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# Holder regularity of solutions of generalized p-Laplacian type parabolic equations

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**Hölder regularity of solutions**

**of generalized $p-$Laplacian type parabolic equations**

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To my parents

# TABLE OF CONTENTS

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## ABSTRACT

Using ideas from the theory of Orlicz spaces, we discuss the Hölder regularity of a bounded weak solution of a $p-$Laplacian type parabolic partial differential equation under generalized structure conditions. To show the Hölder continuity of such solutions, we use the idea of spreading positivity and geometric characters besides the standard De Giorgi's iteration method. For showing Hölder continuity of $Du$, we follow the perturbation argument. Under the generalized structure conditions, we give a uniform method of proof in an intrinsically scaled cylinder without separating degenerate and singular cases.

# CHAPTER 1. Introduction

The prototype of a parabolic $p-$Laplacian partial differential equation (the nonlinear version of the heat equation) is

$$u_t - \text{div}\left(|Du|^{p-2}Du\right) = 0$$

for $p > 1$. We easily observe that the modulus of parabolicity $|Du|^{p-2}$ approaches 0 if $p > 2$ and the quantity diverges if $1 < p < 2$ as $|Du| \to 0$. With the differential equation in nondivergence form,

$$u_t - a_{ij}D_{ij}u = 0,$$

if $1 < p < 2$, then the eigenvalues of the matrix $[a_{ij}]$ tend to infinity whenever $Du \to 0$. For $p > 2$, the eigenvalues of the matrix tend to zero whenever $Du \to 0$. Often, $p-$Laplacian equations are classified as degenerate ($p > 2$) and singular ($1 < p < 2$) and are studied separately. We have made the interesting observation that solutions of both degenerate and singular equations share similar regularity results by assuming boundedness of solutions. Here we try to prescribe a uniform method for showing regularity theory, especially Hölder continuity of $u$ and $Du$, without separating degenerate and singular equations.

The prototype of a generalized $p-$Laplacian equation is

$$u_t - \text{div}\left(\frac{g(|Du|)}{|Du|}Du\right) = 0$$

for a nonnegative and nondecreasing function $g \in C[0, \infty)$ with $g(0) = 0$. For the antiderivative of $g$, say $G$, we impose so called $\nabla_2$ and $\Delta_2$ conditions that

$$g_0 G(s) \leq sg(s) \leq g_1 G(s)$$

for any $s \in [0, \infty)$ and for some constants $1 < g_0 \leq g_1 < \infty$. When $g_0 = g_1$, we obtain the prototype of $p-$Laplacian. Not only does the $p-$Laplacian equation become a special case, but also we can explain a wider group of functions, those which are increasing between two power functions such as $s \mapsto s\log(s+1)$.

Basically, we show that a bounded solution of the generalized $p-$Laplacian equation behaves in a similar way to the heat equation in a properly tailored domain (intrinsically scaled cylinder). We use the idea of 'spreading positivity' in our approach, which idea came from the theory of Harnack estimates. By using strong geometric characters, we avoid argument studying two alternatives (two cases depending on the size of a solution) separately. The standard De Giorgi iteration methods are used after calculating local energy estimates with carefully selected test functions. To show the Hölder continuity of $Du$, we follow a perturbation argument comparing two solutions, one with full structures and another with rather simple structures. The Hölder continuity of $u$ is used and is crucial in estimating the Hölder regularity for $Du$.

This chapter is devoted to the historical background and to an introduction to the generalized structures. In Chapter 2, basic inequalities and local energy estimates are provided. The Hölder regularity of $u$ and $Du$ are proven in Chapters 3 and 4, respectively. In the last chapter, we discuss certain open questions.

## 1.1 Hölder continuity and Harnack estimates

The Hölder continuity was first stated in the doctoral dissertation entitled *Beiträge zur Potentialtheorie* by Otto Hölder in 1882 [26]. For example, existence theory based on Hölder continuity of a solution and its gradient. One significant application of Harnack estimate is Hölder continuity. Throughout this section, Hölder continuity and Harnack estimate are given for simple type of differential equations. Then we also introduce how the Hölder continuity and the Harnack estimate are related.

To introduce local $\alpha-$Hölder continuity for $\alpha \in (0,1)$, we begin with $\alpha-$Hölder norm of a function in the domain $\Omega$. The $\alpha-$Hölder seminorm of a function $u$ in the domain $\Omega$ $[u]^{\alpha}_{\Omega}$ and

$\alpha-$Hölder norm $|u|_{\alpha,\Omega}$ are defined as:

$$\begin{aligned}[u]^{\alpha}_{\Omega} &:= \sup_{x\neq y\in\Omega} \frac{|u(x)-u(y)|}{[\mathrm{dist}(x,y)]^{\alpha}}, \\ |u|_{\alpha,\Omega} &:= \sup_{\Omega} u + [u]^{\alpha}_{\Omega}.\end{aligned} \tag{1.1}$$

The function $u$ is $\alpha-$Hölder continuous in $\Omega$ when $|u|_{\alpha,\Omega}$ is finite. The Hölder continuity can be regarded as fractional differentiability somewhere between Lipschitz continuity (when $\alpha = 1$) and uniform continuity (when $\alpha = 0$). For brevity, let $\Omega$ be a ball with the radius $R$, $B_R$. Then local $\alpha$-Hölder continuous function $u$ satisfies

$$\operatorname*{ess\,osc}_{x\in B_r} u(x) \leq Cr^{\alpha} \tag{1.2}$$

thanks to (1.1), for some constant $C$ and any constant $0 < r < R$.

For the parabolic type of equations which involved with the derivative in terms of the time variable, we need to be careful evaluating the distance between two distinct points in the domain. For example, the proper geometric setting for the heat equation ($u_t = u_{xx}$) is a cylinder $B_R \times [0, R^2]$ roughly because one time derivative is equivalent to two space derivatives which implies

$$\mathrm{dist}\left((x,t),(y,s)\right) = |x-y| + \sqrt{|t-s|}.$$

To derive (1.2), we consider two cases: either for some constants $\epsilon \in (0,1)$ and $C$

$$\operatorname*{ess\,osc}_{B_{\epsilon R}} u(x) \leq CR^{\alpha_1},$$

or

$$\operatorname*{ess\,osc}_{B_{\epsilon R}} u(x) \geq CR^{\alpha_1}.$$

In the first case, Hölder continuity is obvious. In the second case, we hope to show that

$$\operatorname*{ess\,osc}_{B_{\epsilon R}} u(x) \leq \delta \operatorname*{ess\,osc}_{B_R} u(x) \tag{1.3}$$

for some $\delta \in (0,1)$. We can find, for any $r < R$, a positive integer $n$ such that $\epsilon^{n+1}R < r \leq \epsilon^n R$. This will then lead us to obtain

$$\operatorname*{ess\,osc}_{x\in B_r} u(x) \leq C\left(\frac{r}{R}\right)^{\log_{\epsilon}\delta} \operatorname*{ess\,osc}_{x\in B_R} u(x);$$

we then choose $\epsilon < \delta$ and $\alpha = \max\{\alpha_1, \log_\epsilon \delta\}$. Therefore, obtaining the inequality (1.3) is the key to showing the Hölder continuity of a solution; that is, establishing the existence of compact subset in the domain where strictly less oscillation occur, compared to the oscillation in the domain, leads to the Hölder continuity of a solution.

The Harnack inequality was first stated in the book entitled *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*, written by the mathematician Carl Gustav Axel von Harnack (1851 - 1888) [25], who devoted himself to potential theory in his last years. In this section, we consider Harnack inequalities of elliptic partial differential equations in divergence form.

First, suppose that $w$ is a nonnegative weak solution of an elliptic equation in the domain $B_R \subset \mathbb{R}^N$; then the Harnack estimate requires that we find, for $r < R$ (which means $B_r$ is a compact subset of $B_R$), a constant $C = C(N, r, R) > 1$ such that

$$\operatorname*{ess\,sup}_{B_r} w(x) \leq C \operatorname*{ess\,inf}_{B_r} w(x). \tag{1.4}$$

We then replace $w$ by $\operatorname{ess\,sup}_{B_R} u - u(x)$ and $u(x) - \operatorname{ess\,inf}_{B_R} u$. Owing to (1.4), the two solutions generate

$$\operatorname*{ess\,sup}_{B_R} u - \operatorname*{ess\,inf}_{B_r} u \leq C \operatorname*{ess\,sup}_{B_R} u - \operatorname*{ess\,sup}_{B_r} u,$$

and

$$\operatorname*{ess\,sup}_{B_r} u - \operatorname*{ess\,inf}_{B_R} u \leq C \operatorname*{ess\,sup}_{B_r} u - \operatorname*{ess\,sup}_{B_R} u.$$

By subtracting these two inequalities it follows that

$$\operatorname*{ess\,osc}_{B_r} u \leq \frac{C-1}{C+1} \operatorname*{ess\,osc}_{B_R} u.$$

This gives (1.3) and eventually leads to Hölder continuity.

For the heat equation, we set up two subcylinders about a point $(x_0, t_0)$ with a fixed constant $\sigma \in (0, 1)$ and for a given constant $R > 0$

$$Q^+ = K_{\sigma R}(x_0) \times (t_0 + (\sigma R)^2, t_0 + R^2],$$

and

$$Q^- = K_{\sigma R}(x_0) \times (t_0 - R^2, t_0 - (\sigma R)^2].$$

Then the Harnack estimate is

$$\operatorname*{ess\,sup}_{Q^-} u(x,t) \leq \gamma(\sigma) \operatorname*{ess\,inf}_{Q^+} u(x,t)$$

for a positive constant $\gamma$ depending on $\sigma$. We notice that two cylinders $Q^-$ and $Q^+$ have a strictly positive time gap, $2(\sigma R)^2$, that is essential. The Harnack inequality for parabolic differential equations is much more subtle than the elliptic version because it is necessary to have a little bit of waiting time. In some case (particulary, $p-$Laplacian with $1 < p < p^* < 2$ for some critical number $p^*$), the little bit of waiting time may not be allowed, which results in the failure of the Harnack estimates.

## 1.2 Literature review

Before 1950's regularity theory was essentially based on perturbation arguments such as Schauder's estimates which, roughly speaking, guarantee that the solutions of (1.5) is in $C^{k+1,\alpha}$ if all coefficients are in the space $C^{k,\alpha}$. Back then, finding minimum requirement of $a_{ij}$, which guarantees the Hölder continuity and boundedness of a weak solution, was one of topics in the theory of partial differential equations. In late 1950's and 1960's, De Giorgi(1957, [6]), Nash (1958, [51]), and Moser (1960, [48], 1964, [49]) found the nonperturbation arguments that was a big turning point concerning regularity and a priori theories of elliptic and parabolic differential equations in divergence form. Below we outline nonperturbation arguments, first with elliptic equations and then discuss the transition from elliptic to second order parabolic to $p-$Laplacian type of parabolic equations.

### 1.2.1 Elliptic equations

First, we consider a linear second order elliptic equation using summation convention

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)u_{x_j}\right) = 0. \tag{1.5}$$

The function $u \in W^{1,2}(\Omega)$ is called a weak solution of (1.5) if $u$ satisfies

$$\int_{\Omega} a_{ij}(x,t) u_{x_j} \varphi_{x_i} \, dx = 0$$

for any test function $\varphi \in C_c^{\infty}(\Omega)$. Similar type of linear second order parabolic equation is:

$$u_t - \frac{\partial}{\partial x_i}\left(a_{ij}(x,t) u_{x_j}\right) = 0. \tag{1.6}$$

A weak solution $u \in V^{1,2}(\Omega_T)$ where $\Omega_T = \Omega \times [0,T]$ of (1.6) is defined as:

$$\iint_{\Omega_T} u_t \varphi(x,t) \, dx\, dt - \iint_{\Omega_T} a_{ij}(x,t) u_{x_j} \varphi_{x_i} \, dx = 0$$

for any test function $\varphi$ in an appropriate space. We add assumptions on the coefficients $a_{ij}$, uniform ellipticity and boundeness,

$$a_{ij}\xi_i\xi_j \geq \lambda |\xi|^2, \quad \text{for some } \lambda > 0, \text{ for all } \xi \in \mathbb{R}^N, \tag{1.7a}$$

and

$$|a_{ij}| \leq \Lambda, \quad \text{for some } \Lambda \geq \lambda, \text{ for all } i, j. \tag{1.7b}$$

Nash [51] developed a method that gives the Hölder continuity of a weak solution of (1.6) under (1.7). The equation (1.5) is a steady state solution of (1.6). He laid the foundation for detecting new connections between the properties of the coefficients and the equations and the properties of their solutions. But the complexity and difficulty of Nash's proof does not permit one to essentially sharpen his results for other types of equations.

Nonperturbation arguments developed by De Giorgi and Moser became essential methods for later regularity theory; these have been adopted and modified by many mathematicians mainly because those methods do not rely on the linearity of the differential eqautions. Here we briefly describe the famous De Giorgi's and Moser's iterations of a weak solution of the linear elliptic equations (1.5) under (1.7).

We note first that De Giorgi's iteration [6] is based on local energy estimates derived from the differential equation and Sobolev embedding inequality between two balls $B_R \in \mathbb{R}^N$ and $B_r \in \mathbb{R}^N$ such that $B_r \subset B_R$. For any $r < R$ and $h > k$, one can find

$$\int_{B_r \cap \{u > h\}} u^2 \, dx \leq \frac{C}{(R-r)^2 (h-k)^{\frac{4}{N}}} \left[ \int_{B_R \cap \{u > k\}} u^2 \, dx \right]^{1+\frac{2}{N}}.$$

Then iteration based on infinitely many disjoint rings in between $B_r$ and $B_R$ eventually gives

$$\operatorname*{ess\,osc}_{B_r} u(x) \leq C \operatorname*{ess\,osc}_{B_R} u(x)$$

which leads to the Hölder continuity of $u$.

We also note that Moser's iteration is based on choosing appropriate convex test functions ($f(u) = |u|^p$ for $p \geq 1$ and $f(u) = \left(\log u^{-1}\right)_+$) for a nonnegative weak solution to show that

$$\left(\int_{B_r} u^{p_1}(x)\, dx\right)^{\frac{1}{p_1}} \leq C \left(\int_{B_R} u^{p_2}(x)\, dx\right)^{\frac{1}{p_2}}, \tag{1.8a}$$

and

$$\int_{B_r} \exp\left(p_0 \log w\right)\, dx \leq C, \tag{1.8b}$$

where $p_1 > p_2$, $r < R$, and for some constant $p_0$. From (1.8a) we estimate $\operatorname{ess\,sup}_{B_r} u$, $\operatorname{ess\,inf}_{B_r} u$ through iteration that increase $p$ and eventually sending $p$ to $\pm\infty$. The inequality (1.8b) gives a relationship of two quantities, which is (1.4).

Neither of these iteration schemes relies on the linearity of the elliptic equation. This permits an extension of these results to quasilinear equations: for $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, the differential equation is given

$$\operatorname{div} A(x, u, Du) = B(x, u, Du), \tag{1.9}$$

under structure conditions

$$A(x, u, Du) \cdot Du \geq |Du|^p - \varphi_0(x, t), \tag{1.10a}$$

$$|A(x, u, Du)| \leq C_1 |Du|^{p-1} + \varphi_1(x, t), \tag{1.10b}$$

$$|B(x, u, Du)| \leq C_2 |Du|^p + \varphi_2(x, t), \tag{1.10c}$$

for positive constants $C_1, C_2$ and nonnegative functions $\varphi_0, \varphi_1, \varphi_2$ in appropriate function spaces. The prototype of equation is

$$\operatorname{div}\left(|Du|^{p-2} Du\right) = 0, \quad \text{for } p > 1.$$

As $|Du|$ approaches 0, the modulus of ellipticity $|Du|^{p-2}$ vanishes if $p > 2$ and becomes infinite if $1 < p < 2$. Because of their differences we classify equations as degenerate if $p > 2$ and singular if $1 < p < 2$. Often, the two types of equations have been studied separately.

Ladyženskaja and Ural'ceva [34], [39] established that a weak solution of (1.9) under (1.10) is Hölder continuous following the methods of De Giorgi. They define the De Giorgi classes, bounded functions in $W^{1,p}(\Omega)$ satisfying a certain integral inequality, where a membership in the De Giorgi classes guarantees the Hölder continuity. Serrin [53] and Trudinger [55] extend Moser's results to the equation (1.9). Moreover, DiBenedetto & Trudinger [22] and DiBenedetto [10] are able to prove that all functions in the De Giorgi class directly satisfy the Harnack inequality.

### 1.2.2 Parabolic equations

It is natural to ask similar regularity questions for parabolic partial differential equations. For second order linear and quasilinear equations, arguments similar to those for elliptic equations work well because the geometric setting $\Omega_T = B_R \times [0, R^2]$ gives homogeneous local energy estimates, which are essential for both De Giorgi and Moser iterations. However, regularity theory for $p-$Laplacian type of parabolic equations requires careful geometric techniques, so-called intrinsic scaling, to resolve the nonhomogeneity. Moreover degenerate, $p > 2$, and singular, $1 < p < 2$, equations show quite different behaviors. The most significant difference is that a solution of the degenerate equation with positive initial data keeps its positivity for all later times, while a solution of the singular equation may become zero in finite time. In fact, the singular cases are subdivided into subcritical ($p^* < p < 2$) and supercritial ($1 < p < p^*$) types for some constant $p^* \in (1, 2)$. These two types of equations show distinct behaviors.

De Giorgi's iteration was extended by series of papers by Ladyženskaja & Ural'ceva [35], [36], [37] (refer a book written by Ladyženskaja, Solonnikov, and Ural'ceva [38]) for second order linear and quasilinear parabolic equations which is very similar to studies by Ladyženskaja and Ural'ceva [34], [39] on the elliptic equations. Two papers [35], [36] are devoted to the study of the first boundary problem for linear and quasilinear second-order parabolic equations with divergence structure. Ladyženskaja & Ural'ceva first showed that every classical solution is Hölder continuous with Hölder constant and exponents determined by $\|u\|_{\infty,\Omega_T}$ and that the same is true for $Du$ under additional structure hypotheses. In the paper [37], authors general quasilinear second-order parabolic equations as well as linear systems.

The first parabolic version of the Harnack inequality was due to Hadamard and Pini and applies to nonnegative solutions of the heat equation. It follows that, for a nonnegative solution $u$ of the heat equation, there exists a constant $C$ such that

$$u(x_0, t_0) \geq \gamma \sup_{B_R(x_0)} u(x, t_0 - R^2). \tag{1.11}$$

For the parabolic Harnack inequality, it is essential to involve two distinct times. For the equation (1.6) under (1.7), the most influential contribution is made by Moser [49] [50] using John–Nirenberg inequality. Next question is that wether there is a similar Harnack inequality for second order quasilinear parabolic equations or not. First result in this direction was obtained in a paper by Aronson & Serrin [1] in which all complete proofs are contained and several consequences of the Harnack inequality are discussed such as asymptotic behavior of solutions in unbounded domains. On the other hand, some of the ideas in the paper [1] were simultaneously considered by several other authors: Ivanov [32] [33], Kurihara [31], and Trudinger [56].

Now consider quasilinear type parabolic equations of $p-$Laplacian type; that is, for $u \in V^{1,p}_{\text{loc}}(\Omega_T)$, the differential equation is given by

$$u_t - \operatorname{div} A(x, u, Du) = B(x, u, Du), \tag{1.12}$$

under the structure conditions

$$A(x, u, Du) \cdot Du \geq |Du|^p - \varphi_0(x, t), \tag{1.13a}$$

$$|A(x, u, Du)| \leq C_1 |Du|^{p-1} + \varphi_1(x, t), \tag{1.13b}$$

$$|B(x, u, Du)| \leq C_2 |Du|^p + \varphi_2(x, t), \tag{1.13c}$$

for some positive constants $C_1, C_2$ and nonnegative functions $\varphi_0, \varphi_1, \varphi_2$ in appropriate function spaces. When $p > 1$, but not equal to 2, the situation is quite different because of the lack of homogeneity on energy estimates obtained from (1.12) under (1.13). The prototype is given by

$$u_t - \operatorname{div} \left( |Du|^{p-2} Du \right) = 0. \tag{1.14}$$

With the test function $u\zeta^p$ where $\zeta$ is a linear cutoff function, one can derive the energy estimate

$$\begin{aligned}
&\sup_{t_0\le t\le t_1}\int_{B_R} u^2\zeta^p\,dx+\int_{t_0}^{t_1}\int_{B_R}|Du|^p\zeta^p\,dx\,dt\\
&\le\int_{B_R\times\{t_0\}} u^2\zeta^p\,dx+C\int_{t_0}^{t_1}\int_{B_R} u^p|D\zeta|^p\,dx\,dt\\
&\quad+p\int_{t_0}^{t_1}\int_{B_R}|u|^2\zeta^{p-1}\zeta_t\,dx\,dt
\end{aligned}$$

which carries two types of integral norms powered by 2 or $p$ (nonhomogeneous energy estimate). This equation (1.14) is called degenerate ($p>2$) and singular ($1<p<2$) because the modulus of parabolicity $|Du|^{p-2}$ behaves differently as $|Du|$ goes to 0. Actually it appears that unlike the elliptic case the degeneracy or singularity of the principal part plays a peculiar role. For example, a solution of the degenerate equation with positive initial conditions keeps its positivity in any later time while a solution of the singular equation may become extinct.

