac{\sum_{IJKL} Y_{IJK'KL} - Y_{IJK'''L} - Y_{IJK''L} + Y_{IJK'L}}{ABCD(A-1)(B-1)(C-2)(D-1)(C-3)} \]

Suppose now that factors B and C are hierarchically arranged within factor A but factor D is crossed with the other factors. This structure may be represented by the following diagram:

```
 µ
  / \    /
A ○ ------- ○ B D
  |  \  /   |
  v  v    v
C ○ ------- ○ D
   \      \ µ
    \     \ ε
```

\[
\text{µ} \quad \text{ε}
\]
Then the three symmetric means, after imposing the additional restrictions on the subscripts due to the nesting involved, become

\[(4.2a) \sum_{ijkl} \frac{Y_{ijkl} Y_{i'j'k'l'}}{A B^2 C^2 D} (A-1)\]

\[(4.2b) \sum_{ijkl} \frac{Y^2_{ijkl} Y_{ijkl'}}{ABCD} (C-1) (D-1)\]

\[(4.2c) \sum_{ijkl} \frac{Y^2_{ijkl} Y_{ijkl' k'} Y_{i'j'k''l'} Y_{i'j'k''l'}}{AB^2 C^2 D} (A-1) (C-1)^2 (D-1)\]

In (4.2a), since A is not nested in another factor and there are two differently primed subscripts, I and I', the divisor with respect to factor A is A(A-1). Factor B is nested in factor A and hence the subscript J' is free so the divisor with respect to factor B is B^2. Since factor C is nested in factors A and B, the subscript for C in the second Y must be different from that in the first Y, hence the K is primed. Since this is now a free subscript the divisor with regard to this factor is C^2. The subscripts L for factor D remain unchanged because factor D is not nested and since both subscripts are alike the appropriate divisor is D. The g.s.m. of (4.2b) is the same as that of (4.1b) since factors C and D are crossed within the same levels of factors A and B, the factors nesting factor C. In (4.2c) we find an example of subscripts being group-wise free. The first two Y's are alike with respect to the subscripts I and J. The second two Y's are alike with respect to these
two subscripts but different from the first two Y's in that both the I and J are primed. The J is primed in the second group because the subscript I which nests it is primed in (4.1c). Since the I and J are primed, the subscript K must be different in the last two Y's. This condition is already satisfied in (4.1c) hence the subscripts K remain primed in the same manner in (4.2c). The subscript L is primed the same in (4.1c) and (4.2c) since it corresponds to a factor which is not nested by another other-factor. Thus the subscripts K and L are restricted within each of two groups; in the first group K and K', L and L' must remain unequal in the summation, hence the divisors C(C-1) and D(D-1), and in the second group L and L', K'' and K''' must remain unequal so the divisors C(C-1) and D(D-1) appear again. Thus the overall divisor with regard to the factors C and D is \( C^2 D^2 (C-1)^2 (D-1)^2 \).

**Alternate notation**

Thus far we have seen two different notations used for generalized symmetric means from a two-way structure, viz., the matrix notation and the "secondary notation," both introduced by Hooke (1956a). Since in the previous example four factors are involved the matrix notation cannot be used to designate the g.s.m.'s and the analog would have to be a four dimensional matrix, the representation of which would be rather difficult. In addition, neither representation seems to be completely adequate for arbitrary structures. The "secondary" notation for the three g.s.m.'s represented by (4.1a), (4.1b) and (4.1c) would be
where the different partitions of the same set of letters form an ordered quadruple. We could name the g.s.m.'s in (4.2a), (4.2b) and (4.2c) in a similar manner on the basis of the priming of subscripts but the nature of nesting and range of subscripts would be obscure. For the crossed populations this notation is very useful, however, when forming subpartitions of a given partition as it is readily seen between which letters commas can be inserted. An alternate notation which can be modified easily for arbitrary structures and has the advantage of being readily adapted for use in electronic computers for enumeration purposes, is now proposed.

For any crossed population the letters A, B, C, ..., are used to denote subscripts which are primed alike, beginning with A for no primes, B for one prime, etc. A matrix is formed with as many rows as there are factors. If the degree of the g.s.m. is k, then there will be k letters in each row. The g.s.m.'s in (4.1a), (4.1b) and (4.1c) would thus be represented as

\[
\begin{array}{c|cc}
   & A & B \\
\hline
   A & B \\
   A & B \\
   A & A \\
   A & A \\
\end{array}
\]
The letters A, B, C and D here should not be confused with the factors A, B, C, and D. The corresponding polykays, to be defined later, would be represented by enclosing the elements of the matrix in parentheses. When only two or three factors are involved it may be more convenient to place the groups of letters on a single line separated by slash marks as in the secondary notation of Hooke, e.g., <AAAB/ABCC/ABCD>. This form will be used when it is deemed more desirable from an organizational viewpoint.

The notation arranged as a matrix, however, appears to be much better when checking two g.s.m.'s or polykays for identity, the subject of the next section, or when imposing restrictions on the letters of a g.s.m. of a crossed population to obtain the corresponding g.s.m. for an arbitrary structure as discussed in the next paragraph.

The alternate notation may be modified slightly to indicate the nature of nesting and crossing in an arbitrary structure. As we saw earlier, the X's of the g.s.m. may be placed into groups on the basis of the likeness of
the combinations of all subscripts which nest a given subscript. To denote these different groups, the letters of the row corresponding to a nested subscript will be subscripted with numbers 1, 2, 3, ..., denoting the different groups of nesting subscripts. As before, if the subscripts have a restricted range within a group, this will be indicated by use of different letters A, B, C, ..., where A will represent no primes, B one prime, etc. To illustrate the flexibility of this notation let us consider the somewhat complex six-factor structure represented by the following diagram:

Let the factors A, B, C, D, E, F be represented by the subscripts I, J, K, L, M, N respectively. A possible population g.s.m. of degree seven, say, for the completely crossed structure is

\[
\sum_{ijklmn} y^2_{ijkmn} y_{ijkmn} y_{ijkmn} y_{ijkmn} y_{ijkmn} y_{ijkmn} y_{ijkmn} y_{ijkmn} \\
\frac{ABCDEF (A-1)(B-1)(C-1)(C-2)(C-3)}{(D-1)(F-1)(F-2)}
\]
The representation of this g.s.m. in the alternate notation would be

<table>
<thead>
<tr>
<th>factor, subscript</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>E</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>

For convenience, the appropriate factor letter and corresponding subscript are shown at the side of each row of the matrix. Imposing the further requirements of Definition 4.2 upon this g.s.m. for the complex structure, the resulting g.s.m. is found to be

<table>
<thead>
<tr>
<th>factor, subscript</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>E</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>

Since the factors A, B, and C are completely crossed the lettering remains the same. In this case each of these factors is not nested by any other and hence receives no subscript (one might subscript all letters with "1" but the
omission of the "1" is preferred by the author). Factor D is nested by factors A and B and the letters of this row are divided into three groups on the basis of the likeness of the IJ combinations. A glance at the corresponding row in the matrix for the completely crossed situation shows that all subscripts in group 1 are to remain alike while in group 2, two subscripts are to remain alike and one is to be different in the sum. Since the letter A was used in group 1, the letters B and C are used to denote this restriction in group 2. The single subscript of group 3 is free and is denoted by the letter D. Factor E is nested by factors A and C and the letters corresponding to this row are divided into five groups on the basis of the likeness of the combinations of IJ subscripts. Factor F is nested by factors A, C, and E but since the letters of the row corresponding to factor E have already been divided into groups, we can divide the letters of the row corresponding to factor F into groups on the basis of the likeness of the subscript M. In general, we need to look only at combinations of subscripts of the immediate nesting subscripts when dividing a particular nested subscript into groups.

The divisors in the g.s.m.'s are easily determined from this notation. For example, the divisor corresponding to factor A is

$$A (A-1) .$$

The divisor corresponding to factor D is

$$D \cdot D(D-1) \cdot D \cdot D = D^4(D-1) .$$
The other divisors are obtained similarly. In the summation notation then, the g.s.m. would be

\[ \sum_{IJKLMN} Y^{IJK'M'N'} Y^{I'J'K''L'M''N''} \]
\[ \sum_{ijklmn} A B C D E F^5 (A-1)(B-1) (C-1)(C-2)(C-3)(D-1) \]

Identity of generalized symmetric means

Because the generalized symmetric means are invariant under permutation of the subscripts of the observations in the mean, the three notations discussed previously, including the alternate notation just proposed, do not represent the g.s.m.'s uniquely, that is to say, there may be several forms of each of the notations which represent a given g.s.m. and the identity of each is often not easily seen unless certain permutations of the subscripts are made. As an illustration, consider the following two g.s.m.'s of degree four from a three-factor completely crossed structure:

\[ \sum_{IJK} Y^{IJK'} Y^{I'J'K'} Y^{I''JK''} Y^{I''J'K''} \]
\[ \sum_{ijkl} A B C (A-1)(A-2)(B-1)(B-2)(C-1)(C-2) \]

These g.s.m.'s could be represented by the following notations:

\[ (4.6a) \quad <p,qs,r/pr,q,s/pq,r,s> \]
\[ (4.6b) \quad <q,rs,p/pq,s,r/pr,s,q> \]
and

\[
\begin{array}{c|cccc}
(4.7a) & A & B & C & B \\
       & A & B & A & C \\
       & A & A & B & C \\
\end{array}
\]

\[
\begin{array}{c|cccc}
(4.7b) & C & A & B & B \\
       & A & A & C & B \\
       & A & C & A & B \\
\end{array}
\]

At first glance these appear to be distinct g.s.m.'s in any of the forms but (4.5b) can be seen to be equivalent to (4.5a) if the second and third Y's are permuted, and I and I'', J' and J'', and K' and K'' are interchanged. (4.6b) can be made identical to (4.6b) by first interchanging the letters q and r wherever they occur then changing the position of the p and r in the partition corresponding to the first factor, then changing the position of the q and s in the second partition and finally changing the position of the r and s in the third position. The corresponding operations in the alternate notation require permuting the second and third columns of (4.7b), then relabeling the letters from left to right beginning with A and using the next letter in the alphabet when there is a change in letters. Similar permutations of the subscripts would be required in checking the identity of two g.s.m.'s from an arbitrary structure.

The method used in obtaining the distinct g.s.m.'s for two-factor and three-factor structures presented in this thesis will be described in the chapters dealing with the polykays of these structures.
Before proceeding to the definition of the generalized polykay let us exploit an interesting and extremely useful notion, that of treating arbitrary population structures as crossed population structures.

**Arbitrary structures as crossed structures**

Consider an arbitrary balanced structure, say \( P \), with some nesting of factors, and the completely crossed structure with the same number of factors and levels of each factor as that of \( P \). As mentioned earlier we have adopted the rule that a nested factor has different subscripts for every combination of levels of all its nesting factors. This convention is consistent with the physical situation implied in a nested population of individuals in that the different units of a nested factor in a particular combination of levels of nesting factors have, in general, no relationship to the units in another combination of levels of the nesting factors. On the other hand, if a factor is crossed with all other factors, the subscripts for this factor are the same, regardless of the combination of levels of the other factors. The physical implication here is obvious.

Through a process we shall call random cross labeling, however, we may form an "artificial" crossed structure from an arbitrary structure involving nesting. This process consists simply of randomly labeling the subscripts of a nested factor as subscripts of a crossed factor, the random labeling being done independently from one combination of levels of the nesting factors to another. Assume in an arbitrary structure we have \( k \)
factors which are nested by other factors. Let $L_i$, $i = 1, \ldots, k$, denote the number of levels of the nested factors. Then the number of ways the levels of the $i$-th nested factor can be cross labeled with respect to a level of one of the nesting factors is $L_i!$, and assuming this nesting factor has $d$ levels there are $(L_i!)^d$ total ways of random cross labeling the levels of the $i$-th factor with respect to this nesting factor. Further let $M_i$ denote the product of the numbers of levels of each of the factors nesting the $i$-th factor. Then the total number of ways of random cross labeling is

$$
\prod_{i=1}^{k} (L_i!)^{M_i}
$$

If a function of the observations of one of the "artificial" populations is formed, we can find the average value of this function in terms of the observations of the arbitrary structure by taking the expectation of the function over all possible ways of random cross labeling. Let $f(P^*)$ denote a function of observations from the artificially induced population. Then the process of averaging $f(P^*)$ over all possible random cross labelings shall be denoted by the symbol

$$
E_{\text{lab}} f(P^*)
$$

Specifically, g.s.m.'s and polykays of $P^*$ can be averaged in this manner to obtain g.s.m.'s and polykays of $P$. As we shall now see in the following theorem, the expectation over random cross labelings of a g.s.m. from $P^*$ yields a g.s.m. of $P$, this latter g.s.m. being the "corresponding" g.s.m. of Definition 4.2.
Theorem 4.1: Let \( g(X^*_\alpha) \) denote a generalized symmetric mean of \( P^*_\alpha \) and \( g(X) \) the corresponding generalized symmetric mean of \( P \) as defined in Definition 4.2. Then

\[ E_{\text{lab}} g(X^*_\alpha) = g(X) \]

Proof: Since the random relabeling is done only over the nested subscripts, we need consider only the expectation of \( g(X^*_\alpha) \) with respect to the nested subscripts, and the averaging over the non-nested subscripts of \( g(X^*_\alpha) \) and \( g(X) \) can be performed after the expectation is taken over all possible ways of relabeling the nested subscripts.

According to Definition 4.1, \( g(X^*_\alpha) \) is an average of terms of the form

\[ ^*a_1 X_{[1]}^* a_2 X_{[2]}^* \ldots^* a_s X_{[s]} \]

The sum indicated in \( g(X^*_\alpha) \) is over both nested and non-nested subscripts. Let \( \Sigma^*_o \) denote the sum over all the non-nested subscripts with the appropriate restrictions and \( \Sigma^*_n^* \) the sum over all the nested subscripts of \( g(X^*_\alpha) \). Further let \( N_o \) and \( N_n^* \) denote the number of terms in the respective sums. Then \( E_{\text{lab}} g(X^*_\alpha) \) may be written as

\[
E_{\text{lab}} g(X^*_\alpha) = \frac{\Sigma^*_o \Sigma^*_n^* \sum \sum E_{\text{lab}} ^*a_1 X_{[1]}^* a_2 X_{[2]}^* \ldots^* a_s X_{[s]}^*}{N_o^* N_n^*} \]

\[
= \frac{\Sigma^*_o \Sigma^*_n^* \sum \sum E_{\text{lab}} X_{[1]}^* X_{[2]}^* \ldots^* X_{[s]}^*}{N_o^* N_n^*} \]

..
Now each value of a given nested subscript has an equal chance of being
relabeled as a possible value of a crossed subscript within each combina-
tion of levels of the nesting subscripts and hence one of the terms
\[ X_{[1]} X_{[2]} \ldots X_{[s]} \]
will be labeled as \[ X_{[1]} X_{[2]} \ldots X_{[s]} \] so that
\[
\text{E} \sum \sum_{\text{lab}} X_{[1]} X_{[2]} \ldots X_{[s]} = \frac{1}{N_n} \sum_{n} X_{[1]} X_{[2]} \ldots X_{[s]},
\]
where \( \sum_{\text{lab}} \) and \( N_n \) denote the sum over the nested subscripts of \( g(X) \) and
the number of terms in this sum respectively. Thus
\[
\text{E} \sum_{\text{lab}} g(X_{[1]}^*) = \frac{\sum_{\text{lab}} \sum_{n} a_1 a_2 \ldots a_s}{N_o N_n}
\]
\[ = g(X_a), \text{ q.e.d.} \]

As a simple example of this random cross labeling process, let us
consider a structure in which one factor, B say, is nested in another factor,
A. Denote an observation from this population by \( Y_{IJ} \) and further assume
the range of each subscript is two units. The four observations might be
arranged in a table as follows:

\[
\begin{array}{cc}
I = 1 & I = 2 \\
Y_{11} & Y_{23} \\
Y_{12} & Y_{24}
\end{array}
\]
Now we randomly cross label the \( J \) subscripts in each \( I \) level thus forming the following possible crossed structures:

<table>
<thead>
<tr>
<th>( I = 1 )</th>
<th>( J = 1 )</th>
<th>( J = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{I} \sqrt{1} \left( Y_{11} \right) )</td>
<td>( \sqrt{I} \sqrt{1} \left( Y_{12} \right) )</td>
<td>( \sqrt{I} \sqrt{1} \left( Y_{11} \right) )</td>
</tr>
<tr>
<td>( \sqrt{I} \sqrt{2} \left( Y_{21} \right) )</td>
<td>( \sqrt{I} \sqrt{2} \left( Y_{22} \right) )</td>
<td>( \sqrt{I} \sqrt{2} \left( Y_{24} \right) )</td>
</tr>
</tbody>
</table>

where the observations in parentheses denote the observations which were randomly labeled \( \sqrt{I} \sqrt{J} \). If we take a g.s.m., say

\[
\frac{\sum Y_{I} Y_{J}}{AB(A-1)} = \frac{1}{4} \left( Y_{11} Y_{21} + Y_{12} Y_{22} + Y_{21} Y_{11} + Y_{22} Y_{12} \right),
\]

and average this over all possible crossed structures, that is, find

\[
\frac{1}{4} \left( \sum_{\text{lab}} Y_{11} Y_{21} + \sum_{\text{lab}} Y_{12} Y_{22} + \sum_{\text{lab}} Y_{21} Y_{11} + \sum_{\text{lab}} Y_{22} Y_{12} \right),
\]

we obtain

\[
\sum_{\text{lab}} Y_{11} Y_{21} = \frac{1}{4} \left( Y_{11} Y_{23} + Y_{11} Y_{24} + Y_{12} Y_{23} + Y_{12} Y_{24} \right),
\]

\[
\sum_{\text{lab}} Y_{12} Y_{22} = \frac{1}{4} \left( Y_{12} Y_{24} + Y_{12} Y_{23} + Y_{11} Y_{24} + Y_{11} Y_{23} \right),
\]

\[
\sum_{\text{lab}} Y_{21} Y_{11} = \frac{1}{4} \left( Y_{23} Y_{11} + Y_{24} Y_{11} + Y_{23} Y_{12} + Y_{24} Y_{12} \right),
\]

and so on.
\[
E \sum_{\text{lab}} Y_{22}^{*12} = \frac{1}{4} (Y_{24} Y_{12} + Y_{23} Y_{12} + Y_{24} Y_{11} + Y_{23} Y_{11}) .
\]

Hence
\[
E \sum_{\text{lab}} Y_{11} Y_{23}^{*12} = \frac{1}{8} (Y_{11} Y_{23} + Y_{23} Y_{11} + Y_{11} Y_{24} + Y_{24} Y_{11} + Y_{12} Y_{23} + Y_{23} Y_{12} + Y_{12} Y_{24} + Y_{24} Y_{12}) = \frac{\sum Y_{II} Y_{I'I'}}{AB^2 (A-1)} .
\]

Similarly we would find that
\[
E \sum_{\text{lab}} Y_{11} Y_{I'I'}^{*12} = \frac{\sum Y_{II} Y_{I'I'}}{AB^2 (A-1)} .
\]

\[\text{B. Generalized Polykays}\]

We now proceed to define the generalized polykays for an n-way crossed structure. Following Hooke (1956a) we define the following dot-multiplication for an n-way crossed structure:

Definition 4.3: Let \(Y_i\), \(i = 1, \ldots, n\), denote arbitrary partitions of the set of symbols \(\{p, q, r, \ldots\}\). Then
\[
<Y_1><Y_2> \cdots <Y_n> = \begin{cases} 
<Y_1|Y_2|\ldots|Y_n> & \text{if the } Y_i, i=1,\ldots, \text{n, consist of the same set of letters} \\
0 & \text{otherwise}
\end{cases}
\]

where the \(<Y_i>\) represent simple symmetric means with respect to the individual factors.

This dot-multiplication is extended by distributivity to provide dot-multiplication for linear combinations of symmetric means. It is obvious
that this multiplication is non-commutative.

Definition 4.4: For an n-way crossed structure the generalized polykay \((\gamma_1 | \gamma_2 \ldots | \gamma_n)\) is defined as

\[ (\gamma_1 | \gamma_2 \ldots | \gamma_n) = (\gamma_1) \bullet (\gamma_2) \ldots \bullet (\gamma_n) \]

where the \((\gamma_i)\), the simple polykays corresponding to the symmetric means \(\langle \gamma_i \rangle\), are first expressed as linear sums of symmetric means by Definition 3.1 before the dot-multiplication is taken.

Definition 4.5: For an arbitrary population \(P\) the possible generalized polykays are defined by reference to the crossed population \(P^*\) obtained by random cross labeling of the nested subscripts of an observation in the following manner:

(i) Take a polykay of the n-way structure and express it as a linear function of n-way g.s.m.'s

(ii) Take the expectation of the polykays over all possible random cross labelings

(iii) Taking this expectation results in a linear function of g.s.m.'s of the arbitrary structure and if the expectation is non-zero, the n-way polykay is renamed, receiving the name of the leading g.s.m. of the new polykay, and this result is the definition of that polykay for the structure \(P\). If the expectation is zero, the n-way polykay of \(P^*\) has no corresponding polykay in \(P\).
Since the expectation of a g.s.m. of $P^*$ is equal to its corresponding g.s.m. of $P$, the polykay of $P$ corresponding to a polykay of $P^*$ may be formed simply by replacing each g.s.m. of $P^*$ by its corresponding g.s.m. of $P$.

Because several g.s.m.'s of the $n$-way polykay may have the same corresponding g.s.m. in the arbitrary structure, the $n$-way polykay is said to collapse under the process of taking the expectation over all possible random cross labelings. Indeed, many of the $n$-way polykays vanish under this collapsing. It will be later observed that only the polykays whose leading g.s.m. contains letters which are primed exactly alike in both the crossed and collapsed cases (though the ranges of the subscripts may be different, i.e., free or restricted) do not vanish.

As a specific example, let us consider a three-way crossed structure and look at a specific polykay of degree three, say $(pq,r/pq,r/pr,q)$, or in the alternate notation,

\[
\begin{pmatrix}
A & A & B \\
A & A & B \\
A & B & A
\end{pmatrix}
\]

According to Definition 3.1 we find

\[
(pq,r) = <pq,r> - <p,q,r>,
\]
\[
(pq,r) = <pq,r> - <p,q,r>,
\]
\[
(pr,q) = <pr,q> - <p,q,r>.
\]
Thus

\[
\begin{pmatrix} A & A & B \\ A & A & B \\ A & B & A \end{pmatrix} = (pq, r/pq, r/pr, q) = [<pq, r> - <p, q, r>] \cdot [<pq, r> - <p, q, r>]
\]

\[
\cdot [<pr, q> - <p, q, r>]
\]

\[
= <pq, r/pq, r/pr, q> - <p, q, r/pq, r/pr, q> - <pq, r/pq, r/pr, q>
\]

\[
+ <p, q, r/pq, r/pr, q> - <pq, r/pq, r/pr, q> + <p, r/pq, r/pr, q>
\]

\[
+ <pq, r/pq, r/pr, q> - <p, q, r/pq, r/pr, q> - <pq, r/pq, r/pr, q>
\]

\[
+ <p, q, r/pq, r/pr, q> - <pq, r/pq, r/pr, q> + <p, r/pq, r/pr, q>
\]

If we have a structure in which the factor C is nested within factor B which is in turn nested in factor A, the polykay then becomes under collapsing,

\[
\begin{array}{ccc|ccc|ccc}
\hline
\end{array}
\]

\[
\begin{array}{ccc|ccc|ccc}
A_1 & A_1 & B_2 & - & A_1 & B_2 & C_3 & - & A_1 & B_1 & C_2 & + & A_1 & B_2 & C_3 \\
A_1 & B_1 & C_2 & A_1 & B_2 & C_3 & A_1 & B_2 & C_3 & & & & \\
\hline
A_1 & A_1 & B_2 & + & A_1 & B_2 & C_3 & + & A_1 & B_1 & C_2 & - & A_1 & B_2 & C_3 \\
A_1 & B_1 & C_2 & A_1 & B_2 & C_3 & A_1 & B_2 & C_3 & & & & \\
\end{array}
\]

\[
= 0 .
\]
If we consider the polykay \((pq/r/pq,r/p,q,r)\) or in the alternate notation,

\[
\begin{pmatrix}
A & A & B \\
A & A & B \\
A & B & C
\end{pmatrix}
\]

for the crossed structure, we have

\[
\begin{pmatrix}
A & A & B \\
A & A & B \\
A & B & C
\end{pmatrix} = (pq/r/pq,r/p,q,r) = [<pq,r>-<p,q,r>] \cdot [<pq,r>-<p,q,r>]
\]

\[
= <pq,r/pq,r/p,q,r> - <p,q,r/pq,r/p,q,r> - <pq,r/p,q,r/p,q,r> + <p,q,r/p,q,r/p,q,r>
\]

\[
= \begin{vmatrix}
A & A & B \\
A & A & B \\
A & B & C
\end{vmatrix} - \begin{vmatrix}
A & A & B \\
A & B & C \\
A & B & C
\end{vmatrix} - \begin{vmatrix}
A & A & B \\
A & B & C \\
A & B & C
\end{vmatrix} + \begin{vmatrix}
A & A & B \\
A & B & C \\
A & B & C
\end{vmatrix}.
\]

Under collapsing we find the polykay is equal to

\[
\begin{vmatrix}
A & A & B \\
A & A & B \\
A & B & C
\end{vmatrix} - \begin{vmatrix}
A_1 & A_1 & B_2 \\
A_1 & B_2 & C_3 \\
A_1 & B_1 & C_2
\end{vmatrix} - \begin{vmatrix}
A_1 & B_1 & C_2 \\
A_1 & B_2 & C_3 \\
A_1 & B_2 & C_3
\end{vmatrix} + \begin{vmatrix}
A_1 & B_2 & C_3 \\
A_1 & B_2 & C_3 \\
A_1 & B_2 & C_3
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
A & A & B \\
A & A & B \\
A_1 & B_1 & C_2
\end{vmatrix} - \begin{vmatrix}
A_1 & A_1 & B_2 \\
A_1 & B_1 & C_2 \\
A_1 & B_2 & C_3
\end{vmatrix}.
\]

Since the polykay does not vanish it is renamed
It is easily seen that to have a unique system of defining the polykays under collapsing, only those which do not vanish should be renamed. If, in our example here, the first polykay which vanishes had been renamed it would have received the name

$$\begin{pmatrix}
A & A & B \\
A_1 & A_1 & B_2 \\
A_1 & B_1 & C_2
\end{pmatrix}$$

but as we saw immediately above we have another polykay with the same name, a polykay which does not vanish. Thus, to avoid such contradictions we avoid renaming vanishing polykays.

D. Some Properties of the Generalized Symmetric Means and Polykays

Some important properties of the g.s.m.'s and generalized polykays will now be presented in the form of theorems.