In early 1980's, DiBenedetto & Friedman [7], [12], [13] had the idea of 'intrinsic scaling' for showing Hölder regularity of $Du$ for the system of equations in the type of (1.12) under (1.13). Later, Chen [3] and DiBenedetto [8] established Hölder continuity of a weak bounded solution of (1.12) when $p>2$ with the aid of intrinsic scaling for the time variable studying two alternatives depending on the size of a solution. Basically they showed that a degenerate solution behaves like a solution of the heat equation in a well-tailored cylinder. Two authors, Chen & DiBenedetto [4], [5], [11] collaborated to show the same results for singular types of equations using intrinsic scaling for the space variable. Especially handling a solution of singular equations is requiring elaborate work to support the same results for degenerate solutions. For example, the Hölder constant becomes unstable as $p$ approach 2 without a special treatment for all constants appearing. Details of intrinsic scaling will be introduce in Section 3.2.

The Harnack inequalities for (1.12) under (1.13) is a more subtle topic in the sense that Harnack inequalities show clear distinctions depending on subcritical ($1<p<p^*$), supercritical ($p^*<p<2$), and degenerate ($p>2$) for the critical number $p^*=2N/(N+2)$. DiBenedetto developed Harnack estimates for degenerate equations [9], [10]. This topic is currently of high interest; the latest achievements have been made by DiBenedetto, Gianazza, and Vespri [21], [14],[16], [18].

## 1.3 Generalization

We go back to the prototype $p-$Laplacian parabolic equation to roughly describe our approach of the generalized structure for the differential equation (1.12). We rewrite (1.14) as

$$u_t - \operatorname{div}\left(|Du|^{p-1}\frac{Du}{|Du|}\right) = 0, \tag{1.15}$$

for $p > 1$. A prototype of a generalized equation can be made by replacing $|Du|^{p-1}$ by the more general function $g(|Du|)$, specifically,

$$u_t - \operatorname{div}\left(g(|Du|)\frac{Du}{|Du|}\right) = 0 \tag{1.16}$$

where $g$ is a nonnegative and nondecreasing function in $C[0,\infty)$. We also need to impose more conditions on the function $g$. First define $G$ as the antiderivative of $g$ that

$$G(s) = \int_0^s g(\sigma)\,d\sigma \quad \text{for } s \geq 0. \tag{1.17}$$

We then impose so called $\Delta_2$ and $\nabla_2$ conditions in Orlicz space which is a generalized $L^p$ space (Section I.3 & I.4 of [30], Section 2.3 of [52]). That is, for some constants $1 < g_0 \leq g_1 < \infty$,

$$g_0 G(s) \leq s g(s) \leq g_1 G(s). \tag{1.18}$$

For the extension, we also need a suitable definition of weak solution. Our approach to this follows. For an arbitrary open set $\Omega_T \subset \mathbb{R}^{n+1}$, we introduce the generalized Sobolev space $W^{1,G}(\Omega_T)$, which consists of all functions $u$ defined on $\Omega_T$ with weak derivative $Du$ satisfying

$$\iint_{\Omega_T} G(|Du|)\,dx\,dt < \infty.$$

Here we consider nonlinear parabolic partial differential equations, for $u \in W^{1,G}(\Omega_T)$,

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \tag{1.19}$$

with structure conditions in the cylinder $Q_R \subset \mathbb{R}^{N+1}$

$$\mathbf{A}(x,t,u,Du)\cdot Du \geq G(|Du|) - G(b_0), \tag{1.20a}$$

$$|\mathbf{A}(x,t,u,Du)| \leq a_1 g(|Du|) + g(b_1), \tag{1.20b}$$

$$|B(x,t,u,Du)| \leq a_2 G(|Du|) + G(b_2), \tag{1.20c}$$

where $a_1, a_2, b_0, b_1, b_2$ are nonnegative constants.

We say that $u \in C_{\text{loc}}(\Omega_T) \cap W^{1,G}(\Omega_T)$ is a weak subsolution of (1.19) if

$$-\iint_{\Omega_T} u\varphi_t \, dx\, dt + \iint_{\Omega_T} \mathbf{A}(x,t,u,Du)\cdot D\varphi \, dx\, dt \leq \iint_{\Omega_T} B(x,t,u,Du)\varphi \, dt\, dx \tag{1.21}$$

for all $\varphi \in C^1(\bar{\Omega}_T)$ which vanish on the parabolic boundary of $\Omega_T$; a weak supersolution is defined by reversing the inequality.

This generalization is inspired by Lieberman's work on elliptic equations [41] that established a uniform proof for both degenerate and singular equations. Specifically, the structures (1.20) are contained in (1.13) as the special case when $g(s) = s^{p-1}$, that is, $g_0 = g_1 = p$ in (1.18). In addition, our structure allows consideration of more general equations; for any $\alpha$ and $\beta$ with $1 < \alpha < \beta < \infty$, we can find a function $g$ satisfying (1.18) such that

$$\limsup_{s\to\infty} \frac{g(s)}{s^\beta} > 0, \quad \liminf_{s\to\infty} \frac{g(s)}{s^\alpha} < \infty.$$

In this way, we are able to consider a class of structure functions $g$ much wider than that of just power functions. Moreover, this generalization does not require any stability arguments when $p$ approaches 2. We refer papers by Mascolo & Papi [45] and Moscariello [47] for elliptic equations under generalized structures using minimizers of the calculus of variations.

Compared to elliptic equation [41], we need a specially constructed single geometric setting for (1.19) under (1.20). The distinction between degenerate ($g_0 > 2$) and singular ($g_1 < 2$) becomes noticeably different.

# CHAPTER 2. Preliminaries

In this Chapter, we begin with introducing notations (basic and standard notations are presented in Appendix). As mentioned in Section 1.3 in Chapter 1, our story is based on the function $g$ and its antiderivative $G$ satisfying the $\nabla_2$ and $\Delta_2$ conditions (2.2). Therefore we are not anymore allowed to use inequalities designed for power functions such as the ordinary Hölder inequality and Young's inequality. In Section 2.1, basic inequalities used throughout the paper will be introduced. In Section 2.2, we derive two local integral estimates that are used importantly in Chapter 3. In the last section, we provide theorems and inequalities that have been developed and will help out proving propositions in later chapters.

- For any constant $k$, we define two types of level sets

$$(u-k)_+ = \max\{0, u-k\},$$
$$(u-k)_- = \max\{0, k-u\}.$$

- The constant $N$ denotes the dimension of the space.

- The set of parameters $\{g_0, g_1, N, a_1, a_2, b_0, b_1, b_2\}$ is referred as data.

- Let $K_\rho^y$ denote the $N-$dimensional cube centered at $y \in \mathbb{R}^N$ with side length $2\rho$, i.e.,

$$K_\rho^y := \left\{ x \in \mathbb{R}^N : \max_{1\le i\le N} |x_i - y_i| < \rho \right\}.$$

  For simpler notation, let $K_\rho := K_\rho^0$.

- For given $(x_0, t_0) \in \mathbb{R}^{N+1}$ and positive constants $r$ and $s$, we name a cylinder

$$\Omega_{r,s}^{x_0,t_0} := K_r^{x_0} \times [t_0 - s, t_0]$$

  to refer to any arbitrarily given cylinder.

- The three constants $m$, $M$, and $\omega$ denote

$$m \leq \operatorname*{ess\,inf}_{\Omega_{r,s}^{x_0,t_0}} u(x,t),$$

$$M \geq \operatorname*{ess\,sup}_{\Omega_{r,s}^{x_0,t_0}} u(x,t),$$

$$\omega \geq \operatorname*{ess\,osc}_{\Omega_{r,s}^{x_0,t_0}} u(x,t).$$

## 2.1 Inequalities

Here we recall that $g$ is a nonnegative and nondecreasing function in $C[0,\infty)$ and the function $G$ is defined as the antiderivative of $g$,

$$G(s) = \int_0^s g(\sigma)\,d\sigma \quad \text{for } s \geq 0. \tag{2.1}$$

Also for some constants $1 < g_0 \leq g_1 < \infty$, $\Delta_2$ and $\nabla_2$ conditions in Orlicz space is given as

$$g_0 G(s) \leq s g(s) \leq g_1 G(s). \tag{2.2}$$

This section is devoted for lemmata with various types of inequalities using properties of the functions $g$ and $G$ and the inequalities from (2.2).

**Lemma 2.1.1.** *For the nonnegative and nondecreasing function $g \in C[0,\infty)$, let $G$ be the antiderivative of $g$. Suppose that $g$ and $G$ satisfy* (2.2) *for some constants $1 < g_0 \leq g_1 < \infty$ and for any $s \geq 0$. Then for any nonnegative real numbers $s$, $s_1$, and $s_2$, we have*

*(a)* $G(s)/s$ *is a monotone increasing function.*

*(b) For* $0 < \epsilon < 1$,

$$\epsilon^{g_1} G(s) \leq G(\epsilon s) \leq \epsilon^{g_0} G(s). \tag{2.3}$$

*(c) For* $\epsilon > 1$,

$$\epsilon^{g_0} G(s) \leq G(\epsilon s) \leq \epsilon^{g_1} G(s). \tag{2.4}$$

*(d)*

$$s_1 g(s_2) \leq s_1 g(s_1) + s_2 g(s_2). \tag{2.5}$$

*(e)* (Young's inequality)

$$s_1 g(s_2) \le \epsilon g_1 G\left(\frac{s_1}{\epsilon}\right) + \epsilon g_1 G(s_2) \quad \textit{for } \epsilon > 0, \tag{2.6a}$$

$$s_1 g(s_2) \le \epsilon^{1-g_1} g_1 G(s_1) + \epsilon g_1 G(s_2) \quad \textit{for } 0 < \epsilon < 1, \tag{2.6b}$$

$$s_1 g(s_2) \le \epsilon^{1-g_0} g_1 G(s_1) + \epsilon g_1 G(s_2) \quad \textit{for } \epsilon > 1. \tag{2.6c}$$

*Proof.* This lemma is quoted directly or modified from the Lemma 1.1 from [41].

(a) For $s > 0$, we easily obtain due to the left hand side inequality of (2.2),

$$\frac{d}{ds}\left(\frac{G(s)}{s}\right) = \frac{sg(s) - G(s)}{s^2} \ge (g_0 - 1)\frac{G(s)}{s^2} > 0,$$

because $g_0 > 1$.

(b) The left inequality of (2.2) is

$$\frac{g_0}{s} \le \frac{g(s)}{G(s)} \text{ for } \ s \in (0, \infty)$$

for any $s > 0$. By taking the integral over $\epsilon s$ to $s$,

$$g_0 \int_{\epsilon s}^{s} \frac{1}{\sigma}\, d\sigma \le \int_{\epsilon s}^{x} \frac{g(\sigma)}{G(\sigma)}\, d\sigma,$$

we obtain

$$g_0 \ln \frac{s}{\epsilon s} \le \ln \frac{G(s)}{G(\epsilon s)}$$

which is the right hand side of (2.3) ($\Delta_2$ condition). Likewise, we obtain the left hand side of (2.3) with the right hand side of (2.2)($\nabla_2$ condition).

(c) The inequality (2.4) is obtained similar to the proof for getting the inequality (2.3) by take integrals over $s$ to $\epsilon s$.

(d) Because $g$ is nondecreasing function, it is clear that either

$$s_1 g(s_2) \le s_1 g(s_1) \quad \text{or} \quad s_1 g(s_2) \le s_2 g(s_2),$$

which leads to the inequality we desire.

(e) For any positive constant $\epsilon$, we obtain

$$s_1 g(s_2) = \epsilon \frac{s_1}{\epsilon} g(s_2) \leq \epsilon \left[ \frac{s_1}{\epsilon} g\left(\frac{s_1}{\epsilon}\right) + s_2 g(s_2) \right],$$

because of the inequality (2.5). Applying the right hand side of the inequality (2.2) leads to

$$s_1 g(s_2) \leq \epsilon \left[ g_1 G\left(\frac{s_1}{\epsilon}\right) + g_1 G(s_2) \right]$$

which is (2.6a). Depending on the range of $\epsilon$, we apply either (2.3) or (2.4) to derive (2.6b) and (2.6c) respectively.

□

**Remark 2.1.1.** *The typical Young's inequality is given for positive constants $a, b, \epsilon$ that*

$$ab \leq \frac{(\epsilon a)^p}{p} + \frac{(b/\epsilon)^{p'}}{p'} \tag{2.7}$$

*where $1/p + 1/p' = 1$. If we let $a = s_1$, $b = s_2^{p-1}$, and $\epsilon = 1$, then (2.7) gives*

$$\begin{aligned} s_1 s_2^{p-1} &\leq \frac{s_1^p}{p} + \frac{p-1}{p} s_2^p \\ &\leq s_1^p + s_2^p, \end{aligned}$$

*which is derived from the inequality (2.5) of Lemma 2.1.1 (d) with the setting of $g(s) = s^{p-1}$ for any $1 < p < \infty$. Therefore we call the inequalities in Lemma 2.1.1 (d) and (e) to be Young's inequality in this context.*

If we assume $g \in C^1(0,\infty) \cap C[0,\infty)$, then Lemma 2.1.2 is unnecessary. However with only assuming $g \in C[0,\infty)$, we define functions $h$ and $H$ which are kind of equivalent to $g$ and $G$, respectively. Moreover $h \in C^1(0,\infty) \cap C[0,\infty)$ which is required to derive the logarithmic energy estimate (2.40). The logarithmic estimate is essential to verify the main lemma.

**Lemma 2.1.2.** *For any $s > 0$, let*

$$h(s) = \frac{1}{s} \int_0^s g(\sigma)\, d\sigma, \tag{2.8a}$$

$$H(s) = \int_0^s h(\sigma)\, d\sigma. \tag{2.8b}$$

*Then we have*

*(a)*

$$g_0 h(s) \le g(s) \le g_1 h(s). \tag{2.9}$$

*(b)*

$$g_0 H(s) \le G(s) \le g_1 H(s). \tag{2.10}$$

*(c)*

$$(g_0 - 1)h(s) \le sh'(s) \le (g_1 - 1)h(s). \tag{2.11}$$

*(d)*

$$\frac{1}{g_1} sh(s) \le H(s) \le \frac{1}{g_0} sh(s). \tag{2.12}$$

*(e) For a constant $\epsilon > 1$,*

$$\epsilon^{g_0} \le \frac{H(\epsilon s)}{H(s)} \le \epsilon^{g_1}. \tag{2.13}$$

*(f) For a constant $0 < \epsilon < 1$,*

$$\epsilon^{g_1} \le \frac{H(\epsilon s)}{H(s)} \le \epsilon^{g_0}. \tag{2.14}$$

*Proof.* Here we note that $h$ acts like $g$ and $H$ acts like $G$.

(a) Dividing the inequality (2.2) by $s$ completes the proof.

(b) The inequality is obtained by taking integrals to the inequality (2.9).

(c) Because

$$h'(s) = \frac{g(s)}{s} - \frac{G(s)}{s^2},$$

applying the inequality (2.2) completes the proof.

(d) By applying integration by parts and the left hand side of the inequality (2.11), we yield that

$$\begin{aligned} H(s) &= sh(s) - \int_0^s sh'(s)\,ds \\ &\leq sh(s) - (g_0 - 1)\int_0^s h(s)\,ds. \end{aligned}$$

Therefore

$$H(s) \leq \frac{1}{g_0} sh(s).$$

Similarly, we obtain by using the right hand side of the inequality (2.11)

$$H(s) \geq \frac{1}{g_1} sh(s).$$

(e) It is similar to the proof of Lemma 2.1.1 (b).

(f) It is similar to the proof of Lemma 2.1.1 (c).

□

## 2.2 Energy Estimates

Here we recall the nonlinear parabolic partial differential equations, for $u \in W^{1,G}(\Omega_T)$,

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \tag{2.15}$$

with structure conditions in the cylinder $Q_R := K_R \times [t_0, t_1]$

$$\mathbf{A}(x,t,u,Du) \cdot Du \geq G(|Du|) - G(b_0), \tag{2.16a}$$

$$|\mathbf{A}(x,t,u,Du)| \leq a_1 g(|Du|) + g(b_1), \tag{2.16b}$$

$$|B(x,t,u,Du)| \leq a_2 G(|Du|) + G(b_2), \tag{2.16c}$$

where $a_1, a_2, b_0, b_1, b_2$ are nonnegative constants.

The purpose of this section is deriving integral estimates (equivalently, energy estimates) from the equations (2.15) under structure conditions (2.16). Then later proofs are based on those integral estimates, not differential equations. Roughly speaking, we may forget about the differential equation once we have energy estimates. To draw out two types of estimates, local energy estimate and logarithmic estimate, first we emphasize that this is an a priori estimate assuming boundedness of a weak solution. Moreover, we need to impose more restrictions to handle lower order terms in the structure conditions (2.16).

For brevity, denote

$$b := \max\{b_0, b_1, b_2\}, \tag{2.17}$$

where $b_0, b_1, b_2$ are from (2.16). Also let $w$ denote any bounded nonnegative weak solution of (2.15) under assumption (2.16), for example, $w$ could represent

$$u - \operatorname*{ess\,inf}_{\Omega_T} u \quad \text{or} \quad \operatorname*{ess\,sup}_{\Omega_T} u - u.$$

Let $w$ be a nonnegative weak subsolution (supersolution) of (2.15) under (2.16) satisfying

$$0 \le w \le M_w < \infty. \tag{2.18}$$

We pick a constant $k$ such that

$$\sup |(w-k)_{\pm}| \le \delta |k| \tag{2.19}$$

for some constant $\delta \in (0,1)$ and

$$M_w \le |k| \le \Lambda M_w \tag{2.20}$$

for some constant $\Lambda \ge 1$. The restriction (2.19) is saying that the level, $k$, has to be chosen somewhat near the maximum or the minimum of a subsolution (supersolution) to control the lower order terms well.

It would be easier if we were to assume the existence of the time derivative $w_t$ for a weak solution. Unfortunately, a solution of (2.15) under assumption (2.16) does not possess such a degree of time regularity. In general, $w_t$ has a meaning only in the sense of distributions. To overcome this limitation, we introduce the *Steklov average* of a function $v \in L^1(\Omega_T)$ such that, for $h \in (0, T)$,

$$v_h := \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)\,d\tau & \text{if} \quad t \in (0, T-h] \\ 0 & \text{if} \quad t \in (T-h, T] \end{cases}, \tag{2.21a}$$

$$v_{\bar{h}} := \begin{cases} \frac{1}{h}\int_{t-h}^{t} v(\cdot,\tau)\,d\tau & \text{if} \quad t \in (h, T] \\ 0 & \text{if} \quad t \in (0, h] \end{cases}. \tag{2.21b}$$

Based on standard $L^p$ space theory, one can prove that for $\Omega \subset \mathbb{R}^N$, $v_h \to v$ in $L^p(\Omega \times (0, T-\delta))$ if $v \in L^p$ and that $Dv_h \to Dv$ in $L^p(\Omega \times (0, T-\delta))$ if $Dv \in L^p$ provided $0 < \delta < T$ and $h \to 0^+$. In addition, if $v \in V_0$, then $v_h \to v$ in $V_0(\Omega \times (0, T-\delta))$ with the same restrictions on $\delta$ and $h$. (The similar story holds for $v_{\bar{h}}$ in $\Omega \times (\delta, T)$ for any $0 < \delta < T$. )

### 2.2.1 The Local Energy Estimate

Because of the setting of a test function to derive the local energy estimate, we first note how to choose three particular constants and those constants are used in the proof of Proposition 2.2.1.

**Remark 2.2.1.** *The choices* (2.24) *is made to satisfy the following inequalities*

$$(r-1)g_0 + (s+1) > 0, \tag{2.22a}$$

$$(r-1)g_1 + (s+1) > 0, \tag{2.22b}$$

$$rg_0 + s > 0, \tag{2.22c}$$

$$rg_1 + s > 0, \tag{2.22d}$$

$$(r-1)g_0 + s \le 0, \tag{2.22e}$$

$$(r-1)g_1 + s \le 0, \tag{2.22f}$$

$$(r-1)g_0 + q > 0, \tag{2.22g}$$

$$(r-1)g_1 + q > 0, \tag{2.22h}$$

$$(r-1)g_0 + q - 1 \ge 0, \tag{2.22i}$$

$$(r-1)g_1 + q - 1 \ge 0. \tag{2.22j}$$

*The inequalities* (2.22a) *and* (2.22b) *are from* (2.28)*. But then those inequality* (2.22b) *guarantees that a map* $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$ *is an increasing function because*

$$\frac{d}{d\sigma} G^{r-1}(\sigma)\sigma^{s+2} \geq \min\left\{(r-1)g_0 + (s+2), (r-1)g_1 + (s+2)\right\} G^{r-1}(\sigma)\sigma^{s+1}.$$

*The inequalities* (2.22c) *and* (2.22d) *are imposed to have an increasing map* $\sigma \mapsto G^r(\sigma)\sigma^s$ *due to*

$$\frac{d}{d\sigma} G^r(\sigma)\sigma^s \geq \min\left\{rg_0 + s, rg_1 + s\right\} G^r(\sigma)\sigma^{s-1}.$$

*For simpler calculation, we add* (2.22e) *and* (2.22f) *which basically saying that the map* $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$ *is a decreasing function because*

$$\frac{d}{d\sigma} G^{r-1}(\sigma)\sigma^s \leq \max\left\{(r-1)g_0 + s, (r-1)g_1 + s\right\} G^r(\sigma)\sigma^{s-1}.$$

*To estimate upper and lower bounds for* $\Xi_2$ *on* (2.30b)*, we need* (2.22g) *and* (2.22h)*. The last two inequalities are given to make sure that we have nonnegative power for cutoff function on every integral estimates appearing for the proof for the local energy estimate (especially* $I_6$*).*

**Proposition 2.2.1.** *Let* $w$ *be a nonnegative bounded weak solution of* (2.15) *under assumption* (2.16) *in a cylinder* $Q_R := K_R \times [t_0, t_1]$*. For a nonnegative constant* $k$ *and a cutoff function* $\zeta$*, there exists constants* $r, s, q,$ *and* $\gamma_i = \gamma_i(\text{data})$*,* $i = 0, 1, 2, 3, 4$ *such that*

$$\begin{aligned}
&\int G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s+2}\zeta^q\,dx\bigg|_{t_0}^{t_1} \\
&\quad + \gamma_0 \iint_{Q_R} G(|Dw|) G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s}\zeta^q\,dx\,dt \\
&\leq \gamma_1 \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s+2}\zeta^{q-1}\zeta_t\,dx\,dt \\
&\quad + \gamma_2 \iint_{Q_R} G\left(\frac{|D\zeta|(w-k)_\pm}{\zeta}\right) G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s}\zeta^q\,dx\,dt \\
&\quad + \gamma_3 \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s}\zeta^q\,dx\,dt \\
&\quad + \gamma_4\delta\Lambda M_w \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s+1}\zeta^q\,dx\,dt.
\end{aligned} \tag{2.23}$$

*Proof.* For simpler notation, let $\bar{w} := (w-k)_-$ which is also a nonnegative subsolution. To prove Proposition 2.2.1, we work with the test function

$$\varphi(x,t) = e^{a_2 w} G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^{s+1}\zeta^q$$

where

$$r = 1 - \frac{1}{g_1}, \quad s = \frac{g_0}{g_1}, \quad \text{and} \quad q = g_1. \tag{2.24}$$

Those particular choices for constants $r$, $s$, and $q$ are explained in Remark 2.2.1.

Now we look at the time integral term

$$\begin{aligned}
\iint_{Q_R} \left(\frac{\partial}{\partial t} w_h\right) \varphi_h \, dx\, dt &= \int_{t_0}^{t_1} \int_{K_R} \frac{d}{dt} \left(F(w_h)\right) \zeta^q \, dx \\
&= \int_{K_R} F(w_h)\zeta^q \, dx\, dt \Big|_{t_0}^{t_1} - q \iint_{Q_R} F(w_h)\zeta^{q-1}\zeta_t \, dx\, dt \\
&\to \int_{K_R} F(w)\zeta^q \, dx\, dt \Big|_{t_0}^{t_1} - q \iint_{Q_R} F(w)\zeta^{q-1}\zeta_t \, dx\, dt.
\end{aligned}$$

Because of (2.18), we define

$$f(\bar{w}) = \int_0^{\bar{w}} G^{r-1}\left(\frac{\zeta\sigma}{R}\right) \sigma^{s+1} \, d\sigma, \tag{2.25}$$

and observe that

$$f(\bar{w}) \le F(w) \le e^{a_2 M_w} f(\bar{w}). \tag{2.26}$$

From the definition (2.25), upper and lower bounds of the function $f$ are derived by using the integration by parts

$$f(\bar{w}) = \int_0^{\bar{w}} f'(\sigma)\, d\sigma = \bar{w} f'(\bar{w}) - \int_0^{\bar{w}} \sigma f''(\sigma)\, d\sigma. \tag{2.27}$$

Note that

$$f'(\sigma) = G^{r-1}\left(\frac{\zeta\sigma}{R}\right) \sigma^{s+1}, \tag{2.28a}$$

$$f''(\sigma) \le \{(r-1)g_0 + (s+1)\}\, G^{r-1}\left(\frac{\zeta\sigma}{R}\right) \sigma^{s}, \tag{2.28b}$$

$$f''(\sigma) \ge \{(r-1)g_1 + (s+1)\}\, G^{r-1}\left(\frac{\zeta\sigma}{R}\right) \sigma^{s}, . \tag{2.28c}$$

Moreover, (2.24) gives

$$(r-1)g_0 + (s+1) = 1 > 0, \tag{2.29a}$$

$$(r-1)g_1 + (s+1) = \frac{g_0}{g_1} > 0. \tag{2.29b}$$

Combination of (2.27), (2.28), and (2.29) implies that

$$\frac{g_0}{g_1}f'(\sigma)\le \sigma f''(\sigma)\le f'(\sigma),$$

from which we derive that

$$\frac{1}{2}\bar{w}f'(\bar{w})\le f(\bar{w})\le \frac{g_1}{g_1+g_0}\bar{w}f'(\bar{w}),$$

which leads to upper and lower bounds of the function $F(u)$

$$\frac{1}{2}G^{r-1}\left(\frac{\zeta\bar{w}}{R}\right)\bar{w}^{s+2}\le F(u)\le \frac{e^{a_2M_w}g_1}{g_1+g_0}G^{r-1}\left(\frac{\zeta\bar{w}}{R}\right)\bar{w}^{s+2}.$$

First we observe that

$$\begin{aligned} D\varphi &= a_2e^{a_2w}G^{r-1}\left(\frac{\zeta\bar{w}}{R}\right)\bar{w}^{s+1}\zeta^q Dw \\ &\quad + e^{a_2w}G^{r-2}\left(\frac{\zeta\bar{w}}{R}\right)\bar{w}^{s}\zeta^q\Xi_1 Dw \\ &\quad + e^{a_2w}G^{r-2}\left(\frac{\zeta\bar{w}}{R}\right)\bar{w}^{s+1}\zeta^{q-1}\Xi_2 D\zeta \end{aligned}$$

with

$$\begin{aligned} \Xi_1 &= (r-1)g\left(\frac{\zeta\bar{w}}{R}\right)\frac{\zeta\bar{w}}{R}+(s+1)G\left(\frac{\zeta\bar{w}}{R}\right) \\ \Xi_2 &= (r-1)g\left(\frac{\zeta\bar{w}}{R}\right)\frac{\zeta\bar{w}}{R}+qG\left(\frac{\zeta\bar{w}}{R}\right). \end{aligned}$$

We note that

$$\frac{g_0}{g_1}G\left(\frac{\zeta\bar{w}}{R}\right)\le \Xi_1\le G\left(\frac{\zeta\bar{w}}{R}\right), \tag{2.30a}$$

$$(g_1-1)G\left(\frac{\zeta\bar{w}}{R}\right)\le \Xi_2\le \left(g_1-\frac{g_0}{g_1}\right)G\left(\frac{\zeta\bar{w}}{R}\right),. \tag{2.30b}$$

Now we have that

$$\begin{aligned} \iint_{Q_R}\mathbf{A}_h\cdot D\varphi_h\,dx\,dt &= a_2\iint_{Q_R}\mathbf{A}_h\cdot Dw_he^{a_2w_h}G^{r-1}\left(\frac{\zeta\bar{w}_h}{R}\right)\bar{w}_h^{s+1}\zeta^q\,dx\,dt \\ &\quad+\iint_{Q_R}\mathbf{A}_h\cdot Dw_he^{a_2w_h}G^{r-2}\left(\frac{\zeta\bar{w}_h}{R}\right)\bar{w}_h^{s}\left(\Xi_1\right)_h\zeta^q\,dx\,dt \\ &\quad+\iint_{Q_R}\mathbf{A}_h\cdot D\zeta e^{a_2w_h}G^{r-2}\left(\frac{\zeta\bar{w}_h}{R}\right)\bar{w}_h^{s+1}\left(\Xi_2\right)_h\zeta^{q-1}\,dx\,dt. \end{aligned} \tag{2.31}$$

By taking $h \to 0$, we obtain, by using the convergence of the Steklov average and Fatou's lemma,

$$\iint_{Q_R} \mathbf{A} \cdot D\varphi \, dx\, dt \geq I_1 + I_2 - I_3 - I_4 - I_5 - I_6 \tag{2.32}$$

where

$$\begin{aligned}
I_1 &= b_1 \iint_{Q_R} e^{a_2 w} G(|Dw|) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^q \, dx\, dt \\
I_2 &= \iint_{Q_R} e^{a_2 w} G(|Dw|) G^{r-2}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \Xi_1 \zeta^q \, dx\, dt \\
I_3 &= b_1 \iint_{Q_R} e^{a_2 w} G(b_0) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^q \, dx\, dt \\
I_4 &= \iint_{Q_R} e^{a_2 w} G(b_0) G^{r-2}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \Xi_1 \zeta^q \, dx\, dt \\
I_5 &= a_1 \iint_{Q_R} e^{a_2 w} g(|Dw|) |D\zeta| G^{r-2}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \Xi_2 \zeta^{q-1} \, dx\, dt \\
I_6 &= \iint_{Q_R} e^{a_2 w} g(b_1) |D\zeta| G^{r-2}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \Xi_2 \zeta^{q-1} \, dx\, dt.
\end{aligned}$$

Due to (2.30a), we obtain that

$$\begin{aligned}
I_2 &\geq \iint_{Q_R} e^{a_2 w} G(|Dw|) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^q \, dx\, dt, \\
I_4 &\leq \iint_{Q_R} e^{a_2 w} G(b_0) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^q \, dx\, dt.
\end{aligned}$$

The inequality (2.30b) and Young's inequality (2.6) lead us to

$$\begin{aligned}
I_5 &\leq \left(g_1 - \frac{g_0}{g_1}\right) a_1 \iint_{Q_R} e^{a_2 w} g(|Dw|) |D\zeta| G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^{q-1} \, dx\, dt \\
&\leq \left(g_1 - \frac{g_0}{g_1}\right) a_1 g_1 \delta_5 \iint_{Q_R} e^{a_2 w} G(|Dw|) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^q \, dx\, dt \\
&\quad + \left(g_1 - \frac{g_0}{g_1}\right) a_1 g_1 \delta_5 \iint_{Q_R} e^{a_2 w} G\left(\frac{|D\zeta| u}{\delta_5 \zeta}\right) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^q \, dx\, dt,
\end{aligned}$$

for any positive constant $\delta_5$. Actually the choice of $\delta_5$ is made to satisfy

$$1 - \left(g_1 - \frac{g_0}{g_1}\right) a_1 g_1 \delta_5 = \frac{1}{2}.$$

Likewise, with the aid of (2.30b) and (2.6), we obtain

$$
\begin{aligned}
I_6 &\le \left(g_1 - \frac{g_0}{g_1}\right) \iint_{Q_R} e^{a_2 w} g(b_1) |D\zeta| G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^{q-1}\, dx\, dt \\
&\le \left(g_1 - \frac{g_0}{g_1}\right)) g_1 \iint_{Q_R} e^{a_2 w} G(b_1) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^{q}\, dx\, dt \\
&\quad + \left(g_1 - \frac{g_0}{g_1}\right) g_1 \iint_{Q_R} e^{a_2 w} G\left(\frac{|D\zeta| u}{\zeta}\right) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^{q}\, dx\, dt.
\end{aligned}
$$

Now we handle

$$
\iint_{Q_R} B\varphi\, dx\, dt \le II_1 + II_2, \tag{2.33}
$$

where

$$
\begin{aligned}
II_1 &= I_1 \\
II_2 &= \iint_{Q_R} e^{a_2 w} G(b_2) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^{q}\, dx\, dt.
\end{aligned}
$$

After canceling out $I_1$ and $II_1$, rearranging integral estimates leads to

$$
\begin{aligned}
&\frac{1}{2+\beta} \int G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+2} \zeta^{q}\, dx \Big|_{t_0}^{t_1} \\
&\quad + \frac{1}{2} \iint_{Q_R} G(|Dw|) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^{q}\, dx\, dt \\
&\le \frac{q e^{a_2 M_w}}{2} \iint_{Q_R} G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+2} \zeta^{q-1} \zeta_t\, dx\, dt \\
&\quad + \gamma_2 \iint_{Q_R} G\left(\frac{|D\zeta| \bar{w}}{\zeta}\right) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^{q}\, dx\, dt \\
&\quad + \gamma_3 \iint_{Q_R} G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s} \zeta^{q}\, dx\, dt \\
&\quad + \gamma_4 \iint_{Q_R} G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^{s+1} \zeta^{q}\, dx\, dt.
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_2 &= g_1 e^{a_2 M_w} \left(g_1 - \frac{g_0}{g_1}\right) \left(1 + a_1 \max\{\delta_5^{1-g_0}, \delta_5^{1-g_1}\}\right) \\
\gamma_3 &= e^{a_2 M_w} \left\{ G(b_0) + \left(g_1 - \frac{g_0}{g_1}\right) g_1 G(b_1) \right\} \\
\gamma_4 &= e^{a_2 M_w} \left\{ a_2 G(b_0) + G(b_2) \right\}.
\end{aligned}
$$

Similarly, we can show Proposition 2.2.1 for $(w-k)_+$ with the test function

$$
\varphi(x,t) = e^{-a_2 w} G^{r-1}\left(\frac{\zeta (w-k)_+}{R}\right) (w-k)_+^{s+1} \zeta^{q},
$$

where $w$ is a nonnegative supersolution of (2.15) under assumption (2.16). □

**Remark 2.2.2.** *The restriction on* $k$ (2.19) *and* (2.20) *give that*

$$I_3 \leq \frac{\delta}{2} \iint_{Q_R} e^{a_2 w} G(b_0) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^s \zeta^q \, dx \, dt$$
$$II_2 \leq \frac{\delta}{2a_2} \iint_{Q_R} e^{a_2 w} G(b_2) G^{r-1}\left(\frac{\zeta \bar{w}}{R}\right) \bar{w}^s \zeta^q \, dx \, dt.$$

*Therefore* (2.23) *is reduced to*

$$\begin{aligned}
&\int G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s+2}\zeta^q \, dx \Big|_{t_0}^{t_1} \\
&\quad + \gamma_0 \iint_{Q_R} G(|Du|) G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^s \zeta^q \, dx \, dt \\
&\leq \gamma_1 \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^{s+2}\zeta^{q-1}\zeta_t \, dx \, dt \\
&\quad + \gamma_2 \iint_{Q_R} G\left(\frac{|D\zeta|(w-k)_\pm}{\zeta}\right) G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^s \zeta^q \, dx \, dt \\
&\quad + \gamma_3 \delta \iint_{Q_R} G(b) G^{r-1}\left(\frac{\zeta(w-k)_\pm}{R}\right)(w-k)_\pm^s \zeta^{q-1} \, dx \, dt.
\end{aligned} \tag{2.34}$$

We introduce ordinary $p-$Laplacian type equations with constants lower order terms to show connection with generalized structures (2.16) with respect to local energy estimates. For $u \in V^{1,p}(\Omega_T)$, the quasilinear $p-$Laplacian equation is given as

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \tag{2.35}$$

satisfying the structure conditions

$$\mathbf{A}(x,t,u,Du) \cdot Du \geq |Du|^p - b_0, \tag{2.36a}$$

$$|\mathbf{A}(x,t,u,Du)| \leq a_1 |Du|^{p-1} + b_1, \tag{2.36b}$$

$$|B(x,t,u,Du)| \leq a_2 |Du|^p + b_2, \tag{2.36c}$$

for $p > 1$, positive constants $a_1, a_2$, and nonnegative constants $b_0, a_2, b_2$. With the test function $(u-k)_\pm \zeta^p$, the energy estimates are obtained as

$$\begin{aligned}
&\sup_t \int_{K_R} (u-k)_\pm^2 \zeta^p \, dx + \gamma_0 \iint_{Q_R} |D(u-k)_\pm \zeta|^p \, dx \, dt \\
&\leq \int_{K_R \times \{t_0\}} (u-k)_\pm^2 \zeta^p \, dx + \gamma_1 \iint_{Q_R} (u-k)_\pm^p |D\zeta|^p \, dx \, dt \\
&\quad + \gamma_2 \iint_{Q_R} (u-k)_\pm^2 \zeta^{p-1}\zeta_t \, dx \, dt + \gamma_3 \iint_{Q_R} b\zeta^p \, dx \, dt,
\end{aligned} \tag{2.37}$$

which is from Section 3(i)(pp. 24) in [11] by replacing lower order terms as constants.

Suppose that we play with $g(s) = s^{p-1}$ and $G(s) = \frac{1}{p}s^p$ for any $s \geq 0$, in case $g_0 = g_1 = p$. Then we observe that the structure conditions (2.16) agree with (2.36). Moreover, the local energy estimates (2.23) and (2.34) exactly coincide with (2.37).

### 2.2.2 The Logarithmic Energy Estimate

For a constant $\upsilon \in (0,1)$, define the function $\Psi(w)$ as

$$\Psi(w) = \begin{cases} \ln^+ \left[\frac{k}{(1+\upsilon)k-(w-k)_-}\right], & \text{for } \ w \geq 0, \ k \geq 0, \\ \ln^+ \left[\frac{k}{(1+\upsilon)k+(w-k)_+}\right], & \text{for } \ w \leq 0, \ k \leq 0, \end{cases} \tag{2.38}$$

for constants $k \in \mathbb{R}$ and $\delta \in (0,1)$. Note that

$$\Psi'(w) = \begin{cases} \frac{-1}{(1+\upsilon)k-(w-k)_-}, & \text{for } \ w \geq 0, \ k \geq 0, \\ \frac{-1}{(1+\upsilon)k+(w-k)_+}, & \text{for } \ w \leq 0, \ k \leq 0, \end{cases}$$

and

$$\Psi''(w) = \begin{cases} \frac{1}{[(1+\upsilon)k-(w-k)_-]^2}, & \text{for } \ w \geq 0, \ k \geq 0, \\ \frac{1}{[(1+\upsilon)k+(w-k)_+]^2}, & \text{for } \ w \leq 0, \ k \leq 0, \end{cases}$$

which provides that

$$\Psi''(w) = \left[\Psi'(w)\right]^2.$$

Moreover,

$$\left[\Psi'(w)\right]^{-1} \leq 2\delta|k| \leq 2\delta\Lambda M_w. \tag{2.39}$$

If $a_2 = 0$, then there is no restriction for $k$.