Theorem 4.2: Consider a given polykay, say $(\gamma_1 | \gamma_2 | \ldots | \gamma_n)$, from an n-way crossed population. Suppose further that $k$ of the $(\gamma_i)$ are of the form $(p, q, r, \ldots)$ in the secondary notation. Then

(i) If $k < n$, the sum of the coefficients of the generalized symmetric means of the polykay is zero
If \( k = n \), the sum of the coefficients of the generalized symmetric means is unity.

Proof: By definition, the polykay \( \gamma_1 | \gamma_2 | \cdots | \gamma_n \), where the \( (\gamma_i) \), \( i = 1, \ldots, n \), are of the form \((p,q,r,\ldots)\) is equal to the generalized symmetric mean \( < \gamma_1 | \gamma_2 \cdots | \gamma_n > \).

On the other hand, it has been shown by Hooke (1956a) that for all \( (\gamma_i) \) not of the form \((p,q,r,\ldots)\) the sum of the coefficients of the simple symmetric means is zero. Thus the dot-multiplication of several such \( (\gamma_i) \) will yield a linear function of generalized symmetric means, the sum of the coefficients of which add to zero. q.e.d.

Corollary 4.2: The sum of the coefficients of the g.s.m.'s of all polykays, except the one consisting of only one g.s.m., of any balanced population equals zero.

This results obtains immediately from the fact that the g.s.m.'s of the polykays from the completely crossed structure are replaced by the corresponding g.s.m.'s for the particular structure in defining the polykays for this structure. (See Definition 4.3).

Sample symmetric means and polykays and inheritance on the average

Thus far we have defined g.s.m.'s and polykays primarily with reference to population structures. Because of the one-to-one correspondence between the population and sample structures, we can define the sample g.s.m.'s exactly the same as in Definition 4.1 and 4.2, replacing population values by sample values. The one-to-one correspondence
between the population and sample structures is further elucidated by the following theorem which relates the sample g.s.m. and the corresponding population g.s.m.

Theorem 4.3: Let $f(X)$ represent a given population g.s.m. from an arbitrary structure $P$. Let $f(x)$ denote the corresponding sample g.s.m. Then

\[
\mathbb{E}_{sa} f(x) = f(X)
\]

where $\mathbb{E}_{sa}$ refers to the expectation over the sampling procedure.

Proof: Suppose $f(x)$ is of the form

\[
f(x) = \frac{1}{d} \sum_{x \neq x[1] x[2] \ldots x[s]}^{a_1} x[2] \ldots x[s] .
\]

Further suppose $k$ factors are involved, denoted by $N_1, N_2, \ldots, N_k$, with sample ranges of $n_1, n_2, \ldots, n_k$ respectively. Let $N_1, N_2, \ldots, N_k$ also denote the population ranges respectively. Consider the factor $N_i$. The $x$'s of $f(x)$ can be divided into several groups according to the likeness of priming of the subscripts corresponding to factors nesting $N_i$. Let the number of such groups be $r_i$. In this context, if the factor $N_i$ is not nested, we let $r_i = 1$. Further assume within each of the $r_i$ groups that there are $q_{\alpha}^i, \alpha = 1, \ldots, r_i$, differently primed subscripts corresponding to the factor $N_i$. Thus the denominator of $f(x)$ is

\[
d = \prod_{i=1}^{k} (n_i - 1 - (n_i - q_{1}^i + 1)(n_i - 1 - (n_i - q_{2}^i + 1)(n_i - 1 - (n_i - q_{r_i}^i + 1))
\]

The total number of possible samples from the population is
and each term
\[ \frac{a_1}{X_1} \frac{a_2}{X_2} \cdots \frac{a_s}{X_s} \]
occur in the same number of samples, this number being
\[
N' = \pi \prod_{i=1}^{k} \left( \frac{(N_i - q_i)}{n_i} \right) \left( \frac{(N_i - q_i)}{n_i - q_i} \right) \cdots \left( \frac{(N_i - q_i)}{n_i - q_i} \right)
\]

Hence
\[
Ef(x) = \frac{N'}{Nd} \sum a_1 \frac{a_2}{X_2} \cdots \frac{a_s}{X_s} \cdots
\]

where the sum is, of course, over the population range of the subscripts.

But
\[
\frac{N'}{Nd} = \prod_{i=1}^{k} \frac{(N_i - q_i) !}{(n_i - q_i) !} \frac{(N_i - q_i) !}{(n_i - q_i) !} \cdots \frac{(N_i - q_i) !}{(n_i - q_i) !}
\]

\[
\frac{\left[ (N_i - n_i) ! \right]^{r_i(n_i !)}}{(N_i !)^{r_i}}
\]

\[
\frac{\prod_{i=1}^{k} (n_i - 1) \cdots (n_i - q_i + 1) (n_i - 1) \cdots (n_i - q_i + 1) \cdots (n_i - 1) \cdots (n_i - q_i + 1)}{\prod_{i=1}^{k} (N_i - 1) \cdots (N_i - q_i + 1) (N_i - 1) \cdots (N_i - q_i + 1) \cdots (N_i - 1) \cdots (N_i - q_i + 1)}
\]
so that finally

\[
E[f(x)] = \sum_{i=1}^{k} \frac{a_1^x X_{[1]} a_2^x X_{[2]} \cdots \cdots X_{[s]}}{(N_i-1) \cdots (N_i-q_1^i + 1)(N_i^1-1) \cdots (N_i-q_2^i + 1) \cdots (N_i^1-1) \cdots (N_i-q_q^i + 1)}
\]

= \text{f}(x), \quad \text{q.e.d.}

The sample polykays for the crossed structure are defined exactly as in Definition 4.4 again replacing population values by sample values. In order to give the sample polykays obtained from an arbitrary structure meaning of their own without reference to the population polykays we envisage the following sampling scheme:

(i) The arbitrary population \( P \) is represented as a completely crossed structure by random cross labeling the levels of the nested factors

(ii) One of the "artificially" induced crossed populations, say \( P_i^* \), is selected randomly

(iii) A crossed sample is now taken from \( P_i^* \) obtaining a sample which would have been obtained by sampling directly from \( P \).

It should be noted that the possible samples from \( P \) occur equally frequently in the totality of crossed samples from all the \( P_i^* \), \( i = 1, 2, \ldots \).

To obtain the expectation of a function of the observations of a sample taken in the manner prescribed above, say \( h(x) \), we visualize a conditional expectation symbolized as follows:

\[
(4.8a) \quad E[h(x)] = E \left[ E \left[ h(x) \mid P^* \right] \right],
\]

lab sa
where $E_{sa}[h(x)|P^*]$ denotes the expectation of $h(x)$ over the possible ways of sampling a crossed sample for a fixed choice of a randomly cross labeled population, and $E_{lab}$ denotes the expectation over random cross labeling as described previously.

It is obvious that if we sample directly from the structure $P$ we can then random cross label the subscripts of the sample observations and define an expectation over the random cross labeling of the sample subscripts. Theorem 4.1 would hold also for sample generalized symmetric means. Let $[h(x)|P']$ denote a sample g.s.m. from $P$ which has been random cross labeled. Then according to this sampling scheme

\[(4.8b) \quad Eh(x) = E_{sa}E_{lab}[h(x)|P'].\]

Corollary 4.8a:

\[(4.8c) \quad E_{lab}E_{sa}[h(x)|P^*] = E_{sa}E_{lab}[h(x)|P'].\]

This result obtains directly from the fact that in both (4.8a) and (4.8b) we have $Eh(x) = h(X)$ by Theorem 4.8.

We can now define sample polykays from an arbitrary structure $P$, without first referring to the population g.s.m.'s and polykays, by the same process described in Definition 4.5, replacing population values by corresponding sample values.

Since each sample or population polykay from a given structure is expressed as a linear sum of g.s.m.'s the following corollary obtains directly from Theorem 4.3 and Equation 4.8c. 
Corollary 4.3: Let \( F(X) \) represent a given population polykay from an arbitrary structure \( P \). Let \( F(x) \) denote the corresponding sample polykays, then

\[
E F(x) = F(X).
\]

**Multiplication of polykays**

The following theorem and corollary indicates a manner in which multiplication of g.s.m.'s and polykays from a completely crossed structure can be effected, and forms the basic groundwork for taking moments of polykays from an arbitrary structure as we shall soon see. The formulation will be in terms of sample g.s.m.'s and polykays since we will be interested in obtaining products, specifically moments, of sample quantities.

Theorem 4.4: Consider an \( n \)-way crossed sample structure. Denote two arbitrary sample g.s.m.'s by

\[
<y_1|y_2|\ldots|y_n> = \frac{1}{B_1B_2\ldots B_n} \sum_{\text{all } i,j,\ldots,p} x_{\theta_1^1\theta_2^1\ldots\theta_n^1}\normalsize_{a_1}^{a_1} \times \theta_1^2\theta_2^2\ldots\theta_n^2\normalsize_{a_2}^{a_2} \times \theta_1^s\theta_2^s\ldots\theta_n^s\normalsize_{a_s}^{a_s} \times \theta_1^s\theta_2^s\ldots\theta_n^s
\]

and

\[
<\delta_1|\delta_2|\ldots|\delta_n> = \frac{1}{C_1C_2\ldots C_n} \sum_{\text{all } i,m,r} x_{\gamma_1^1\gamma_2^1\ldots\gamma_n^1}\normalsize_{b_1}^{b_1} \times \gamma_1^2\gamma_2^2\ldots\gamma_n^2\normalsize_{b_2}^{b_2} \times \gamma_1^s\gamma_2^s\ldots\gamma_n^s\normalsize_{b_s}^{b_s} \times \gamma_1^s\gamma_2^s\ldots\gamma_n^s
\]
where $\sum_{i=1}^{a_1}$ and $\sum_{i=1}^{b_1}$ are not necessarily the same, and the $\gamma^u_v$ have the same meaning with respect to the subscripts $l, m, \ldots, r$ as the $\theta^u_v$ with respect to the subscripts $i, j, \ldots, p$, and are not necessarily different from the latter. The $B_i$ and $C_i$, $i = 1, \ldots, n$, denote the appropriate denominators for the first factor, second factor, and so on. Then

$$<\gamma_1|\gamma_2|\ldots|\gamma_n><\delta_1|\delta_2|\ldots|\delta_n> = [<\gamma_1><\delta_1><\gamma_2><\delta_2><\ldots><\gamma_n><\delta_n>]$$

where the $<\gamma_i><\delta_i>$ represent the product of two simple symmetric means.

Proof:
\[
\sum_{\text{all } i,j,m} \left( \sum_{\text{all } p,r} \frac{a_1 \theta_1^1 \theta_1^2 \cdots \theta_1^s \cdots a_s \theta_s^1 \theta_s^2 \cdots \theta_s^s}{B_1 B_2 C_2} \right)
\]

But the last expression enclosed in braces, where the sum is over all the values of \(p\) and \(r\) with the restriction that subscripts differently primed must remain unequal, disregarding all subscripts but \(p\) and \(r\), represents the product \(<\gamma_n><\delta_n>\). This product results in some linear function of g.s.m.'s with respect to the \(n\)-th factor, each g.s.m. being of degree \(a_s + b_s\) with respect to that factor. The partitioning with respect to the remaining factors is unchanged. Performing the second summation and division by \(B_{n-1} C_{n-1}\) over each of the g.s.m.'s with respect to the \(n\)-th factor is, by the distributivity of dot-multiplication, equivalent to the operation \(<\gamma_{n-1}><\delta_{n-1}>\)
\(<\gamma_{n-1}><\delta_{n-1}>*<\gamma_n><\delta_n>\). Performing the next process of averaging over each of the resulting terms, now g.s.m.'s with respect to the \(n\)-th and \((n-1)\)-th factors, is equivalent to the operation \(<\gamma_{n-2}><\delta_{n-2}>*<\gamma_{n-1}><\delta_{n-1}>*<\gamma_n><\delta_n>\), etc. After the last averaging process, the complete operation
\[ \langle \gamma_1 \rangle \langle \delta_1 \rangle \bullet \langle \gamma_2 \rangle \langle \delta_2 \rangle \bullet \ldots \langle \gamma_n \rangle \langle \delta_n \rangle \] will have been performed, \textit{q.e.d.}

**Corollary 4.4:**

\[
(\gamma_1 | \gamma_2 \quad \ldots | \gamma_n) (\delta_1 \delta_2 \quad \ldots \delta_n)
\]

\[
= (\gamma_1) (\delta_1) \bullet (\gamma_2) (\delta_2) \bullet \ldots \bullet (\gamma_n) (\delta_n)
\]

where the \((\gamma_i) (\delta_i)\) represent products of simple sample polykays.

This result obtains since each simple polykay can be expressed in terms of symmetric means and Theorem 4.4 can be applied to the products of symmetric means.

Although no general formulation will be made here it should be pointed out that each g.s.m. from an arbitrary structure when randomly cross labeled can be expressed as a linear sum of g.s.m.'s from the completely crossed structure. The same is true of polykays. The latter is closely related to the fact that in the analysis of variance certain sums of squares in a nested analysis consist of the "pooling" of sums of squares of crossed factors.

Thus to perform multiplication of g.s.m.'s and polykays from an arbitrary structure, we express these functions as linear sums of g.s.m.'s or polykays from a completely crossed structure, under random cross labeling, then perform the multiplication as indicated in Theorem 4.4 and Corollary 4.4 and then take the expectation of the result over the random cross labeling.
In a later section where we will be interested in obtaining the variances and covariances of certain polykays from arbitrary structures, we shall see that under random cross labeling, these polykays can be expressed very nicely as simple linear sums of polykays from the crossed structure.
V. EQUIVALENCE OF POLYKAYS OF THE SECOND DEGREE AND $\Sigma$'S

Zyskind (1958) recognized that the $\Sigma$'s for a balanced two-way crossed structure and the bipolykays of degree two defined by Hooke were in fact equivalent, the former being expressed in terms of components of variation and the latter in terms of symmetric means, and conjectured that a natural extension of Tukey's polykays to multipolykays will lead to second degree multipolykays identical with the $\Sigma$'s.

This conjecture was further strengthened when Throckmorton (1961) proved that for pure random sampling, the expected value of $\Sigma$'s defined for sample values was equal to the corresponding $\Sigma$'s defined for population values, this of course being the property of inheritance on the average which is shared by the polykays.

In this chapter, a general definition of g.s.m.'s and generalized polykays of degree two are given, these definitions being specific and more explicit cases of the general definitions of Chapter 4. Then the equivalence of these second degree generalized polykays and the $\Sigma$'s, defined in Chapter 3, is demonstrated.

A. Second Degree Generalized Polykays

Recall that for the integer $m=2$, the only partitions are 11 and 2 itself, or in the secondary notation of Chapter 3, $p$, $q$, and $pq$ respectively. The simple polykays of degree two are

$$(pq) = \langle pq \rangle - \langle p, q \rangle$$
and

\[(p,q) = <p,q> .\]

In the alternate notation proposed in Chapter 4, these polykays could be written

\[(AA) = <AA> - <AB>\]

and

\[(AB) = <AB> .\]

Henceforth in this chapter let the partition pq or AA be denoted by α and the partition p,q or AB by β.

Take as a specific example of the dot-multiplication given by Definitions 4.3 and 4.4, a three-way crossed population structure with factors A, B, and C, say. A typical response, according to the discussion in Chapter 3, would be denoted by

\[Y_{IJK} = Y + (Y_I - Y) + (Y_J - Y) + (Y_K - Y) + (Y_{II} - Y_{I} - Y_{J} + Y) + (Y_{II} - Y_{I} - Y_{K} + Y)\]

\[+ (Y_{JK} - Y_{J} - Y_{K} + Y) + (Y_{JK} - Y_{J} - Y_{K} + Y) + (Y_{IJ} - Y_{I} - Y_{J} + Y) + (Y_{IJ} - Y_{I} - Y_{J} + Y)\]

and the structural diagram would be represented as
Then the generalized polykay \((\alpha/\alpha/\beta) = (pq/pq/p,q)\), for example, is defined as

\[
(\alpha/\alpha/\alpha) = (\alpha) \bullet (\alpha) \bullet (\alpha) = [<\alpha\alpha> - <\alpha\beta>] [<\alpha\alpha> - <\alpha\beta>] \bullet <\alpha\beta>
\]

\[
= <\alpha\alpha/\alpha\alpha/\alpha\beta> - <\alpha\beta/\alpha\alpha/\alpha\beta> - <\alpha\alpha/\alpha\beta/\alpha\beta> + <\alpha\beta/\alpha\beta/\alpha\beta>
\]

Note that the g.s.m.'s above occur with a plus or minus sign according to whether the letter in an odd or even number of the simple polykays contained in \((\alpha) \bullet (\alpha) \bullet (\beta)\) have been subpartitioned. The generalization of this result to products of many simple polykays is obvious. Hence for an \(n\)-way crossed structure the following definition of a generalized polykay of degree two in terms of g.s.m.'s may be given:

**Definition 5.1:**

\[
\begin{align*}
(\theta_1 | \theta_2 | \ldots | \theta_n) &= \left< \theta_1 | \theta_2 | \ldots | \theta_n \right> + \sum (-1)^\pi \left< \theta'_1 | \theta'_2 | \ldots | \theta'_n \right> \\
\text{where } \theta_i \text{ equals } \alpha \text{ or } \beta \text{ and } \theta'_i \text{ is a subpartition of } \theta_i; \text{ for example if } \theta_i \text{ is } \alpha \text{ then } \theta'_i \text{ may be } \alpha \text{ or } \beta \text{ and if } \theta_i = \beta, \theta'_i \text{ must be } \beta. \text{ The sum is over all possible subpartitions of the } \theta_i \text{ and } \pi \text{ is the number of changes of } \alpha \text{ to } \beta. 
\end{align*}
\]

A general structure may consist of both nested and crossed factors and the definitions of the g.s.m.'s and polykays must be modified.
accordingly, as we saw in the previous chapter. In the previous chapter we
say that an arbitrary structure could be represented as a crossed structure
by random cross labeling the subscripts corresponding to nested factors.
We also observed that certain g.s.m.'s from such a crossed structure may
have the same expectation over random cross labeling. For example, if we
consider the g.s.m.'s

\[ \Sigma Y_I Y_{I'} I J \]
\[ \text{AB(A-1)} \]

and

\[ \Sigma Y_I Y_{I'} I' J' \]
\[ \text{AB(A-1)(B-1)} \]

obtained from a crossed structure derived by random cross labeling from the
structure in which the factor B, represented by the subscript J, is nested in
factor A, represented by the subscript A, we observe that these g.s.m.'s
have the same expectation over cross labeling, namely

\[ \Sigma Y_I Y_{I'} I' J' \]
\[ \text{AB^2 (A-1)} \]

Because of this result, the polykay

\[ (AA/AA)^* = \langle AA/AA \rangle^* - \langle AB/AA \rangle^* - \langle AA/AB \rangle^* + \langle AB/AB \rangle^* \]

for example, collapses to

\[ (AA/AA) = \langle AA/AA \rangle - \langle AA/AB \rangle = \frac{\Sigma Y_I^2}{AB} - \frac{\Sigma Y_I Y_{I'} I' J'}{AB(B-1)} \]

under the process of taking the expectation over random cross labeling.
With these particular results in mind, we proceed to define generalized symmetric means and generalized polykays of degree two for an arbitrary structure. We first extend the meaning of the partitions \( \alpha \) and \( \beta \) to facilitate this development.

In the alternate notation the partition \( AA \) would be written as \( A_1A_1 \) if this partition corresponded to a nested factor. Similarly, if \( AB \) corresponded to a nested factor, it would be written as \( A_1B_1 \) or \( A_1B_2 \) depending on whether any of the nesting factors were represented by subscripts with none or one prime in the g.s.m. Henceforth we shall assume \( \alpha \) is of the form \( AA \) or \( A_1A_1 \) and \( \beta \) is of the form \( A_1B_1 \) or \( A_1B_2 \). The subscripts may be fixed according to the rules adopted in naming g.s.m.'s and polykay in the alternate notation, after the g.s.m.'s and polykays of degree two have been written out explicitly. The use of only two symbols \( \alpha \) and \( \beta \) greatly simplifies the following definitions.

Definition 5.2: Consider a general balanced population structure with \( n \) factors. Let \( S \) denote the set of subscripts \( I, J, K \ldots \) and let the range of the subscripts be \( N_1, N_2, \ldots, N_n \). Then

\[
<\theta_1 | \theta_2 | \cdots | \theta_n> = \frac{\sum X_{S}X_{S'}}{\theta_1 \theta_2 \cdots \theta_n} , \quad \frac{N_1 N_2 \cdots N_n}{n}
\]

\[
\cdot\cdot\cdot
\]
where

\( \theta_i \) can be \( \alpha \) or \( \beta \) if the \( i \)-th factor is not nested by any other factor

and \( N_i^\alpha = N_i^\beta = N_i (N_i - 1) \)

\( \theta_i \) can be \( \alpha \) or \( \beta \) if all factors which nest the \( i \)-th factor have \( \alpha \) and

then \( N_i^\alpha = N_i^\beta = N_i (N_i - 1) \)

\( \theta_i \) equals \( \beta \) if one or more of the factors which nest the \( i \)-th factor

have \( \beta \), then \( N_i^\beta = N_i^2 \)

and where the sum is over all subscripts of \( S \) and \( S' \) (the set where some, none, or all subscripts are primed) with the restriction that those corresponding to a \( \beta \) partition are unequal. As before, we use the rule that a nested factor has different subscripts for every combination of levels of all its nesting factors.

As an example consider again the case where factor B is nested within factor A. Then

\[
<AA/AA> = \frac{\sum Y_{ij}^2}{AB}
\]

\[
<AA/AB> = \frac{\sum Y_{ij}^2 Y_{ij}'}{AB(B-1)}
\]

\[
<AB/AB> = \frac{\sum Y_{ij}^2 Y_{ij}'}{AB^2(A-1)}
\]

It is obvious then, according to this definition, that there is a one-to-one correspondence of g.s.m.'s of degree two and admissible means. In the previous case the admissible means are \( Y_{i(j)}' \), \( Y_{i(j)}' \), and \( Y \) respectively.
Because of this fact and the convenience of the use of the concept of the rightmost bracket as discussed in Chapter 3, the following definition of the generalized polykays of degree two will be given in these terms.

**Definition 5.3:** Given a population structure $P$ and an admissible mean $Y_{L(R)}$, where $R$ is the set of subscripts of the rightmost bracket of the set $L + R$; denote by $S$ the set of all subscripts of $P$, so that $S = L + R + Q$, $Q$ being the set of remaining subscripts. Then

$$
\begin{align*}
(\theta_1 | \theta_2 | \ldots | \theta_n)^{L(R)} &= <\theta_1 | \theta_2 | \ldots | \theta_n> + \Sigma (-1)^{\pi} <\theta_1' | \theta_2' | \ldots | \theta_n'> \\
\end{align*}
$$

where

$$
\begin{align*}
\theta_i &= \begin{cases} 
\alpha & \text{if } i \in L' \\
\alpha & \text{if } i \in R' \\
\beta & \text{if } i \in Q' \\
\theta_i & \text{if } i \in Q' 
\end{cases} \\
\theta'_i &= \begin{cases} 
\alpha & \text{or } \beta & \text{if } i \in R'
\end{cases}
\end{align*}
$$

and $\pi$ is the number of $\alpha'$s which are changed to $\beta$, and the sum is over all possible subpartitions of $\theta_i$ where $i \in R$.

**B. Equivalence of the Polykays and $\Sigma$'s**

We shall now proceed to prove the equivalence of the generalized polykays of degree two (Definition 5.3) and the $\Sigma$'s (Definition 3.4). In this discussion it should be remembered that the component of variation corresponding to the admissible mean $Y_{L(R)}$ is defined as

$$
\sigma_{L(R)}^2 = \frac{\Sigma [L(R)]^2}{L \prod_{i} (R_i - 1)},
$$

where $L(R)$ represents a typical component in the population identity corresponding to $Y_{L(R)}$, and the sum is over all subscripts of the leading mean of $L(R)$, and $L_1$ and $R_1$ denote the population ranges of the respective subscripts with $\pi L_i = L$ and $\pi R_i = R$.

First, two lemmas must be demonstrated.

Lemma 5.1: A g.s.m. of degree 2 of population values is equal to the sum of corresponding g.s.m.'s of all components, a corresponding g.s.m. being obtained by deleting all factors in the name of the population g.s.m. which do not occur in the name of the component and using the same partition of the remaining factors. For example, in a two-way crossed structure

$$
\frac{\sum_{i,j} Y_{ij} Y_{ii}^t}{AB(B-1)} = \mu^2 + \frac{\sum A_i^2}{A} + \frac{\sum B_i^2}{B(B-1)} + \frac{\sum (AB)_{ij} (AB)_{ii}^t}{AB(B-1)}
$$

Proof: This result is obvious as regards products of like components, when the population identity is substituted for the population values appearing in the generalized symmetric mean. All products consisting of unlike components will vanish, for there will be at least one subscript not common to the rightmost brackets of the two components of a product and the sum over this subscript will be zero by definition of the components.

Lemma 5.2: Each numerator of a non-vanishing g.s.m. of degree 2 of any component can be expressed as a sum of squares of that component with either a plus or minus sign. For example
\[ \sum_{j} A(B)_{I(j)} A(B)_{I'(j)} = 0 \quad \text{but} \quad \Sigma A_i A_i' = - A_i^2. \]

Proof: If the name of a g.s.m. of a component contains \( B \) partitions corresponding to factors in the non-rightmost bracket of the component, that component g.s.m. vanishes because the factors of the non-rightmost bracket will nest at least one factor contained in the rightmost bracket, and the summation over nested subscripts is independent for different combinations of the nesting subscripts. Thus the name of a g.s.m. of a component contains \( \alpha \) or \( \beta \) partitions for the factors of the rightmost bracket of the set of factors involved in the component. So with regard to the \( \alpha \) partitioned factors contained in both the rightmost bracket and the non-rightmost bracket we have a sum of squares over levels. With regard to a factor in the rightmost bracket for which the name of the g.s.m. involves \( B \) we have a sum over all pairs of unequal levels. But, for example,

\[ \sum_{j} x_j x_j' = - x_j^2 \quad \text{if} \quad \Sigma x_j = 0. \]

So with regard to the numerator of a component g.s.m., each \( B \) in the name of the g.s.m. for a rightmost bracket factor can be replaced by \( \alpha \) with multiplication by \((-1)\). The numerator then becomes a sum of squares of the components of the particular type with a coefficient of \((-1)^r\), where \( r \) is the number of such \( B \)'s corresponding to factors in the rightmost bracket.

Consider again the population structure \( P \) and the admissible mean \( Y \). As before let \( S \) denote the set of all subscripts of \( P \) where
S = L + R + Q, R being the set of rightmost bracket subscripts of L + R and Q the set of subscripts not contained in the name of the admissible mean.

Then we have the following theorem:

Theorem 5.1: The generalized polykays, as defined in Definition 5.3, whose name contains α for the factors of L, α for the factors of R, and β for the factors of Q is equal to \(\Sigma_{L(R)}\).