**Proposition 2.2.2.** *Let $w$ be a bounded weak solution of* (2.15) *under assumption* (2.16) *in a cylinder $Q_R := K_R \times [t_0, t_1]$ and $k \in \mathbb{R}$. For a cutoff function $\zeta$ independent of $t$, then there are constants $\gamma_1, \gamma_2, \gamma_3$ and $b > 0$ depending on data such that*

$$\begin{aligned} \int_{K_R\times\{t_1\}} H(\Psi^2)\zeta^{g_1}\,dx \leq \int_{K_R\times\{t_0\}} & H(\Psi^2)\zeta^{g_1}\,dx \\ & + \gamma_1 \iint_{Q_R} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ & + \gamma_2 \iint_{Q_R} G(b)h(\Psi^2)\left(1+\Psi\right)|\Psi'|^2\zeta^{g_1}\,dx\,dt. \end{aligned} \tag{2.40}$$

*Proof.* We work with the test function

$$\varphi(w) = 2h(\Psi^2)\Psi\Psi'\zeta^{g_1}.$$

Owing to limitations on the level sets (2.19) and , we consider either $0 \le (w-k)_- \le \delta k$ for $k > 0$ or $0 \le (w-k)_+ \le -\delta k$ for $k < 0$. Therefore $\Psi$ and $\Psi'$ are bounded which means $\varphi(w)$ belongs to $L^\infty$ that is admissible space for a test function.

First, we observe that

$$\begin{aligned}\int_{t_0}^{t_1}\int_{K_R}\left(\frac{\partial}{\partial t}w_h\right)\varphi_h\,dx\,dt &= \int_{t_0}^{t_1}\int_{K_R}\left[\frac{d}{dt}H(\Psi^2(w_h))\right]\zeta^{g_1}\,dx\,dt\\ &= \int_{K_R}H(\Psi^2(w_h))\zeta^{g_1}\,dx\Big|_{t_0}^{t_1}\\ &\to \int_{K_R}H(\Psi^2)\zeta^{g_1}\,dx\Big|_{t_0}^{t_1}.\end{aligned}$$

To estimate other parts, note that

$$\begin{aligned}D\varphi = {} & h'(\Psi^2)\left(2\Psi\Psi'\right)^2\zeta^{g_1}Dw\\ & + 2h(\Psi^2)\left[1+\Psi\right](\Psi')^2\zeta^{g_1}Dw\\ & + 2qh(\Psi^2)\Psi\Psi'\zeta^{g_1-1}D\zeta.\end{aligned}$$

Thus

$$\begin{aligned}&\iint_{Q_R}\mathbf{A}_h\cdot D\varphi_h\,dx\,dt\\ &= \iint_{Q_R}\mathbf{A}_h\cdot Dw_h h'(\Psi^2(w_h))\left(2\Psi(w_h)\Psi'(w_h)\right)^2\zeta^{g_1}\,dx\,dt\\ &\quad + 2\iint_{Q_R}\mathbf{A}_h\cdot Dw_h h(\Psi^2(w_h))\left[1+\Psi(w_h)\right](\Psi'(w_h))^2\zeta^{g_1}\,dx\,dt\\ &\quad + 2q\iint_{Q_R}\mathbf{A}_h\cdot D\zeta h(\Psi^2(w_h))\Psi(w_h)\Psi'(w_h)\zeta^{g_1-1}\,dx\,dt.\end{aligned} \tag{2.41}$$

By sending $h \to 0$, we yield, using the convergence of the Steklov average and Fatou's lemma,

$$\iint_{Q_R}\mathbf{A}\cdot D\varphi\,dx\,dt \ge I_1 + I_2 - I_3 - I_4 - I_5 - I_6, \tag{2.42}$$

where

$$I_1 = \iint_{Q_R} G(|Dw|)h'(\Psi^2)\left(2\Psi\Psi'\right)^2 \zeta^{g_1}\,dx\,dt,$$
$$I_2 = 2\iint_{Q_R} G(|Dw|)h(\Psi^2)\Psi(\Psi')^2\zeta^{g_1}\,dx\,dt,$$
$$I_3 = \iint_{Q_R} G(b_0)h'(\Psi^2)\left(2\Psi\Psi'\right)^2 \zeta^{g_1}\,dx\,dt,$$
$$I_4 = 2\iint_{Q_R} G(b_0)h(\Psi^2)\Psi(\Psi')^2\zeta^{g_1}\,dx\,dt,$$
$$I_5 = 2q\iint_{Q_R} g(|Dw|)|D\zeta|h(\Psi^2)\Psi|\Psi'|\zeta^{g_1-1}\,dx\,dt,$$
$$I_6 = 2q\iint_{Q_R} g(b_1)|D\zeta|h(\Psi^2)\Psi|\Psi'|\zeta^{g_1-1}\,dx\,dt.$$

The inequality $(d)$ from Lemma 2.1.2, we observe that

$$4(g_0-1)h(\Psi^2)(\Psi')^2 \le h'(\Psi^2)\left(2\Psi\Psi'\right)^2 \le 4(g_1-1)h(\Psi^2)(\Psi')^2$$

and note that

$$\iint_{Q_R} G(|Dw|)h(\Psi^2)(\Psi')^2\zeta^{g_1}\,dx\,dt \ge 0.$$

Therefore

$$I_1 \ge 4(g_0-1)\iint_{Q_R} G(|Dw|)h(\Psi^2)|\Psi'|^2\zeta^{g_1}\,dx\,dt \ge 0,$$

and

$$\begin{aligned} I_3 &\le 4(g_1-1)\iint_{Q_R} G(b_0)h(\Psi^2)|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ &\le 4(g_1-1)\iint_{Q_R} G(b_0)h(\Psi^2)\left(1+\Psi\right)|\Psi'|^2\zeta^{g_1}\,dx\,dt. \end{aligned}$$

By applying Young's inequality (2.6), we obtain that

$$\begin{aligned} I_5 \le\ & 2qg_1\delta_5\iint_{Q_R} G(|Dw|)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt, \\ & + 2qg_1\delta_5^{1-g_1}\iint_{Q_R} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right)|D\zeta|h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt, \end{aligned}$$

for any positive constant $\delta_5$. In particular, we determine $\delta_3$ such that

$$2qg_1\delta_5 = 2(1-\delta)$$

owing to $II_1$ appearing a bit later. Similarly, by Young's inequality (2.6), we have

$$\begin{aligned} I_6 \leq 2q &\iint_{Q_R} G(b_1)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ &+ 2q \iint_{Q_R} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt. \end{aligned}$$

Also we consider

$$\iint_{Q_R} B\varphi\,dx\,dt \leq II_1 + II_2,$$

where

$$\begin{aligned} II_1 &= 2a_2 \iint_{Q_R} G(|Dw|)h(\Psi^2)\Psi|\Psi'|\zeta^{g_1}\,dx\,dt \\ II_2 &= 2 \iint_{Q_R} G(b_2)h(\Psi^2)\Psi|\Psi'|\zeta^{g_1}\,dx\,dt. \end{aligned}$$

Owing to (2.39), we obtain that

$$\begin{aligned} II_1 &\leq 2\delta \iint_{Q_R} G(|Dw|)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ II_2 &\leq 2\delta\Lambda M_w \iint_{Q_R} G(b_2)h(\Psi^2)\left(1+\Psi\right)|\Psi'|^2\zeta^{g_1}\,dx\,dt. \end{aligned}$$

Upper bound of $II_1$ and $I_5$ with a particular choice of $\delta_5$ cancels out $I_2$. Now, rearrangements of all estimates gives that

$$\begin{aligned} &\int H(\Psi^2)\zeta^{g_1}dx\Big|_{t_0}^{t_1} \\ &\leq [2(qg_1)^{g_1} + 2q] \iint_{Q_R} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ &\quad + \left[4(g_1-1) + 2q + \frac{1}{a_2}\right] \iint_{Q_R} G(b)h(\Psi^2)\left(1+\Psi\right)|\Psi'|^2\zeta^{g_1}\,dx\,dt \end{aligned}$$

because $\delta_t \leq (qg_1)^{-1}$ since $0 < \delta < 1$. □

The logarithmic energy estimate for (2.35) under the structure conditions (2.36) is

$$\begin{aligned} \int_{K_R\times\{t_1\}} \Psi^2\zeta^p dx \leq &\int_{K_R\times\{t_0\}} \Psi^2\zeta^p dx \\ &+ \gamma_1 \iint_{Q_R} \Psi|\Psi'|^{2-p}|D\zeta|^p\,dx\,dt \\ &+ \gamma_2 \iint_{Q_R} b\Psi|\Psi'|^2\zeta^p\,dx\,dt \end{aligned} \tag{2.43}$$

which derived from the test function $\varphi(u) = \left[\Psi^2\right]' \zeta^p$ (refer Section 3 (ii) pp. 28 from [11]). If $g(s) = s^{p-1}$, then $G(s) = \frac{1}{p}s^p$, $h(s) = \frac{1}{p}s^{p-1}$, and $H(s) = \frac{1}{p^2}s^p$. Hence (2.40) becomes

$$\begin{aligned}\int_{K_R\times\{t_1\}} \Psi^{2p}\zeta^p dx \leq \int_{K_R\times\{t_0\}} &\Psi^{2p}\zeta^p dx \\ &+ \gamma_1 \iint_{Q_R} \Psi^{2p-1}|\Psi'|^{2-p}|D\zeta|^p \, dx\, dt \\ &+ \gamma_2 \iint_{Q_R} G(b)\Psi^{2p-2}\left(1+\Psi\right)|\Psi'|^2\zeta^p \, dx\, dt\end{aligned}$$

that is similar to (2.43) but all integrals are weighted by the quantity $\Psi^{2p-2}$.

### 2.2.3 Energy estimates under slightly different structure conditions

If we replace structure condition (2.16c) by

$$|B(x,t,u,Du)| \leq \frac{a_2}{R}g(|Du|) + \frac{1}{R}g(b_2), \tag{2.44}$$

then still we obtain similar types of energy estimates (2.23) and (2.40) without restrictions on level (2.19) and (2.20). The Young's inequality (2.6) generates

$$\begin{aligned}&\int G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s+2}\zeta^q \, dx\Big|_{t_0}^{t_1} \\ &\quad + \gamma_0 \iint_{Q_R} G(|Du|)G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s}\zeta^q \, dx\, dt \\ &\leq \gamma_1 \iint_{Q_R} G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s+2}\zeta^{q-1}\zeta_t \, dx\, dt \\ &\quad + \gamma_2 \iint_{Q_R} G\left(\frac{|D\zeta|(u-k)_+}{\zeta}\right)G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s}\zeta^q \, dx\, dt \\ &\quad + \gamma_3 \iint_{Q_R} G\left(\frac{(u-k)_+}{R\zeta}\right)G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s}\zeta^q \, dx\, dt \\ &\quad + \gamma_4 \iint_{Q_R} G(b)G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s}\zeta^{q-1} \, dx\, dt.\end{aligned}$$

as the local energy estimate with test function

$$\varphi(x,t) = G^{r-1}\left(\frac{\zeta(u-k)_+}{R}\right)(u-k)_+^{s+1}\zeta^q.$$

Because two quantities in the estimate

$$G\left(\frac{|D\zeta|(u-k)_+}{\zeta}\right) \quad \text{and} \quad G\left(\frac{(u-k)_+}{R\zeta}\right)$$

are equivalent except constant multiple difference because $|D\zeta| \le \frac{c}{R}$ for some constant $c$ depending on data.

Moreover, the logarithmic estimate with (2.44) is now

$$\begin{aligned}\int_{K_R\times\{t_1\}} H(\Psi^2)\zeta^{g_1}dx \le \int_{K_R\times\{t_0\}} & H(\Psi^2)\zeta^{g_1}dx \\ & + \gamma_1 \iint_{Q_R} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ & + \gamma_2 \iint_{Q_R} G\left(\frac{1}{R|\Psi'|}\right) h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt \\ & + \gamma_3 \iint_{Q_R} G(b)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt.\end{aligned}$$

Because $|D\zeta| \le c/R$ for some constant $c$, two quantities

$$G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) \quad \text{and} \quad G\left(\frac{1}{R|\Psi'|}\right)$$

are equivalent with only constant multiple difference.

## 2.3 Energy estimates with boundary conditions

Here we obtain the similar energy estimates like (2.23) and (2.40) in a local sense where places near the lateral boundary $S_T$ and near the initial boundary $t = 0$. For a point $(x_0, t_0)$ on $S_T$, construct a cylinder

$$Q_\tau = K_\rho^{x_0} \times [t_0 - \tau, t_0], \quad \text{for } t_0 > 0$$

by choosing $\tau > 0$ is small enough such that $t_0 - \tau > 0$.

### 2.3.1 The Dirichlet boundary value problem

Consider the Dirichlet problem

$$\begin{aligned} u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = B(x,t,u,Du), & \qquad \text{in} \quad \Omega_T \\ u(x,t) = g(x,t), & \qquad \text{if} \quad x \in \partial\Omega,\ t \in (0,T) \\ u(x,0) = u_0(x), & \end{aligned} \tag{2.45}$$

where **A, B** satisfy (2.16). Assume that $u_0$ is a continuous function. Also we require regularity of the function $g$ in terms of trace.

A function $u$ is a weak subsolution of (2.45) if $u$ satisfy

$$\begin{aligned}\int_{K_\rho^{x_0}} u\varphi\,dx - \iint_{Q_\tau} \left[u\varphi_t + \mathbf{A}\cdot D\varphi\right]\,dx\,dt \\ \le \int_{K_\rho^{x_0}} u_0\varphi(x,0)\,dx + \iint_{Q_\tau} B\varphi\,dx\,dt,\end{aligned}$$

and reversing the inequality gives a weak supersolution. Because of the lack of regularity in terms of the time variable, we apply the Steklov average argument that gives

$$u_h(x,0) - u_0(x) \to 0.$$

Therefore we only consider the integral inequality

$$\iint_{Q_\tau} u_t\varphi\,dx\,dt + \iint_{Q_\tau} \mathbf{A}\cdot D\varphi\,dx\,dt \le \iint_{Q_\tau} B\varphi\,dx\,dt,$$

for $0 < t < T - h$.

To derive a similar integral inequality to (2.23), we set the cutoff function $\zeta$ vanish on the parabolic boundary of $Q_\tau$. By choosing

$$k \ge \operatorname*{ess\,sup}_{Q_\tau\cap S_T} g, \qquad \text{for} \quad (u-k)_+,$$

or

$$k \le \operatorname*{ess\,inf}_{Q_\tau\cap S_T} g, \qquad \text{for} \quad (u-k)_-,$$

which make $(u-k)_\pm$ to vanish on $Q_\tau \cap S_T$.

### 2.3.2 The Neumann boundary value problem

Consider the Neumann problem

$$\begin{aligned} u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du), \qquad &\text{in} \quad \Omega_T \\ \mathbf{A}(x,t,u,Du)\cdot\mathbf{n} = \psi(x,t,u), \qquad &\text{on} \quad S_T \\ u(x,0) = u_0(x), \end{aligned} \tag{2.46}$$

where **A, B** satisfy (2.16). A weak subsolution $u$ of (2.46) holds the below integral inequality

$$\int_{K_\rho^{x_0}} u\varphi\,dx\bigg|_{t_0-\tau}^{t_0} + \iint_{Q_\tau} \left[-u\varphi_t + \mathbf{A}\cdot D\varphi\right]\,dx\,dt$$
$$\le \iint_{Q_\tau} B\varphi\,dx\,dt + \int_{t_0-\tau}^{t_0}\int_{K_\rho^{x_0}\cap\partial\Omega} \psi\varphi\,dS\,dt$$

where $dS$ denotes the surface measure on $\partial\Omega$. Reversing the inequality yields the definition for a weak super solution. Taking the Steklov average leads us

$$\iint_{Q_\tau} u_t\varphi\,dx\,dt + \iint_{Q_\tau} \mathbf{A}\cdot D\varphi\,dx\,dt$$
$$\le \iint_{Q_\tau} B\varphi\,dx\,dt + \int_{t_0-\tau}^{t_0}\int_{K_\rho^{x_0}\cap\partial\Omega} \psi\varphi\,dS\,dt.$$

To obtain an energy estimate like (2.23), $\psi$ is differentiated in terms of $x$ and $u$ that those derivatives are related the function spaces for the lower order terms. Since here we only handle constant lower order terms, we impose the limitation on Neumann type data to be constant to obtain the similar energy estimate for the interior.

## 2.4 Integral estimates

### 2.4.1 Collecting positivity

**Lemma 2.4.1.** *Let $u(\cdot,\tau)\in W^{1,1}(K_\rho)$ for all $\tau$ and satisfy*

$$\int_{K_\rho\times\{\tau\}} |Du|\,dx \le \gamma\rho^{N-1} \text{ and } \operatorname{meas}\{x\in K_\rho : u(x,\tau)>1\} \ge \alpha\,|K_\rho|$$

*for some $\gamma>0$ and $\alpha\in(0,1)$. Then for every $\delta\in(0,1)$ and $0<\lambda<1$, there exist $x_0\in K_\rho$ and $\eta=\eta(\alpha,\delta,\gamma,\lambda,N)\in(0,1)$ such that*

$$\operatorname{meas}\left\{x\in K_{\eta\rho}^{x_0} : u(x,\tau)>\lambda\right\} > (1-\delta)\left|K_{\eta\rho}^{x_0}\right|.$$

This lemma is from [15].

### 2.4.2 Poincare type inequality

**Theorem 2.4.2.** *Suppose that $u\in W^{1,1}(K_\rho)$ with $u(x)=0$ on some set $\Sigma_0$ of positive measure. Then for any measurable set $\Sigma$ from $K_\rho$, the inequality holds*

$$\int_\Sigma u(x)\varphi(x)\,dx \le \beta\frac{\rho^N}{|\Sigma_0|}|\Sigma|^{\frac{1}{N}}\int_{K_\rho} |Du(x)|\varphi(x)\,dx.$$

This theorem is appearing on Section 2.5 from [38].

**Corollary 2.4.3.** *Let $v \in W^{1,1}\left(K_\rho^{x_0}\right) \cap C\left(K_\rho^{x_0}\right)$ for some $\rho > 0$ and some $x_0 \in \mathbb{R}^N$ and let $k$ and $l$ be any pair of real numbers such that $k < l$. Then there exists a constant $\gamma$ depending only upon $N, p$ and independent of $k, l, v, x_0, \rho$, such that*

$$\begin{aligned}&(l-k)\operatorname{meas}\left\{x \in K_\rho^0 : v(x) > l\right\}\\&\leq \gamma \frac{\rho^{N+1}}{\operatorname{meas}\left(K_\rho^0 \setminus \left\{x \in K_\rho^0 : v(x) < k\right\}\right)} \int_{\left\{x \in K_\rho^0 : k < v(x) < l\right\}} |Dv|\, dx.\end{aligned}$$

This lemma is appearing on page 5 from [11].

### 2.4.3 Embedding theorem

**Theorem 2.4.4.** *For a nonnegative function $v \in W_0^{1,1}(Q)$ where $Q = K \times [t_0, t_1]$, $K \subset \mathbb{R}^N$, we have*

$$\begin{aligned}\iint_Q v\, dx\, dt \leq C(N) &\left[\iint_Q \chi_{\{v>0\}}\, dx\, dt\right]^{\frac{1}{N+1}} \times\\&\left[\operatorname*{ess\,sup}_{t_0 \leq t \leq t_1} \int_K v\, dx\right]^{\frac{1}{N+1}} \left[\iint_Q |Dv|\, dx\, dt\right]^{\frac{N}{N+1}}.\end{aligned} \tag{2.47}$$

*Proof.* First, by the Hölder inequality, we obtain

$$\iint_Q v\, dx\, dt \leq \left[\iint_Q \chi_{\{v>0\}}\, dx\, dt\right]^{\frac{1}{N+1}} \left[\iint_Q v^{\frac{N+1}{N}}\, dx\, dt\right]^{\frac{N}{N+1}}. \tag{2.48}$$

Second, by Hölder inequality and Sobolev inequality for $p = 1$, we have

$$\begin{aligned}\int_K v^{\frac{N+1}{N}}\, dx &\leq \left[\int_K v^{\frac{N}{N-1}}\, dx\right]^{\frac{N-1}{N}} \left[\int_K v\, dx\right]^{\frac{1}{N}}\\&\leq C(N) \int_K |Dv|\, dx \left[\int_K v\, dx\right]^{\frac{1}{N}}.\end{aligned} \tag{2.49}$$

Combining two inequalities (2.48) and (2.49) produces the inequality (2.47). □

We also introduce a form of the weighted Sobolev inequality which is used in Chapter 4.