Proof: Consider a given component g.s.m. whose corresponding component of variation is denoted by \(\sigma^2_{S'}\), where \(S' = L' + R'\), R' being the set of rightmost bracket factors in the name of the given component. When the generalized polykay above is expressed in terms of the component g.s.m.'s (by Lemma 5.1) it will be shown that g.s.m.'s of all components vanish except those whose subscripts \(S'\) are of the form \(S' = L + R + Q^*\) where \(Q^*\) is a subset of Q and lies exclusively in \(R'\). These component g.s.m.'s will then be expressed as components of variation of the form \(\sigma^2_{L(R+Q^*)}\) with the coefficient given in the definition of \(\Sigma_{L(R)}\).

Case 1. Consider first the case where the set of subscripts \(S'\) is null. Then the component g.s.m. \(Y = \sigma^2_S = \sigma^2_{\emptyset}\) occurs in each g.s.m. of the polykay and by Theorem 4.2 vanishes for all polykays except \((\emptyset/\emptyset/\ldots/\emptyset)\) in which case it occurs with a coefficient of (+1).

Now consider g.s.m.'s of components whose set of subscripts \(S'\) is non-empty. Then we distinguish the following three cases of component g.s.m.'s according to the relationship of the subscripts \(S'\) to the subscripts R of the generalized polykay.
Case 2. Assume $R$ is non-empty and take g.s.m.'s of components whose subscripts $S'_2$ do not contain any of the subscripts $R$, i.e.,

$$S'_2 \subseteq L + Q,$$

where either $L$ or $Q$ may be empty. By Definition 5.3 the partitions corresponding to the factors in $L$ and $Q$ are $\alpha$ and $\beta$ respectively in each g.s.m. of the generalized polykay and by Theorem 4.2 (i.e., the sum of the coefficients of the polykays is zero) the g.s.m.'s of components vanish.

Case 3. Consider again $R$ to be non-empty and component g.s.m.'s whose subscripts $S'_3$ contain some, but not all, of the subscripts of $R$, i.e.,

$$S'_3 \subseteq L + Q + R^*$$

where $R^* \subseteq R$. In the polykay any g.s.m. has a corresponding g.s.m. differing with regard to one subscript in $R - R^*$ and therefore in sign but these two g.s.m.'s give the same component g.s.m. and therefore cancel each other.

Case 4. Now take the components whose subscripts $S'_4$ contain all the subscripts $R$, where $R$ may be null as in the case of the polykay ($\beta/\beta/\ldots/\beta$). Since the subscripts of $R$ are contained in $S'_4$ so must be the set $L$, and hence $S'_4 = L + R + Q^*$, where $Q^* \subseteq Q$. This case may be subdivided into the following two cases depending on whether $Q^*$ lies exclusively or not in the set $R'_4$, the set of rightmost subscripts of $S'_4$.

Case 4a. If the set $Q^*$ is non-empty and does not lie exclusively in $R'_4$, then a subset of $Q^*$ must be contained in $L'_4$, the non-rightmost bracket part of $S'_4$, and according to the proof of Lemma 5.2 the component g.s.m.'s whose subscripts are of this form vanish.
Case 4b. Consider now component g.s.m.'s whose subset of subscripts $Q^*$ is either null or lies exclusively in $R^*_4$. Then each component g.s.m. may be expressed as a sum of squares of the corresponding component by Lemma 5.2, preceded by a plus or minus sign. In fact, the sum of squares will appear with like sign in each g.s.m. of the polykay, positive if the number of subscripts in $Q^*$ is even and negative if the number is odd. For suppose $Q^*$ has an odd number of subscripts. The original polykay has $\alpha$ partitions for the set $L$, $\alpha$ partitions for the set $R$ and $\beta$ partitions for the set $Q^*$, and when the polykay is expressed in terms of g.s.m.'s we get the leading term appearing with a plus sign and with the same number of $\alpha$'s and $\beta$'s as in the name of the polykay. But since $Q^*$ contains an odd number of subscripts the sign of the sum of squares of the component in question will be $(+1)(-1) = -1$ in the leading g.s.m. by Lemma 5.2. Now a second g.s.m. of the polykay will have one $\alpha$ of $R$ changed to $\beta$ and thus preceded by a negative sign because of the additional $\beta$. But the component g.s.m. now has an additional $\beta$ partition so that the sum of squares will be preceded by a $(-1)(-1)(-1) = -1$ again. Another g.s.m. will have two of the $\alpha$'s changed to $\beta$ and hence preceded by a plus one so that the sum of squares will be preceded by a $(+1)(-1)(-1)(-1) = -1$ and so on. A similar argument shows the sum of squares of a component in this case will occur with a plus sign if the number of subscripts in $Q^*$ is even.

Thus we see that the sum of squares of components in Case 4b do not vanish in the polykay whose name contains $\alpha$ partitions for $L$ and $R$ and $\beta$
partitions for \( Q \). We must now find the proper coefficient for these sums of squares.

Let a component of the type in question be denoted by \( \Gamma (\Psi) \) and the corresponding sum of squares by \( \Sigma [\Gamma (\Psi)]^2 \) where \( \Gamma \) contains \( \rho \) subscripts and \( \Psi \) contains \( k \) subscripts. Let \( \Gamma_i \) denote the population range of the corresponding subscripts with \( \Gamma = \sum_{i=1}^{\rho} \Gamma_i \). Similarly let \( \Psi = \sum_{i=1}^{k} \Psi_i \). Further let \( N = (\prod_{i=1}^{\rho} \Gamma_i)(\prod_{j=1}^{k} \Psi_j) = \Gamma \Psi \).

Suppose \( \Psi_i, i=1, \ldots, \nu \) corresponds to subscripts which are contained in \( R \) and \( \Psi_i, i=\nu+1, \ldots, k \) correspond to subscripts in \( Q^* \). Then the partitions corresponding to the latter are of the form \( \beta \), while those of the former are subpartitioned according to Definition 5.1. By the argument given in Case 4b the sign of the component is \((-1)^{k-\nu}\), so the coefficient of \( \Sigma [\Gamma (\Psi)]^2 \) will be:

\[
(-1)^{k-\nu} \left[ \frac{1}{N \pi \nu + 1} + \frac{1}{N (\Psi - 1) \nu + 1} + \ldots \right]
\]

\[
+ \frac{1}{N (\Psi - 1) \nu + 1} + \frac{k}{N (\Psi - 1) (\Psi_2 - 1) \nu + 1} + \ldots
\]
\[
\left[ \frac{1}{N(Y-1)\ldots(Y-j)} \right]_{\nu+1} = (-1)^{k-v} N TT (Y-1) ^{13} \\
= (-1)^{k-v} \left\{ \frac{1}{k} \right\} \begin{cases} 
1 + \sum_{i=1}^{\nu} \frac{1}{(Y_i-1)} + \sum_{i<j}^{\nu} (Y_i-1)(Y_j-1) + \ldots 
+ \frac{1}{(Y_1-1)\ldots(Y_{\nu}-1)} 
\end{cases} \\
= (-1)^{k-v} \frac{\sum_{i=1}^{\nu} \frac{1}{(Y_i-1) + 1}}{k \pi (Y_i-1)} = (-1)^{k-v} \frac{\sum_{i=1}^{\nu} \frac{1}{(Y_i-1) + 1}}{k N \pi (Y_i-1)} \\
= (-1)^{k-v} \frac{1}{k} \frac{p}{k} \frac{k}{k} \frac{(\pi Y_i)(\pi Y_j)}{\nu+1} \frac{(Y_i-1)}{1} \frac{(Y_j-1)}{1} 
\]

Thus

\[
\frac{(-1)^{k-v} \sum [\Gamma(Y)]^2}{k p k (\pi Y_i)(\pi Y_j) p (Y_i-1) \nu+1} = \frac{(-1)^{k-v}}{\sigma \Gamma(Y)} \frac{2}{k} 
\]

which is the typical term in the expansion of \( \sigma \Gamma(Y) \) according to Definition 3.4, q.e.d.

As an illustration of the preceding theorem we consider a four-factor structure illustrated by the following diagram:
i.e., a structure where factors A and B are completely crossed with each other but A nests two other factors, C and D, one nested within the other and B is crossed with C within each level of A and crossed with D within each level of C.

Let \( Y_{IJKL} \) denote a typical response where I corresponds to factor A and \( I = 1, \ldots, A \); J corresponds to factor B and \( J = 1, \ldots, B \); K corresponds to factor C and has a range of C; L corresponds to factor D and has a range of D.

The admissible means are then \( Y_{I}^I , Y_{J}^J , Y_{IJ}^I , Y_{I(K)}^I , Y_{IK(L)}^I , Y_{IJK}^I , Y_{IKL}^I \) and \( Y \), and the population identity may be written

\[
Y_{IJKL} = Y + (Y_{I} - Y) + (Y_{J} - Y) + (Y_{IJ} - Y_{I} - Y_{J} + Y) + (Y_{I(K)} - Y_{I}) + (Y_{IK(L)} - Y_{I(K)}) + (Y_{I(JK)} - Y_{I(J)} - Y_{I(K)} + Y_{I}) + (Y_{IKL} - Y_{I(K)}) - Y_{I(K)} + Y_{I} \\
+ A_I + B_J + (AB)_{IJ} + A(C)_{I(K)} + AC(D)_{IK(L)} + A(BC)_{I(JL)} + AC(BD)_{IK(L)}
\]
say. Then the $\Sigma$'s corresponding to the above means are respectively:

$$
\Sigma_\emptyset = \sigma^2 - \frac{1}{A} \sigma^2 - \frac{1}{B} \sigma^2 + \frac{1}{AB} \sigma^2
$$

$$
\Sigma_A = \sigma^2 - \frac{1}{A} \sigma^2 - \frac{1}{AB} \sigma^2
$$

$$
\Sigma_B = \sigma^2 - \frac{1}{A} \sigma^2
$$

$$
\Sigma_{AB} = \sigma^2
$$

$$
\Sigma_{A(C)} = \sigma^2 - \frac{1}{D} \sigma^2
$$

$$
\Sigma_{AC(D)} = \sigma^2 - \frac{1}{AC(BD)} \sigma^2
$$

$$
\Sigma_{A(BC)} = \sigma^2
$$

$$
\Sigma_{AC(BD)} = \sigma^2
$$

The corresponding generalized polykays are

$$
(AB/AB/A_1B_2/A_1B_2) = <AB/AB/A_1B_2/A_1B_2> = \frac{\Sigma_{IJKL}Y_{I'J'K'L'}}{ABC^2D^2(A-1)(B-1)}
$$

$$
(AA/AB/A_1B_1/A_1B_2) = <AA/AB/A_1B_1/A_1B_2> = \frac{\Sigma_{IJKL}Y_{I'J'K'L'}}{ABCD^2(B-1)(C-1)}
$$

$$
(AB/AB/A_1B_2/A_1B_2) = <AB/AB/A_1B_2/A_1B_2> = \frac{\Sigma_{IJKL}Y_{I'J'K'L'}}{ABC^2D^2(A-1)(B-1)}
$$
\[
<\sum_{a}^{b} A_C - <\sum_{c}^{b} A_C > + <\sum_{a}^{b} A_C > = (i-a)(i-b) \sum_{c}^{b} A_C
\]

\[
\frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} - \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} = \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}}
\]

\[
<\sum_{a}^{b} A_C - <\sum_{c}^{b} A_C > + <\sum_{a}^{b} A_C > = (i-a)(i-b) \sum_{c}^{b} A_C
\]

\[
\frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} - \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} = \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}}
\]

\[
<\sum_{a}^{b} A_C - <\sum_{c}^{b} A_C > + <\sum_{a}^{b} A_C > = (i-a)(i-b) \sum_{c}^{b} A_C
\]

\[
\frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} - \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}} = \frac{(A_C)(B-C)(1-D)}{I_{\lambda} K_{\lambda}}
\]

\[
<\sum_{a}^{b} A_C - <\sum_{c}^{b} A_C > + <\sum_{a}^{b} A_C > = (i-a)(i-b) \sum_{c}^{b} A_C
\]
\[
\frac{\text{ABCD} (B-I) (D-I)}{\lambda \neq 3} + \\
\frac{\text{ABCD} (B-I) (D-I)}{\lambda \neq 3} - \\
\frac{\text{ABCD} (B-I) (D-I)}{\lambda \neq 3} - \\
\frac{\text{ABCD} (B-I) (D-I)}{\lambda \neq 3} = \\
<\AA/\AA/\AA/\AA> + <\AA/\AA/\AA/\AA> - \\
<\AA/\AA/\AA/\AA> - <\AA/\AA/\AA/\AA> = (1/1/1/1)
\]
VI. POLYKAYS FOR TWO-FACTOR STRUCTURES

A. Preliminaries

Before presenting the g.s.m.'s and generalized polykays for the two-factor balanced structures, viz., the two-factor crossed and two-factor nested structures, we first present the simple polykays of the first four degrees which are used in forming the necessary dot-products. These will be given in both Hooke's secondary notation and the alternate notation introduced in Chapter 4.

According to Definition 3.1, the only simple polykay of degree one is

\[(A) = \langle A \rangle\]

For degree two we have the following equations

\[\langle pq \rangle = (pq) + (p,q) ,\]
\[\langle p,q \rangle = (p,q) ,\]

which lead to

\[(pq) = \langle pq \rangle - \langle p,q \rangle ,\]
\[(p,q) = \langle p,q \rangle ,\]

or, in the alternate notation,

\[(AA) = \langle AA \rangle - \langle AB \rangle ,\]
\[(AA) = \langle AB \rangle .\]

According to Definition 3.1, we obtain the following equations for the third degree polykays:

\[\langle pqr \rangle = (pq,r) + (p,q,r) + (pr,q) + (p,r,q) + (p,q,r) ,\]
\[ <pq, r> = (pq, r) + (p, q, r), \]
\[ <p, qr> = (p, qr) + (p, q, r), \]
\[ <pr, q> = (pr, q) + (p, q, r), \]
\[ <p, q, r> = (p, q, r). \]

These lead to the following:

\[ (pqr) = <pqr> - <pq, r> - <p, qr> - <pr, q> + 2 <p, q, r>, \]
\[ (pq, r) = <pq, r> - <p, q, r>, \]
\[ (p, qr) = <p, qr> - <p, q, r>, \]
\[ (pr, q) = <pr, q> - <p, q, r>, \]
\[ (p, q, r) = <p, q, r>, \]

which written in the alternate notation become

\[ (AAA) = <AAA> - <AAB> - <ABB> - <A\overline{A}A > + 2 <ABC>, \]
\[ (AAB) = <AAB> - <ABC>, \]
\[ (ABB) = <ABB> - <ABC>, \]
\[ (ABA) = <ABA> - <ABC>, \]
\[ (ABC) = <ABC>. \]

The appropriate equations for degree four are

\[ <pqrs> = (pqrs) + (pqr, s) + (pq, rs) + (p, qrs) + (prs, q) + (pq, rs) + (pq, rs) + (pr, qs) + (ps, qr) + (pq, rs) + (p, q, rs) + (p, q, r, s), \]
\[ <pqr, s> = (pqr, s) + (pq, r, s) + (pr, q, s) + (p, s, qr) + (p, q, r, s), \]
\[ <pq, s> = (pq, s) + (pq, r, s) + (ps, q, r) + (p, q, r, s), \]
\[ <pqs, r> = (pqs, r) + (pq, r, s) + (ps, q, r) + (p, r, qs) + (p, q, r, s), \]
\[ <p, qrs> = (p, qrs) + (p, s, qr) + (p, q, rs) + (p, r, qs) + (p, q, r, s). \]
\[
<\text{prs}, q> = (\text{prs}, q) + (\text{ps}, q, r) + (\text{pr}, q, s) + (p, q, rs) + (p, q, r, s),
\]
\[
<\text{pq}, rs> = (\text{pq}, rs) + (\text{pq}, r, s) + (p, q, rs) + (p, q, r, s),
\]
\[
<\text{pr}, qs> = (\text{pr}, qs) + (\text{pr}, q, s) + (p, r, qs) + (p, q, r, s),
\]
\[
<\text{ps}, qr> = (\text{ps}, qr) + (\text{ps}, q, r) + (p, s, qr) + (p, q, r, s),
\]
\[
<\text{pq}, r, s> = (\text{pq}, r, s) + (p, q, r, s),
\]
\[
<\text{pr}, q, s> = (\text{pr}, q, s) + (p, q, r, s),
\]
\[
<\text{ps}, q, r> = (\text{ps}, q, r) + (p, q, r, s),
\]
\[
<\text{p}, s, qr> = (\text{p}, s, qr) + (p, q, r, s),
\]
\[
<\text{p}, r, qs> = (\text{p}, r, qs) + (p, q, r, s),
\]
\[
<\text{p}, q, rs> = (\text{p}, q, rs) + (p, q, r, s),
\]
\[
<\text{p}, q, r, s> = (p, q, r, s).
\]

Solving these for the polykays in terms of symmetric means we obtain

\[
(pqrs) = <pqrs> - <pqr, s> - <pq, rs> - <p, qrs> - <prs, q>
- <pq, rs> - <pr, qs> - <ps, qr> + 2 <pq, r, s> + 2 <pr, q, s>
+ 2 <ps, q, r> + 2 <p, s, qr> + 2 <p, r, qs> + 2 <p, q, rs>
- 6 <p, q, r, s>,
\]
\[
(pqr, s) = <pqr, s> - <pq, r, s> - <pr, q, s> - <p, s, qr> + 2 <p, q, r, s>,
\]
\[
(pqs, r) = <pqs, r> - <pq, r, s> - <pr, q, s> - <p, r, qs> + 2 <p, q, r, s>,
\]
\[
(p, qrs) = <p, qrs> - <p, r, qs> - <p, q, rs> - <p, s, qr> + 2 <p, q, r, s>,
\]
\[
(prs, q) = <prs, q> - <ps, q, r> - <pr, q, s> - <p, qr, s> + 2 <p, q, r, s>,
\]
\[
(pq, rs) = <pq, rs> - <pq, r, s> - <p, q, rs> + <p, q, r, s>,
\]
\[
(pr, qs) = <pr, qs> - <pr, q, s> - <p, r, qs> + <p, q, r, s>,
\]
\[
(ps, qr) = <ps, qr> - <ps, q, r> - <p, s, qr> + <p, q, r, s>,
\]
(pq, r, s) = <pq, r, s> - <p, q, r, s>,
(pr, q, s) = <pr, q, s> - <p, q, r, s>,
(ps, q, r) = <ps, q, r> - <p, q, r, s>,
(p, s, qr) = <p, s, qr> - <p, q, r, s>,
(p, r, qs) = <p, r, qs> - <p, q, r, s>,
(p, q, rs) = <p, q, rs> - <p, q, r, s>,
(p, q, r, s) = <p, q, r, s>.

Written in the alternate notation these equations become

(AAAA) = <AAAA> - <AAAB> - <AABA> - <ABBB> - <ABAA> - <AABB>
- <ABAB> - <ABBA> + 2 <AABC> + 2 <ABAC> + 2 <ABCA>
+ 2 <ABBC> + 2 <ABCB> + 2 <ABCC> - 6 <ABCD>,

(AAAB) = <AAAB> - <AABC> - <ABAC> - <ABBC> + 2 <ABCD>,

(AABA) = <AABA> - <AABC> - <ABCA> - <ABCB> + 2 <ABCD>,

(ABBB) = <ABBB> - <ABCB> - <ABCC> - <ABBC> + 2 <ABCD>,

(ABAA) = <ABAA> - <ABCA> - <ABCAB> - <ABCC> + 2 <ABCD>,

(AABB) = <AABB> - <AABC> - <ABCC> + <ABCD>,

(ABAB) = <ABAB> - <AABC> - <ABAC> + <ABCD>,

(ABBA) = <ABBA> - <ABCA> - <ABBC> + <ABCD>,

(AABC) = <AABC> - <ABCD>,

(ABAC) = <ABAC> - <ABCD>,

(ABCA) = <ABCA> - <ABCD>,

(ABBC) = <ABBC> - <ABCD>,

(ABCB) = <ABCB> - <ABCD>. 
\[(ABCC) = \langle ABCC \rangle - \langle ABCD \rangle,\]

\[(ABCD) = \langle ABCD \rangle.\]

If we consider sampling from a one-factor structure or simple random sampling, the number of distinct polykays is not so great as indicated above. Using Tukey's primary notation the distinct polykays of degree one are in this case,

\[(1) = \langle 1 \rangle,\]

while those for degree two are

\[(2) = \langle 2 \rangle - \langle 11 \rangle,\]
\[(11) = \langle 11 \rangle,\]

and for degree three,

\[(3) = \langle 3 \rangle - 3\langle 21 \rangle + 2\langle 111 \rangle,\]
\[(21) = \langle 21 \rangle - \langle 111 \rangle,\]
\[(111) = \langle 111 \rangle.\]

The distinct polykays of degree four are

\[(4) = \langle 4 \rangle - 4\langle 31 \rangle - 3\langle 22 \rangle + 12\langle 211 \rangle - 6\langle 1111 \rangle,\]
\[(31) = \langle 31 \rangle - 3\langle 21 \rangle + 2\langle 1111 \rangle,\]
\[(22) = \langle 22 \rangle - 2\langle 211 \rangle + \langle 1111 \rangle,\]
\[(211) = \langle 211 \rangle - \langle 1111 \rangle,\]
\[(1111) = \langle 1111 \rangle.\]
B. Two-factor Structures

Two-factor crossed structure

In this situation we envisage the following population identity:

\[ Y_{IJ} = Y + (Y_I - Y) + (Y_J - Y) + Y_{IJ} - Y_I - Y_J + Y \]

\[ = \mu + A_I + B_J + (AB)_{IJ}, \text{ say} \]

where \( I = 1, \ldots, A; J = 1, \ldots, B \). The population structure can be represented as

\[
\begin{align*}
Y & \quad \text{u} \\
A & \quad \text{a} \\
B & \quad \text{b} \\
\varepsilon & \quad \text{e}
\end{align*}
\]

G.s.m.'s and polykays of degree two

According to Definition 4.1 or 5.2 we have the following distinct generalized symmetric means of degree two:

\[
\begin{align*}
<AA/AA> &= \frac{\Sigma Y_{II}^2}{AB} \\
<AA/AB> &= \frac{\Sigma Y_{II} Y_{I'I'}}{AB(B-1)} \\
<AB/AA> &= \frac{\Sigma Y_{II} Y_{I'I'}}{AB(A-1)} \\
<AB/AB> &= \frac{\Sigma Y_{II} Y_{I'I'}}{AB(A-1)(B-1)}
\end{align*}
\]
By Definitions 4.4 we obtain the corresponding generalized polykays by dot-multiplication. Thus

\[
\begin{align*}
(AA/AA) &= (AA) \cdot (AA) \\
&= [<AA> - <AB>] [ <AA> - <AB> ] \\
&= <AA/AA> - <AB/AA> - <AA/AB> + <AB/AB> , \\

(AA/AB) &= (AA) \cdot (AB) \\
&= [<AA> - <AB>] \cdot <AB> \\
&= <AA/AB> - <AB/AB> , \\

(AB/AA) &= (AB) \cdot (AA) \\
&= <AB> \cdot [<AA> - <AB>] \\
&= <AB/AA> - <AB/AB> \\
\end{align*}
\]

and

\[
\begin{align*}
(AB/AB) &= (AB) \cdot (AB) \\
&= <AB/AB> .
\end{align*}
\]

These generalized polykays of degree two could have been obtained directly from Definition 5.1 and are, as we saw in Chapter 5, the functions \(\Sigma_{AB}'\), \(\Sigma_A'\), \(\Sigma_B'\) and \(\Sigma_{\emptyset}'\) respectively.

**G.s.m.'s and polykays of degree three** The distinct g.s.m.'s of degree three are found to be

\[
<AAA/AAA> = \frac{\Sigma_{Y_1} Y_1^3}{AB} , \quad <AAB/AAA> = \frac{\Sigma_{Y_1} Y_1^2 Y_1 I}{AB(A-1)} ,
\]
Again by dot-multiplication of the simple polykays in Section A of this chapter, we obtain the following generalized polykays of degree three:

\[
\begin{align*}
\langle AAA/AAA \rangle &= \langle AAA/AAA \rangle - 3 \langle AAB/AAA \rangle - 3 \langle AAA/AAB \rangle + 3 \langle AAB/AAB \rangle \\
&\quad + 6 \langle AAB/ABA \rangle + 2 \langle AAA/ABC \rangle + 2 \langle ABC/AAA \rangle \\
&\quad - 6 \langle ABC/AAB \rangle - 6 \langle AAB/ABB \rangle + 4 \langle ABC/ABO \rangle,
\langle AAB/AAA \rangle &= \langle AAB/AAA \rangle - \langle AAB/AAB \rangle - \langle ABC/AAA \rangle + 2 \langle AAB/ABB \rangle \\
&\quad + 3 \langle ABC/AAB \rangle - 2 \langle AAB/ABA \rangle - 2 \langle ABC/ABC \rangle,
\langle AAA/AAB \rangle &= \langle AAA/AAB \rangle - \langle AAB/AAB \rangle - \langle AAA/ABC \rangle + 2 \langle ABC/AAB \rangle \\
&\quad + 3 \langle AAB/ABB \rangle - 2 \langle AAB/ABA \rangle - 2 \langle ABC/ABC \rangle,
\langle AAB/AAB \rangle &= \langle AAB/AAB \rangle - \langle ABC/AAB \rangle - \langle AAB/ABB \rangle + \langle ABC/ABC \rangle,
\langle AAB/ABA \rangle &= \langle AAB/ABA \rangle - \langle ABC/AAB \rangle - \langle AAB/ABB \rangle + \langle ABC/ABC \rangle,
\langle ABC/AAA \rangle &= \langle ABC/AAA \rangle - 3 \langle ABC/AAB \rangle + 2 \langle ABC/ABC \rangle,
\langle AAA/ABC \rangle &= \langle AAA/ABC \rangle - 3 \langle AAB/ABB \rangle + 2 \langle ABC/ABC \rangle,
\langle ABC/AAB \rangle &= \langle ABC/AAB \rangle - 3 \langle AAB/ABB \rangle + 2 \langle ABC/ABC \rangle,
\langle ABC/AAB \rangle &= \langle ABC/AAB \rangle - \langle ABC/ABC \rangle,
\end{align*}
\]
(AAB/ABB) = \(<AAB/ABB> - <ABC/ABC>.

(ABC/ABC) = <ABC/ABC>.