**Theorem 2.4.5.** *Let $u \in V_0^p(Q_T)$ where $Q_T := \Omega \times [t_0, t_1] \subset \mathbb{R}^N \times \mathbb{R}$ for some $p \geq 1$. Then*

$$\iint_{Q_T} |u|^{p(N+p)/N}\, dx\, dt \leq C(N,p) \left(\max_s \int_\Omega |u(x,s)|^p\, dx\right)^{p/N} \iint_{\Omega_T} |Du|^p\, dx\, dt.$$

*Moreover, for any nonnegative function* $\lambda \in L^\infty(\Omega_T)$*, if* $k = p$ *for* $N = 1$*,* $k > p$ *for* $1 < N \le p$*, and* $k = N$ *for* $N > p$*, and if* $v = |u|^p \lambda^{k/p}$*, then*

$$\iint_{Q_T} |u|^{p(N+p)/N}\,dx\,dt \le C(N,p)\left(\max_s \int_\Omega v(x,s)\,dx\right)^{p/N} \times \left(\iint_{\Omega_T} |Du|^p\,dx\,dt\right)^{N/k}\left(\iint_{\Omega_T} |u|^p\,dx\,dt\right)^{1-N/k}.$$

This theorem is quoted from [42], Theorem 6.11 on pp.112.

### 2.4.4 Iteration

**Lemma 2.4.6.** *Let* $\{Y_n\}$*,* $n = 0, 1, 2, \ldots,$ *be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \le C b^n Y_n^{1+\alpha}$$

*where* $C, b > 1$ *and* $\alpha > 0$ *are given numbers. If*

$$Y_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

*then* $\{Y_n\}$ *converges to zero as* $n \to \infty$*.*

This lemma is in Section I.4 from [11].

**Lemma 2.4.7.** *Let* $\omega$ *and* $\sigma$ *be increasing functions on an interval* $(0, R_0]$ *and suppose that there are positive constants* $\alpha$*,* $\delta$*, and* $\tau$ *with* $\tau < 1$ *and* $\delta < \alpha$ *such that*

$$r^{-\delta}\sigma(r) \le s^{-\delta}\sigma(s) \quad \text{if } 0 < s \le r \le R_0,$$

*and*

$$\omega(\tau r) \le \tau^\alpha \omega(r) + \sigma(r) \quad \text{if } 0 < r \le R_0.$$

*Then there is a constant* $C = C(\alpha, \delta, \tau)$ *such that*

$$\omega(r) \le C[\left(\frac{r}{R_0}\right)^\alpha \omega(R_0) + \sigma(r)].$$

This lemma is Lemma 4.6 from [42].

**Lemma 2.4.8.** *Let $F$ be a nonnegative and nondecreasing function defined on $[0,1]$. Suppose that*

$$F(\rho) \leq \gamma_0 \left(\frac{\rho}{R}\right)^l F(R) + \gamma_0 R^{l-\sigma\kappa}$$

*holds for all $\rho$ and $R$ such that $0 < \rho < R \leq 1$, where $\gamma_0$, $l$, $\kappa$, and $\sigma \in (0,1)$ with $\kappa < l$. Then there exist constants $0 < \delta < 1$, $q > 1$, and $\gamma_1$ such that*

$$F(\rho) \leq \gamma_1 \rho^\delta \left(R^{-lq} F(R) + 1\right).$$

This lemma is a modified version of Lemma 6 of [46].

# CHAPTER 3. Hölder continuity of $u$

## 3.1 Introduction

This chapter is devoted to showing Hölder continuity of a bounded weak solution, $u \in W^{1,G}(\Omega_T)$, of the differential equation

$$u_t - \text{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \tag{3.1}$$

with structure conditions

$$\mathbf{A}(x,t,u,Du) \cdot Du \geq G(|Du|) - G(b_0), \tag{3.2a}$$

$$|\mathbf{A}(x,t,u,Du)| \leq a_1 g(|Du|) + g(b_1), \tag{3.2b}$$

$$|B(x,t,u,Du)| \leq a_2 G(|Du|) + G(b_2), \tag{3.2c}$$

where $a_1, a_2, b_0, b_1, b_2$ are nonnegative constants. We first show the modulus of continuity of $u$ using the functions $g$ and $G$, then locally $\alpha-$Hölder continuous follows provided by properties of the functions $g$ and $G$.

Roughly speaking, we work in a cylinder about $(x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$ where a nonnegative weak solution does not vanish. Since we are trying to understand behavior of a solution without distinguishing degenerate ($2 < g_0 \leq g_1$) or singular ($1 < g_0 \leq g_1 < 2$), we need to concern about going upward (beyond the given time) where a solution of singular equation may become extinct in finite time (refer [27] and [11]). To not cause any of vanishing solution issue, we work inside of a given cylinder and the goal is finding a subcylinder about $(x_0, t_0)$. Constructing a sequence of nested and shrinking subcylinders about the point $(x_0, t_0)$ will yield to Hölder continuity.

Our method of proof uses some recent ideas of Gianazza, Surnachev, and Vespri [23], who gave a different proof for the Hölder continuity in [3],[8]. While [3],[8] examine an alternative based on the size of the set on which $|u|$ is close to its maximum, the method in [23] using a geometric approach from regularity theory and Harnack estimates and the geometry from [23] is an important ingredient of our proof. On the other hand, [23] takes advantage of the nonvanishing of nonnegative solutions of degenerate equations for all time, so we need to uses some ideas from [4],[5] to analyze the corresponding behavior of more general equations. The proof is based on studying two cases separately. Either a bounded weak solution $u$ is close to its maximum at least half of a cylinder around $(x_0, t_0)$ or not. In either case, the conclusion is that the essential oscillation of $u$ is smaller in a subcylinder centered at $(x_0, t_0)$. Basically, our goal is reached using geometric characters of $u$ with two integral estimates, local and logarithmic estimates (2.23), (2.40).

### 3.2 Intrinsic Scaling

The intrinsic scaling is originated from DiBenedetto & Friedman [12], [13] to show Hölder continuity of $Du$ of systems of degenerate ($p > 2$) type of equations. DiBenedetto [8] and Chen & DiBenedetto [4], [5], [11] used intrinsic scaling to over come the lack of homogeneity of energy estimates and eventually show Hölder continuity of a bounded weak solutions of parabolic $p-$Laplacian type equations (3.3).

To introduce detailed idea about the intrinsic scaling, we go back to the prototype of parabolic $p-$Laplacian

$$u_t - \operatorname{div}|Du|^{p-2}Du = 0 \tag{3.3}$$

and the local energy estimate of (3.3) derived with the test function $(u-k)_\pm \zeta^p$ is

$$\begin{aligned} &\sup_{t_0 \le t \le t_1} \int_{K_R \times \{t\}} (u-k)_\pm^2 \zeta^p \, dx + \iint_{Q_R} |D(u-k)_\pm|^p \, \zeta^p \, dx \, dt \\ &\le \int_{K_R \times \{t_0\}} (u-k)_\pm^2 \zeta^p \, dx + C \iint_{Q_R} (u-k)_\pm^p |D\zeta|^p \, dx \, dt \\ &\quad + p \iint_{Q_R} (u-k)_\pm^2 \zeta^{p-1} \zeta_t \, dx \, dt \end{aligned} \tag{3.4}$$

for some constant $C$ where $\zeta$ is a cutoff function in the cylinder $Q_R := K_R \times [t_0, t_1]$ ( Proposition 2.4 in [58]). When $p = 2$, the equation (3.3) is the heat equation and (3.4) is homogenous; that is, all integral norms are corresponding to the same power $p = 2$. The iterative procedure works well in the standard parabolic cylinder $K_R \times [t_0, t_0 + R^2]$. When $1 < p < \infty$ but $p \neq 2$, the energy estimate (3.4) is now nonhomogeneous because integral norms are matching two different powers $p$ and 2. Therefore neither $K_R \times [t_0, t_0 + R^2]$ nor $K_R \times [t_0, t_0 + R^p]$ works properly for iterations. Indeed, the cylinder $K_R \times [t_0, t_0 + R^p]$ is working for a homogeneous equation

$$\left(u^{p-1}\right)_t - \text{div}|Du|^{p-2}Du = 0 \tag{3.5}$$

because an energy estimate by (3.5) with the test function $(u-k)_{\pm}\zeta^p$ involves the integral norms powered by only $p$.

Here we point out that intrinsic scaling is modified differently for degenerate ($p > 2$) and singular ($1 < p < 2$) equations in the original work by DiBenedetto[8] and chen & DiBenedetto [4] [5]. First, we consider (3.3) with $p > 2$. If we assume positive initial data, a solution never vanish in any finite time. Therefore intrinsic scaling for the time variable is natural for degenerate type of equations. The cylinder $K_R \times [t_0, t_0 + R^p]$ is replaced by intrinsically scaled cylinder

$$K_R \times [t_0, t_0 + \left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p] \tag{3.6}$$

where a constant $\lambda$ to be determined somewhat large. Due to $p > 2$, the quantity

$$\left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p > R^p$$

which implies the intrinsically rescaled cylinder has stretched the time axis.

Second, for singular equations (3.3) with $1 < p < 2$, a weak solution may become vanishing in finite time but positivity spreads over the space easily. To take advantage of singular solutions behavior, Chen & DiBenedetto approached with intrinsic scaling in the space length (side length of $K_R$). The cylinder $K_R \times [t_0, t_0 + R^p]$ is now replaced by the cylinder

$$K_{sR} \times [t_0, t_0 + R^p], \quad s = \left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}}$$

where $\lambda$ will be chosen later large enough. Because of $1 < p < 2$, we have

$$\left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}} R > R$$

which means that stretch of the side length is made for singular equations.

To introduce the intrinsic scaling, we rewrite (3.3) into

$$\frac{1}{p-1}u^{2-p}\left(u^{p-1}\right)_t - \operatorname{div}|Du|^{p-2}Du = 0. \tag{3.7}$$

This tells us that the homogeneity can be recovered at the expense of a scaling factor that looks like $u^{2-p}$, which is $\omega^{2-p}$ where $\omega := \operatorname{ess\,osc}_{Q_R} u$.

To deliver a uniform geometric setting, we go back to the prototype of generalized $p-$Laplacian equation (1.16) and note a local energy estimate is

$$\begin{aligned}
&\sup_{t_0\le t\le t_1}\int_{K_R\times\{t\}} G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s+2}\zeta^q\,dx \\
&\quad+\gamma_0\iint_{Q_R} G(|Du|)G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s}\zeta^q\,dx\,dt \\
&\le \int_{K_R\times\{t_0\}} G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s+2}\zeta^q\,dx \\
&\quad+\gamma_1\iint_{Q_R} G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s+2}\zeta^{q-1}\zeta_t\,dx\,dt \\
&\quad+\gamma_2\iint_{Q_R} G\left(\frac{|D\zeta|(u-k)_\pm}{\zeta}\right)G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s}\zeta^q\,dx\,dt.
\end{aligned} \tag{3.8}$$

From (3.8), we observe that there are two integral norms to compare. Therefore we want to set up intrinsic scaling for the time length to satisfy that

$$G^r\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s} \sim G^{r-1}\left(\frac{\zeta(u-k)_\pm}{R}\right)(u-k)_\pm^{s+2}\zeta^{q-1}\zeta_t \tag{3.9}$$

by assuming that

$$|D\zeta| \le \frac{c}{R}$$

for some constant $c$. The (3.9) leads to

$$G\left(\frac{\omega}{R}\right) \sim \frac{\omega^2}{T}$$

which says

$$T \sim \omega^2 G\left(\frac{\omega}{R}\right)^{-1}.$$

If $G(s) = s^p$, then

$$T \sim w^2\left(\frac{\omega}{R}\right)^{-p} = \omega^{2-p}R^p$$

which coincides with intrinsically scaled time length for $p-$Laplacian (3.3) with $p > 2$.

To deliver a uniform geometric setting that capture behaviors of solutions of degenerate and singular types of equations, we define a standard cylinder for generalized equation such that

$$T_{k,\rho} := \theta k^2 G\left(\frac{k}{\rho}\right)^{-1},$$
$$Q_{k,\rho}^{x_0,t_0} := K_\rho^{x_0} \times [t_0 - T_{k,\rho}, t_0],$$
$$Q_{k,\rho} := Q_{k,\rho}^{0,0},$$

for given $(x_0, t_0) \in \mathbb{R}^{N+1}$, given positive constants $\rho$ and $k$, and a positive constant $\theta$ to be determined later depending on data. Note that $\rho > 0$ and $k > 0$ in both $T_{k,\rho}$ and $Q_{k,\rho}$ are replaceable. Details for $T_{k,\rho}$ and $\theta$ is following.

## 3.3 The main lemma and Hölder estimates

The main lemma says that a nonnegative solution $u$ is strictly positive in a subcylinder if $u$ is near to the maximum value in more than a half of cylinder.

Denote

$$b = \max\{b_0, b_1, b_2\}$$

where nonnegative constants $b_0$, $b_1$, and $b_2$ are given in (3.2).

**Lemma 3.3.1.** *(Main Lemma) For a given constant $\theta > 0$, suppose that $w$ is a nonnegative bounded weak solution of* (3.1) *under assumption* (3.2) *in a cylinder*

$$Q_{M,4R}^{x_0,t_0} := K_{4R}^{x_0} \times [t_0 - \theta M^2 G\left(\frac{M}{4R}\right)^{-1}, t_0]$$

*where constants $M$ and $R > 0$ satisfy*

$$\operatorname{ess\,sup} |(w - M)_\pm| \le \delta M \quad \textit{for a constant } \delta \in (0,1), \tag{3.10a}$$
$$M_w \le M \le \Lambda M_w, \tag{3.10b}$$
$$R \le \lambda M, \tag{3.10c}$$

*for some constant* $\Lambda \geq 1$ *and* $\lambda > 0$. *There exist positive constants* $\theta$, $\mu \in (0,1)$, *and* $\lambda \in (0,1)$ *depending on data such that, if* $w$ *satisfies*

$$\text{meas}\left\{(x,t) \in Q^{x_0,t_0}_{M,2R} : w(x,t) > 2M\right\} > \frac{1}{2}\left|Q^{x_0,t_0}_{M,2R}\right|, \tag{3.11}$$

*then*

$$\operatorname*{ess\ inf}_{K^{x_0}_R \times [t_0 - \lambda T_{M,R}, t_0]} w(x,t) \geq \mu M.$$

*Proof.* The proof of the main lemma will be presented in the end of Section 3.4. □

**Remark 3.3.1.** *Under the same hypothesis on Lemma 3.3.1 for* $w$ *and* $k$, *suppose that for some constant* $\alpha > 0$

$$\text{meas}\left\{(x,t) \in Q^{x_0,t_0}_{k,2R} : w(x,t) > \alpha k\right\} > \frac{1}{2}\left|Q^{x_0,t_0}_{k,2R}\right|,$$

*then the constants* $\theta$, $\mu \in (0,1)$, *and* $\lambda \in (0,1)$ *depending on data and* $\alpha$ *such that*

$$\operatorname*{ess\ inf}_{K^{x_0}_R \times [t_0 - \lambda T_{k,R}, t_0]} w(x,t) \geq \mu k.$$

**Lemma 3.3.2.** *Suppose that* $u$ *is a weak bounded solution of* (3.1) *under assumption* (3.2) *(say* $|u| \leq M$*) in in a cylinder* $Q^{x_0,t_0}_{k,4R}$ *for some positive constants* $k$, $R$, *and* $\Lambda$ *satisfying* (3.10). *Also say that* $|u| \leq M$ *for a positive constant* $M$. *Then there exist positive constants* $\eta \in (0,1)$, $\lambda \in (0,1)$ *and* $c$ *such that*

$$\operatorname*{ess\ osc}_{K^{x_0}_R \times [t_0 - \lambda T_{k,R}]} u(x,t) \leq \eta \operatorname*{ess\ osc}_{Q^{x_0,t_0}_{k,4R}} u(x,t) + cR. \tag{3.12}$$

*Proof.* Without loss of generality, let $(x_0, t_0) := (0,0)$. For simple notation, denote $Q := K_R \times [-\lambda T_{k,R}, 0]$. If $\text{ess osc}_{Q_{k,4R}} u = 0$, then clearly $\text{ess osc}_Q u = 0$. Hence (3.12) is true. Assume that $\text{ess osc}_{Q_{k,4R}} u(x,t) > 0$ and $w = u - \text{ess inf}_{Q_{k,4R}} u(x,t)$ which is a nonnegative bounded weak solution of (3.1) under assumption (3.2).

When (3.11) holds, then Lemma 3.3.1 directly says that there exist $\mu \in (0,1)$ and $\lambda \in (0,1)$ such that if $w$ satisfies (3.11)

$$\operatorname*{ess\ inf}_{Q} w(x,t) \geq \mu k$$

equivalently

$$\operatorname*{ess\,inf}_{Q} u(x,t) \geq \operatorname*{ess\,inf}_{Q_{k,4R}} u(x,t) + \mu k. \tag{3.13}$$

The inequality (3.13) implies

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q} u(x,t) &= \operatorname*{ess\,sup}_{Q} u(x,t) - \operatorname*{ess\,inf}_{Q} u(x,t) \\
&\leq \operatorname*{ess\,sup}_{Q_{k,4R}} u(x,t) - \operatorname*{ess\,inf}_{Q_{k,4R}} u(x,t) - \mu k \qquad (3.14) \\
&\leq \operatorname*{ess\,osc}_{Q_{k,4R}} u(x,t) - \frac{\mu k}{2M} \operatorname*{ess\,osc}_{Q_{k,4R}} u(x,t)
\end{aligned}$$

because of the boundedness that $|u| \leq M$ (which says $\operatorname{ess\,osc}_{Q_{k,4R}} u(x,t) \leq 2M$) for a positive constant $M$.

When (3.11) fails, then we consider two cases; either

$$\operatorname*{ess\,osc}_{Q_{k,4R}} w(x,t) \leq \sigma k, \tag{3.15a}$$

$$\operatorname*{ess\,osc}_{Q_{k,4R}} w(x,t) > \sigma k, \tag{3.15b}$$

for some $\sigma \in (1-\mu, 1)$.

In case (3.15a) holds, it is clear to find

$$\operatorname*{ess\,osc}_{Q} w(x,t) \leq \sigma k,$$

equivalently

$$\operatorname*{ess\,sup}_{Q} u(x,t) \leq \operatorname*{ess\,inf}_{Q_{k,4R}} u(x,t) + \sigma k.$$

Therefore

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q} u(x,t) &\leq \operatorname*{ess\,inf}_{Q_{k,4R}} u(x,t) - \operatorname*{ess\,inf}_{Q_{k,4R}} u(x,t) + \sigma k \\
&\leq \sigma k \qquad (3.16) \\
&\leq \sigma b R
\end{aligned}$$

using (**??**) which satisfies (3.12).

The inequality (3.15b) implies

$$(1-\sigma)k + \operatorname*{ess\,sup}_{Q_{k,4R}} u(x,t) - u \geq \frac{k}{2}.$$

Therefore Lemma 3.3.1 says that

$$(1-\sigma)k + \operatorname*{ess\,sup}_{Q_{k,4R}} u(x,t) - \operatorname*{ess\,inf}_{Q} u(x,t) \geq \mu k,$$

which yields

$$\begin{aligned} \operatorname*{ess\,sup}_{Q} u(x,t) &\leq \operatorname*{ess\,sup}_{Q_{k,4R}} u - (\sigma+\mu-1)k \\ &\leq \left[1 - \frac{(\sigma+\mu-1)k}{2M}\right] \operatorname*{ess\,osc}_{Q_{k,4R}} u. \end{aligned} \tag{3.17}$$

Note that $\sigma + \mu > 1$.

Therefore, we choose $\eta \in (0,1)$ such that

$$\eta = \max\left\{1 - \frac{\mu k}{2M}, \sigma, 1 - \frac{(\sigma+\mu-1)k}{2M}\right\}$$

and note that $M$ can be controlled large enough so $\eta \in (0,1)$. Moreover

$$c = \max\left\{(1+\delta)b, \sigma b\right\}.$$

Hence we reach to our conclusion (3.14), (3.16), and (3.17). □

Now based on Lemma 3.3.2, we find a sequence of shrinking and nested cylinders that corresponds to a strictly decreasing sequence of oscillations.