G.s.m.'s and polykays of degree four  

By now the reader should be familiar with the alternate notation so that we can henceforth omit giving the meaning of the symbolic notation. Using \( f_i \) and \( F_i \) (\( i = 1, \ldots, 33 \)) to denote the g.s.m.'s and polykays of degree four respectively (after Hooke), we find the distinct g.s.m.'s

<table>
<thead>
<tr>
<th>( f_i )</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{33} )</td>
<td>&lt;AAAA/AAAA&gt;</td>
</tr>
<tr>
<td>( f_{32} )</td>
<td>&lt;AAAA/AAAA&gt;</td>
</tr>
<tr>
<td>( f_{31} )</td>
<td>&lt;AAAA/AAAB&gt;</td>
</tr>
<tr>
<td>( f_{30} )</td>
<td>&lt;AABD/AAAA&gt;</td>
</tr>
<tr>
<td>( f_{29} )</td>
<td>&lt;AAAA/AABB&gt;</td>
</tr>
<tr>
<td>( f_{28} )</td>
<td>&lt;AAAB/AAAB&gt;</td>
</tr>
<tr>
<td>( f_{27} )</td>
<td>&lt;AABB/AABB&gt;</td>
</tr>
<tr>
<td>( f_{26} )</td>
<td>&lt;AAAB/AABA&gt;</td>
</tr>
<tr>
<td>( f_{25} )</td>
<td>&lt;AABB/AAAB&gt;</td>
</tr>
<tr>
<td>( f_{24} )</td>
<td>&lt;AAAB/AABB&gt;</td>
</tr>
<tr>
<td>( f_{23} )</td>
<td>&lt;AABC/AAAA&gt;</td>
</tr>
<tr>
<td>( f_{22} )</td>
<td>&lt;AAAA/AABC&gt;</td>
</tr>
<tr>
<td>( f_{21} )</td>
<td>&lt;AABC/AABC&gt;</td>
</tr>
<tr>
<td>( f_{20} )</td>
<td>&lt;AAAB/AABC&gt;</td>
</tr>
<tr>
<td>( f_{19} )</td>
<td>&lt;AABC/AABC&gt;</td>
</tr>
<tr>
<td>( f_{18} )</td>
<td>&lt;AAAB/AABC&gt;</td>
</tr>
<tr>
<td>( f_{17} )</td>
<td>&lt;AABB/ABAB&gt;</td>
</tr>
<tr>
<td>( f_{16} )</td>
<td>&lt;AABC/ABBB&gt;</td>
</tr>
<tr>
<td>( f_{15} )</td>
<td>&lt;AAAB/ABCA&gt;</td>
</tr>
<tr>
<td>( f_{14} )</td>
<td>&lt;AABC/ABAB&gt;</td>
</tr>
<tr>
<td>( f_{13} )</td>
<td>&lt;AABB/ABAC&gt;</td>
</tr>
<tr>
<td>( f_{12} )</td>
<td>&lt;AABC/AABC&gt;</td>
</tr>
<tr>
<td>( f_{11} )</td>
<td>&lt;AABC/ABCC&gt;</td>
</tr>
<tr>
<td>( f_{10} )</td>
<td>&lt;AAAC/ABAC&gt;</td>
</tr>
<tr>
<td>( f_{9} )</td>
<td>&lt;ABCD/AAAA&gt;</td>
</tr>
<tr>
<td>( f_{8} )</td>
<td>&lt;AAAA/ABCD&gt;</td>
</tr>
<tr>
<td>( f_{7} )</td>
<td>&lt;ABCD/AAAB&gt;</td>
</tr>
<tr>
<td>( f_{6} )</td>
<td>&lt;AAAB/ABCD&gt;</td>
</tr>
<tr>
<td>( f_{5} )</td>
<td>&lt;ABCD/AABB&gt;</td>
</tr>
<tr>
<td>( f_{4} )</td>
<td>&lt;AABB/ABCD&gt;</td>
</tr>
</tbody>
</table>
\[ f_3 = \langle \text{ABCD/ABCD} \rangle, \quad f_2 = \langle \text{AABC/ABCD} \rangle, \quad f_1 = \langle \text{ABCD/ABCD} \rangle \]

and the generalized polykays, obtained by dot-multiplication,

\[
F_{33} = f_{33} - 4f_{32} - 4f_{31} - 3f_{30} - 3f_{29} + 4f_{28} + 3f_{27} + 12f_{26} + 12f_{25} + 12f_{24} + 12f_{23} \\
+ 12f_{22} - 24f_{21} - 24f_{20} - 12f_{19} - 12f_{18} - 12f_{17} - 6f_6 - 24f_{15} - 24f_{14} \\
- 24f_{13} - 24f_{12} + 24f_{11} + 24f_{10} + 96f_9 - 6f_8 - 6f_7 + 24f_6 + 24f_5 + 18f_4 \\
+ 18f_3 - 72f_2 + 36f_1,
\]

\[
F_{32} = f_{32} - 28f_{28} - 26f_{26} - 24f_{24} - 23f_{23} + 6f_{21} + 6f_{20} + 2f_{19} + 6f_{18} + 6f_{17} + 6f_{16} - 6f_{12} \\
- 6f_{11} - 24f_{10} + 2f_9 - 8f_7 - 6f_6 - 6f_5 + 24f_4 + 18f_2 - 12f_1,
\]

\[
F_{31} = f_{31} - 28f_{28} - 26f_{26} - 25f_{25} + 6f_{21} + 6f_{20} + 2f_{19} + 6f_{18} + 6f_{17} + 6f_{16} - 6f_{12} \\
- 6f_{11} - 24f_{10} + 2f_9 - 8f_7 - 6f_6 - 6f_5 + 24f_4 + 18f_2 - 12f_1,
\]

\[
F_{30} = f_{30} - 28f_{27} - 25f_{25} - 22f_{22} + 2f_{19} + 4f_{18} - 2f_{17} + 4f_{16} + 4f_{14} + 8f_{13} - 4f_{12} \\
- 4f_{11} - 16f_{10} + f_9 - 4f_7 - 6f_5 + 12f_3 + 12f_2 - 6f_1,
\]

\[
F_{29} = f_{29} - 27f_{27} - 24f_{24} - 22f_{22} + 2f_{19} + 4f_{18} - 2f_{17} + 4f_{15} + 8f_{14} + 4f_{13} - 4f_{12} \\
- 4f_{11} - 16f_{10} + f_9 - 4f_7 - 5f_4 + 12f_3 + 12f_2 - 6f_1,
\]

\[
F_{28} = f_{28} - 3f_{27} - 3f_{26} + 3f_{20} + 6f_{19} + 2f_7 + 2f_6 - 6f_3 - 6f_2 + 4f_1,
\]

\[
F_{27} = f_{27} - 2f_{19} + 2f_{18} + 2f_{17} + 2f_{16} + 2f_{15} + 2f_{14} + 2f_{13} + 2f_{12} + f_9 + f_8 + f_7 + f_6 + f_5 + f_4 + f_3 + f_2 + f_1,
\]

\[
F_{26} = f_{26} - f_{21} - 2f_{20} - 2f_{19} - 2f_{18} + 2f_{17} + 2f_{16} + 6f_{15} + 7f_6 + 6f_5 + 6f_4 + 4f_3 + 4f_2 + 4f_1,
\]

\[
F_{25} = f_{25} - f_{21} - f_{18} - 2f_{17} + 4f_{15} + 12f_{11} + 4f_{10} + f_9 + 2f_8 - 3f_7 - 4f_6 + 2f_5 + 2f_4 + 2f_3 - 6f_2 + f_1,
\]

\[
F_{24} = f_{24} - f_{20} - f_{19} - 15f_6 + 12f_{12} + 14f_{14} + 4f_0 + 2f_7 + 2f_6 - 3f_3 - 4f_2 + 2f_1,
\]

\[
F_{23} = f_{23} - 2f_{21} - f_{19} - 2f_{16} + 2f_{12} + 2f_8 + 8f_7 + f_6 + 4f_5 + 3f_3 - 6f_2 - 6f_1,
\]

\[
F_{22} = f_{22} - 2f_{20} - f_{18} - 2f_{15} + 12f_{13} + 2f_{12} + 2f_8 + 8f_7 + f_6 + 4f_5 + 3f_3 - 6f_2 - 6f_1,
\]
\[ F_{21} = f_{21} - \frac{f_{12}}{3} - 2f_{10} - f_7 + 3f_3 + 2f_2 - 2f_1, \]
\[ F_{20} = f_{20} - \frac{f_{12}}{6} - f_7 + 2f_3 + 3f_2 - 2f_1, \]
\[ F_{19} = f_{19} - \frac{f_{12}}{3} - f_5 + 2f_3 + f_2 - f_1, \]
\[ F_{18} = f_{18} - \frac{f_{12}}{3} f_4 + 2f_2 - f_1, \]
\[ F_{17} = f_{17} - 2f_{14} - 2f_{13} + 4f_9 + f_5 + 4f_4 + 2f_3 - 2f_2 + f_1, \]
\[ F_{16} = f_{16} - \frac{f_{11}}{7} - 2f_{10} - f_7 + 3f_3 + 2f_2 - 2f_1, \]
\[ F_{15} = f_{15} - \frac{f_{11}}{6} - 2f_{10} - f_6 + 2f_3 + 3f_2 - 2f_1, \]
\[ F_{14} = f_{14} - 2f_{10} - f_5 + 2f_3 + f_2 - f_1, \]
\[ F_{13} = f_{13} - 2f_{10} - f_4 + f_3 + f_2 - f_1, \]
\[ F_{12} = f_{12} - f_3 - f_2 + f_1, \]
\[ F_{11} = f_{11} - f_3 - f_2 + f_1, \]
\[ F_{10} = f_{10} - f_3 - f_2 + f_1, \]
\[ F_{9} = f_{9} - 4f_7 - 3f_5 + 12f_3 - 6f_1, \]
\[ F_{8} = f_{8} - 4f_6 - 3f_4 + 12f_2 - 6f_1, \]
\[ F_{7} = f_{7} - 3f_3 + 2f_1, \]
\[ F_{6} = f_{6} - 3f_2 + 2f_1, \]
\[ F_{5} = f_{5} - 2f_3 + f_1, \]
\[ F_{4} = f_{4} - 2f_2 + f_1, \]
\[ F_{3} = f_{3} - f_1, \]
\[ F_{2} = f_{2} - f_1, \]
\[ F_{1} = f_{1}. \]
Since all the g.s.m.'s and polykays of this section were previously given by Hooke the problem of determining the distinct g.s.m.'s and hence distinct polykays did not arise. However, the general computer programs developed to perform this task in the three-way crossed structure could have been applied if necessary. A brief description of this enumerative procedure is given in the next chapter where g.s.m.'s and polykays of degree four are discussed.

Two-factor nested structure

In this structure we shall assume factor B is nested in factor A. The population identity is

\[ Y_{i(j)} = Y + (Y_{r-i}) + (Y_{i(j)} - Y_i) = u + A_i + A(B)_{ij}, \]

where I has the range 1, ..., A and J has the range of B units. The diagram below illustrates the structure.

\[ \text{u} \]
\[ \text{Ao} \]
\[ \text{Bo} \]
\[ \text{Io} \]

G.s.m.'s and polykays of degree two According to Definition 4.2 we obtain from the g.s.m.'s of the two-factor crossed structure the following distinct g.s.m.'s of degree two of the two-factor nested structure:

\[ <AA/A_1A_1> \]
\[ <AA/A_1B_1> \]
These g.s.m.'s could also have been obtained from Definition 5.2. The generalized polykays, derived from the polykays of the crossed structure by Definition 4.5, or obtained more directly from Definition 5.3, are

\[
\begin{align*}
(AA/A_1A_1) &= <AA/A_1A_1> - <AA/A_1B_1>, \\
(AA/A_1B_1) &= <AA/A_1B_1> - <AB/A_1B_2>, \\
\end{align*}
\]

and

\[
(AB/A_1B_2) = <AB/A_1B_2>,
\]

which correspond to \(\Sigma_{A(B)}\), \(\Sigma_A\) and \(\Sigma_B\) respectively. Since the polykay \((AB/AA)\) of the crossed structure vanishes under collapsing, it has no corresponding polykay in the nested structure. This is consistent with the fact that \(\Sigma_B\) is not defined.

**G.s.m.'s and polykays of degree three**  The distinct g.s.m.'s of degree three are found to be

\[
\begin{align*}
<AAA/A_1A_1A_1> , \\
<AAA/A_1A_1B_1> , \\
<AAB/A_1A_1B_2> , \\
<AAA/A_1B_1C_1> , \\
<AAB/A_1B_1C_2> , \\
<ABC/A_1B_2C_3> , \\
\end{align*}
\]

The generalized polykays of degree three are again obtained from the polykays of same degree of the crossed structure by Definition 4.5. They are
\[(\text{AAA}/A_1A_1A_1) = <\text{AAA}/A_1A_1A_1> - 3<\text{AAA}/A_1A_1B_2> + 2<\text{AAA}/A_1B_1C_1> , \]

\[(\text{AAA}/A_1A_1B_1) = <\text{AAA}/A_1A_1B_1> - 2<\text{AAB}/A_1A_1B_2> - <\text{AAA}/A_1B_1C_1> + <\text{AAB}/A_1B_1C_2> , \]

\[(\text{AAB}/A_1A_1B_2) = <\text{AAB}/A_1A_1B_2> - <\text{AAB}/A_1B_1C_2> , \]

\[(\text{AAA}/A_1B_1C_1) = <\text{AAA}/A_1B_1C_1> - 3<\text{AAB}/A_1B_1C_2> + 2<\text{ABC}/A_1B_2C_3> , \]

\[(\text{AAB}/A_1B_1C_2) = <\text{AAB}/A_1B_1C_2> - <\text{ABC}/A_1B_2C_3> , \]

\[(\text{ABC}/A_1B_2C_3) = <A_1B_2C_3> . \]

The polykays \((\text{AAB}/\text{AAA}), (\text{AAB}/\text{ABA}), (\text{ABC}/\text{AAA})\) and \((\text{ABC}/\text{AAB})\) from the cross structure vanish under collapsing and hence have no corresponding polykays in the nested structure. As an example of this vanishing let us consider the polykay \((\text{AAB}/\text{AAA})\) from the crossed structure. From the previous section we have

\[(\text{AAB}/\text{AAA}) = <\text{AAB}/\text{AAA}> - <\text{AAB}/\text{AAB}> - <\text{ABC}/\text{AAA}> + 2<\text{AAB}/\text{ABB}> + 3<\text{ABC}/\text{AAB}> - 2<\text{AAB}/\text{ABA}> - 2<\text{ABC}/\text{ABC}> . \]

Now under collapsing this becomes equal to

\[
<\text{AAB}/A_1A_1B_2> - <\text{AAB}/A_1A_1B_2> - <\text{ABC}/A_1B_2C_3> \\
+ 2<\text{AAB}/A_1B_1C_2> + 3<\text{ABC}/A_1B_2C_3> - 2<\text{AAB}/A_1B_1C_2> \\
- 2<\text{ABC}/A_1B_2C_3> \\
= 0 .
\]

**G.s.m.'s and polykays of degree four** Let us now look at the g.s.m.'s and generalized polykays of degree four. Denote by \(g_1\) the g.s.m.'s and by \(G_1\) the corresponding polykays. We find the following distinct g.s.m.'s:
\[ g_{33} = <AAAA/A_1A_1A_1A_1>, \quad g_{31} = <AAAA/A_1A_1A_1B_1>, \]
\[ g_{29} = <AAAA/A_1A_1B_1B_1>, \quad g_{28} = <AAAB/A_1A_1A_1B_2>, \]
\[ g_{27} = <AABB/A_1A_1B_2B_2>, \quad g_{22} = <AAAA/A_1A_1B_1C_1>, \]
\[ g_{20} = <AAAB/A_1A_1B_1C_2>, \quad g_{18} = <AABB/A_1A_1B_2C_2>, \]
\[ g_{12} = <AABC/A_1A_1B_2C_3>, \quad g_{8} = <AAAA/A_1B_1C_1D_1>, \]
\[ g_{6} = <AAAB/A_1B_1C_1D_2>, \quad g_{4} = <AABB/A_1B_1C_2D_2>, \]
\[ g_{2} = <AABC/A_1B_1C_2D_3>, \quad g_{1} = <ABCD/A_1B_2C_3D_4>. \]

The generalized polykays of degree four for the nested structure are

\[ G_{33} = g_{33} - 4g_{31} - 3g_{29} + 12g_{22} - 6g_{8}, \]
\[ G_{31} = g_{31} - g_{28} - 3g_{22} + 3g_{20} + 2g_{8} - 2g_{6}, \]
\[ G_{29} = g_{29} - g_{27} - 2g_{22} + 2g_{18} + g_{8} - g_{4}, \]
\[ G_{28} = g_{28} - 3g_{20} + 2g_{6}, \]
\[ G_{27} = g_{27} - 2g_{18} + g_{4}, \]
\[ G_{22} = g_{22} - 2g_{20} - g_{18} + 2g_{12} - g_{6} + 2g_{4} + g_{4} - 2g_{2}, \]
\[ G_{20} = g_{20} - g_{12} - g_{6} + g_{2}, \]
\[ G_{18} = g_{18} - g_{12} - g_{4} + g_{2}, \]
\[ G_{12} = g_{12} - g_{2}, \]
\[ G_{8} = g_{8} - 4g_{6} - 3g_{4} + 12g_{2} - 6g_{1}, \]
\[ G_{6} = g_{6} - 3g_{2} + 2g_{1}, \]
\[ G_{4} = g_{4} - 2g_{2} + g_{1}, \]
\[ G_{2} = g_{2} - g_{1}, \]
\[ G_{1} = g_{1}. \]
The polykays $F_{32}'$, $F_{30}'$, $F_{26}'$, $F_{25}'$, $F_{24}'$, $F_{23}'$, $F_{21}'$, $F_{19}'$, $F_{17}'$, $F_{16}'$, $F_{15}'$
$F_{14}'$, $F_{13}'$, $F_{11}'$, $F_{10}'$, $F_{9}'$, $F_{7}'$, $F_{5}'$ and $F_{3}'$ vanish under collapsing and hence corresponding polykays for the nested structure are not defined.
VII. POLYKAYS FOR THREE-FACTOR STRUCTURES

In this chapter we will examine the distinct g.s.m.'s and polykays for the five possible balanced three-factor structures. One structure is the completely crossed structure which we shall call structure U and the other four are balanced structures involving different nesting and crossing of the three factors, these structures being denoted by the symbols V, W, X, Z.

A. Structure U

In this situation all three factors, A, B, and C are completely crossed. The population identity is

\[
Y_{ijk} = \mu + (Y_i - \mu) + (Y_j - \mu) + (Y_k - \mu) + (Y_{ij} - Y_i - Y_j + \mu) + (Y_{ik} - Y_i - Y_k + \mu) + (Y_{jk} - Y_j - Y_k + \mu) + (Y_{ijk} - Y_i - Y_j - Y_k + \mu) + (Y_{i} - \mu) + (Y_{j} - \mu) + (Y_{k} - \mu) + (Y_{ij} - Y_i - Y_j + \mu) + (Y_{ik} - Y_i - Y_k + \mu) + (Y_{jk} - Y_j - Y_k + \mu) + (Y_{ijk} - Y_i - Y_j - Y_k + \mu),
\]

where \( I = 1, \ldots, A; J = 1, \ldots, B; K = 1, \ldots, C \). The population structural diagram is

```
    u
   /|
  /  |
A  B
 / |
/  |
C
```

\[ u \]

\[ e \]
G.s.m.'s and polykays of degree two

The distinct g.s.m.'s of degree two are

\[ <AA/AA/AA>, \quad <AB/AA/AA> , \]
\[ <AA/AB/AA>, \quad <AA/AA/AB> , \]
\[ <AB/AB/AA>, \quad <AB/AA/AB> , \]
\[ <AA/AB/AB>, \quad <AB/AB/AB> . \]

These are found easily by application of either Definition 4.1 or 5.2.

The generalized polykays of degree two, which may be obtained directly from Definition 5.1, are

\[ (AA/AA/AA) = <AA/AA/AA> - <AA/AA/AB> - <AA/AB/AA> \]
\[ \quad - <AB/AA/AA> + <AA/AB/AB> + <AB/AA/AB> + <AB/AB/AA> - <AB/AB/AB> , \]
\[ (AB/AA/AA) = <AB/AA/AA> - <AB/AA/AB> - <AB/AB/AA> \]
\[ \quad + <AB/AB/AB> , \]
\[ (AA/AB/AA) = <AA/AB/AA> - <AA/AB/AB> - <AB/AB/AA> \]
\[ \quad + <AB/AB/AB> , \]
\[ (AA/AA/AB) = <AA/AA/AB> - <AA/AB/AB> - <AB/AA/AB> \]
\[ \quad + <AB/AB/AB> , \]
\[ (AB/AB/AA) = <AB/AB/AA> - <AB/AB/AB> , \]
\[ (AB/AA/AB) = <AB/AA/AB> - <AB/AB/AB> , \]
\[ (AA/AB/AB) = <AA/AB/AB> - <AB/AB/AB> , \]
\[ (AB/AB/AB) = <AB/AB/AB> . \]
These polykavs correspond to $\Sigma_{ABC}$, $\Sigma_{BC}$, $\Sigma_{AC}$, $\Sigma_{AB}$, $\Sigma_C$, $\Sigma_D$, $\Sigma_A$ and $\Sigma_G$ respectively.

**G.s.m.'s and polykays of degree three**

Employing Definition 4.1 directly we find the following distinct g.s.m.'s of degree three. (Because of the relatively large number of g.s.m.'s and polykays we shall denote the g.s.m.'s by $b_i$ and the corresponding polykays by $B_i$.)

\[
\begin{align*}
    b_{37} &= \langle AAA/AAA/AAA \rangle, & b_{36} &= \langle AAA/AAB/AAA \rangle, \\
    b_{35} &= \langle AAB/AAA/AAA \rangle, & b_{34} &= \langle AAA/AAA/AAB \rangle, \\
    b_{33} &= \langle AAB/ABA/AAA \rangle, & b_{32} &= \langle AAB/AAB/AAA \rangle, \\
    b_{31} &= \langle AAB/AAA/ABA \rangle, & b_{30} &= \langle AAB/AAA/AAB \rangle, \\
    b_{29} &= \langle AAA/ABA/AAA \rangle, & b_{28} &= \langle AAA/AAB/AAB \rangle, \\
    b_{27} &= \langle AAB/ABA/ABB \rangle, & b_{26} &= \langle AAB/AAB/ABA \rangle, \\
    b_{25} &= \langle AAB/ABA/ABA \rangle, & b_{24} &= \langle AAB/ABA/AAB \rangle, \\
    b_{23} &= \langle AAB/AAB/AAB \rangle, & b_{22} &= \langle ABC/AAA/AAA \rangle, \\
    b_{21} &= \langle AAA/ABC/AAA \rangle, & b_{20} &= \langle AAA/AAA/ABC \rangle, \\
    b_{19} &= \langle AAB/ABC/AAA \rangle, & b_{18} &= \langle ABC/AAB/AAA \rangle, \\
    b_{17} &= \langle ABC/AAA/AAB \rangle, & b_{16} &= \langle AAA/ABC/AAB \rangle, \\
    b_{15} &= \langle AAB/AAA/ABC \rangle, & b_{14} &= \langle AAA/AAB/ABC \rangle, \\
    b_{13} &= \langle AAB/ABC/ABA \rangle, & b_{12} &= \langle AAB/ABC/AAB \rangle, \\
    b_{11} &= \langle ABC/AAB/ABA \rangle, & b_{10} &= \langle ABC/AAB/AAB \rangle, \\
    b_{9} &= \langle AAB/ABA/ABC \rangle, & b_{8} &= \langle AAB/AAB/ABC \rangle,
\end{align*}
\]
The generalized polykays of degree three are

\[
B_{37} = \begin{array}{c}
    b_{37} - 3b_{36} - 3b_{35} - 3b_{34} + 6b_{33} + 3b_{32} + 6b_{31} + 3b_{30} + 6b_{29} + 3b_{28} - 6b_{27} \\
    -6b_{26} - 6b_{25} - 6b_{24} - 3b_{23} + 2b_{22} + 2b_{21} + 2b_{20} - 6b_{19} - 6b_{18} - 6b_{17} \\
    -6b_{16} - 6b_{15} - 6b_{14} - 12b_{13} + 6b_{12} + 12b_{11} + 6b_{10} + 12b_{9} + 6b_{8} + 4b_{7} \\
    + 4b_{6} + 4b_{5} - 12b_{4} - 2b_{3} - 2b_{2} + 8b_{1} \\
\end{array}
\]

\[
B_{36} = \begin{array}{c}
    b_{36} - 2b_{33} - b_{32} - 2b_{29} - b_{28} + 2b_{27} + 2b_{26} + 2b_{25} + 2b_{24} + b_{23} - b_{21} + 3b_{19} \\
    + 2b_{18} + 3b_{16} + 2b_{14} - 6b_{13} - 3b_{12} - 4b_{11} - 2b_{10} - 4b_{9} - 2b_{8} - 2b_{7} - 2b_{5} \\
    + 6b_{4} + 6b_{3} + 4b_{2} - 4b_{1} \\
\end{array}
\]

\[
B_{35} = \begin{array}{c}
    b_{35} - 2b_{33} - b_{32} - 2b_{31} - b_{30} + 2b_{27} + 2b_{26} + 2b_{25} + 2b_{24} + b_{23} - b_{22} + 2b_{19} \\
    + 3b_{18} + 3b_{17} + 2b_{15} - 4b_{13} - 2b_{12} - 6b_{11} - 3b_{10} - 4b_{9} - 2b_{8} - 2b_{7} - 2b_{6} \\
    + 6b_{4} + 6b_{3} + 4b_{2} - 4b_{1} \\
\end{array}
\]

\[
B_{34} = \begin{array}{c}
    b_{34} - 2b_{31} - b_{30} - 2b_{29} - b_{28} + 2b_{27} + 2b_{26} + 2b_{25} + 2b_{24} + b_{23} - b_{20} + 2b_{17} \\
    + 2b_{16} + 3b_{15} + 3b_{14} - 4b_{13} - 2b_{12} - 4b_{11} - 2b_{10} - 6b_{9} - 3b_{8} - 2b_{6} - 2b_{5} \\
    + 4b_{4} + 6b_{3} + 6b_{2} - 4b_{1} \\
\end{array}
\]

\[
B_{33} = \begin{array}{c}
    b_{33} - b_{27} - b_{25} - b_{24} - b_{19} - b_{18} + 2b_{13} + b_{12} + 2b_{11} + b_{10} + 2b_{9} + b_{7} - 3b_{4} \\
    - 2b_{3} - 2b_{2} + 2b_{1} \\
\end{array}
\]

\[
B_{32} = \begin{array}{c}
    b_{32} - 2b_{26} - b_{23} - b_{19} + 2b_{13} + b_{12} + 2b_{11} + b_{10} + 2b_{8} + b_{7} - 3b_{4} \\
    - 2b_{3} - 2b_{2} + 2b_{1} \\
\end{array}
\]
\[ B_{31} = b_{31} - b_{27} - b_{26} - b_{25} - b_{17} - b_{15} + 2b_{13} + 2b_{11} + b_{10} + 2b_9 + b_8 + b_6 - 2b_4 - 3b_3 - 2b_2 + 2b_1, \]
\[ B_{30} = b_{30} - 2b_{24} - b_{23} - b_{17} - b_{15} + 2b_{12} + 2b_{11} + b_{10} + 2b_9 + b_8 + b_6 - 2b_4 - 3b_3 - 2b_2 + 2b_1, \]
\[ B_{29} = b_{29} - b_{27} - b_{26} - b_{24} - b_{16} - b_{14} + 2b_{13} + b_{12} + 2b_{11} + 2b_9 + b_8 + b_5 - 2b_4 - 2b_3 - 3b_2 + 2b_1, \]
\[ B_{28} = b_{28} - 2b_{25} - b_{24} - b_{16} - b_{14} + 2b_{13} + b_{12} + 2b_{10} + 2b_9 + b_8 + b_5 - 2b_4 - 2b_3 - 3b_2 + 2b_1, \]
\[ B_{27} = b_{27} - b_{13} - b_{11} - b_9 + b_4 + b_3 + b_2 - b_1, \]
\[ B_{26} = b_{26} - b_{13} - b_{11} - b_9 + b_4 + b_3 + b_2 - b_1, \]
\[ B_{25} = b_{25} - b_{13} - b_{10} - b_9 + b_4 + b_3 + b_2 - b_1, \]
\[ B_{24} = b_{24} - b_{12} - b_{11} - b_9 + b_4 + b_3 + b_2 - b_1, \]
\[ B_{23} = b_{23} - b_{12} - b_{10} - b_9 + b_4 + b_3 + b_2 - b_1, \]
\[ B_{22} = b_{22} - 3b_{18} - 3b_{17} + 6b_{16} + 3b_{15} + 3b_{10} + 2b_7 + 2b_6 - 6b_4 - 6b_3 + 4b_1, \]
\[ B_{21} = b_{21} - 3b_{19} - 3b_{18} + 6b_{17} + 3b_{16} + 3b_{12} + 2b_7 + 2b_5 - 6b_4 - 6b_2 + 4b_1, \]
\[ B_{20} = b_{20} - 3b_{15} - 3b_{14} + 6b_9 + 3b_8 + 2b_6 + 2b_5 - 6b_3 - 6b_2 + 4b_1, \]
\[ B_{19} = b_{19} - 2b_{13} - b_{12} - b_7 + 3b_4 + 2b_2 - 2b_1, \]
\[ B_{18} = b_{18} - b_{11} - b_{10} - b_7 + 3b_4 + 2b_3 - 2b_1, \]
\[ B_{17} = b_{17} - 2b_{11} - b_{10} - b_6 + 2b_4 + 3b_3 - 2b_1, \]
\[ B_{16} = b_{16} - 2b_{13} - b_{12} - b_5 + 2b_4 + 3b_2 - 2b_1, \]
\[ B_{15} = b_{15} - 2b_{11} - b_8 + 3b_3 + 2b_2 - 2b_1, \]
\[ B_{14} = b_{14} - 2b_9 - b_5 + 2b_3 + 3b_2 - 2b_1, \]
\[ B_{13} = b_{13} - b_4 - b_2 + b_1, \]
\[ B_{12} = b_{12} - b_4 - b_2 + b_1, \]
\[ B_{11} = b_{11} - b_4 - b_3 + b_1, \]
\[ B_{10} = b_{10} - b_4 - b_3 + b_1, \]
\[ B_9 = b_9 - b_3 - b_2 + b_1, \]
\[ B_8 = b_8 - b_3 - b_2 + b_1, \]
\[ B_7 = b_7 - 3b_4 + 2b_1, \]
\[ B_6 = b_6 - 3b_3 + 2b_1, \]
\[ B_5 = b_5 - 3b_2 + 2b_1, \]
\[ B_4 = b_4 - b_1, \]
\[ B_3 = b_3 - b_1, \]
\[ B_2 = b_2 - b_1, \]
\[ B_1 = b_1. \]

**G.s.m.'s and polykays of degree four**

Obtaining the distinct g.s.m.'s of degree four from Definition 4.1 proved much too tedious to be done by simple enumeration as in all the previous cases. In view of this a computer program was developed which enumerated all possible g.s.m.'s then checked these for identity by a process like that discussed in Chapter 4, i.e., by permuting columns of the name matrix, then re-lettering from left to right. In all, since there are 15 simple polykays \(<AAAA>, \ldots, <ABCD>\) of degree four, there are \(15^3\) possible g.s.m.'s of which 285 were found to be distinct. Because of
the space required to present the 285 g.s.m.'s, they are included in Appendix 1.

In order to generate the polykays of degree four for the completely crossed structure, another computer program was developed which, in essence, performed the appropriate dot-multiplication, checked each resulting g.s.m. for identity with the other g.s.m.'s and collected all like g.s.m.'s with the appropriate coefficients. The presentation of all these polykays would require a prohibitive amount of space in this thesis because of the number of lengthy equations involved, and hence the author feels justified in presenting only a few representative polykays, viz., $U_{285}$, $U_{201}$, $U_{41}$, and $U_{5}$ to illustrate the nature of these functions. These also are included in Appendix 1. Copies of the full set of equations will be made and will be available to anyone interested at request.

B. Structure V

In this structure we assume factor C is nested in factor B and factor B is nested in factor A. The population identity is

$$Y_{ijk} = Y + (Y_i - Y) + (Y_{i(j)} - Y_i) + (Y_{ij(k)} - Y_i(j))$$

$$= u + a_i + a(B)_{ij} + AB(C)_{ijk},$$

where $i=1, \ldots, A$; $j$ has a range of $B$ units; $C$ has a range of $C$ units.

The structural diagram is
G.s.m.'s and polykays of degree two

The distinct g.s.m.'s of degree two are obtainable from Definitions 4.2 or 5.2. They are

<AA/A_1A_1/A_1A_1>,
<AA/A_1A_1/A_1B_1>,
<AA/A_1B_1/A_1B_2>,
<AB/A_1B_2/A_1B_2>.

The polykays, obtainable from the polykays of the crossed structure or directly from Definition 5.3, are

\[
\begin{align*}
(AA/A_1A_1/A_1A_1) &= <AA/A_1A_1/A_1A_1> - <AA/A_1A_1/A_1B_1>, \\
(AA/A_1A_1/A_1B_1) &= <AA/A_1A_1/A_1B_1> - <AA/A_1B_1/A_1B_2>, \\
(AA/A_1B_1/A_1B_2) &= <AA/A_1B_1/A_1B_2> - <AB/A_1B_2/A_1B_2>, \\
(AB/A_1B_2/A_1B_2) &= <AB/A_1B_2/A_1B_2>.
\end{align*}
\]

These polykays correspond to $\sum_{AB(C)}$, $\sum_{A(B)}$, $\sum_A$, and $\sum_B$ respectively.

G.s.m.'s and polykays of degree three

The distinct g.s.m.'s of degree three are found by composing the additional restrictions of Definition 4.2 on the g.s.m.'s of the completely
crossed structure. Let \( m_i \) denote the g.s.m.'s and \( M_i \) the corresponding polykays. We have the following distinct g.s.m.'s:

\[
\begin{align*}
m_{37} & = <AAA/A_1A_1A_1/A_1A_1A_1>, & m_{34} & = <AAA/A_1A_1A_1/A_1A_1B_1>, \\
m_{28} & = <AAA/A_1A_1B_1/A_1A_1B_2>, & m_{23} & = <AAB/A_1A_1B_2/A_1A_1B_2>, \\
m_{20} & = <AAA/A_1A_1A_1/A_1B_1C_1>, & m_{14} & = <AAA/A_1A_1B_1/A_1B_1C_2>, \\
m_{8} & = <AAB/A_1A_1B_2/A_1B_1C_2>, & m_{5} & = <AAA/A_1B_1C_1/A_1B_2C_3>, \\
m_{2} & = <AAB/A_1B_1C_2/A_1B_2C_3>, & m_{1} & = <ABC/A_1B_2C_3/A_1B_2C_3>.
\end{align*}
\]

Under collapsing, the polykays of the crossed structure yield the following polykays:

\[
\begin{align*}
M_{37} & = m_{37} - 3m_{34} + 2m_{20}, \\
M_{34} & = m_{34} - m_{28} - m_{20} + m_{14}, \\
M_{28} & = m_{28} - m_{14} + m_{8} - m_{1}, \\
M_{23} & = m_{23} - m_{8}, \\
M_{20} & = m_{20} - 3m_{14} + 2m_{5}, \\
M_{14} & = m_{14} - m_{8} - m_{5} + m_{2}, \\
M_{8} & = m_{8} - m_{2}, \\
M_{5} & = m_{5} - 3m_{2} + 2m_{1}, \\
M_{2} & = m_{2} - m_{1}, \\
M_{1} & = m_{1}.
\end{align*}
\]
G.s.m.'s and polykays of degree four

The g.s.m.'s obtained by imposing the additional restrictions of Definition 4.2 on the g.s.m.'s of the crossed structure are included in Appendix 2. In addition, the polykays $V_{285}$, $V_{201}$, $V_{41}$ and $V_{5}$, obtained from $U_{285}$, $U_{201}$, $U_{41}$ and $U_{5}$ under collapsing, are included in this appendix.

Another computer program, included as a subroutine of the crossed polykay generating program, was used to replace each g.s.m. of the crossed polykay by its corresponding g.s.m. for structure $V$ thus obtaining the collapsed polykays. Such a program was used to obtain the fourth degree polykays of all the structures $V$, $W$, $X$ and $Z$.

C. Structure W

In this structure we assume factors $A$ and $B$ are crossed and factor $C$ is nested in the combination of levels of $A$ and $B$. The structural diagram is

\[
\begin{align*}
  &u \\
  &\downarrow A \\
  &\downarrow B \\
  &\downarrow C \\
  &\varepsilon
\end{align*}
\]

and the population identity is written as

\[
Y_{IJK} = Y + (Y_I - Y) + (Y_J - Y) + (Y_{IJ} - Y_I - Y_J + Y) + (Y_{IJK} - Y_{IJ})
\]

\[
= \mu + A_I + B_J + (AB)_{IJ} + AB(C)_{IJK}
\]
where $i = 1, \ldots, A; \ j = 1, \ldots, B$, and $K$ has the range of $C$ units.

**G.s.m.'s and polykays of degree two**

The distinct g.s.m.'s of degree two are found to be

\[
\begin{align*}
&<AA/AA/A_1A_1>, \\
&<AA/AA/A_1B_1>, \\
&<AB/AA/A_1B_2>, \\
&<AA/AB/A_1B_2>, \\
&<AB/AB/A_1B_2>.
\end{align*}
\]

The polykays of degree two which are obtainable by the collapsing of the polykays from the completely crossed structure are found to be

\[
\begin{align*}
(AA/AA/A_1A_1) &= <AA/AA/A_1A_1> - <AA/AA/A_1B_1>, \\
(AA/AA/A_1B_1) &= <AA/AA/A_1B_1> - <AA/AB/A_1B_2> - <AB/AA/A_1B_2> + <AB/AB/A_1B_2>, \\
(AB/AA/A_1B_2) &= <AB/AA/A_1B_2> - <AB/AB/A_1B_2>, \\
(AB/AB/A_1B_2) &= <AB/AB/A_1B_2>. \\
\end{align*}
\]

These polykays correspond to $\Sigma_{AB(C)}$, $\Gamma_{AB}$, $\Sigma_A$, and $\Sigma_B$ respectively.

**G.s.m.'s and polykays of degree three**

Let us denote the g.s.m.'s of degree three for this structure by $p_i$ and the polykays by $P_i$. We have the following distinct g.s.m.'s:
The generalized polykays of degree three expressed in terms of the g.s.m.'s are

\[
\begin{align*}
p_{37} &= \langle AAA/AAA/A_1A_1A_1 \rangle, \\
p_{34} &= \langle AAA/AAA/A_1A_1B_1 \rangle, \\
p_{30} &= \langle AAB/AAA/A_1A_1B_2 \rangle, \\
p_{28} &= \langle AAA/AAB/A_1A_1B_2 \rangle, \\
p_{27} &= \langle AAB/ABA/A_1B_2B_3 \rangle, \\
p_{23} &= \langle AAB/ABA/A_1A_1B_1 \rangle, \\
p_{20} &= \langle AAA/AAA/A_1B_1C_1 \rangle, \\
p_{15} &= \langle AAB/AAA/A_1B_1C_2 \rangle, \\
p_{14} &= \langle AAA/AAB/A_1B_1C_2 \rangle, \\
p_{9} &= \langle AAB/ABA/A_1B_2C_3 \rangle, \\
p_{8} &= \langle AAB/AAB/A_1B_2C_2 \rangle, \\
p_{6} &= \langle ABC/AAA/A_1B_2C_3 \rangle, \\
p_{5} &= \langle AAA/ABC/A_1B_2C_3 \rangle, \\
p_{3} &= \langle ABC/AAB/A_1B_2C_3 \rangle, \\
p_{2} &= \langle AAB/ABC/A_1B_2C_3 \rangle, \\
p_{1} &= \langle ABC/ABC/A_1B_2C_3 \rangle.
\end{align*}
\]
\[ P_5 = p_5 - 3p_2 + 2p_1 , \]
\[ P_3 = p_3 - p_1 , \]
\[ P_2 = p_2 - p_1 , \]
\[ P_1 = p_1 . \]

**G.s.m.'s and polykays of degree four**

The distinct g.s.m.'s of degree four, denoted by the symbol \( w_1 \), as well as the polykays \( W_{285}, W_{201}, W_{41} \) and \( W_5 \) are included in Appendix 3.

**D. Structure X**

In this situation we envisage factors A and B crossed and factor C nested in the levels of factor A only. The structural diagram is

\[
\begin{align*}
&\begin{array}{c}
A \\
B \\
C \\
\end{array} \\
&\begin{array}{c}
A \\
B \\
C \\
\end{array}
\end{align*}
\]

The population identity is

\[
Y_{ijk} = \Delta + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta)
\]

\[
+ (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta) + (Y - \Delta)
\]

\[
= \Delta + \alpha_i + \beta_j + \gamma_{ij} + \alpha_{i(k)} + \beta_{i(k)} + \gamma_{ijk} .
\]
The distinct g.s.m.'s of degree two are obtainable from Definitions 4.2 or 5.2. These g.s.m.'s are

\[<AA/AA/A_1A_1>,\]
\[<AA/AB/A_1A_1>,\]
\[<AA/AA/A_1B_1>,\]
\[<AB/AA/A_1B_2>,\]
\[<AA/AB/A_1B_1>,\]
\[<AB/AB/A_1B_2>.\]

The polykays of degree two for this structure, which as usual are obtained from either Definition 5.3 or from the polykays of the crossed structure, are

\[(AA/AA/A_1A_1) = \left<AA/AA/A_1A_1\right> - \left<AA/AA/A_1B_1\right> - \left<AA/AB/A_1A_1\right> + \left<AA/AB/A_1B_1\right>,\]
\[(AA/AB/A_1A_1) = \left<AA/AB/A_1A_1\right> - \left<AA/AB/A_1B_1\right>,\]
\[(AA/AA/A_1B_1) = \left<AA/AA/A_1B_1\right> - \left<AA/AB/A_1B_1\right> - \left<AB/AA/A_1B_2\right> + \left<AB/AB/A_1B_2\right>,\]
\[(AB/AA/A_1B_2) = \left<AB/AA/A_1B_2\right> - \left<AB/AB/A_1B_2\right>,\]
\[(AA/AB/A_1B_1) = \left<AA/AB/A_1B_1\right> - \left<AB/AB/A_1B_2\right>,\]
\[(AB/AB/A_1B_2) = \left<AB/AB/A_1B_2\right>.\]

These polykays are seen to be equivalent to the functions \(\Sigma_{A(BC)}', \Sigma_{A(C)}', \Sigma_{AB}', \Sigma_{A'}\) and \(\Sigma_{\emptyset}\) respectively.
G.s.m.'s and polykays of degree three

Denoting the g.s.m.'s of degree three by \( q_i \) and the polykays of degree three by \( Q_j \), we obtain distinct g.s.m.'s for this structure,

\[
\begin{align*}
q_{37} &= <\text{AAA/AAA/A}_1A_1A_1>, & q_{36} &= <\text{AAA/AAB/A}_1A_1A_1>, \\
q_{34} &= <\text{AAA/AAA/A}_1A_1B_1>, & q_{33} &= <\text{AAB/ABA/A}_1A_1A_2>, \\
q_{30} &= <\text{AAB/AAA/A}_1A_1B_2>, & q_{29} &= <\text{AAA/AAB/A}_1B_1A_1>, \\
q_{28} &= <\text{AAA/AAB/A}_1A_1B_1>, & q_{24} &= <\text{AAB/ABA/A}_1A_1B_2>, \\
q_{23} &= <\text{AAB/AAB/A}_1B_1C_1>, & q_{21} &= <\text{AAA/ABC/A}_1A_1A_1>, \\
q_{20} &= <\text{AAA/AAA/A}_1B_1C_1>, & q_{16} &= <\text{AAA/ABC/A}_1A_1B_1>, \\
q_{15} &= <\text{AAB/AAA/A}_1B_1C_2>, & q_{14} &= <\text{AAA/AAB/A}_1B_1C_1>, \\
q_{12} &= <\text{AAB/ABC/A}_1A_1B_2>, & q_{9} &= <\text{AAB/ABA/A}_1B_1C_2>, \\
q_{8} &= <\text{AAB/AAB/A}_1B_1C_2>, & q_{6} &= <\text{ABC/AAA/A}_1B_2C_3>, \\
q_{5} &= <\text{AAB/ABC/A}_1B_1C_1>, & q_{3} &= <\text{ABC/ABA/A}_1B_2C_3>, \\
q_{2} &= <\text{AAB/ABC/A}_1B_1C_2>, & q_{1} &= <\text{ABC/ABC/A}_1B_2C_3>,
\end{align*}
\]

and the polykays

\[
\begin{align*}
Q_{37} &= q_{37}^2 + 3q_{36} - 3q_{34} - 6q_{29} + 3q_{28} + 2q_{21} + 2q_{20} - 6q_{16} + 6q_{14} + 4q_5, \\
Q_{36} &= q_{36}^2 - 2q_{29} - q_{29} - q_{28} - q_{21} + 3q_{16} + 2q_{14} - 2q_5, \\
Q_{34} &= q_{34}^2 - q_{30} - 2q_{29} - q_{28} + 2q_{24} + q_{23} - q_{20} + 2q_{16} + q_{15} + 3q_{14} - 2q_{12} - 2q_9 \\
& - q_8 - 2q_5 + 2q_2, \\
Q_{33} &= q_{33}^2 - q_{24}, \\
Q_{30} &= q_{30}^2 - 2q_{24} - q_{23} - q_{15} + 2q_{12} + 2q_9 + q_8 - 2q_2, \\
Q_{29} &= q_{29}^2 - q_{24} - q_{16} - q_1 + q_{12} + q_9 + q_5 - q_2.
\end{align*}
\]
\[ Q_{28} = q_{28} - q_{16} - q_{14} + q_{12} + q_{8} - q_{6} + q_{5} + q_{2}, \]
\[ Q_{24} = q_{24} - q_{12} - q_{9} + q_{2}, \]
\[ Q_{23} = q_{23} - q_{12} - q_{8} + q_{2}, \]
\[ Q_{21} = q_{21} - 3q_{15} - 2q_{5}, \]
\[ Q_{20} = q_{20} - 3q_{15} - 3q_{14} + 6q_{9} + 3q_{8} + 2q_{6} + 2q_{5} - 6q_{3} - 6q_{2} + 4q_{1}, \]
\[ Q_{16} = q_{16} - q_{12} - q_{5} + q_{2}, \]
\[ Q_{15} = q_{15} - 2q_{9} - q_{8} - q_{5} + 2q_{3} + 3q_{2} - 2q_{1}, \]
\[ Q_{14} = q_{14} - 2q_{9} - q_{8} - q_{5} + 2q_{3} + 3q_{2} - 2q_{1}, \]
\[ Q_{12} = q_{12} - q_{2}, \]
\[ Q_{9} = q_{9} - q_{3} - q_{2} + q_{1}, \]
\[ Q_{8} = q_{8} - q_{3} - q_{2} + q_{1}, \]
\[ Q_{6} = q_{6} - 3q_{3} + 2q_{1}, \]
\[ Q_{5} = q_{5} - 3q_{2} + 2q_{1}, \]
\[ Q_{3} = q_{3} - q_{1}, \]
\[ Q_{2} = q_{2} - q_{1}, \]
\[ Q_{1} = q_{1}. \]

**G.s.m.'s and polykays of degree four**

The g.s.m.'s of degree four, denoted by \( x_{1} \), and the polykays \( X_{285} \), \( X_{201} \), \( X_{41} \) and \( X_{5} \) are presented in Appendix 4.

**E. Structure Z**

In this structure factors B and C are crossed but each is nested in factor A. The structural diagram is
The population identity is written as

\[ Y_{IJK} = Y + (Y_I - Y) + (Y_J - Y) + (Y_K - Y) + (Y_{IK} - Y_I) - Y_{IK} \]

\[ = u + A_I + A(B)_{JJ} + A(C)_{KK} + A(BC)_{IJJK}. \]

G.s.m.'s and polykays of degree two

The distinct g.s.m.'s of degree two are

\[ <AA/A_1A_1/A_1A_1>, \]
\[ <AA/A_1B_1/A_1A_1>, \]
\[ <AA/A_1A_1/A_1B_1>, \]
\[ <AA/A_1B_1/A_1B_1>, \]
\[ <AB/A_1B_2/A_1B_2>. \]

The generalized polykays, corresponding to \( \Sigma_{A(BC)'} \), \( \Sigma_{A(C)'} \), and \( \Sigma_A' \) respectively, are

\[ (AA/A_1A_1/A_1A_1) = <AA/A_1A_1/A_1A_1> - <AA/A_1A_1/A_1B_1> - <AA/A_1B_1/A_1A_1> + <AA/A_1B_1/A_1B_1>. \]
(AA/A_1B_1/A_1A_1) = <AA/A_1B_1/A_1A_1> - <AA/A_1B_1/A_1B_1>.

(AA/A_1A_1/A_1B_1) = <AA/A_1A_1/A_1B_1> - <AA/A_1B_1/A_1A_1>.

(AB/A_1B_2/A_1B_2) = <AB/A_1B_2/A_1B_2>.

**G.s.m.'s and polykays of degree three**

Let us denote the distinct g.s.m.'s of degree three by the symbols \( r_i \) and the corresponding polykays by \( R_i \). Employing Definition 4.2 we obtain the following g.s.m.'s:

\[
\begin{align*}
\ r_{37} &= <AAA/A_1A_1A_1/A_1A_1A_1>, \\
\ r_{34} &= <AAA/A_1A_1A_1/A_1A_1B_1>, \\
\ r_{28} &= <AAA/A_1A_1B_1/A_1A_1B_1>, \\
\ r_{21} &= <AAA/A_1B_1C_1/A_1A_1A_1>, \\
\ r_{16} &= <AAA/A_1B_1C_1/A_1A_1B_1>, \\
\ r_{12} &= <AAB/A_1B_1C_2/A_1A_1B_2>, \\
\ r_{5} &= <AAA/A_1B_1C_1/A_1B_1C_1>, \\
\ r_{1} &= <ABC/A_1B_2C_3/A_1B_2C_3>.
\end{align*}
\]

The polykays of degree three are

\[
\begin{align*}
\ R_{37} &= r_{37}^{3}r_{36}^{-3}r_{34}^{-3}r_{29}^{-6}r_{28}^{3}r_{21}^{2}r_{20}^{2}r_{16}^{-6}r_{14}^{6}r_{12}^{4}r_{5}, \\
\ R_{36} &= r_{36}^{2}r_{29}^{-2}r_{28}^{2}r_{21}^{2}r_{16}^{2}r_{14}^{-4}r_{5}, \\
\ R_{34} &= r_{34}^{2}r_{29}^{-2}r_{28}^{2}r_{21}^{2}r_{16}^{2}r_{14}^{-4}r_{5}, \\
\ R_{29} &= r_{29}^{2}r_{16}^{2}r_{14}^{4}r_{5}, \\
\ R_{28} &= r_{28}^{2}r_{16}^{2}r_{14}^{4}r_{12}^{2}r_{8}^{2}r_{5}^{2}r_{2}^{2}r_{1}, \\
\ R_{23} &= r_{23}^{2}r_{12}^{-2}r_{8}^{2}r_{5}^{2}r_{2}^{2}.
\end{align*}
\]
\[ R_{21} = r_{21} - 3r_{16} + 2r_5, \]
\[ R_{20} = r_{20} - 3r_{14} + 2r_5, \]
\[ R_{16} = r_{16} - r_{12} - r_5 + r_2, \]
\[ R_{14} = r_{14} - r_8 + r_5 + r_2, \]
\[ R_{12} = r_{12} - r_2, \]
\[ R_8 = r_8 - r_2, \]
\[ R_5 = r_5 - 3r_2 + 2r_1, \]
\[ R_2 = r_2 - r_1, \]
\[ R_1 = r_1. \]

G.s.m.'s and polykays of degree four

The distinct g.s.m.'s of degree four, denoted by the symbol \( z_4 \), as well as the polykays \( Z_{285}', Z_{201}', Z_{41}' \) and \( Z_5' \), are included in Appendix 5.
VIII. EXPANSION OF POWERS OF SAMPLE MEANS
IN TERMS OF POLYKAYS

A. Introduction

An interesting result observed in the process of this current research, and one which gave considerable encouragement to work with polykays rather than some other function of symmetric means, is that when sample means from different populations are raised to a given power, say $p$, and expressed as linear functions of symmetric means, the expressions appearing as coefficients of terms of the form

$$\frac{1}{\text{(product of sample ranges of subscripts)}}$$

are in fact polykays, or generalized polykays, of degree $p$. Specifically, the expressions obtained from the square of a sample mean are generalized polykays of degree two or as was shown in Chapter 5, the $\Sigma$'s. Zyskind (1958) dealt with the second degree expansion of admissible means in terms of $\Sigma$'s and White (1963) gave expansions of products of certain admissible means in terms of $\Sigma$'s.