**Lemma 3.3.3.** *Suppose that $u$ is a bounded weak solution of* (3.1) *under assumption* (3.2) *in a cylinder $\Omega_{r,s}^{x_0,t_0}$. There exists a family of shrinking and nested cylinders $\{Q_n\}_{n=0}^{\infty}$ such that*

$$\operatorname*{ess\,osc}_{Q_n} u(x,t) \leq \eta^n \operatorname*{ess\,osc}_{\Omega_{r,s}^{x_0,t_0}} u(x,t) + \frac{c}{1-\eta} r$$

*where constants $\eta \in (0,1)$ and $c$ are defined in Lemma 3.3.2.*

*Proof.* For a positive constant $k \leq \frac{1}{2a_2}$, choose $R > 0$ such that

$$K_R^{x_0} \times [t_0 - \theta k^2 G\left(\frac{k}{R}\right)^{-1}, t_0] \subset \Omega_{r,s}^{x_0,t_0}$$

which means

$$R \leq \min\left\{r, \frac{k}{G^{-1}\left(\theta k^2 s^{-1}\right)}\right\}.$$

Also determine $\epsilon \in (0,1)$ such that

$$K^{x_0}_{\epsilon^{n+1}R} \times [t_0 - \theta k^2 G\left(\frac{k}{\epsilon^{n+1}R}\right)^{-1}, t_0] \subset K^{x_0}_{\epsilon^n R/4} \times [t_0 - \theta k^2 G\left(\frac{k}{\epsilon^n R/4}\right)^{-1}, t_0], \tag{3.18}$$

that is,

$$4\epsilon \leq \min\left\{1, \lambda^{\frac{1}{g_0}}, 4\eta\right\}. \tag{3.19}$$

The necessity of inequality $\epsilon < \eta$ will become clear in Theorem 3.3.4.

By letting

$$Q_n := K^{x_0}_{\epsilon^n R} \times [t_0 - \theta k^2 G\left(\frac{k}{\epsilon^n R}\right)^{-1}, t_0], \tag{3.20}$$

the relationship (3.18) and Lemma 3.3.1 says that there exist $\eta \in (0,1)$ and $c$ such that

$$\operatorname*{ess\,osc}_{Q_{n+1}} u(x,t) \leq \eta \operatorname*{ess\,osc}_{Q_n} u(x,t) + cR.$$

Throughout iteration, we obtain that

$$\operatorname*{ess\,osc}_{Q_n} u(x,t) \leq \eta^n \operatorname*{ess\,osc}_{Q_0} u(x,t) + c\left(\sum_{i=0}^{i=n-1} \eta^i\right) R$$

which leads to our conclusion because $Q_0 \subset \Omega^{x_0,t_0}_{r,s}$ and $R \leq r$. □

For a bounded weak solution $u$ of (3.1) under (3.2) in a cylinder $\Omega_T \subset \mathbb{R}^{N+1}$, suppose there are two distinct points $(x_1, t_1)$ and $(x_2, t_2)$ in $\Omega_T$. Then we define the length between $t_1$ and $t_2$ by

$$\|t_1 - t_2\|_G = \frac{\|u\|^2_{\infty,\Omega_T}}{G^{-1}\left(\theta\|u\|^2_{\infty,\Omega_T}/|t_1 - t_2|\right)}. \tag{3.21}$$

Moreover, we define distance in between two sets in $\mathbb{R}^{N+1}$ such that

$$\operatorname{dist}(\mathcal{K}_1; \mathcal{K}_2) := \inf_{(x_1,t_1)\in\mathcal{K}_1, (x_2,t_2)\in\mathcal{K}_2} \left(|x_1 - x_2| + \|t_1 - t_2\|_G\right).$$

Because of the function $G$, it is natural to figure out the modulus of continuity of $u$ with presence of $G$ (Theorem 3.3.4). Then later, we are able to derive Hölder estimate written in terms of powers involving $g_0$ and $g_1$ under an extra assumption (Corollary 3.3.5).

**Theorem 3.3.4.** *Suppose that $u$ is a bounded weak solution of* (3.1) *under assumption* (3.2) *in a cylinder $\Omega_T \subset \mathbb{R}^{N+1}$. Also assume that there are two distinct points $(x_1, t_1)$ and $(x_2, t_2)$*

*in a cylinder* $\Omega_{r,s}^{x_0,t_0}$ *which is a subset of* $\Omega_T$ *strictly away from* $\partial_p\Omega_T$*. Then* $(x,t) \to u(x,t)$ *has modulus of continuity. Moreover, there exist positive constants* $\gamma$*,* $\beta$*, and* $\alpha \in (0,1)$ *depending upon data such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma\,(\text{data}, \|u\|_{\infty,\Omega_T}) \left( \frac{|x_1 - x_2| + \|t_1 - t_2\|_G}{\text{dist}(\Omega_{r,s}^{x_0,t_0}; \partial_p\Omega_T)} \right)^{\alpha}.$$

*Proof.* Without loss of generality, let $(x_0,t_0) := (0,0)$. From Lemma 3.3.3, we build a sequence of shrinking and nested cylinders $\{Q_n\}_{n=0}^{\infty}$ such that

$$\operatorname*{ess\,osc}_{Q_n} u(x,t) \leq \eta^n \operatorname*{ess\,osc}_{\Omega_{r,s}^{x_0,t_0}} u(x,t) + \frac{c}{1-\eta} r$$

for some constants $\eta \in (0,1)$ and $c$. Because of (3.20), for positive constants $R$, $k$, and $\epsilon$, there exist nonnegative integers $m_1$ and $m_2$ such that

$$\epsilon^{m_1+1} R < |x_1 - x_2| \leq \epsilon^{m_1} R, \tag{3.22a}$$

$$\theta k^2 G \left( \frac{k}{\epsilon^{m_2+1} R} \right)^{-1} < |t_1 - t_2| \leq \theta k^2 G \left( \frac{k}{\epsilon^{m_2} R} \right)^{-1}. \tag{3.22b}$$

Denote $m := \min\{m_1, m_2\}$. Indeed, $(x_1,t_1)$ and $(x_2,t_2)$ are in the cylinder $Q_m$.

Then inequality (3.22b) actually implies that

$$c(k, \|u\|_\infty)\epsilon^{m_2+1} R < \|t_1 - t_2\|_G \leq C(k, \|u\|_\infty)\epsilon^{m_2} R \tag{3.23}$$

for some constants $c$ and $C$ depending on data and $k$ and $\|u\|_{\infty,\Omega_{r,s}^{x_0,t_0}}$.

From Lemma 3.3.3, we obtain that

$$\operatorname*{ess\,osc}_{Q_m} u(x,t) \leq \eta^m \operatorname*{ess\,osc}_{\Omega_{r,s}^{x_0,t_0}} u(x,t) + \frac{c}{1-\eta} r.$$

Note

$$\epsilon^m = \eta^{m \log_\eta \epsilon}$$

and the choice of $\epsilon$ in (3.19) gives that

$$\alpha := \log_\epsilon \eta > 0.$$

The left hand side of inequality (3.22a) gives that

$$\eta^m < \epsilon^{(m-m_1-1)\alpha} \left( \frac{r}{R} \right)^{\alpha} \left( \frac{|x_1 - x_2|}{r} \right)^{\alpha}.$$

Moreover, the left hand side of inequality (3.23) yields

$$\eta^m < c\epsilon^{(m-m_2-1)\alpha}\left(\frac{r}{R}\right)^{\alpha}\left(\frac{\|t_1-t_2\|_G}{r}\right)^{\alpha}$$

for some constant $c$ depending on data, $k$, and $\|u\|_{\infty,\Omega_{r,s}^{x_0,t_0}}$.

In addition, we observe that

$$r = r\left(\frac{|x_1-x_2|}{r}\right)^{-\alpha}\left(\frac{|x_1-x_2|}{r}\right)^{\alpha}.$$

Therefore, we complete the proof by choosing

$$\gamma = \max\left\{\epsilon^{(m-m_1-1)\alpha}\left(\frac{r}{R}\right)^{\alpha}, c\epsilon^{(m-m_2-1)\alpha}\left(\frac{r}{R}\right)^{\alpha}, r\left(\frac{|x_1-x_2|}{r}\right)^{-\alpha}\right\}$$

and noting that

$$r \geq \mathrm{dist}(\Omega_{r,s}^{x_0,t_0}; \partial_p\Omega_T).$$

□

**Corollary 3.3.5.** *Suppose that $u$ is a bounded weak solution of* (3.1) *under assumption* (3.2) *in a cylinder $\Omega_T \subset \mathbb{R}^{N+1}$. Also assume that there are two distinct points $(x_1, t_1)$ and $(x_2, t_2)$ in a cylinder $\Omega_{r,s}^{x_0,t_0}$ which is a subset of $\Omega_T$ strictly away from $\partial_p\Omega_T$. Then $(x,t) \to u(x,t)$ is locally Hölder continuous. Moreover, there exist positive constants $\gamma$, $\beta$, and $\alpha \in (0,1)$ depending upon data such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma\,(\mathrm{data}, \|u\|_{\infty,\Omega_T})\left(\frac{|x_1-x_2| + \beta\theta^{-\frac{1}{g_0}}\|u\|_{\infty,\Omega_T}^{\frac{g_0-2}{g_0}}|t_1-t_2|^{\frac{1}{g_0}}}{Hdist(\Omega_{r,s}^{x_0,t_0}; \partial_p\Omega_T)}\right)^{\alpha} \tag{3.24}$$

*where*

$$Hdist(\Omega_{r,s}^{x_0,t_0}, \partial_p\Omega_T) = \inf_{(x,t)\in\Omega_{r,s}^{x_0,t_0},(y,s)\in\partial_p\Omega_T}\left[|x-y| + \beta\theta^{-\frac{1}{g_0}}\|u\|_{\infty,\Omega_T}^{\frac{g_0-2}{g_0}}|t-s|^{\frac{1}{g_0}}\right]. \tag{3.25}$$

*Proof.* For simplicity, let $\beta$ be a positive constant such that

$$G(\beta^{-1}) = 1. \tag{3.26}$$

For a positive constant $k \leq \frac{1}{2a_2}$ (without loss of generality, let $k \leq \|u\|_{\infty,\Omega_T}$), we determine $R > 0$ such that

$$R \leq \beta k, \tag{3.27a}$$

$$K_R^{x_0} \times [t_0 - \theta k^2 G\left(\frac{k}{R}\right)^{-1}, t_0] \subset K_r^{x_0} \times [t_0 - s, t_0]. \tag{3.27b}$$

Therefore (3.27b) generates $R \leq r$ and

$$\theta k^2 G\left(\frac{k}{R}\right)^{-1} \leq s. \tag{3.28}$$

The inequality (3.28) is guaranteed if

$$\theta k^2 \leq s\left(\frac{\beta k}{R}\right)^{g_0}$$

because of (3.26), (3.27a), and (2.4). Therefore we derive that

$$R \leq \min\left\{r, \beta k, \beta\theta\left(\frac{k^{g_0-2}s}{\theta}\right)^{\frac{1}{g_0}}\right\}$$

which indeed inspires (3.25).

Here we recall the left hand side of (3.22b)

$$\theta k^2 G\left(\frac{k}{\epsilon^{m_2+1}R}\right)^{-1} < |t_1 - t_2|. \tag{3.29}$$

Owing to (3.26) and (3.27a), the inequality (3.29) becomes true if

$$\theta k^2 < \left(\frac{\beta k}{\epsilon^{m_2+1}R}\right)^{g_0} |t_1 - t_2|$$

which implies

$$\epsilon^m < \epsilon^{m_2+1-m}\beta\left(\frac{k^{g_0-2}}{\theta R}\right)^{\frac{1}{g_0}} |t_1 - t_2|^{\frac{1}{g_0}}.$$

This complete our conclusion. □

## 3.4 Proof of the Main Lemma

Throughout this section, let $w$ to be a bounded nonnegative weak solution of (3.1) with structure conditions (3.2). The proof of Lemma 3.3.1 will be given at the end of this section which composed with four propositions under the assumption that $w$ is somewhat large at least half of a cylinder. Then Proposition 3.4.1 implies that a spatial cube at some fixed time level is found on which $w$ is away from its minimum (zero value) on arbitrary fraction of the spatial cube. From the spatial cube, positive information spread in both the time and the space variables carrying time limitations (Proposition 3.4.2 and Proposition 3.4.4). The constant $\theta$ appearing intrinsically scaled time length is determined at the end of this section in order to control time length properly for spreading positivity. From Proposition 3.4.2 & Proposition 3.4.4, a subcylinder is obtained which shares the same center of the originally givencylinder. Moreover, the place where $w$ is close to its minimum (zero) is controlled with arbitrary fraction of the subcylinder. From this subcylinder, De Giorgi iteration (Proposition 3.4.5) leads to the main lemma (Lemma 3.3.1).

Throughout this section, we assume a positive constant $k$ to satisfy

$$b \le c\frac{k}{\rho} \tag{3.30}$$

where $b = \max\{b_0, b_1, b_2\}$ from (3.1) under (3.2) and for a constant $c$ to be determined later depending on the data. Also for positive constants $k$, $\rho$, and $\theta$, we recall that

$$Q_{k,\rho} = K_\rho \times [-T_{k,\rho}, 0]$$

where

$$T_{k,\rho} = \theta k^2 G\left(\frac{k}{2\rho}\right)^{-1}.$$

The techniques verifying Proposition 3.4.1 borrow some ideas from Proposition 3.7 in [23], Lemmata III.7.1, IV.10.1 in [11], and concerning the equation (4.2) on page 35 in [58].

**Proposition 3.4.1.** *For given constants $k > 0$ and $\rho > 0$, suppose that $w$ is a nonnegative weak solution of* (3.1) *under assumption* (3.2) *on the cylinder $Q_{k,2\rho}$ satisfying*

$$\text{meas}\{(x,t) \in Q_{k,\rho} : w(x,t) > k\} \ge \frac{1}{2}|Q_{k,\rho}| \tag{3.31}$$

*Then for any $\nu_1 \in (0,1)$ and $\delta_1 \in (0,1)$, there exist $y \in K_\rho$, $\tau_1 \in [T_{k,\rho}/16, T_{k,\rho}]$ and $\eta = \eta(\text{data}, k) \in (0,1)$ such that $K^y_{\eta\rho} \subset K_\rho$ and*

$$\text{meas}\left\{x \in K^y_{\eta\rho} : w(x, -\tau_1) < \delta_1 k\right\} < (1-\nu_1)\left|K^y_{\eta\rho}\right|.$$

*Proof.* We apply the local energy estimate (2.34) with a piecewise linear cutoff function

$$\zeta = \begin{cases} 1 & \text{inside } \ Q_{k,\rho} \\ 0 & \text{on the parabolic boundary of } \ Q_{k,2\rho} \end{cases}$$

with

$$|D\zeta| \le \frac{1}{\rho}, \quad \zeta_t \le \frac{1}{(2^{g_0}-1)\,\theta k^2} G\left(\frac{k}{\rho}\right).$$

Because of the support of $\zeta$ and positivity of integral, we ignore the first term on the left hand side of (2.34). Then we have

$$\begin{aligned}
&\int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G\left(|D(w-k)_-|\right) G^{r-1}\left(\frac{\zeta(w-k)_-}{\rho}\right)(w-k)_-^s \zeta^q \, dx\, dt \\
&\le \gamma_1 \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G^{r-1}\left(\frac{\zeta(w-k)_-}{\rho}\right)(w-k)_-^{s+2}\zeta^{q-1}\zeta_t \, dx\, dt \\
&\quad + \gamma_2 \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G^{r}\left(\frac{\zeta(w-k)_-}{\rho}\right)(w-k)_-^{s}\zeta^{q-1-2g_1} \, dx\, dt \\
&\quad + \gamma_3 \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G(b) G^{r-1}\left(\frac{\zeta(w-k)_-}{\rho}\right)(w-k)_-^{s}\zeta^{q-1-2g_1} \, dx\, dt,
\end{aligned}$$

for some constants $\gamma_1$, $\gamma_2$, and $\gamma_3$. Note that

$$(w-k)_- = \max\{0, (k-w)\} \le k,$$

and the maps $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$, $\sigma \mapsto G^r(\sigma)\sigma^s$ are increasing and the map $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$ is decreasing.

Therefore we obtain that

$$\begin{aligned}
&\int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G\left(|D(w-k)_-|\right) \zeta^q \, dx\, dt \\
&\le \left\{\gamma_1 k^2 \frac{G(k/\rho)}{(2^{g_0-1})\theta k^2} + \gamma_2 G\left(\frac{k}{\rho}\right) + \gamma_3 G(b)\right\} |K_{2\rho} \times [-T_{k,2\rho}, 0]| \\
&\le \tilde{\gamma} G\left(\frac{k}{\rho}\right) |K_{2\rho} \times [-T_{k,2\rho}, 0]| \,.
\end{aligned}$$

Then Jensen's inequality provides

$$\int_{-T_{k,\rho}}^0 \int_{K_\rho} |D(w-k)_-| \, dx\, dt \le \frac{\gamma k}{\rho} |K_\rho \times [-T_{k,\rho}, 0]| \tag{3.32}$$

for some constant $\gamma$ depending on data and $k$.

Now we say that there exists $\tau_1 \in [-T_{k,\rho}, -T_{k,\rho}/16]$ satisfying both

$$\int_{K_\rho \times \{-\tau_1\}} |D(w-k)_-| \, dx \leq \frac{16\gamma k}{\rho} |K_\rho|, \tag{3.33a}$$

$$|\{w(x,-\tau_1) \geq k\} \cap K_\rho| \geq \frac{5}{8} |K_\rho|. \tag{3.33b}$$

If the inequality (3.33a) fails in the time set with more than $T_{k,\rho}/16$ measure, then clearly it produces contradiction to the inequality (3.32). If (3.33b) fails in the set with more than $T_{k,\rho}/8$ measure set, then we derive

$$\begin{aligned} \text{meas}\{Q_{k,\rho} : w(x,t) \geq k\} &= |Q_{k,\rho}| - \text{meas}\{Q_{k,\rho} : w(x,t) < k\} \\ &\leq \left(1 - \frac{5}{8}(1 - \frac{1}{8})\right) |Q_{k,\rho}| \\ &= \frac{29}{64} |Q_{k,\rho}| < \frac{1}{2} |Q_{k,\rho}|, \end{aligned}$$

that contradicts to our assumption (3.31). Therefore in the set $[-T_{k,\rho}, 0]$, the inequality (3.33a) holds in more than set with measure $15T_{k,\rho}/16$ and the inequality (3.33b) is true in more that set with measure $7T_{k,\rho}/8$. Thus, there exists $\tau_1 \in [-T_{k,\rho}, 0]$ where (3.33) hold. Our conclusion is made after applying Lemma 2.4.1 which is quoted from [15] that for any $\delta \in (0,1)$ and $\lambda \in (0,1)$, there exist $y \in K_\rho$ and $\eta = \eta(k, \text{data}) \in (0,1)$ such that

$$\text{meas}\left\{x \in K^y_{\eta\rho} : (w-k)_- > \lambda k\right\} > (1-\delta)\left|K^y_{\eta\rho}\right|. \tag{3.34}$$

By choosing $\lambda = 1 - \delta_1$ and $\delta = 1 - \nu_1$ for any $\delta_1 \in (0,1)$ and $\nu_1 \in (0,1)$, the equation (3.34) implies that

$$\text{meas}\left\{x \in K^y_{\eta\rho} : w < \delta_1 k\right\} < (1-\nu_1)\left|K^y_{\eta\rho}\right|.$$

□

The proposition 3.4.2 is similar to Lemmata III.4.1, III.7.2, IV.10.2 from [11]. If $g_0 > 2$, then the next proposition can be replaced by Corollary 3.4 from [23] which does not involve the logarithmic energy estimate.

**Proposition 3.4.2.** *For given constants $\nu \in (0,1)$, $k > 0$, and $\rho > 0$ and any $\epsilon \in (0,1)$, there exists a nonnegative integer $j = j(\nu, N, g_1, \epsilon) \in (0,1)$ such that, if*

$$\text{meas}\left\{x \in K^y_\rho : w(x,-\tau) < k\right\} < (1-\nu)\left|K^y_\rho\right| \tag{3.35}$$

*for*

$$\begin{cases} \tau \le k^2 G\left(\frac{k}{\rho}\right)^{-1}, & \text{if} \quad g_0 \ge 2, \\ \tau \le \left(2^{-j}k\right)^2 G\left(\frac{2^{-j}k}{\rho}\right)^{-1}, & \text{if} \quad g_1 \le 2, \end{cases} \tag{3.36}$$

*then*

$$\text{meas}\left\{x \in K_\rho^y : w(x,-t) < 2^{-j}k\right\} < (1-(1-\epsilon)\nu)\,|K_\rho^y|$$

*for any* $t \in [0,\tau)$.