B. Samples from One-Factor Structures

As an illustration of these findings, consider a simple random sample $\{x_i\}, i = 1, \ldots, n$, from a known population $\{X_I\}, I = 1, \ldots, N$. Then
\[ x. = \frac{\sum_{i=1}^{n} x_i}{n} = \langle 1 \rangle = k_1, \text{ a sample polykay,} \]

\[ x.^2 = \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2 = \frac{1}{n} \left[ \sum x_i^2 + \sum x_i x_i \right] = \frac{1}{n} \langle 2 \rangle + \frac{n-1}{n} \langle 11 \rangle \]

\[ = \frac{1}{n} [\langle 2 \rangle - \langle 11 \rangle] + \langle 11 \rangle \]

\[ = \frac{1}{n} k_2 + k_{11} \]

\[ x.^3 = \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^3 = \frac{1}{n} \left[ \sum x_i^3 + 3 \sum x_i^2 x_i + \sum x_i x_i x_i \right] \]

\[ = \frac{1}{n} \langle 3 \rangle + \frac{3}{2} \frac{(n-1)}{n} \langle 21 \rangle + \frac{(n-1)(n-2)}{n^2} \langle 111 \rangle \]

\[ = \frac{1}{n} [\langle 3 \rangle - 3 \langle 21 \rangle + 2 \langle 111 \rangle] + \frac{3}{n} [\langle 21 \rangle - \langle 111 \rangle] + \langle 111 \rangle \]

\[ = \frac{1}{n^2} k_3 + \frac{3}{n} k_{21} + k_{111} \]

Similarly we find

\[ x.^4 = \frac{1}{n^3} k_4 + \frac{4}{n^2} k_{31} + \frac{3}{n^2} k_{22} + \frac{6}{n} k_{2111} + k_{111} \]

and
\[ x^5 = \frac{1}{n} k_5 + \frac{5}{n^3} k_{41} + \frac{10}{n^3} k_{32} + \frac{10}{n^2} k_{311} + \frac{15}{n^2} k_{211} + \frac{10}{n} k_{1111} \]

The following general expression is suggested from these examples:

\[ x^t = \sum_{n=\rho}^{t} \frac{t!}{n^{t-\rho}} \left( \frac{1}{p_1!} \cdots \frac{1}{p_s!} \right) \]

where

\[ p_1 \pi_1 + p_2 \pi_2 + \cdots + p_s \pi_s = t, \]

and

\[ \rho = \pi_1 + \pi_2 + \cdots + \pi_s, \]

and the sum is over all partitions of t subject to the above restrictions.

Since the polykays are inherited on the average the above expressions would yield the t-th moment of the mean about the origin if the sample polykays are replaced by the corresponding population polykays. An induction proof of this relationship has been tried but found very difficult. Further consideration of this proof does not seem justifiable at this time. A result similar to the above but for central moments of the mean was proved by Barton and David (1961).
Two-way crossed structure

Let us now consider selecting a random sample \( \{x_{ij}\} \), \( i = 1, \ldots, a \); \( j = 1, \ldots, b \), from a two-way crossed population. We find

\[
x_{\cdots} = \frac{\Sigma x_{ij}}{ab} = \frac{a}{a}
\]

\[
x_{\cdots}^2 = \left( \frac{\Sigma x_{ij}}{ab} \right)^2
\]

\[
= \frac{1}{ab} \left( \Sigma x_{ij}^2 + \Sigma x_{ij} x_{ij} + \Sigma x_{ij} x_{ij} + \Sigma x_{ij} x_{ij} \right)
\]

\[
= \frac{1}{ab} \left[ <aa/aa> + (b-1)<aa/ab> + (a-1)<ab/aa> + (a-1)(b-1)<ab/ab> \right]
\]

\[
= \frac{1}{ab} \left[ <aa/aa>- <aa/ab>- <ab/aa> + <ab/ab> \right]
\]

\[
+ \frac{1}{a} \left[ <aa/ab>- <ab/ab> \right] + \frac{1}{b} \left[ <ab/aa>- <ab/ab> \right]
\]

\[
+ <ab/ab>
\]

\[
= \frac{1}{ab} (aa/aa) + \frac{1}{a} (aa/ab) + \frac{1}{b} (ab/aa) + (ab/ab),
\]

where small letters replace capital letters in the names of the polykays to distinguish sample polykays from population polykays. Thus the coefficients of the terms involving sample size are in this case generalized.
polykays of degree two (bipolykays) instead of simple polykays. These
polykays are of course the functions $\Sigma_{ab}$, $\Sigma_a$, $\Sigma_b$ and $\Sigma_\emptyset$ respectively.

In this respect the $\Sigma$'s with small letters for subscripts are defined exactly
as in Definition 3.4, replacing population values by sample values.

Taking now the third power of $x$, we obtain

$x^3 = \left( \sum_{i,j} x_{ij} \right)^3$

$$= \frac{1}{a^3 b^3} \left( \sum_{i,j} x_{ij}^3 + 3 \sum_{i,j} x_{ij} x_{i'j} + 3 \sum_{i,j} x_{ij} x_{ij'} + 3 \sum_{i,j} x_{ij} x_{ij'} + 3 \sum_{i,j} x_{ij} x_{ij'} \\
+ \sum_{i,j} x_{ij} x_{ij} x_{ij} + \sum_{i,j} x_{ij} x_{ij} x_{ij} + 3 \sum_{i,j} x_{ij} x_{ij} x_{ij} \\
+ 3 \sum_{i,j} x_{ij} x_{ij} x_{ij} + 6 \sum_{i,j} x_{ij} x_{ij} x_{ij} + \sum_{i,j} x_{ij} x_{ij} x_{ij} \\
+ 3 \sum_{i,j} x_{ij} x_{ij} x_{ij} + (a-1)(a-2) <aaa/abc> + (b-1)(b-2) <aab/abc> \right)$$

$$= \frac{1}{a^2 b^2} \left( <aaa/aaa> + 3(a-1) <aab/aaa> + 3(b-1) <aaa/aab> \\
+ 3(a-b)(b-1) <aab/aab> + (a-1)(a-2) <abc/aaa> \\
+ (b-1)(b-2) <aaa/abc> + 3(a-1)(a-2)(b-1) <abc/aab> \\
+ 3(a-1)(b-1)(b-2) <aab/abc> + 6(a-1)(b-1) <aab/aba> \\
+ (a-1)(a-2)(b-1)(b-2) <abc/abc> \right)$$

$$= \frac{1}{a^2 b^2} \left( <aaa/aaa> - 3 <aab/aaa> - 3 <aaa/aab> \\
+ 3 <aab/aab> + 6 <aab/aba> + 2 <aaa/abc> + 2 <abc/aaa> \right)$$
\(-6<\text{abc/aab}> - 6<\text{aab/abc}> + 4<\text{abc/abc}>\)

\[+ \frac{3}{a^2b} (\text{aaa/aab} - 2\text{aab/aab} - \text{aaa/abc} + 2\text{abc/aab})\]

\[+ 3\text{aab/abc} - 2\text{aab/aba} - 2\text{abc/abc}\]

\[+ \frac{3}{ab^2} (\text{aab/aab} - \text{aab/aaa} - 3\text{abc/aaa} + 2\text{aab/abc})\]

\[+ 3\text{abc/aab} - 2\text{aab/aba} - 3\text{abc/abc}\]

\[+ \frac{3}{ab} (\text{aab/aab} - \text{abc/abc} - \text{aab/abc} + \text{abc/abc})\]

\[+ \frac{2}{ab} (\text{aab/aba} - \text{abc/abc} - \text{aab/abc} + \text{abc/abc})\]

\[+ \frac{1}{a^2} (\text{aaa/abc} - 3\text{aab/abc} + 2\text{abc/abc})\]

\[+ \frac{3}{b} (\text{abc/aab} - \text{abc/abc}) + \frac{3}{a} (\text{aab/abc})\]

\[\text{It is of interest to note that since there are two distinct g.s.m.'s of the form }<21/21>, \text{ viz., }<\text{aab/aab}> \text{ and } <\text{aab/aba}>, \text{ the coefficient of } 1/ab\]

\[\text{consists of two generalized polykays of degree three.}\]
The following result obtains when the mean $x_{..}$ is expanded to the fourth power:

$$x_{..}^4 = \frac{1}{a^3 b^3} (aaaa/aaaa) + \frac{1}{a^3 b^2} [4(aaaa/aaab) + 3(aaaa/aabb)]$$

$$+ \frac{6}{a^3 b} (aaaa/aabc) + \frac{1}{a^3} (aaaa/abcd) + \frac{1}{a^2 b^3} (aaab/aaaa)$$

$$+ 3(aabb/aaaa) + \frac{6}{ab} (aabc/aaaa) + \frac{1}{b^3} (abcd/aaaa)$$

$$+ \frac{1}{a^2 b^2} [4(aaab/aaab) + 12(aaab/aaba) + 12(aabb/aaab)$$

$$+ 12(aaab/aabb) + 3(aabb/aabb) + 6(aabb/abab)]$$

$$+ \frac{1}{ab} [12(aabc/aaab) + 12(aabc/abaa) + 6(aabc/aabb)$$

$$+ 12(aabc/abab)]$$

$$+ \frac{1}{a^2 b} [12(aaab/aabc) + 12(aaab/abca) + 6(aabb/aabc)$$

$$+ 12(aabb/abac)]$$

$$+ \frac{1}{ab} [6(aabc/abcc) + 24(aabc/abac) + 6(aabc/aabc)]$$

$$+ \frac{1}{b^2} [4(abcd/aaab) + 3(abcd/aabb)] + \frac{6}{b} (abcd/abcd)$$

$$+ \frac{1}{a} [aaab/abcd) + 3(aabb/abcd)] + \frac{6}{a} (aabc/abcd)$$

$$+ (abcd/abcd).$$
Another fact of interest in these expansions is that the polykays sub-partitioned in all possible ways on the letters representing column subscripts, for a row partition of the form (aaaa) or (abcd), or vice versa, appear with coefficients as in the expansion of the mean in simple random sampling. For example, consider the polykays whose name contains all sub-partitions of four for rows and a constant column partition (aaaa) in the expansion of \( x^4 \). We find that the polykays appear in the form

\[
\frac{1}{3} \left[ \frac{1}{3} (aaaa/aaaa) + \frac{4}{2} (aaab/aaaa) + \frac{3}{2} (aabb/aaaa) + \frac{6}{a} (aabc/aaaa) + (abcd/aaaa) \right].
\]

Similarly the polykays whose name contains partitioning of the columns for a constant row partition (aaaa) appear in the form

\[
\frac{1}{3} \left[ (aaaa/aaaa) + \frac{4}{b} (aaaa/aaab) + \frac{3}{b} (aaaa/aabb) + \frac{6}{b} (aaaa/aabc) + (aaaa/abcd) \right].
\]

Both forms are analogous to the expansion

\[
x^4 = \frac{1}{n^3} k_4 + \frac{4}{n^2} k_{31} + \frac{3}{n^2} k_{22} + \frac{6}{n} k_{211} + k_{1111}.
\]

Similar results obtain where the constant row or column partition is of the form (abcd). If any other constant partition is considered this simple result does not follow but instead more complex form arise, a nice general pattern of which is not apparent.
Two-factor nested structure

Let \( \{x_{ij}\} \), where \( i = 1, \ldots, a \) and \( j \) has a range of \( b \) units, denote the sample observations from a two-factor structure in which factor B is nested in factor A. Expanding the sample mean of these observations yields results similar to those of the previous section. For,

\[
x_{..} = \frac{\sum x_{ij}}{ab} = \langle a/a \rangle
\]

\[
x_{\ldots}^2 = \left( \frac{\sum x_{ij}}{ab} \right)^2
\]

\[
= \frac{1}{ab} \left[ \frac{\sum x_{ij}^2}{ab} + \frac{\sum \sum x_{ij}x_{ij'} + \sum \sum x_{ij}x_{ij'}}{ab(b-1)} + b(a-1) \frac{\sum \sum x_{ij}x_{ij'}}{ab^2(a-1)} \right]
\]

\[
= \frac{1}{ab} \left[ \langle aa/a_1a_1 \rangle + (b-1)\langle aa/a_1b_1 \rangle + b(a-1) \langle ab/a_1b_2 \rangle \right]
\]

\[
= \frac{1}{ab} \left[ \langle aa/a_1a_1 \rangle - \langle aa/a_1b_1 \rangle + \frac{1}{a} (\langle aa/a_1b_1 \rangle - \langle ab/a_1b_2 \rangle) + \langle ab/a_1b_2 \rangle \right]
\]

\[
= \frac{1}{ab} (aa/a_1a_1) + \frac{1}{a} (aa/a_1b_1) + (ab/a_1b_2)
\]

These polykays correspond to the functions \( \Sigma_a(b) \), \( \Sigma_a \) and \( \Sigma_b \) respectively.

It should be observed that the term \( 1/b \) does not occur in this expansion.

In forming the g.s.m.'s above we see that the coefficient \((a-1)(b-1)\) does not occur [instead, \( b(a-1) \)] because of the sample range of \( \sum \sum x_{ij}x_{ij'} \).
and hence the term $a/ab = 1/b$ cannot occur. Recalling the coefficient of
$1/b$ in the expansion of the two-factor crossed sample mean, viz., $(ab/aa)$
we note that this polykay vanishes under collapsing and hence there is no
Corresponding polykay in the nested structure. This is consistent with the
fact that there is no $1/b$ in this expansion. It is also consistent with the
fact that $\Sigma_b$ is not defined in the case of nesting.

Similarly, expanding the mean $x$ to the third power, we obtain

$$x^3 = \frac{1}{a_1b} (aaa/a_1^1a_1^1a_1^1) + \frac{3}{a_2b} (aaa/a_1^1a_1^1b_1^1) + \frac{3}{ab} (aab/a_1^1a_1^1b_2^1)$$

$$+ \frac{1}{a_2} (aaa/a_1^1b_1^1c_1^1) + \frac{3}{a} (aab/a_1^1b_1^1c_2^1) + (abc/a_1^1b_2^1c_3^1)$$

and for the expansion to the fourth power,

$$x^4 = \frac{1}{a_3b} (aaa/a_1^1a_1^1a_1^1a_1^1) + \frac{1}{a_2b^2} [4(aaa/a_1^1a_1^1a_1^1b_1^1)$$

$$+ 3(aaa/a_1^1a_1^1b_1^1b_1^1)] + \frac{6}{a_3b} (aaa/a_1^1a_1^1b_1^1c_1^1)$$

$$+ \frac{1}{a_3} (aaa/a_1^1b_1^1c_1^1d_1^1) + \frac{1}{a_2b^2} [4(aaab/a_1^1a_1^1a_1^1b_2^1)$$

$$+ 3(aab/a_1^1a_1^1b_2^1b_2^1)] + \frac{1}{a_2b} [12(aaab/a_1^1a_1^1b_1^1c_2^1)$$

$$+ 6(aab/a_1^1a_1^1b_2^1c_2^1)] + \frac{6}{ab} (aab/a_1^1a_1^1b_2^1c_3^1)$$

$$+ \frac{1}{a_2} [4(aaab/a_1^1b_1^1c_1^1d_1^1) + 3(aab/a_1^1b_1^1c_2^1d_2^1)]$$
+ \frac{6}{a} \left( a_{abc} / a_1 b_2 c_2 d_2 \right) + (abcd / a_1 b_2 c_3 d_4) .

D. Samples from Three-Factor Structures

Only one example will be given here, that of the expansion of the sample mean from a three-way crossed structure to the second degree. Let \( \{x_{ijk}\} \) with \( i = 1, \ldots, a; j = 1, \ldots, b; k = 1, \ldots, c \), denote the sample observations. Then

\[
x^2 = \sum_{ijk} \frac{x_{ijk}^2}{abc}
\]

\[
= \frac{1}{abc} \left[ <aa/aa/aa> + (c-1) <aa/aa/ab> + (b-1) <aa/ab/aa>
+ (a-1) <ab/aa/aa> + (b-1) <aa/ab/ab>
+ (a-1)(c-1) <ab/aa/ab> + (a-1)(b-1) <ab/ab/aa>
+ (a-1)(b-1)(c-1) <ab/ab/ab> \right]
\]

\[
= \frac{1}{ab} \left[ <aa/aa/aa> - <aa/aa/ab> - <aa/ab/aa> - <ab/aa/aa>
+ <aa/ab/ab> + <ab/aa/ab> + <ab/ab/aa> - <ab/ab/ab> \right]
\]

\[
+ \frac{1}{ac} \left[ <aa/aa/ab> - <aa/ab/ab> - <ab/aa/ab> + <ab/ab/ab> \right]
\]

\[
+ \frac{1}{bc} \left[ <aa/ab/aa> - <aa/ab/ab> - <ab/ab/aa> + <ab/ab/ab> \right]
\]
+ \frac{1}{bc} [ <ab/aa/aa> - <ab/ab/al> - <ab/aa/ab> + <ab/ab/ab> ]
+ \frac{1}{a} [ <aa/ab/ab> - <ab/ab/ab> ] + \frac{1}{b} [ <ab/aa/ab> - <ab/ab/ab> ]
+ <ab/ab/ab>

= \frac{1}{abc} (aa/aa/aa) + \frac{1}{ab} (aa/aa/ab) + \frac{1}{ac} (aa/ab/aa)
+ \frac{1}{bc} (ab/aa/aa) + \frac{1}{a} (aa/ab/ab) + \frac{1}{b} (ab/aa/ab)
+ \frac{1}{c} (ab/ab/aa) + (ab/ab/ab).

These generalized polykays of degree two are, of course, the \( \Sigma \)'s, \( \Sigma_{abc} \),
\( \Sigma_{ab} \), \( \Sigma_{ac} \), \( \Sigma_{bc} \), \( \Sigma_a \), \( \Sigma_b \), \( \Sigma_c \) and \( \Sigma_{g} \) respectively. The pattern for the
third and fourth powers of \( x \) should be obvious.

It is worth mentioning here that the expansion of the means from arbitrary two and
three-factor structures can be obtained from the expansion of the mean from the two
and three-factor completely crossed structures by first random cross labeling the
levels of the observations from the arbitrary structures, then taking the powers of
the means and finally taking the expectation over random cross labelings. This has
the effect of simply replacing the polykays from the crossed structure by the
Corresponding polykays of the arbitrary structure, if they are defined.

The generalization of the results of this chapter to many factors and
higher powers should follow similarly, though the formalization and proof
of this has not been obtained.
IX. VARIANCES AND COVARIANCES OF ESTIMATES OF COMPONENTS OF VARIATION

One application of the general results of Chapter 4 is that of obtaining the variances and covariances of estimates of components of variation from the analysis of variance of balanced sample structures. We have seen that the Σ's of Zyskind (1958), which are defined in terms of components of variation, are equivalent to generalized polykays of degree two. Working with the definition of the Σ's, we can solve for the components of variation in terms of the Σ's or polykays and hence obtain the variances and covariances of the components of variation by obtaining the variances and covariances of the sample Σ's or generalized polykays of degree two. These results will be functions of generalized polykays of degree four.

Denote by F(x) a sample polykay of degree two from a completely crossed structure. Then the variance of F(x) is

\[ E \left( F(x) \right)^2 - \left( E[F(X)] \right)^2 \]

\[ = E \left( F(x) \right)^2 - \left( F(X) \right)^2, \]

where F(X) is the population polykay corresponding to F(x). Because of the property of inheritance on the average, \([F(X)]^2\) can be obtained from \([F(x)]^2\) simply by replacing the sample polykays of degree four by the corresponding population polykays and replacing all sample values appearing as coefficients of the fourth degree polykays by corresponding population values.
Consider now a generalized sample polykay of degree two from an arbitrary structure $P$. Call this sample polykay $H(x)$. As we shall see in specific examples, under random cross labeling of sample observations from this arbitrary structure, $H(x)$ can be expressed as a linear sum of polykays of degree two from a crossed structure. Assume then that $H(x)$ is of the form

$$H(x) = \sum_{i} [F_i(x) | P']$$

where $P'$ represents the sample structure $P$ after random cross labeling (in this context $P$ may refer to either the sample or population structure) and $[F_i(x) | P']$ represents a second degree sample polykay from the crossed structure $P'$. Thus

$$EH(x)^2 = E\left\{ \sum_{i} [F_i(x) | P']^2 \right\}$$

$$= \sum_{i} E[F_i(x) | P']^2 + \sum_{i \neq i'} E[F_i(x) | P'] \cdot [F_i'(x) | P']$$

$$= \sum_{i \text{ sa lab}} E[F_i(x) | P']^2 + \sum_{i \neq i'} E[F_i(x) | P'] \cdot [F_i'(x) | P']$$

from Equation 4.3b. But the $[F_i(x) | P']^2$ and $[F_i(x) | P'] \cdot [F_i'(x) | P']$ are obtained by the multiplication formula for crossed polykays and hence the desired result is obtained by taking the expectations of these products over random cross labeling and sampling.

In this chapter, we find the variances and covariances of estimates of components of variation for the two-factor structures [the results for the
two-way crossed structure have been given previously by Hooke (1956b)]
and the five three-factor structures discussed in previous chapters.

A. Basic Products of Polykays Used

As we indicated in the previous section, we need only find the squares
and cross products of generalized polykays from the two-factor and three-
factor crossed sample structures in order to obtain the desired variances and
covariances.

From Chapter 6, we have the following distinct sample polykays of
degree two for the two-factor crossed structure:

(aa/aa),
(aa/ab),
(ab/aa),
(ab/ab).

Also, we found the distinct polykays of degree two for the three-factor
crossed structure in Chapter 7. These are

(aa/aa/aa),
(ab/aa/aa),
(aa/ab/aa),
(aa/aa/ab),
(ab/ab/aa),
(ab/aa/ab),
(aa/ab/ab),
(ab/ab/ab).
According to Corollary 4.4 of Theorem 4.4, the variances and covariances of these second degree polykays will involve dot-multiplication of the terms

\[(aa)^2, (aa)(ab), (ab)(aa), (ab)^2\]

only. Thus we first find these basic products of simple polykays. For generality, let us assume the simple polykays here are based upon a sample size of \(n\). Then we obtain

\[(aa)^2 = (<aa> - <ab>)^2 = <aa>^2 - <aa><ab> - <ab><aa> + <ab>^2.\]

The products of the simple g.s.m.'s in the last equation are given by Tukey (1956a) in a slightly different form and are modified for our purposes. These products are

\[<aa>^2 = (1 - \frac{1}{n})<aabb> + \frac{1}{n} <aaaa>,\]

\[<aa><ab> = \frac{1}{n} (<aaab> + <aaba>) + (1 - \frac{2}{n}) <aabc>,\]

\[<ab><aa> = \frac{1}{n} (<abbb> + <abaa>) + (1 - \frac{2}{n}) <abcc>,\]

\[<ab>^2 = (1 - \frac{4}{n} + \frac{2}{n(n-1)}) <abcd> + \left( \frac{1}{n} - \frac{1}{n(n-1)} \right) <abac>\]

\[+ \left( \frac{1}{n} - \frac{1}{n(n-1)} \right) <abca> + \left( \frac{1}{n} - \frac{1}{n(n-1)} \right) <abbc>\]

\[+ \left( \frac{1}{n} - \frac{1}{n(n-1)} \right) <abcb> + \frac{1}{n(n-1)} <abab> + \frac{1}{n(n-1)} <abba>\]
Collecting like terms we find the following expression for \((aa)^2\):

\[
(aa)^2 = (aabb) + \frac{1}{n} (aaaa) + \frac{1}{n-1} [(abba) + (abab)].
\]

Similarly, we obtain

\[
(aa)(ab) = (aabc) - \frac{1}{n(n-1)} [(abab) + (abba)] + \frac{1}{n} [(aaab) + (aaba)],
\]

\[
(ab)(aa) = (abcc) - \frac{1}{n(n-1)} [(abab) + (abba)] + \frac{1}{n} [(aaba) + (abbb)],
\]

and

\[
(ab)^2 = (abcd) + \frac{1}{n} [(abac) + (abca) + (abbc) + abcb)] + \frac{1}{n(n-1)} [(abab) + (abba)].
\]

To obtain the variance of \((aa/ab)\) from the two-factor crossed structure, for example, we note that

\[
\text{Var} (aa/aa) = E (aa/ab)^2 - (AA/AB)^2
\]

\[
= E (aa)^2 \cdot (ab)^2 - (AA)^2 \cdot (AB)^2.
\]

Taking the dot-product of \((aa)^2\) and \((ab)^2\), utilizing the products of simple polykays just presented, we obtain sample polykays of degree four with coefficients involving the sample ranges of the two factors. Upon taking the expectation of this product, the sample polykays are replaced by their expected values, the corresponding population polykays. The product \((AA)^2 \cdot (AB)^2\) is then obtained by replacing all coefficients involving sample
values by the corresponding population values. These results are presented in the following sections.

The analysis of variance tables presented in the following sections illustrate the simple manner in which the Σ's appear in the expected mean squares. General results and theorems pertaining to this use of the Σ's are given by Zyskind (1958) and Throckmorton (1961) and will not be elaborated on here.

B. Two-Factor Structures

Two-factor crossed structure

The population identity and structural diagram for this structure were presented in Chapter 6. The ANOVA (analysis of variance table) for the sample observations is presented in Table 1 below.

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>Σ_{AB} + b Σ_A</td>
</tr>
<tr>
<td>B</td>
<td>b-1</td>
<td>Σ_{AB} + a Σ_B</td>
</tr>
<tr>
<td>AB</td>
<td>(a-1)(b-1)</td>
<td>Σ_{AB}</td>
</tr>
</tbody>
</table>

The use of the capital letters to denote the source should be self-explanatory. The Σ's are defined as
\[ \Sigma_{AB} = \sigma_{AB}^2, \]
\[ \Sigma_A = \sigma_A^2 - \frac{1}{B} \sigma_{AB}^2, \]
\[ \Sigma_{AB}' = \sigma_{AB}'^2 - \frac{1}{A} \sigma_{AB}^2. \]

Solving for the \( \sigma^2 \)'s in terms of \( \Sigma \)'s we obtain

\[ \sigma_{AB}^2 = \Sigma_{AB}, \]
\[ \sigma_A^2 = \Sigma_A + \frac{1}{B} \Sigma_{AB}, \]
\[ \sigma_{AB}'^2 = \Sigma_{AB}' + \frac{1}{A} \Sigma_{AB}. \]

The estimates of the \( \sigma^2 \)'s are obtained by replacing the population \( \Sigma \)'s by the sample \( \Sigma \)'s. Thus

\[ \text{var} (\hat{\sigma}_{AB}^2) = \text{var} (\Sigma_{ab}), \]
\[ \text{var} (\hat{\sigma}_A^2) = \text{var} (\Sigma_A) + \frac{1}{2} \text{var} (\Sigma_{ab}) + \frac{2}{B} \text{cov} (\Sigma_A, \Sigma_{ab}), \]
\[ \text{var} (\hat{\sigma}_B^2) = \text{var} (\Sigma_B) + \frac{1}{2} \text{var} (\Sigma_{ab}) + \frac{2}{A} \text{cov} (\Sigma_B, \Sigma_{ab}), \]
\[ \text{cov} (\hat{\sigma}_A^2, \hat{\sigma}_{AB}'^2) = \text{cov} (\Sigma_A, \Sigma_{ab}) + \frac{1}{B} \text{var} (\Sigma_{ab}), \]
\[ \text{cov} (\hat{\sigma}_B^2, \hat{\sigma}_{AB}'^2) = \text{cov} (\Sigma_B, \Sigma_{ab}) + \frac{1}{A} \text{var} (\Sigma_{ab}), \]
\[ \text{cov} (\hat{\sigma}_A^2, \hat{\sigma}_B^2) = \text{cov} (\Sigma_A, \Sigma_B) + \frac{1}{A} \text{cov} (\Sigma_A, \Sigma_{ab}) + \frac{1}{B} \text{cov} (\Sigma_B, \Sigma_{ab}) \]
\[ + \frac{1}{AB} \text{var} (\Sigma_{ab}). \]
The variances and covariances of the $\Sigma$'s, or generalized polykays of degree two are obtained by the procedure indicated in the previous section, i.e., by replacing, for example, $\Sigma_{ab}$ by its equivalent form $(aa/aa)$ and obtaining the variance of this form utilizing the basic product $(aa)^2$ given earlier.