*Proof.* Here we apply the logarithmic energy estimate (2.40) on the cylinder $K_\rho^y \times [-\tau, 0]$. Denote a piecewise linear cutoff function independent of the time variable as

$$\zeta = \begin{cases} 1 & \text{inside } K_{(1-\sigma)\rho}^y \times [-\tau, 0] \\ 0 & \text{on the lateral boundaries of } \; K_\rho^y \times [-\tau, 0]. \end{cases}$$

with

$$|D\zeta| \le \frac{1}{\sigma\rho}$$

where $\sigma \in (0,1)$ will be determined later. For positive constants $\gamma_1$ and $\gamma_2$ and any $t \in [0,\tau)$, we have

$$\begin{aligned} \int_{K_\rho^y \times \{-t\}} H(\Psi^2)\zeta^{g_1}\,dx &\le \int_{K_\rho^y \times \{-\tau\}} H(\Psi^2)\zeta^{g_1}\,dx \\ &+ \gamma_1 \int_{-\tau}^{-t}\int_{K_\rho^y} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)|\Psi||\Psi'|^2\zeta^{g_1}\,dx\,ds \\ &+ \gamma_2 \int_{-\tau}^{-t}\int_{K_\rho^y} G(b)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,ds, \end{aligned} \tag{3.37}$$

for any $t \in (-\tau, 0]$ where $h$ and $H$ are defined in (2.8a) and (2.8b).

Let $2^{-j}$ where $j$ to be determined large enough. We recall (2.38) that

$$\Psi = \ln^+\left[\frac{k}{(1+2^{-j})k - (w-k)_-}\right], \quad \Psi' = \frac{-1}{(1+2^{-j})k - (w-k)_-}.$$

Since $0 \le (w-k)_- \le k$, we have

$$\Psi \le \ln^+ 2^j = j\ln 2, \quad \frac{1}{(1+2^{-j})k} \le |\Psi'| \le \frac{1}{2^{-j}k}.$$

Moreover, in the set $\{w < 2^{-j}k\}$, we obtain a lower bound

$$\Psi \ge \ln^+(2\cdot 2^{-j})^{-1} = (j-1)\ln 2.$$

When we work with supersolution $w \le 0$, the function

$$\Psi(w) = \ln^+\left[\frac{k}{(1+2^{-j})k + (w-k)_+}\right], \quad \Psi' = \frac{-1}{(1+2^{-j})k + (w-k)_+},$$

is used with $k \le 0$. The property that $0 \le (w-k)_+ \le -k$ delivers

$$\Psi \le \ln^+ 2^j = j \ln 2, \quad \frac{-1}{(1+2^{-j})k} \le |\Psi'| \le \frac{-1}{2^{-j}k}.$$

In the set $\{w > 2^{-j}k\}$,

$$\Psi \ge \ln^+(2 \cdot 2^{-j})^{-1} = (j-1)\ln 2.$$

The left hand side of the inequality (3.37) is lower bounded

$$\begin{aligned} &\int_{K_\rho^y \times \{t\}} H(\Psi^2)\zeta^{g_1}\,dx \\ &\ge H\left((j-1)^2(\ln 2)^2\right) \operatorname{meas}\left\{x \in K^y_{(1-\sigma)\rho} : w(x,t) < \upsilon k\right\}. \end{aligned}$$

Due to (3.35), the first integral term on the right hand side of (3.37) is bounded by

$$\int_{K_\rho^y \times \{-\tau\}} H(\Psi^2)\zeta^{g_1}\,dx \le H\left(j^2(\ln 2)^2\right)(1-\nu)|K_\rho^y|.$$

To estimate a upper bound of the second integral on the right hand side of (3.37), we observe first that if $g_1 \le 2$, then

$$\frac{1/\Psi'}{\upsilon k} \ge 1,$$

and therefore

$$\begin{aligned} &G\left(\frac{|D\zeta|}{\zeta\Psi'}\right)\Psi'^2(\upsilon k)^2 G\left(\frac{\upsilon k}{\rho}\right)^{-1} \\ &\le \left(\frac{1/\Psi'}{\upsilon k}\right)^{g_1-2}(\zeta\sigma)^{-g_1} \\ &\le \frac{1}{(\zeta\sigma)^{g_1}}. \end{aligned}$$

When $g_0 \ge 2$, then

$$\begin{aligned} &G\left(\frac{|D\zeta|}{\zeta\Psi'}\right)\Psi'^2 k^2 G\left(\frac{k}{\rho}\right)^{-1} \\ &\le \upsilon^{g_0-2}(\zeta\sigma)^{-g_1} \\ &\le \frac{1}{(\zeta\sigma)^{g_1}}. \end{aligned}$$

Hence

$$\int_{-\tau}^{-t}\int_{K_\rho^y} G\left(\frac{|D\zeta|}{\zeta|\Psi'|}\right) h(\Psi^2)|\Psi||\Psi'|^2\zeta^{g_1}\,dx\,dt$$
$$\leq \sigma^{-g_1} h\left(j^2(\ln 2)^2\right) j\ln 2|K_\rho^y|$$
$$\leq g_1\sigma^{-g_1}\frac{H\left(j^2(\ln 2)^2\right)}{j\ln 2}|K_\rho^y|.$$

Moreover, by imposing

$$b \leq \frac{\upsilon k}{\rho},$$

we obtain that

$$\int_{-\tau}^{-t}\int_{K_\rho^y} G(b)h(\Psi^2)\Psi|\Psi'|^2\zeta^{g_1}\,dx\,dt$$
$$\leq g_1\frac{H\left(j^2(\ln 2)^2\right)}{j\ln 2}|K_\rho^y|,$$

because if $g_1 \leq 2$

$$\left|\Psi'\right|^2\tau \leq G\left(\frac{\upsilon k}{\rho}\right)^{-1},$$

and if $g_0 \geq 2$

$$\left|\Psi'\right|^2\tau \leq \upsilon^{g_0-2}G\left(\frac{\upsilon k}{\rho}\right)^{-1} \leq G\left(\frac{\upsilon k}{\rho}\right)^{-1}.$$

Therefore, we derive

$$H\left((j-1)^2(\ln 2)^2\right)\text{meas}\left\{x\in K_{(1-\sigma)\rho}^y : w(x,t) < \upsilon k\right\}$$
$$\leq H\left(j^2(\ln 2)^2\right)(1-\nu)|K_\rho^y|$$
$$+ g_1\left[\frac{\gamma_1}{\sigma^{g_1}}+\gamma_2\right]\frac{H\left(j^2(\ln 2)^2\right)}{j\ln 2}|K_\rho^y|.$$

For simplicity let

$$H' = \frac{H\left(j^2(\ln 2)^2\right)}{H\left((j-1)^2(\ln 2)^2\right)},$$
$$\gamma = \max\{\gamma_1,\gamma_2\},$$

then

$$\text{meas}\left\{x\in K_\rho^y : w(x,t) < \upsilon k\right\}$$
$$\leq \text{meas}\left\{x\in K_{(1-\sigma)\rho}^y : w(x,t) < \upsilon k\right\} + K_\rho^y/K_{(1-\sigma)\rho}^y \tag{3.38}$$
$$\leq \left[H'(1-\nu) + g_1\gamma\left\{1+\left(\frac{1}{\sigma}\right)^{g_1}\right\}\frac{H'}{j\ln 2} + N\sigma\right]|K_\rho^y|.$$

For a fixed $\epsilon \in (0,1)$, we choose two constants $\sigma \in (0,1)$ and $j$ such that

$$H' \leq 1 + \epsilon\nu, \tag{3.39a}$$

$$g_1 \left\{ 1 + \left(\frac{2}{\sigma}\right)^{g_1} \right\} \frac{H'}{j \ln 2} \leq \frac{\epsilon\nu^2}{2}, \tag{3.39b}$$

$$N\sigma \leq \frac{\epsilon\nu^2}{2}. \tag{3.39c}$$

If we assume $\sigma$ and $j$ satisfy (3.39), then (3.38) yields

$$\text{meas}\left\{x \in K_\rho^y : w(x,t) < \upsilon k\right\} \leq (1 - (1-\epsilon)\nu)\, |K_\rho^y|$$

which leads to our conclusion.

Now we return to (3.39) to find appropriate $\sigma$ and $j$. First we fix

$$\sigma = \frac{\epsilon\nu^2}{2N}$$

from (3.39c). Assuming (3.39a), the equation (3.39b) implies that

$$j \ln 2 \geq (1 + \epsilon\nu) \frac{4 g_1 \gamma}{\nu^2} \left[ 1 + \left(\frac{2N}{\epsilon\nu^2}\right)^{g_1} \right],$$

which implies that

$$j \geq \gamma(\text{data}) \left(\epsilon\nu^2\right)^{-1-g_1}.$$

Finally, due to (2.13), the inequality

$$\left(\frac{j}{j-1}\right)^{g_1} \leq 1 + \epsilon\nu. \tag{3.40}$$

Because

$$(1 + \epsilon\nu)^{\frac{1}{g_1}} \leq 1 + (\epsilon\nu)^{\frac{1}{g_1}},$$

the inequality (3.40) provides

$$1 + \frac{1}{j-1} \leq 1 + (\epsilon\nu)^{\frac{1}{g_1}},$$

therefore,

$$j \geq 1 + (\epsilon\nu)^{-\frac{1}{g_1}}.$$

Hence we choose a positive integer $j$ large enough such that

$$j \geq \max\left\{ \gamma(\text{data}) \left(\epsilon\nu^2\right)^{-1-g_1}, 1 + (\epsilon\nu)^{-\frac{1}{g_1}} \right\}.$$

$\square$

We can show similar type of argument as in Proposition 3.4.2 relying only the local energy estimate.

**Corollary 3.4.3.** *For given constants $\nu \in (0,1)$, $k > 0$, and $\rho > 0$ and any $\epsilon \in (0,1)$, there exists a nonnegative integer $j = j(\nu, N, g_1) \in (0,1)$ such that, if*

$$\operatorname{meas}\left\{x \in K_\rho^y : w(x,-\tau) < k\right\} < (1-\nu)\left|K_\rho^y\right| \tag{3.41}$$

*for some*

$$\tau \le 2^{-j} k^2 G\left(\frac{k}{\rho}\right)^{-1}, \tag{3.42}$$

*then*

$$\operatorname{meas}\left\{x \in K_\rho^y : w(x,t) < 2^{-j}k\right\} < (1-(1-\epsilon)\nu)\,|K_\rho^y|$$

*for any $t \in (-\tau, 0]$.*

*Proof.* Here we rely on the local energy estimate (2.23) on the cylinder $K_\rho^y \times [-\tau, 0]$. Denote a piecewise linear cutoff function independent of the time variable as

$$\zeta = \begin{cases} 1 & \text{inside } K_{(1-\sigma)\rho}^y \times [-\tau, 0] \\ 0 & \text{on the lateral boundaries of } \ K_\rho^y \times [-\tau, 0]. \end{cases}$$

with

$$|D\zeta| \le \frac{1}{\sigma\rho}, \quad \zeta_t = 0$$

where $\sigma \in (0,1)$ will be determined later. By ignoring the second term on the left hand side of (2.23), we obtain the below inequality for any $t \in (-\tau, 0]$

$$\begin{aligned}
&\int_{K_\rho^y \times \{t\}} G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^{s+2}\zeta^q\,dx \\
&\le \int_{K_\rho^y \times \{-\tau\}} G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^{s+2}\zeta^q\,dx \\
&\quad + \gamma_2 \iint_{Q_R} G\left(\frac{\zeta(w-k)_-}{\sigma\rho}\right) G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^s\zeta^q\,dx \\
&\quad + \gamma_3 \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^s\zeta^q\,dx\,dt \\
&\quad + \gamma_4\delta\Lambda M_w \iint_{Q_R} G^{r-1}\left(\frac{\zeta(w-k)_-}{R}\right)(w-k)_-^{s+1}\zeta^q\,dx\,dt.
\end{aligned}$$

Since $(w-k)_- \leq k$ and increasing functions $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$ and $\sigma \mapsto G^r(\sigma)\sigma^s$, a decreasing function $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$, we have cancelation of $G^{r-1}(k/\rho)k^s$ that leads to an inequality

$$\begin{aligned}\int_{K_\rho^y\times\{t\}} (w-k)_-^2 \zeta^q \, dx \leq \int_{K_\rho^y\times\{-\tau\}} & k^2\zeta^q \, dx \\ & + \gamma \iint_{Q_R} G\left(\frac{k}{\sigma\rho}\right) dx\, dt,\end{aligned} \tag{3.43}$$

for some constant $\gamma = 3\max\{\gamma_2, \gamma_3, \gamma_4\delta\Lambda M_w\}$. In the set $\{w < 2^{-j}k\}$ for a positive integer $j$ to be determined later, we obtain

$$\int_{K_\rho^y\times\{t\}} (w-k)_-^2 \zeta^q \, dx \geq \left(1-2^{-j}\right)^2 k^2 \operatorname{meas}\left\{x \in K_{(1-\sigma)\rho}^y : w(x,t) < 2^{-j}k\right\}.$$

Moreover, the assumption (3.41) at the initial time level gives

$$\int_{K_\rho^y\times\{-\tau\}} k^2\zeta^q \, dx \leq k^2(1-\nu)\left|K_\rho^y\right|,$$

and the limitation of the time length (3.42) implies that

$$\iint_{Q_R} G\left(\frac{k}{\sigma\rho}\right) dx\, dt \leq k^2 2^{-j}\sigma^{-g_1}\left|K_\rho^y\right|.$$

As a results, the inequality (3.43) is simplified

$$\begin{aligned}&\left(1-2^{-j}\right)^2 \operatorname{meas}\left\{x \in K_{(1-\sigma)\rho}^y : w(x,t) < 2^{-j}k\right\} \\ &\leq \left\{(1-\nu) + \gamma 2^{-j}\sigma^{-g_1}\right\}\left|K_\rho^y\right|,\end{aligned}$$

which implies

$$\begin{aligned}&\operatorname{meas}\left\{x \in K_\rho^y : w(x,t) < 2^{-j}k\right\} \\ &\leq \left\{\frac{(1-\nu)}{\left(1-2^{-j}\right)^2} + \frac{\gamma 2^{-j}}{\sigma^{g_1}\left(1-2^{-j}\right)^2} + N\sigma\right\}\left|K_\rho^y\right|.\end{aligned}$$

Hence for a fixed $\epsilon \in (0,1)$, we choose $j$ large enough such that

$$\frac{1}{\left(1-2^{-j}\right)^2} \leq 1 + \epsilon\nu, \tag{3.44a}$$

$$\frac{\gamma 2^{-j}}{\sigma^{g_1}\left(1-2^{-j}\right)^2} \leq \frac{\epsilon\nu^2}{2}, \tag{3.44b}$$

$$N\sigma \leq \frac{\epsilon\nu^2}{2}, \tag{3.44c}$$

which leads to our conclusion. The inequalities (3.44) delivers that

$$\sigma = \frac{\epsilon\nu^2}{2N},$$
$$j \geq \log_2 \max\left\{\gamma(\text{data})\nu^{-2}(\epsilon\nu^2)^{-g_1}, 1 + 1/\sqrt{\epsilon\nu}\right\}.$$

□

Proposition 3.4.4 begins with a cylinder in which strictly positive occurs in strictly positive measure place. With appropriate time length, a cylinder can be stretched spatially. Moreover, the place where a solution is near its minimum in the stretched cylinder can be controlled by any fraction of the cylinder. Appropriate time is necessary to handle degenerate type of equations ($g_1 > 2$). Otherwise ($1 < g_1 \leq g_1 < 2$), spreading positivity over the space is natural behavior of a solution. The Proposition 3.4.4 is analogous to Lemma 3.5 from [23], Theorem 1.1 from [17], Proposition 6.1 from [16], and Lemma IV.11.1 from [11].

**Proposition 3.4.4.** *For given $k > 0$, $\rho > 0$, $y \in K_\rho^y$, $\eta \in (0,1)$, and $\alpha \in (0,1)$, suppose that $K_{\eta\rho}^y \subset K_\rho$. Then for any $\nu \in (0,1)$, there exists a positive integer $j^* = j^*(N, \alpha, g_1, \eta, \nu)$ such that, if*

$$\text{meas}\left\{x \in K_{\eta\rho}^y : w(x,t) < k\right\} < (1-\alpha)|K_{\eta\rho}^y| \tag{3.45}$$

*for all $t \in (-2\tau, 0]$ where*

$$\begin{cases} \tau \geq k^2 G\left(\frac{k}{\rho}\right)^{-1}, & \text{for } 1 < g_1 \leq 2, \\ \tau \geq \left(2^{-j^*}k\right)^2 G\left(\frac{2^{-j^*}k}{\rho}\right)^{-1}, & \text{for } g_1 > 2, \end{cases} \tag{3.46}$$

*then*

$$\text{meas}\left\{(x,t) \in K_\rho \times [-\tau, 0] : w(x,t) < 2^{-j^*}k\right\} < \nu\left|K_\rho \times [-\tau, 0]\right|.$$

*Proof.* Let $k_j = 2^{-j}k$ for $j = 0, 1, 2, \ldots, j^*$ with $j^*$ to be determined later. We work with a piecewise linear cutoff function that

$$\zeta = \begin{cases} 1 & \text{inside of } \; K_\rho \times [-\tau, 0] \\ 0 & \text{on the parabolic boundary of } \; K_{2\rho} \times [-2\tau, 0] \end{cases}$$

with

$$|D\zeta| \le \frac{1}{\rho}, \quad \zeta_t \le \frac{1}{\tau}.$$

Here we apply the local energy estimate (2.34) that

$$\begin{aligned}
&\int_{-2\tau}^{0}\int_{K_{2\rho}} G\left(|D(w-k_j)_-|\right) G^{r-1}\left(\frac{\zeta(w-k_j)_-}{\rho}\right)(w-k_j)_-^s\zeta^q\,dx\,dt\\
&\le \gamma_1\int_{-2\tau}^{0}\int_{K_{2\rho}} G^{r-1}\left(\frac{\zeta(w-k_j)_-}{\rho}\right)(w-k_j)_-^{s+2}\zeta^{q-1}\zeta_t\,dx\,dt\\
&\quad+\gamma_2\int_{-2\tau}^{0}\int_{K_{2\rho}} G^{r}\left(\frac{\zeta(w-k_j)_-}{\rho}\right)(w-k_j)_-^{s}\zeta^{q-1-2g_1}\,dx\,dt\\
&\quad+\gamma_3\int_{-2\tau}^{0}\int_{K_{2\rho}} G(b)G^{r-1}\left(\frac{\zeta(w-k_j)_-}{\rho}\right)(w-k_j)_-^{s}\zeta^{q-1-2g_1}\,dx\,dt.
\end{aligned}$$

Using that $(w-k_j)_- \le k_j$, increasing functions $\sigma \mapsto G^r(\sigma)\sigma^s$ and $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$, and a decreasing function $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$, we obtain that

$$\begin{aligned}
&\int_{-2\tau}^{0}\int_{K_{2\rho}} G\left(|D(w-k_j)_-|\right)\chi_{\{x_{j+1}\le u<k_j\}}\zeta^q\,dx\,dt\\
&\le \int_{-2\tau}^{0}\int_{K_{2\rho}}\left[\gamma_1 k_j^2\zeta_t+\gamma_2 G\left(\frac{k_j}{\rho}\right)+\gamma_3\delta G(b)\right]dx\,dt,
\end{aligned} \tag{3.47}$$

where $\delta \in (0,1)$ is from (**??**) which is determined small enough. Owing to (2.3), we obtain for any nonnegative integer $j = 0, 1, \ldots, j^*$

$$\delta G\left(\frac{k}{\rho}\right) \le \delta 2^{jg_1} G\left(\frac{k_j}{\rho}\right) \le G\left(\frac{k_j}{\rho}\right)$$

by choosing

$$\delta \le 2^{-j^* g_1}.$$

From (3.46), we evaluate that if $1 < g_1 \le 2$

$$\begin{aligned}
k_j^2\zeta_t &\le 2^{-2j} G\left(\frac{k}{\rho}\right)\\
&\le 2^{-j(2-g_1)} G\left(\frac{k_j}{\rho}\right)\\
&\le G\left(\frac{k_j}{\rho}\right)
\end{aligned}$$

for any $j$. If $g_1 > 2$, then using $G$ is an increasing function we have

$$\begin{aligned}
k_j^2\zeta_t &\le 2^{-2j}2^{2j^*} G\left(\frac{k_{j^*}}{\rho}\right)\\
&\le G\left(\frac{k_j}{\rho}\right)
\end{aligned}$$

for any $j$. Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Then the inequality (3.47) is reduced to

$$\begin{aligned} &\int_{-2\tau}^{0} \int_{K_{2\rho}} G\left(|D(w-k_j)_-|\right) \chi_{\{w \le k_j\}} \zeta^q \, dx \, dt \\ &\le \gamma G\left(\frac{k_j}{\rho}\right) |K_{2\rho} \times [-2\tau, 0]| \, . \end{aligned} \tag{3.48}$$