The following results are then obtained:

\[
\text{Var} (\Sigma_a) = \left( \frac{2}{a-1} - \frac{2}{A-1} \right) F_4 + \left( \frac{1}{a} - \frac{1}{A} \right) F_8 + \left[ \left( \frac{4}{b} - \frac{4}{B} \right) \right] F_{13}
\]

\[
+ \left[ \left( \frac{2}{b(a-1)} - \frac{2}{B(A-1)} \right) \right] F_{17}
\]

\[
+ \left[ \left( \frac{2}{b(b-1)} - \frac{2}{B(B-1)} \right) + \left( \frac{2}{b(a-1)(b-1)} - \frac{2}{B(A-1)(B-1)} \right) \right] F_{17}
\]

\[
+ \left( \frac{4}{b(a-1)} - \frac{4}{B(A-1)} \right) F_{18} \left( \frac{4}{ab} - \frac{4}{AB} \right) F_{22}
\]

\[
+ \left( \frac{2}{b(a-1)(b-1)} - \frac{2}{B(A-1)(B-1)} \right) F_{27}
\]

\[
+ \left( \frac{2}{ab(b-1)} - \frac{2}{AB(B-1)} \right) F_{29}
\]

\[
\text{Var} (\Sigma_b) = \left( \frac{2}{b-1} - \frac{2}{B-1} \right) F_5 + \left( \frac{1}{b} - \frac{1}{B} \right) F_9
\]

\[
+ \left[ \left( \frac{4}{a} - \frac{4}{A} \right) + \left( \frac{2}{a(b-1)} - \frac{2}{A(B-1)} \right) \right] F_{14}
\]

\[
+ \left[ \left( \frac{2}{a(a-1)} - \frac{2}{A(A-1)} \right) + \left( \frac{2}{a(a-1)(b-1)} - \frac{2}{A(A-1)(B-1)} \right) \right] F_{17}
\]

\[
+ \left( \frac{4}{a(b-1)} - \frac{4}{A(B-1)} \right) F_{19} \left( \frac{4}{ab} - \frac{4}{AB} \right) F_{23}
\]
\[ V_{\text{ar}}(\Sigma_{ab}) = \left[ \left( \frac{2}{a-1} - \frac{2}{A-1} \right) + \left( \frac{2}{b-1} - \frac{2}{B-1} \right) + \left( \frac{2}{(a-1)(b-1)} - \frac{2}{(A-1)(B-1)} \right) \right] F_{17} + \left( \frac{2}{(a-1)(b-1)} - \frac{2}{(A-1)(B-1)} \right) F_{27} \]

\[ + \left[ \left( \frac{1}{a} - \frac{1}{A} \right) + \left( \frac{2}{a(b-1)} - \frac{2}{A(B-1)} \right) \right] F_{29} + \left[ \left( \frac{1}{b} - \frac{1}{B} \right) + \left( \frac{2}{b(a-1)} - \frac{2}{B(A-1)} \right) \right] F_{30} + \left( \frac{1}{ab} - \frac{1}{AB} \right) F_{33} \]

\[ C_{\text{ov}}(\Sigma_a, \Sigma_b) = -\left( \frac{2}{a(a-1)} - \frac{2}{A(A-1)} \right) F_{13} - \left( \frac{2}{b(b-1)} - \frac{2}{B(B-1)} \right) F_{14} + \left( \frac{2}{a} - \frac{2}{A} \right) F_{15} + \left( \frac{2}{b} - \frac{2}{B} \right) F_{16} + \left( \frac{2}{ab(a-1)(b-1)} - \frac{2}{AB(A-1)(B-1)} \right) F_{17} - \left( \frac{4}{ab(b-1)} - \frac{4}{AB(B-1)} \right) F_{24} - \left( \frac{4}{ab(a-1)} - \frac{4}{AB(A-1)} \right) F_{25} + \left( \frac{4}{ab} - \frac{4}{AB} \right) F_{26} + \left( \frac{2}{ab(a-1)(b-1)} - \frac{2}{AB(A-1)(B-1)} \right) F_{27} \]

\[ C_{\text{ov}}(\Sigma_a, \Sigma_{ab}) = \left( \frac{2}{a-1} - \frac{2}{A-1} \right) F_{13} - \left[ \left( \frac{2}{b(a-1)(b-1)} - \frac{2}{B(A-1)(B-1)} \right) \right] F_{17} + \left( \frac{1}{a} - \frac{1}{A} \right) F_{22} + \left[ \left( \frac{2}{b} - \frac{2}{B} \right) + \left( \frac{4}{b(a-1)} - \frac{4}{B(A-1)} \right) \right] F_{25} - \left( \frac{2}{b(a-1)(b-1)} - \frac{2}{B(A-1)(B-1)} \right) F_{27} - \left( \frac{2}{ab(b-1)} - \frac{2}{AB(B-1)} \right) F_{29} + \left( \frac{2}{ab} - \frac{2}{AB} \right) F_{31} \]
\[ \text{Cov} (\Sigma_b, F_{ab}) = \left( \frac{2}{b-1} - \frac{2}{B-1} \right) F_{14} \left[ \frac{2}{a(a-1)(b-1)} - \frac{2}{A(A-1)(B-1)} \right] + \left( \frac{2}{b(b-1)} - \frac{2}{B(B-1)} \right) F_{17} + \left( \frac{1}{b} - \frac{1}{B} \right) F_{23} + \left( \frac{2}{a} - \frac{2}{A} \right) F_{27} + \left( \frac{2}{a(b-1)} - \frac{2}{A(B-1)} \right) F_{24} - \left( \frac{2}{a(a-1)(b-1)} - \frac{2}{A(A-1)(B-1)} \right) F_{27} - \left( \frac{2}{a(b(a-1))} - \frac{2}{A B(A-1)} \right) F_{30} + \left( \frac{2}{a b} - \frac{2}{A B} \right) F_{32}, \]

when the \( F_i \) denote the generalized population polykays of degree four and are defined explicitly in Appendix 1. It should be noted here that these results have also been given by Hooke (1956a). Unbiased estimates of these variances and covariances may be obtained by replacing the population polykays by the sample polykays though in practice this may not be the most efficient way. Indeed, additional work needs to be done on the actual computing of the generalized polykays. The variances and covariances of the \( \sigma^2 \)'s may now be obtained by simple substitution. Because the resulting expressions are long and do not appear to simplify the variances and covariances of the \( \sigma^2 \)'s will not be exhibited explicitly.

Two-factor nested structure

The ANOVA for the sample observations from this structure is presented in Table 2.
Table 2. ANOVA (two-factor nested structure)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>$\bar{\Sigma}_{A(B)} + b\Sigma_A$</td>
</tr>
<tr>
<td>A(B)</td>
<td>a(b-1)</td>
<td>$\Sigma_A(B)$</td>
</tr>
</tbody>
</table>

The $\Sigma$'s are defined as

$$
\Sigma_{A(B)} = \sigma_{AB}^2,
$$

$$
\Sigma_A = \sigma_A^2 - \frac{1}{B} \sigma_{A(B)}^2.
$$

Thus

$$
\sigma_{A(B)}^2 = \Sigma_{A(B)},
$$

$$
\sigma_A^2 = \Sigma_A + \frac{1}{B} \Sigma_{A(B)}
$$

and

$$
\text{Var}(\hat{\sigma}_{A(B)}^2) = \text{Var}(\Sigma_a(b))
$$

$$
\text{Var}(\hat{\sigma}_A^2) = \text{Var}(\Sigma_a) + \frac{1}{B} \text{Cov}(\Sigma_a, \Sigma_a(b)) + \frac{1}{B^2} \text{Var}(\Sigma_a(b))
$$

$$
\text{Cov}(\hat{\sigma}_A^2, \hat{\sigma}_{A(B)}^2) = \text{Cov}(\Sigma_a, \Sigma_a(b)) + \frac{1}{B} \text{Var}(\Sigma_a(b)).
$$

To obtain the variances and covariances of the $\Sigma$'s in this situation we first note that under random cross labeling we can write

$$
\Sigma_a = \Sigma_a',
$$

$$
\Sigma_a(b) = \Sigma_b' + \Sigma_{ab}'.
$$
where the primes denote $\Sigma'$s for a crossed structure. This result is easily verified by observing that the sum of squares for $A(B)$ is actually a pooled sum of squares (under cross labeling of subscripts) i.e.,

$$S.S. A(B) = S.S. B' + S.S. AB',$$

where $S.S.$ stands for sum of squares. Representing the sums of squares by sample $\Sigma'$s leads to the desired result.

Now the variances can be taken on these $\Sigma'$s from the crossed population, which have already been obtained in the previous section. The variances and covariances of the $\Sigma'$s from the nested structure are then obtained by taking the expectation of the variances and covariances of the $\Sigma'$s from the crossed structure over random cross labeling. This, of course, is equivalent to replacing the fourth degree polykays of the crossed structure by the corresponding polykays of the nested structure. Thus

\[
V_{\text{ar}}(\Sigma_{a}(b)) = \mathbb{E}_{\text{lab}} \left[ V_{\text{ar}}(\Sigma'_{b}) + V_{\text{ar}}(\Sigma'_{ab}) + 2C_{\text{ov}}(\Sigma'_{b}, \Sigma'_{ab}) \right],
\]

\[
V_{\text{ar}}(\Sigma_{a}) = \mathbb{E}_{\text{lab}} (\Sigma'_{a})
\]

\[
C_{\text{ov}}(\Sigma_{a}', \Sigma_{a}(b)) = \mathbb{E}_{\text{lab}} \left[ C_{\text{ov}}(\Sigma'_{a}, \Sigma'_{b}) + C_{\text{ov}}(\Sigma'_{a}, \Sigma'_{ab}) \right].
\]

Performing the indicated operation leads to the following:

\[
V_{\text{ar}}(\Sigma_{a}(b)) = \left( \frac{2}{a(b-1)} - \frac{2}{A(B-1)} \right) G_{27} + \left[ \frac{1}{a} - \frac{1}{A} \right] + \left( \frac{2}{a(b-1)} - \frac{2}{A(B-1)} \right) G_{29}
\]

\[
+ \left( \frac{1}{ab} - \frac{1}{AB} \right) G_{33},
\]
\[ V_{\text{ar}}(\Sigma_a) = \left( \frac{2}{a-1} - \frac{2}{A-1} \right) G_4 + \left( \frac{1}{a} - \frac{1}{A} \right) G_8 + \left( \frac{4}{b(a-1)} - \frac{4}{B(A-1)} \right) G_{18} \]

\[ + \left( \frac{4}{ab} - \frac{4}{AB} \right) G_{22} + \left( \frac{2}{b(a-1)(b-1)} - \frac{2}{B(A-1)(B-1)} \right) G_{27} \]

\[ + \left( \frac{2}{ab(b-1)} - \frac{2}{AB(B-1)} \right) G_{29} . \]

\[ C_{\text{ov}}(\Sigma_a, \Sigma_a(b)) = \left( \frac{1}{a} - \frac{1}{A} \right) G_{22} - \left( \frac{2}{ab(b-1)} - \frac{2}{AB(B-1)} \right) G_{27} \]

\[ - \left( \frac{2}{ab(b-1)} - \frac{2}{AB(B-1)} \right) G_{29} + \left( \frac{2}{ab} - \frac{2}{AB} \right) G_{31} . \]

where, of course, the \( G_i \) represent generalized polykays of degree four for the two-factor nested structure and are defined explicitly in Chapter 6.

C. Three-Factor Structures

Structure U

The population identity and structural diagram for structure U, the completely crossed three-factor structure, were given in Chapter 7. The ANOVA table is presented in the following table:
### Table 3. ANOVA (structure U)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>$\Sigma_{ABC}^2 + b\Sigma_{AC}^2 + c\Sigma_{AB}^2 + bc\Sigma_A$</td>
</tr>
<tr>
<td>B</td>
<td>b-1</td>
<td>$\Sigma_{ABC}^2 + a\Sigma_{BC}^2 + c\Sigma_{AB}^2 + ac\Sigma_B$</td>
</tr>
<tr>
<td>C</td>
<td>c-1</td>
<td>$\Sigma_{ABC}^2 + b\Sigma_{AC}^2 + a\Sigma_{BC}^2 + ab\Sigma_C$</td>
</tr>
<tr>
<td>AB</td>
<td>(a-1)(b-1)</td>
<td>$\Sigma_{ABC}^2 + c\Sigma_{AB}^2$</td>
</tr>
<tr>
<td>AC</td>
<td>(a-1)(c-1)</td>
<td>$\Sigma_{ABC}^2 + b\Sigma_{AC}^2$</td>
</tr>
<tr>
<td>BC</td>
<td>(b-1)(c-1)</td>
<td>$\Sigma_{ABC}^2 + a\Sigma_{BC}^2$</td>
</tr>
<tr>
<td>ABC</td>
<td>(a-1)(b-1)(c-1)</td>
<td>$\Sigma_{ABC}$</td>
</tr>
</tbody>
</table>

The $\Sigma$'s are defined as

$$
\Sigma_{ABC}^2 = \sigma_{ABC}^2 \\
\Sigma_{AB}^2 = \sigma_{AB}^2 - \frac{1}{C} \sigma_{ABC}^2 \\
\Sigma_{AC}^2 = \sigma_{AC}^2 - \frac{1}{B} \sigma_{ABC}^2 \\
\Sigma_{BC}^2 = \sigma_{BC}^2 - \frac{1}{A} \sigma_{ABC}^2 \\
\Sigma_A^2 = \sigma_B^2 - \frac{1}{A} \sigma_{AB}^2 - \frac{1}{C} \sigma_{AC}^2 + \frac{1}{BC} \sigma_{ABC}^2 \\
\Sigma_B^2 = \sigma_C^2 - \frac{1}{A} \sigma_{AB}^2 - \frac{1}{C} \sigma_{BC}^2 + \frac{1}{AC} \sigma_{ABC}^2 \\
\Sigma_C^2 = \sigma_A^2 - \frac{1}{A} \sigma_{AC}^2 - \frac{1}{B} \sigma_{BC}^2 + \frac{1}{AB} \sigma_{ABC}^2
$$

and the $\sigma^2$'s are found to be

$$
\sigma_{ABC}^2 = \Sigma_{ABC}^2
$$
The products needed for obtaining covariances are easily obtained from these equations but are omitted here to conserve space. The variances and covariances of the estimates of these $\sigma^2$'s, obtained as before by replacing the population $\Sigma$'s by sample $\Sigma$'s, are found by obtaining the variances and covariances of the constituent sample $\Sigma$'s.

The variances and covariances of the sample $\Sigma$'s are found as in the two-factor case by appropriate dot-multiplication of the forms $(aa)^2$, $(aa)(ab)$, $(ab)(aa)$ and $(ab)^2$. By symmetry of this dot-multiplication the variances of $\Sigma_b$ and $\Sigma_c$ are obtainable from $V_{ar}(\Sigma_a)$ by interchanging the groups of letters for factors B and A, and factors C and A respectively in the names of the fourth degree polykays and by interchanging the same letters in the coefficients. For example, one of the terms in $V_{ar}(\Sigma_a)$ is

$$\left(\frac{2}{bc(b-1)(c-1)} - \frac{2}{BC(B-1)(C-1)}\right) \text{(AABB/ABAB/ABBA)},$$

and the corresponding term in $V_{ar}(\Sigma_b)$ is found to be...
by interchanging the role of the factors A and B. The last polykay would ordinarily be rewritten as (AABB/ABAB/ABBA) by permuting the second and third columns of the name of this polykay when expressed in matrix form. In fact, it can be easily seen that all variances and covariances of the $\Sigma$'s can be obtained by symmetry from the variances and covariances given below.

\[
\begin{align*}
V_{ar}(\Sigma_a), & V_{ar}(\Sigma_{ab}), V_{ar}(\Sigma_{abc}), C_{ov}(\Sigma_a, \Sigma_b), C_{ov}(\Sigma_a, \Sigma_{ab}), \\
C_{ov}(\Sigma_a, \Sigma_{bc}), & C_{ov}(\Sigma_a, \Sigma_{abc}), C_{ov}(\Sigma_{ab}, \Sigma_{abc}).
\end{align*}
\]

Only these variances and covariances will be presented.

If we examine the basic products used in the dot-multiplication to obtain the variances and covariances of the second degree polykays from the crossed structure, we observe that the resulting fourth degree polykays must have coefficients containing the factors

\[
1, \frac{1}{n}, \frac{1}{n-1}, \frac{1}{n(n-1)}
\]

where $n$ refers to the sample range of a particular subscript, and corresponding factors involving population ranges. In view of this we adopt the following notation for the coefficients appearing in the variance formulas.

Let

\[
A_{i,j,k} = \left( \frac{1}{n_1n_2n_3} - \frac{1}{N_1N_2N_3} \right).
\]
where \( a_1 = 1, \ldots, 4; \ j = 1, \ldots, 4; \ k = 1, \ldots, 4 \) and

\[
 n_1 = \begin{cases} 
1 & \text{if } i = 1 \\
\frac{1}{a} & \text{if } i = 2 \\
\frac{1}{a-1} & \text{if } i = 3 \\
\frac{1}{a(a-1)} & \text{if } i = 4 
\end{cases} \quad n_2 = \begin{cases} 
1 & \text{if } j = 1 \\
\frac{1}{b} & \text{if } j = 2 \\
\frac{1}{b-1} & \text{if } j = 3 \\
\frac{1}{b(b-1)} & \text{if } j = 4 
\end{cases} \quad n_3 = \begin{cases} 
1 & \text{if } k = 1 \\
\frac{1}{c} & \text{if } k = 2 \\
\frac{1}{c-1} & \text{if } k = 3 \\
\frac{1}{c(c-1)} & \text{if } k = 4 
\end{cases} 
\]

The \( \mathbf{N}_\alpha, \alpha = 1, 2, 3, \) are defined similarly in terms of capital letters. Thus, for example,

\[
a_{134} = \left( \frac{1}{c(b-1)(c-1)} - \frac{1}{c(B-1)(C-1)} \right). 
\]

Upon appropriate dot-multiplication of the basic products used in obtaining the variances and covariances of generalized polykays of degree two we obtain the following variances and covariances of the relevant \( \Sigma \)'s:

\[
\text{Var}(\Sigma_{abc}) = 2a_{333}(U_{83} + U_{84} + U_{85} + U_{86}) + 2a_{113}(U_{84} + U_{87}) + 2a_{131}(U_{85} + U_{87}) + 2a_{311}(U_{86} + U_{87}) + a_{121}(U_{237} + U_{238}) + a_{211}(U_{237} + U_{238}) + 2a_{233}(U_{237} + U_{238}) + 2a_{313}(U_{237} + U_{238}) + a_{231}(U_{239} + U_{240}) + 2a_{321}(U_{241} + U_{242}) + 2a_{312}(U_{241} + U_{242}) + a_{112}(U_{241} + U_{242}) + 2a_{322}(U_{281} + U_{285}) + a_{221}(U_{279} + U_{279} + U_{280} + U_{280} + U_{122} + U_{281}) + a_{122}(U_{281} + a_{222}(U_{281} + a_{222}(U_{285}) \\
\text{Var}(\Sigma_{ab}) = 2a_{331}(U_{62} + U_{63}) + 2a_{131}(U_{63} + U_{63}) + 2a_{311}(U_{63} + U_{63}) + 4a_{322}(U_{68} + U_{69} + U_{70} + U_{71}) + 4a_{112}(U_{69} + U_{70} + U_{72}) + 4a_{312}(U_{70} + U_{72}) + 4a_{132}(U_{71} + U_{72})
\]
\[ V_{ar}(\Sigma_a) = 4a_{121}(U_{29} + U_{30}) + 4a_{121}U_{30} + 2a_{311}U_{31} + 4a_{312}(U_{31} + U_{32}) + 4a_{112}U_{32} + 4a_{322}(U_{41} + U_{42} + U_{43} + U_{44} + U_{45} + U_{46} + U_{47}) + 4a_{122}(U_{45} + 2U_{46} + U_{47}) + 2a_{341}(U_{62} + U_{63}) + 2a_{414}U_{63} + 2a_{114}U_{65} + 4a_{142}(U_{71} + U_{72}) + 4a_{124}(U_{76} + U_{77}) + 2a_{144}(U_{86} + U_{87}) + 4a_{221}U_{204} + a_{211}U_{205} + 4a_{212}+U_{205} + 4a_{222}(U_{210} + 2U_{211} + U_{212}) + 2a_{241}U_{219} + 2a_{214}U_{220} + 4a_{242}(U_{225} + U_{226}) + 4a_{224}(U_{227} + U_{228}) + 2a_{444}(U_{237} + U_{238}) \]

\[ C_{ov}(\Sigma_a, \Sigma_b) = 2a_{121}(U_{6} + U_{7}) + 2a_{112}(U_{14} + U_{15}) + 4a_{122}(U_{18} + U_{19} + U_{20} + U_{21}) - 2a_{411}U_{30} - 2a_{141}U_{34} - 2a_{412}(U_{45} + 2U_{46} + U_{47}) - 2a_{142}(U_{51} + 2U_{53} + U_{54}) + 2a_{114}U_{56} + 4a_{124}(U_{58} + U_{59}) + 4a_{441}(U_{62} + U_{63}) + 4a_{64} + U_{65} + U_{66} + 4a_{442}(U_{68} + U_{69} + U_{70} + U_{71}) - 4a_{414}U_{75} - 2a_{144}(U_{81} + U_{82}) - 2a_{444}(U_{83} + U_{84} + U_{85} + U_{86}) + a_{211}U_{91} + 2a_{212}(U_{105} + U_{106}) - 2a_{241}U_{127} - 4a_{421}U_{128} + 2a_{a11}U_{133} - 4a_{242}(U_{139} + U_{140}) - 4a_{422}(U_{143} + U_{144} + U_{145} + U_{146}) + 2a_{214}U_{148} - 2a_{244}(U_{181} + U_{182}) - 4a_{424}(U_{181} + U_{182}) + 4a_{224}U_{187} \]
\[ C_{ov}(\Sigma_a, \Sigma_{ab}) = 2a_{311}U_{29} + 2a_{312}(U_{41} + U_{42} + U_{43}) + 2a_{112}U_{42} - 4a_{341}U_{62} - 2a_{141}U_{63} - 2a_{142}(U_{68} + U_{69} + U_{70} + U_{71}) - 2a_{342}(U_{68} + U_{69} + U_{70} + U_{71}) - 2a_{142}(U_{68} + U_{69} + U_{70} + U_{71}) - 2a_{342}(U_{68} + U_{69} + U_{70} + U_{71}) + 2a_{121}U_{128} + 4a_{321}U_{128} + 2a_{122}(U_{143} + U_{144} + U_{145} + U_{146}) + 4a_{322}(U_{143} + U_{144} + U_{145} + U_{146}) + 2a_{124}(U_{183} + U_{184}) + 4a_{324}(U_{183} + U_{184}) + 2a_{211}U_{204} + 4a_{212}U_{211} - 2a_{241}U_{219} - 2a_{242}(U_{225} + U_{226}) + 2a_{214}U_{228} - 2a_{244}(U_{237} + U_{238}) + 2a_{221}U_{243} + 4a_{222}(U_{249} + U_{250}) + 4a_{224}U_{261}, \]

\[ C_{ov}(\Sigma_a, \Sigma_{bc}) = -2a_{411}U_{41} - 2a_{141}U_{51} - 2a_{114}U_{56} + 4a_{441}U_{68} + 4a_{212}U_{69} + 4a_{414}U_{73} + 4a_{144}U_{81} + 2a_{211}U_{103} + 2a_{121}U_{111} + 2a_{112}U_{119} - 4a_{241}U_{140} - 4a_{421}U_{143} - 4a_{214}U_{148} - 4a_{412}U_{151} - 2a_{124}(U_{155} + U_{156}) - 4a_{142}U_{161} + 4a_{221}U_{163} + 4a_{122}U_{175} + 2a_{444}(U_{189} + U_{182}) + 4a_{244}(U_{181} + U_{182}) + 4a_{244}(U_{183} + U_{184}) + 4a_{442}(U_{185} + U_{186}) - 2a_{224}(U_{187} + U_{186} + U_{189}) - 2a_{242}(U_{190} + 2U_{191} + U_{192}) - 2a_{422}(U_{193} + 2U_{194} + U_{195}) + 2a_{222}(U_{196} + U_{197} + U_{198} + U_{199}), \]

\[ C_{ov}(\Sigma_a, \Sigma_{abc}) = 2a_{311}U_{45} - 2a_{341}(U_{69} + U_{72}) - 2a_{141}U_{70} - 2a_{114}U_{74} - 2a_{314}(U_{75} + U_{77}) + 2a_{344}(U_{83} + U_{84} + U_{85} + U_{86}) + 2a_{184}(U_{86} + U_{87}) + 2a_{121}U_{143} + 4a_{321}U_{144} + 2a_{112}U_{151} - 4a_{324}(U_{183} + U_{184}) - 4a_{124}U_{184} - 4a_{142}U_{185} - 4a_{342}(U_{185} + U_{186}) + 4a_{312}U_{186} + 2a_{122}(U_{193} + U_{194}) + 2a_{322}(U_{193} + U_{195}) + 2a_{211}U_{210} - 2a_{241}(U_{225} + U_{226}) - 2a_{214}U_{228} + 2a_{244}(U_{237} + U_{238}) + 2a_{221}U_{249} + 2a_{212}U_{251} - 4a_{224}U_{261} - 4a_{242}U_{262}, \]
where the $U_i$ are the generalized polykays of degree four. As was noted in a previous chapter, these polykays are defined explicitly in terms of g.s.m.'s but only $U_{285}$, $U_{201}$, $U_{41}$ and $U_{5}$ are given in this thesis because of the vast amount of space which would be required to present the full set of polykays.

**Structure V**

The ANOVA table for this structure is given in Table 4 below.

**Table 4. ANOVA (structure V)**

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$a-1$</td>
<td>$\Sigma_{AB(C)} + c \Sigma A(B) + bc \Sigma A$</td>
</tr>
<tr>
<td>A(B)</td>
<td>$ab(c-1)$</td>
<td>$\Sigma_{AB(C)} + c \Sigma A(B)$</td>
</tr>
<tr>
<td>AB(C)</td>
<td>$ab(c-1)$</td>
<td>$\Sigma_{AB(C)}$</td>
</tr>
</tbody>
</table>

...
The $\Sigma$'s are defined as

$$
\Sigma_{AB(C)} = \sigma_{AB(C)}^2,
$$

$$
\Sigma_A(B) = \sigma_A^2 - \frac{1}{C} \sigma_{AB(C)}^2,
$$

$$
\Sigma_A = \sigma_A^2 - \frac{1}{B} \sigma_A(B),
$$

and hence the $\sigma^2$'s are

$$
\sigma_{AB(C)}^2 = \Sigma_{AB(C)},
$$

$$
\sigma_A^2 = \gamma_A(B) + \frac{1}{C} \Sigma_{AB(C)},
$$

$$
\sigma_A^2 = \Sigma_A + \frac{1}{B} \gamma_A(B) + \frac{1}{BC} \Sigma_{AB(C)}.
$$

The variances and covariances of the estimates of these components of variation are obtained in the same manner as in the two-factor nested structure, i.e., by expressing the sample $\Sigma$'s in terms of sample $\Sigma$'s from the three-factor crossed structure, utilizing the results already obtained for the latter structure, and finally replacing the fourth degree polykay for the crossed structure by the corresponding polykays for structure $V$. The necessary relationships are easily verified to be

$$
\Sigma_a = \Sigma_e',
$$

$$
\Sigma_a(b) = \Sigma_b + \Sigma_{ab}',
$$

$$
\Sigma_{ab}(c) = \Sigma_c + \Sigma_{ac} + \gamma_{bc} + \gamma_{abc}',
$$

where the primed $\Sigma$'s represent the $\gamma$'s defined for the three-factor crossed structure.
The detailed exposition of the variances and covariances of the $\Sigma$'s or second degree polykays, for structures V, W, X and Z is omitted herein because of the lengthy equations involved and because there seems to be very little structural relationship between the expressions from different structures.

Structure W

Following the order of presentation of the previous sections we give the ANOVA table for structure W.

Table 5. ANOVA (structure W)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>$\Sigma_{AB(C)} + \Sigma_{AB} + bc\Sigma_A$</td>
</tr>
<tr>
<td>B</td>
<td>b-1</td>
<td>$\Sigma_{AB(C)} + \Sigma_{AB} + ac\Sigma_B$</td>
</tr>
<tr>
<td>AB</td>
<td>(a-1)(b-1)</td>
<td>$\Sigma_{AB(C)} + \Sigma_{AB}$</td>
</tr>
<tr>
<td>AB(C)</td>
<td>ab(c-1)</td>
<td>$\Sigma_{AB(C)}$</td>
</tr>
</tbody>
</table>

The $\Sigma$'s are defined to be

$$\Sigma_{AB(C)} = \sigma_{AB(C)}^2,$$

$$\Sigma_{AB} = \sigma_{AB}^2 - \frac{1}{C} \sigma_{AB(C)}^2,$$

$$\Sigma_A = \sigma_A^2 - \frac{1}{B} \sigma_{AB}^2,$$

$$\Sigma_B = \sigma_B^2 - \frac{1}{A} \sigma_{AB}^2.$$
The $\sigma^2$'s are thus found to be

\[ \sigma_{AB(C)}^2 = \Sigma_{AB(C)} \]
\[ \sigma_{AB}^2 = \Sigma_{AB} + \frac{1}{C} \Sigma_{ABC} \]
\[ \sigma_A^2 = \Sigma_A + \frac{1}{B} \Sigma_{AB} + \frac{1}{BC} \Sigma_{ABC} \]
\[ \sigma_B^2 = \Sigma_B + \frac{1}{A} \Sigma_{AB} + \frac{1}{AC} \Sigma_{ABC} \]
\[ \sigma_{AB}^2 = \Sigma_{AB} + \frac{1}{C} \Sigma_{ABC} \]

The pertinent relationships of the sample $\Sigma$'s of structure $W$ in terms of $\Sigma$'s from the crossed structure are

\[ \Sigma_a = \Sigma'_a \]
\[ \Sigma_b = \Sigma'_b \]
\[ \Sigma_{ab} = \Sigma'_{ab} \]
\[ \Sigma_{ab(c)} = \Sigma'_c + \Sigma'_{ac} + \Sigma'_{bc} + \Sigma'_{abc} \]

We now have the basic ingredients for obtaining the desired variances and covariances.

**Structure X**

The ANOVA for structure X is given in Table 6.
Table 6. ANOVA (structure X)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>$\sum A(BC) + c\Sigma AB + b\sum A(C) + bc\Sigma A$</td>
</tr>
<tr>
<td>B</td>
<td>b-1</td>
<td>$\sum A(BC) + c\Sigma AB + ac\Sigma B$</td>
</tr>
<tr>
<td>AB</td>
<td>(a-1)(b-1)</td>
<td>$\sum A(BC) + c\Sigma AB$</td>
</tr>
<tr>
<td>A(C)</td>
<td>a(c-1)</td>
<td>$\sum A(BC) + b\sum A(C)$</td>
</tr>
<tr>
<td>A(BC)</td>
<td>a(b-1)(c-1)</td>
<td>$\sum A(BC)$</td>
</tr>
</tbody>
</table>

The $\Sigma$'s are defined as

\[
\begin{align*}
\Sigma_{A(BC)} &= \sigma_{A(BC)}^2, \\
\Sigma_{A(C)} &= \sigma_{A(C)}^2 - \frac{1}{B} \sigma_{A(BC)}^2, \\
\Sigma_{AB} &= \sigma_{AB}^2 - \frac{1}{C} \sigma_{A(BC)}^2, \\
\Sigma_{B} &= \sigma_{B}^2 - \frac{1}{A} \sigma_{AB}^2, \\
\Sigma_{A} &= \sigma_{A}^2 - \frac{1}{B} \sigma_{AB}^2 - \frac{1}{C} \sigma_{A(C)}^2 + \frac{1}{BC} \sigma_{A(BC)}^2.
\end{align*}
\]

Solving for the $\sigma^2$'s in terms of the $\Sigma$'s, we obtain

\[
\begin{align*}
\sigma_{A(BC)}^2 &= \Sigma_{A(BC)}^2, \\
\sigma_{A(C)}^2 &= \Sigma_{A(C)}^2 + \frac{1}{B} \Sigma_{A(BC)}^2, \\
\sigma_{AB}^2 &= \Sigma_{AB}^2 + \frac{1}{C} \Sigma_{A(BC)}^2, \\
\sigma_{A}^2 &= \Sigma_{A}^2 + \frac{1}{B} \Sigma_{AB}^2 + \frac{1}{C} \Sigma_{A(C)}^2 + \frac{1}{BC} \Sigma_{A(BC)}^2.
\end{align*}
\]
Upon random cross labeling the subscripts of the sample observations we obtain the following relationships required for obtaining the variances and covariances of the sample S's:

\[
\begin{align*}
\Sigma_a &= \Sigma'_a , \\
\Sigma_b &= \Sigma'_b , \\
\Sigma_{ab} &= \Sigma'_{ab} , \\
\Sigma_{a(c)} &= \Sigma'_c + \Sigma'_{ac} , \\
\Sigma_{a(bc)} &= \Sigma'_{bc} + \Sigma'_{abc} .
\end{align*}
\]

Structure Z

The ANOVA for structure Z is presented in Table 7.

Table 7. ANOVA (structure Z)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>E.M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>(\Sigma_{A(BC)} + c\Sigma_{A(B)} + b\Sigma_{A(C)} + bc\Sigma_A)</td>
</tr>
<tr>
<td>A(B)</td>
<td>a(b-1)</td>
<td>(\Sigma_{A(BC)} + c\Sigma_{A(B)})</td>
</tr>
<tr>
<td>A(C)</td>
<td>a(c-1)</td>
<td>(\Sigma_{A(BC)} + b\Sigma_{A(C)})</td>
</tr>
<tr>
<td>A(BC)</td>
<td>a(b-1)(c-1)</td>
<td>(\Sigma_{A(BC)})</td>
</tr>
</tbody>
</table>

The \(\Sigma\)'s are defined as

\[
\begin{align*}
\Sigma_{A(BC)} &= \sigma^2_{A(BC)} , \\
\Sigma_{A(C)} &= \sigma^2_{A(C)} - \frac{1}{B} \sigma^2_{A(BC)} , \\
\Sigma_{A(B)} &= \sigma^2_{A(B)} - \frac{1}{C} \sigma^2_{A(BC)} .
\end{align*}
\]
\[ \Sigma_A = \sigma_A^2 - \frac{1}{B} \sigma_{A(B)}^2 - \frac{1}{C} \sigma_{A(C)}^2 + \frac{1}{BC} \sigma_{A(BC)}^2, \]

The \( \sigma^2 \)'s in terms of the \( \Sigma \)'s are

\[ \sigma_{A(BC)}^2 = \Sigma_{A(BC)}', \]
\[ \sigma_{A(B)}^2 = \Sigma_{A(B)} + \frac{1}{C} \Sigma_{A(BC)}', \]
\[ \sigma_{A(C)}^2 = \Sigma_{A(C)} + \frac{1}{B} \Sigma_{A(BC)}', \]
\[ \sigma_A^2 = \Sigma_A + \frac{1}{B} \Sigma_{A(B)} + \frac{1}{C} \Sigma_{A(C)} + \frac{1}{BC} \Sigma_{A(BC)}'. \]

The sample \( \Sigma \)'s from structure \( Z \) expressed in terms of the \( \Sigma \)'s from the crossed structure are

\[ \Sigma_a = \Sigma'_a, \]
\[ \Sigma_{a(b)} = \Sigma'_b + \Sigma'_{ab}, \]
\[ \Sigma_{a(c)} = \Sigma'_c + \Sigma'_{ac}, \]
\[ \Sigma_{a(bc)} = \Sigma'_{bc} + \Sigma'_{abc}. \]
X. SUMMARY

In this thesis a generalization is made of the simple polykays and bipolykays to completely generalized polykays for an arbitrary balanced population or sample structure in pure random sampling situations. This general development includes the generalization of generalized symmetric means, an introduction of the concept of random cross labeling of the subscripts of observations from an arbitrary structure and finally, a general definition of the generalized polykays.

In Chapter 5 a more explicit definition of generalized symmetric means and polykays of degree two is given and the equivalence of generalized polykays of degree two and a class of functions called $\Sigma$'s is then shown.

In Chapter 6 the generalized symmetric means and polykays of degrees two, three and four for two two-factor structures are exhibited. In Chapter 7 the generalized symmetric means and polykays of degrees two and three are given for five three-factor structures. The fourth degree generalized symmetric means and some of the polykays are given in the appendices.

Presented in Chapter 8 by way of examples is an interesting observation that a sample mean from balanced structures raised to a given power can be expressed as a linear sum of generalized polykays with coefficients depending only on sample size.
The rest of the thesis is concerned with the application of the general results to obtaining the variances and covariances of estimates of components of variation in models with finite populations and with no assumptions of independence or distribution of components. These variances and covariances are exhibited for the two two-factor structures and the five three-factor structures.


XII. ACKNOWLEDGMENTS

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Acknowledgment is also to be made to the Aeronautical Research Laboratory, Air Force Research Division, who, for the most part, supported this research under contract number AF33(616)—8269.

My appreciation is also extended to Professor H. O. Hartley for his helpful suggestions during the preparation of this manuscript, and to Professor T. A. Bancroft who made it possible for me to pursue my training at Iowa State.

Finally, I would like to acknowledge the person who kindled my interest in statistics and encouraged me to return to school to pursue this degree, Professor C. B. Godbey.
**XIII. APPENDIX I**

**A. Generalized Symmetric Means of Degree Four for Structure U**

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\end{align*}
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B. Some Generalized Polykays of Degree Four for Structure U

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- u_{274} - 6u_{273} + 12u_{272} - 3u_{271} + 12u_{270} + 12u_{269} + 12u_{268} + 12u_{267} + 12u_{266} + 12u_{265} \\
+ 12u_{264} + 12u_{263} + 12u_{262} - 12u_{261} - 24u_{260} + 24u_{259} - 24u_{258} - 24u_{257} - 24u_{256} \\
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+ 48u_{145} + 48u_{144} + 24u_{143} + 24u_{142} + 48u_{141} + 48u_{140} + 24u_{139} - 24u_{138} - 24u_{137} - 24u_{136} \\
- 24u_{135} - 24u_{134} + 24u_{133} - 24u_{132} - 24u_{131} - 24u_{130} - 24u_{129} - 24u_{128} - 24u_{127} - 24u_{126} \\
- 48u_{125} - 96u_{124} - 96u_{123} - 96u_{122} - 96u_{121} - 48u_{120} - 48u_{119} - 48u_{118} - 96u_{117} - 96u_{116} \\
- 48u_{115} - 48u_{114} - 48u_{113} - 96u_{112} - 48u_{111} - 48u_{110} + 48u_{109} - 96u_{108} - 96u_{107} - 48u_{106} \\
+ 96u_{105} - 96u_{104} + 48u_{103} + 144u_{102} + 144u_{101} + 144u_{100} + 144u_{99} + 144u_{98} + 144u_{97} \\
+ 144u_{96} + 144u_{95} + 144u_{94} + 144u_{93} + 144u_{92} + 144u_{91} + 144u_{90} - 144u_{89} - 144u_{88} + 144u_{87} \\
- 6u_{86} - 6u_{85} - 6u_{84} - 3u_{83} + 24u_{82} + 24u_{81} + 24u_{80} + 24u_{79} + 12u_{78} + 24u_{77} + 24u_{76} + 24u_{75} + 24u_{74} \\
+ 12u_{73} + 24u_{72} + 24u_{71} + 24u_{70} + 24u_{69} + 12u_{68} - 36u_{67} - 18u_{66} + 36u_{65} + 18u_{64} - 36u_{63}
\[-18u_{62} + 48u_{61} - 24u_{60} - 96u_{59} - 96u_{58} - 96u_{57} - 48u_{56} - 24u_{55} - 48u_{54} - 96u_{53} - 96u_{52} - 48u_{51} - 24u_{50} - 96u_{49} - 24u_{48} + 48u_{47} - 96u_{46} - 48u_{45} - 96u_{44} - 24u_{43} - 96u_{42} - 24u_{41} + 144u_{40} + 72u_{39} + 144u_{38} + 72u_{37} + 144u_{36} + 72u_{35} + 144u_{34} + 72u_{33} + 144u_{32} + 72u_{31} + 144u_{30} + 72u_{29} - 108u_{28} - 108u_{27} - 108u_{26} + 48u_{25} + 192u_{24} + 48u_{23} + 192u_{22} + 192u_{21} + 192u_{20} + 192u_{19} + 192u_{18} + 192u_{17} + 48u_{16} + 192u_{15} + 48u_{14} - 144u_{13} - 576u_{12} - 144u_{11} - 144u_{10} - 57u_{9} - 144u_{8} - 144u_{7} - 57u_{6} - 144u_{5} + 432u_{4} + 432u_{3} + 432u_{2} - 216u_{1}.

u_{201} = u_{201} - 4u_{88} + 3u_{26} + 12u_{2} - 6u_{1}.

u_{41} = u_{41} - u_{31} - u_{29} + u_{26} - u_{25} - u_{14} + u_{11} + u_{10} + u_{8} + u_{7} + u_{5} - 2u_{2} + u_{1}.

u_{5} = u_{5} - u_{3} - u_{2} + u_{1}.
XIV. APPENDIX II

A. Generalized Symmetric Means of Degree Four
for Structure V

\[ v_{285} = \langle \text{AAAA} / A_1 A_1 A_1 / A_1 A_1 A_1 \rangle, \]
\[ v_{279} = \langle \text{AAAA} / A_1 A_1 A_1 / A_1 A_1 B_1 \rangle, \]
\[ v_{273} = \langle \text{AAAA} / A_1 A_1 A_1 / A_1 B_1 C_1 \rangle, \]
\[ v_{249} = \langle \text{AAAA} / A_1 A_1 B_1 / A_1 B_1 A_1 \rangle, \]
\[ v_{237} = \langle \text{AAAA} / A_1 A_1 B_1 / A_1 B_1 B_1 \rangle, \]
\[ v_{219} = \langle \text{AAAA} / A_1 A_1 B_1 / A_1 B_1 C_1 \rangle, \]
\[ v_{204} = \langle \text{AAAA} / A_1 A_1 B_1 / A_1 B_1 C_1 D_1 \rangle, \]
\[ v_{196} = \langle \text{AAAA} / A_1 A_1 A_1 B_1 / A_1 A_1 B_1 \rangle, \]
\[ v_{193} = \langle \text{AAAA} / A_1 A_1 A_1 B_1 / A_1 A_1 B_1 C_1 \rangle, \]
\[ v_{91} = \langle \text{AAAA} / A_1 A_1 B_1 C_1 / A_1 B_1 C_1 D_1 \rangle, \]
\[ v_{63} = \langle \text{AAAA} / A_1 A_1 B_1 C_1 / A_1 B_1 C_1 D_1 \rangle, \]
\[ v_{29} = \langle \text{AAAA} / A_1 A_1 B_1 C_1 / A_1 B_1 C_1 D_1 \rangle, \]
\[ v_{14} = \langle \text{AAAA} / A_1 A_1 B_1 C_1 / A_1 A_1 B_1 C_1 \rangle, \]
\[ v_2 = \langle \text{AAAA} / A_1 A_1 B_1 C_1 / A_1 A_1 B_1 C_1 D_1 \rangle. \]

B. Some Generalized Polykays of Degree Four
for Structure V

\[ v_{285} = v_{285} - 4v_{282} - 3v_{279} + 12v_{276} + 6v_{273}. \]
\[ v_{201} = v_{201} - 4v_{201} + 3v_{26} + 12v_2 - 6v_1. \]
\[ v_{41} = v_{41} - v_{29} - v_{14} + v_5. \]
\[ v_5 = v_5 - v_2. \]
A. Generalized Symmetric Means of Degree Four
for Structure W

\[ w_{285} = \langle AAAA/AAAA/A_1 A_2 A_3 \rangle, \]
\[ w_{279} = \langle AAAA/AAAA/A_1 A_2 B_3 \rangle, \]
\[ w_{273} = \langle AAAA/AAAA/A_1 B_2 C_3 D_4 \rangle, \]
\[ w_{267} = \langle AAAA/AAAB/A_1 A_2 B_3 \rangle, \]
\[ w_{249} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{243} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 D_4 \rangle, \]
\[ w_{237} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{225} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{219} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 D_4 \rangle, \]
\[ w_{210} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 D_4 \rangle, \]
\[ w_{204} = \langle AAAA/AAAB/A_1 A_2 B_3 C_3 D_4 \rangle, \]
\[ w_{201} = \langle AAAA/ABCD/A_1 A_2 B_3 C_3 D_4 \rangle, \]
\[ w_{165} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{143} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{134} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{128} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{111} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{94} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{92} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{89} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{83} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{63} = \langle AAAB/AAAB/A_1 A_2 B_3 C_3 \rangle, \]
\[ w_{282} = \langle AAAA/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{276} = \langle AAAA/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{269} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{253} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{245} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{239} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{229} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{221} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{213} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{206} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{202} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{196} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{163} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{139} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{133} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{127} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{103} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{93} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{91} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{88} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{68} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{62} = \langle AAAB/AAAA/A_1 A_2 A_3 B_3 \rangle, \]
\[ w_{16} = <\text{AABC}/\text{ABB}/A_1 A_2 B_3 C_4>, \]
\[ w_{34} = <\text{AABC}/\text{ABAB}/A_1 A_2 B_3 C_4 D_1>, \]
\[ w_{30} = <\text{AABB}/\text{ABAB}/A_1 A_2 B_3 C_4 D_1>, \]
\[ w_{27} = <\text{ABCD}/\text{AABB}/A_1 A_2 B_3 C_4 D_1>, \]
\[ w_{14} = <\text{AABC}/\text{ABBC}/A_1 A_2 B_3 C_4>, \]
\[ w_{6} = <\text{AABC}/\text{ABAB}/A_1 A_2 B_3 C_4 D_1>, \]
\[ w_{3} = <\text{ABCD}/\text{AABB}/A_1 A_2 B_3 C_4 D_1>, \]
\[ w_{1} = <\text{ABCD}/\text{ABCD}/A_1 A_2 B_3 C_4 D_1>. \]

B. Some Generalized Polykays of Degree Four for Structure W

\[ w_{285} = w_{285} - 4w_{282} - 3w_{279} + 12w_{276} - 6w_{273}. \]
\[ w_{201} = w_{201} - 4w_{201} - 3w_{26} + 12w_{26} - 6w_{21}. \]
\[ w_{41} = w_{41} - w_{29} - w_{14} + w_{10} - w_{5} - w_{2}. \]
\[ w_{5} = w_{5} - w_{3} - w_{2} + w_{1}. \]
A. Generalized Symmetric Means of Degree Four for Structure X

\[ x_{289} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1A_1>, \]
\[ x_{282} = <\text{AAAA}/\text{AAAA}/A_1A_1B_1>, \]
\[ x_{279} = <\text{AAAA}/\text{AAAA}/A_1A_1B_1B_1>, \]
\[ x_{276} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1>, \]
\[ x_{273} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1D_1>, \]
\[ x_{268} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1A_1>, \]
\[ x_{266} = <\text{AAAA}/\text{AAAA}/A_1B_1B_1B_1>, \]
\[ x_{261} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1A_1>, \]
\[ x_{252} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1A_1>, \]
\[ x_{250} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1A_1>, \]
\[ x_{245} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1D_2>, \]
\[ x_{244} = <\text{AAAA}/\text{AAAA}/A_1B_1D_1>, \]
\[ x_{239} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1B_2>, \]
\[ x_{237} = <\text{AAAA}/\text{AAAA}/A_1B_1B_1>, \]
\[ x_{228} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1D_1>, \]
\[ x_{226} = <\text{AAAA}/\text{AAAA}/A_1B_1A_1C_1>, \]
\[ x_{221} = <\text{AAAA}/\text{AAAA}/A_1B_1C_2D_2>, \]
\[ x_{219} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1D_1>, \]
\[ x_{212} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1C_1>, \]
\[ x_{210} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1C_1>, \]
\[ x_{205} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1C_1>, \]
\[ x_{202} = <\text{AAAA}/\text{AAAA}/A_1B_1C_1D_4>, \]
\[ x_{283} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1A_1>, \]
\[ x_{280} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1A_1>, \]
\[ x_{277} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1A_1>, \]
\[ x_{274} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1A_1>, \]
\[ x_{269} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_2>, \]
\[ x_{267} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{262} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{253} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{251} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{249} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{245} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{243} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{238} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{229} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{227} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{225} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{220} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{213} = <\text{AAAA}/\text{AAAA}/A_1A_1A_1B_1>, \]
\[ x_{211} = <\text{AAAA}/\text{AAAA}/A_1A_1B_2C_3>, \]
\[ x_{206} = <\text{AAAA}/\text{AAAA}/A_1A_1B_2C_3>, \]
\[ x_{204} = <\text{AAAA}/\text{AAAA}/A_1A_1B_2C_3>, \]
\[ x_{201} = <\text{AAAA}/\text{AAAA}/A_1A_1B_2C_3>, \]
\[ x_{129} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{190} = <\text{AAAB}/\text{ABCA}/A_{11}A_{11}B_{2}, \]
\[ x_{172} = <\text{AAAB}/\text{ABBA}/A_{11}A_{11}B_{2}, \]
\[ x_{166} = <\text{AAAB}/\text{ABBA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{163} = <\text{AAAB}/\text{ABBA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{143} = <\text{AAAB}/\text{ABBA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{139} = <\text{AAAB}/\text{ABBA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{134} = <\text{AAAB}/\text{AAAB}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{128} = <\text{AAAB}/\text{AAAB}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{115} = <\text{AAAB}/\text{ABCA}/A_{11}A_{11}B_{2}, \]
\[ x_{109} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{104} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{95} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{93} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{91} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{88} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{83} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{73} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{68} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{63} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{51} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{44} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{41} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{33} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{30} = <\text{AAAB}/\text{ABCA}/A_{11}B_{1}A_{11}C_{2}, \]
\[ x_{126} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{123} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{127} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{111} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{107} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{103} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{94} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{92} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{89} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{85} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{75} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{70} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{64} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{62} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{48} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{43} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{34} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{31} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
\[ x_{29} = <\text{AAAB}/\text{AAAB}/A_{11}A_{11}B_{2}, \]
B. Some Generalized Polykays of Degree Four

for Structure X

\[ x_{285} = x_{285} - 4x_{283} - 3x_{282} + 3x_{280} + 12x_{279} + 12x_{277} - 6x_{276} - 6x_{274} + 12x_{261} \]
\[ - 24x_{252} - 24x_{251} - 24x_{250} - 24x_{249} - 24x_{248} + 24x_{246} + 6x_{238} - 3x_{237} - 24x_{228} \]
\[ - 12x_{227} - 24x_{226} - 12x_{225} + 18x_{220} + 18x_{219} + 24x_{212} + 96x_{211} + 24x_{210} - 72x_{205} \]
\[ - 72x_{204} + 36x_{201}. \]

\[ x_{201} = x_{201} - 4x_{200} - 3x_{200} + 12x_{20} - 6x_{1}. \]

\[ x_{41} = x_{41} - x_{31} - x_{26} + x_{14} + x_{6} + x_{5} - x_{2}. \]

\[ x_{5} = x_{5} - x_{3} - x_{2} + x_{1}. \]
### A. Generalized Symmetric Means of Degree Four for Structure Z

| $z_{285}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{282}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{279}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{276}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{273}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{267}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{261}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{251}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{249}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{243}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{237}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{227}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{225}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{219}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{211}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{205}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{201}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{196}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{169}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{135}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{104}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
| $z_{95}$ | $<\text{AAAA}/A_1A_1A_1A_1/A_1A_1A_1A_1>$ |
B. Some Generalized Polykays of Degree Four
for Structure Z

\[ z_{285} = z_{285} - 4z_{203} - 4z_{282} - 3z_{260} - 3z_{279} + 12z_{271} + 12z_{276} - 6z_{274} - 6z_{273} + 12z_{268} \]
\[ + 267 - 12z_{262} + 12z_{261} - 24\z_{252} - 24z_{251} - 24z_{250} - 24z_{249} + 24z_{244} + 24z_{243} \]
\[ + 6z_{238} + 3z_{237} - 24z_{228} - 12z_{227} - 24z_{226} - 12z_{225} + 18z_{220} + 18z_{219} + 24z_{212} \]
\[ + 96z_{211} + 24z_{210} - 72z_{205} - 72z_{204} + 36z_{201} \]

\[ z_{201} = z_{201} - 4z_{88} - 3z_{26} + 12z_{2} - 6z_{1} \]

\[ z_{41} = z_{41} - z_{31} - 29 + 26 - 14 + 8 + 8 + 5 - 2 \]

\[ z_{5} = z_{5} - z_{2} \]