Due to the assumption (3.45), we apply a Poincare type inequality, Corollary 2.4.3. For any $t \in [-\tau, 0]$, it follows that

$$\begin{aligned} &(k_j - k_{j+1}) \operatorname{meas}\{x \in K_\rho : w(x,t) > k_{j+1}\} \\ &\le \frac{\rho^{N+1}}{\alpha(\eta\rho)^N} \int_{K_\rho \cap \{k_{j+1} \le u < k_j\}} |D(w-k_j)_-| \, dx. \end{aligned} \tag{3.49}$$

For brief notation, let

$$\Omega_\tau := K_\rho \times [-\tau, 0],$$

$$A_j := \{(x,t) \in \Omega_\tau : u(x,t) < k_j\} \, .$$

Note that $k_j - k_{j+1} = k_{j+1}$. To the inequality (3.49), take integration in terms of time from $-\tau$ to 0 and take division by $\rho$, it follows that

$$\frac{k_{j+1}}{\rho} |A_{j+1}| \le \frac{1}{\alpha\eta^N} \iint_{A_j \setminus A_{j+1}} |D(w-k_j)_-| \, dx \, dt. \tag{3.50}$$

After another division by $|A_j \setminus A_{j+1}|$, taking Jensen's inequality generates that

$$G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right) \le \frac{1}{\alpha\eta^N |A_j \setminus A_{j+1}|} \iint_{A_j \setminus A_{j+1}} G\left(|D(w-k_j)_-|\right) \, dx \, dt. \tag{3.51}$$

With the aid of the inequality (3.48), the integral inequality (3.51) now becomes

$$G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right) \le \frac{\gamma 2^{N+1} |\Omega_\tau|}{\alpha\eta^N |A_j \setminus A_{j+1}|} G\left(\frac{k_j}{\rho}\right). \tag{3.52}$$

For any $j = 0, 1, \ldots, j^*$, we have two cases to study: either

$$|A_{j+1}| \le |A_j \setminus A_{j+1}|, \tag{3.53a}$$

$$|A_{j+1}| \le |A_j \setminus A_{j+1}|. \tag{3.53b}$$

First, if we assume (3.53a), then we observe that

$$\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|}\right)^{g_1} 2^{-g_1} G\left(\frac{k_{j+1}}{\rho}\right) \le G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right).$$

Therefore we derive from (3.52) that

$$\left(\frac{|A_{j+1}|}{|\Omega_\tau|}\right)^{\frac{g_1}{g_1-1}} \leq \left(\frac{\gamma 2^{N+1+g_1}}{\alpha\eta^N}\right)^{\frac{1}{1-g_1}} \frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}. \tag{3.54}$$

In case (3.53b) holds, we first evaluate that

$$\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|}\right)^{g_0} 2^{-g_1} G\left(\frac{k_{j+1}}{\rho}\right) \leq G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|}\frac{k_{j+1}}{\rho}\right).$$

Therefore we derive from (3.52) that

$$\left(\frac{|A_{j+1}|}{|\Omega_\tau|}\right)^{\frac{g_0}{g_0-1}} \leq \left(\frac{\gamma 2^{N+1+g_1}}{\alpha\eta^N}\right)^{\frac{1}{1-g_0}} \frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}. \tag{3.55}$$

Now we take the sum of $j$ from 0 to $j^*-1$ of (3.52), then owing to (3.54) and (3.55). It follows that

$$\begin{aligned} j^* \min&\left\{\left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_1}{g_1-1}}, \left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_0}{g_0-1}}\right\} \\ &\leq \max\left\{\beta^{\frac{1}{g_1-1}}, \beta^{\frac{1}{g_0-1}}\right\}, \end{aligned}$$

where

$$\beta = \frac{\gamma 2^{N+1+g_1}}{\alpha\eta^N}.$$

Because $g_0 \leq g_1$ and $\beta \geq 1$,

$$\max\left\{\beta^{\frac{1}{g_1-1}}, \beta^{\frac{1}{g_0-1}}\right\} = \beta^{\frac{1}{g_0-1}}.$$

Moreover, $g_0 \leq g_1$ implies that

$$\frac{g_0}{g_0-1} \geq \frac{g_1}{g_1-1},$$

from which we derive

$$\min\left\{\left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_1}{g_1-1}}, \left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_0}{g_0-1}}\right\} = \left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_0}{g_0-1}}.$$

Hence we determine $j^*$ such that

$$j^* \left(\frac{|A_{j^*}|}{|\Omega_\tau|}\right)^{\frac{g_0}{g_0-1}} \leq \beta^{\frac{1}{g_0-1}}. \tag{3.56}$$

From the previous inequality, we have

$$|A_{j^*}| \leq \left[\frac{\beta^{\frac{1}{g_0-1}}}{j^*}\right]^{\frac{g_0-1}{g_0}} |\Omega_\tau|.$$

For any fixed $\nu \in (0,1)$, we pick a positive integer $j^*$ large enough satisfying

$$|A_{j^*}| \leq \nu |\Omega_\tau|,$$

that is true if $j^*$ holds

$$\left[\frac{\beta^{\frac{1}{g_0-1}}}{j^*}\right]^{\frac{g_0-1}{g_0}} \leq \nu.$$

Therefore we reach to our conclusion by determining

$$j^* \geq \nu^{\frac{g_0}{1-g_0}} \beta^{\frac{1}{g_0-1}}.$$

□

**Remark 3.4.1.** *Assume that* $\tau$ *in* (3.46) *is replaced by* $\theta\tau$ *for some constant* $\theta > 0$*. Then* $\gamma = \theta^{-1}\gamma_1 + \gamma_2 + \gamma_3$ *which implies that any constant* $\theta \geq 1$ *gives the same* $j^*$*. If we have* $\theta < 1$*, the*

$$j^* \geq \nu^{\frac{g_0}{1-g_0}} \beta^{\frac{1}{g_0-1}}$$

*where*

$$\beta = \frac{\gamma 2^{N+1+g_1}}{\theta \alpha \eta^N}.$$

Proposition 3.4.5 provides modified De Giorgi iteration for parabolic $p-$Laplacian type of equations (3.1) under generalized structure conditions (3.2). The iteration is made by comparing the local energy estimate (2.23) and Sobolev inequality (Theorem 2.4.4). To overcome nonhomogeneity, intrinsic scaling for the time variable and change of time variable are used. Roughly speaking, in a cylinder which has positive portion of where a weak solution is strictly positive, Proposition 3.4.5 gives a subcylinder where a solution is strictly positive almost everywhere. Basically, Proposition 3.4.5 is equivalent to Lemmata III.4.1, III.9.1, IV.4.1 from [11].

**Proposition 3.4.5.** *For given constants* $k > 0$ *and* $\rho > 0$*, there exists* $\nu_0 = \nu_0\left(\min\left\{\theta^N, \theta^{-1}\right\}, \text{data}\right) \in (0,1)$ *such that, if*

$$\text{meas}\left\{(x,t) \in Q_{k,2\rho} : w(x,t) < k\right\} < \nu_0 \left|Q_{k,2\rho}\right|,$$

*then*

$$\operatorname*{ess\,inf}_{Q_{k,\rho}} w(x,t) \geq \frac{k}{2}.$$

*Proof.* We first construct two sequences $\{\rho_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ such that

$$\rho_n = \rho + \frac{\rho}{2^n} \text{ and } k_n = \frac{k}{2} + \frac{k}{2^{n+1}} \text{ for } n = 0, 1, \ldots.$$

Note that $\rho/2 \leq \rho_n \leq \rho$ for any nonnegative integer $n$ which means

$$\frac{1}{2} \leq \frac{\rho_{n+1}}{\rho_n} \leq 1.$$

Because $G(\sigma)$ is an increasing function, a sequence $\{Q_n\}_{n=0}^{\infty}$ by setting

$$Q_n = K_{\rho_n} \times [-T_{k,\rho_n}, 0]$$

gives nested and shrinking family of cylinders. Let us take a sequence of piecewise linear cutoff functions $\{\zeta_n\}_{n=0}^{\infty}$ such that

$$\zeta_n = \begin{cases} 1 & \text{inside of } \; Q_{n+1} \\ 0 & \text{on the parabolic boundary of } Q_n, \end{cases}$$

satisfying

$$|D\zeta_n| \leq \frac{2^{n+1}}{\rho} = \frac{2^{n+1}+2}{\rho_n},$$
$$0 \leq (\zeta_n)_t \leq \frac{2^n}{g_0\theta} k^{-2} G\left(\frac{k}{\rho_n}\right).$$

Particulary, $(\zeta_n)_t$ is derived from the below inequalities;

$$\begin{aligned} 1 - \left(\frac{\rho_n}{\rho_{n+1}}\right)^{-g_0} &= \frac{g_0}{\rho_n^{g_0}} \int_{\rho_{n+1}}^{\rho_n} s^{g_0-1}\, ds \\ &\geq \frac{g_0}{\rho_n^{g_0}} \rho_{n+1}^{g_0-1} (\rho_n - \rho_{n+1}) \\ &= g_0 \left(\frac{2^n+1}{2^n+2}\right)^{g_0-1} \frac{1}{2^n+2}, \end{aligned}$$

which generates

$$\begin{aligned} (\zeta_n)_t &\leq \left\{1 - \left(\frac{\rho_n}{\rho_{n+1}}\right)^{-g_0}\right\}^{-1} \theta^{-1} k^{-2} G\left(\frac{k}{2\rho_n}\right) \\ &\leq \frac{2^n+2}{g_0} \left(\frac{2^n+2}{2^n+1}\right)^{g_0-1} \frac{1}{2^{g_0}} \theta^{-1} k^{-2} G\left(\frac{k}{2\rho_n}\right) \\ &\leq \frac{2^n}{g_0\theta} k^{-2} G\left(\frac{k}{2\rho_n}\right), \end{aligned}$$

because

$$\frac{2^n+2}{2^n+1} < 2.$$

Note that

$$G(|D\zeta_n|\zeta_n(w-k_n)_-) \leq \left(2^{n+1}+2\right)^{g_1} G\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right).$$

Therefore, the local energy estimate (2.34) yields, for some constants $\gamma_0$, $\gamma_1$, and $\gamma_2$, that

$$\begin{aligned}
&\sup_t \int_{K_{\rho_n}} G^{r-1}\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right)(w-k_n)_-^{s+2}\zeta_n^q\,dx \\
&\quad + \iint_{Q_n} G\left(|D(w-k_n)_-|\right) G^{r-1}\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right)(w-k_n)_-^s\zeta_n^q\,dx\,dt \\
&\leq \gamma_0 \iint_{Q_n} G^{r-1}\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right)(w-k_n)_-^{s+2}\zeta_n^{q-1}(\zeta_n)_t\,dx\,dt \\
&\quad + \gamma_1\left(2^{n+1}+2\right)^{g_1} \iint_{Q_n} G^{r}\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right)(w-k_n)_-^s\zeta_n^{q-1-2g_1}\,dx\,dt \\
&\quad + \gamma_2 \iint_{Q_n} G(b)G^{r-1}\left(\frac{\zeta_n(w-k_n)_-}{\rho_n}\right)(w-k_n)_-^s\zeta_n^q\,dx\,dt
\end{aligned} \tag{3.57}$$

With a property from the level set,

$$(w-k_n)_- = \max\{0, k_n - w\} \leq k_n \leq k,$$

and the maps that $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$, $\sigma \mapsto G^r(\sigma)\sigma^s$ are increasing, the right hand side of (3.57) is bounded by

$$\begin{aligned}
RHS \leq &\left\{\gamma_0\frac{2^{g_0+n+1}}{g_0\theta}G\left(\frac{k}{\rho_n}\right) + \gamma_1(2^{n+1}+2)^{g_1}G\left(\frac{k}{\rho_n}\right) + \gamma_2 G(b)\right\} \\
&\times G^{r-1}\left(\frac{k}{\rho_n}\right)k^s \iint_{Q_n} \chi_{\{w<k_n\}}\,dx\,dt.
\end{aligned} \tag{3.58}$$

We impose

$$G\left(b\right) \leq G\left(\frac{k}{\rho_n}\right), \tag{3.59}$$

which is provided by the restriction on the level set $k$,

$$\rho_n \leq \rho, \quad b \leq \frac{k}{\rho} \leq \frac{k}{\rho_n}.$$

Then the inequality (3.58) is reduced

$$RHS \leq \left\{\gamma_0\frac{2^{g_0+n+1}}{g_0\theta} + \gamma_1(2^{n+1}+2)^{g_1} + \gamma_2\right\} G^r\left(\frac{k}{\rho_n}\right)k^s \iint_{Q_n} \chi_{\{w<k_n\}}\,dx\,dt. \tag{3.60}$$

To find out a lower bound of the left hand side of (3.57), we consider the set $Q_n \cap \{w < k_{n+1}\}$. Indeed in the set $\{w < k_{n+1}\}$, we observe that

$$(w-k_n)_- = \max\{0, k_n - w\} \geq k_n - k_{n+1} = \frac{k}{2^{n+2}},$$
$$(w-k_n)_-^2 G\left(\frac{\zeta_n (w-k_n)_-}{\rho_n}\right)^{-1} \geq 2^{-(n+2)} k^2 G\left(\frac{k}{\rho_n}\right)^{-1} \zeta_n^{-g_1},$$

because the map $\sigma \mapsto \sigma$ is increasing and the map $\sigma \mapsto \sigma G(\sigma)^{-1}$ is decreasing. Let $w_n := (w-k_n)_-$ for simpler notations. Thus, we obtain that

$$\begin{aligned} &2^{-(n+2)} k^2 G\left(\frac{k}{\rho_n}\right)^{-1} \sup_t \int_{K_{\rho_n}} G^r\left(\frac{\zeta_n w_n}{\rho_n}\right) w_n^s \zeta_n^{q-g_1}\, dx \\ &+ \iint_{Q_n} G\left(|Dw_n|\right) G^{r-1}\left(\frac{\zeta_n w_n}{\rho_n}\right) w_n^s \zeta_n^q \, dx\, dt \\ &\leq \gamma 2^{n g_1} G^r\left(\frac{k}{\rho_n}\right) k^s \iint_{Q_n} \chi_{\{w<k_n\}}\, dx\, dt, \end{aligned} \tag{3.61}$$

by setting

$$\gamma = \frac{\gamma_0}{g_0 \theta} + 3\gamma_1 + \gamma_2.$$

To transform the time coordinate, denote

$$d_n := k^2 G\left(\frac{k}{\rho_n}\right)^{-1},$$
$$\bar{t} := \frac{t}{d_n}$$

which leads a mapping

$$Q_n \mapsto \bar{Q}_n = K_{\rho_n} \times [-\theta, 0].$$

Set up

$$w(\cdot, t) = \bar{w}(\cdot, t_1 + d_n \bar{t}),$$

$$\zeta_n(\cdot, t) = \bar{\zeta}_n(\cdot, t_1 + d_n \bar{t}).$$

Let $\bar{w}_n := (\bar{w} - k_n)_-$ for simpler notations. Divide the inequality (3.61) by $2^{-(n+2)} d_n$ and make transformation with respect to the time variable from $t$ to $\bar{t}$. As a result, we get

$$\begin{aligned} &\sup_t \int_{K_{\rho_n} \cap \{\bar{w}<k_{n+1}\}} G^r\left(\frac{\bar{\zeta}_n \bar{w}_n}{\rho_n}\right) \bar{w}_n^s \bar{\zeta}_n^{q-g_1}\, dx \\ &+ \iint_{Q_n \cap \{\bar{w}<k_{n+1}\}} G\left(|D\bar{w}_n|\right) G^{r-1}\left(\frac{\bar{\zeta}_n \bar{w}_n}{\rho_n}\right) \bar{w}_n^s \bar{\zeta}_n^q \, dx\, d\bar{t} \\ &\leq \gamma 2^{n(g_1+1)} G^r\left(\frac{k}{\rho_n}\right) k^s \iint_{Q_n} \chi_{\{\bar{w}<k_n\}}\, dx\, d\bar{t}. \end{aligned} \tag{3.62}$$

To provide the proper Sobolev embedding theorem, we apply Theorem 2.4.4 with

$$v = G^r\left(\frac{\bar{\zeta}_n \bar{w}_n}{2\rho_n}\right)\bar{w}_n^s \bar{\zeta}_n^q.$$

After taking the derivative of $v$ and applying inequalities of $g$ and $G$ from Lemma 2.1.1, for some constants $c_0$ and $c_1$, we derive

$$|Dv| \le \frac{c_0}{\rho_n} G(|D\bar{w}_n|) G^{r-1}\left(\frac{\bar{w}_n}{2\rho_n}\right)\bar{w}_n^s + \frac{c_1 2^n}{\rho_n} v.$$

Hence, from the inequality (3.62) and Theorem 2.4.4, it follows

$$\begin{aligned}&\iint_{\bar{Q}_n \cap \{\bar{w}<k_{n+1}\}} G^r\left(\frac{\bar{\zeta}_n \bar{w}_n}{\rho_n}\right)\bar{w}_n^s \bar{\zeta}_n^q \, dx\, d\bar{t} \\ &\le C\rho_n^{-\frac{N}{N+1}} 2^{n(g_1+2)} G^r\left(\frac{k}{\rho_n}\right) k^s \left[\iint_{\bar{Q}_n} \chi_{\{\bar{w}<k_n\}} \, dx\, d\bar{t}\right]^{1+\frac{1}{N+1}},\end{aligned} \tag{3.63}$$

where $C = (c_0 + c_1)\gamma$. The left hand side of (3.63) has a lower bound

$$LHS \ge k^s G^{r-1}\left(\frac{k}{\rho_n}\right) G\left(\frac{k_n - k_{n+1}}{\rho_n}\right)\bar{\zeta}_n^{q+rg_0} \tag{3.64}$$

because the map $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$ is a nonincreasing function and $\sigma \mapsto G(\sigma)$ is an increasing function. As some cancelation, we obtain

$$\begin{aligned}&\iint_{\bar{Q}_n} \chi_{\{\bar{w}<k_{n+1}\}} \bar{\zeta}_n^{q+rg_0} \, dx\, d\bar{t} \\ &\le C\rho_n^{-\frac{N}{N+1}} 2^{2n(g_1+1)} \left[\iint_{\bar{Q}_n} \chi_{\{\bar{w}<k_n\}} \, dx\, d\bar{t}\right]^{1+\frac{1}{N+1}}.\end{aligned} \tag{3.65}$$

Now divide the inequality (3.65) by $|\bar{Q}_n|$ with notice that

$$|\bar{Q}_n| = |K_{\rho_n} \times [-\theta, 0]| = \theta \rho_n^N,$$

which is equivalent to

$$\rho_n^{\frac{N}{N+1}} = \theta^{-\frac{1}{N+1}} \left|\bar{Q}_n\right|^{\frac{1}{N+1}}$$

that leads us to the inequality

$$\begin{aligned}&\frac{\iint_{\bar{Q}_n} \chi_{\{\bar{w}<k_{n+1}\}} \bar{\zeta}_n^{q+rg_0} \, dx\, d\bar{t}}{|\bar{Q}_n|} \\ &\le C 2^{2n(g_1+1)} \theta^{\frac{1}{N+1}} \left[\frac{\iint_{\bar{Q}_n} \chi_{\{\bar{w}<k_n\}} \, dx\, d\bar{t}}{|\bar{Q}_n|}\right]^{1+\frac{1}{N+1}}.\end{aligned} \tag{3.66}$$

We go back to the original time coordinate $t$ from $\bar{t}$, then we apply below two inequalities,

$$\iint_{Q_n} \chi_{\{w<k_{n+1}\}}\zeta_n^{q+rg_0}\, dx\, dt \geq \iint_{Q_{n+1}} \chi_{\{w<k_{n+1}\}}\, dx\, dt,$$

$$|Q_n| \leq 2^{N+g_1}|Q_{n+1}|,$$

to the estimate (3.66). Eventually, for some constant $C$ depending on data and $\theta^{-1}$, we derive

$$\frac{\iint_{Q_{n+1}} \chi_{\{w<k_{n+1}\}}\, dx\, dt}{|Q_{n+1}|} \leq C2^{2n(g_1+1)}\theta^{\frac{1}{N+1}}\left[\frac{\iint_{Q_n} \chi_{\{w<k_n\}}\, dx\, dt}{|Q_n|}\right]^{1+\frac{1}{N+1}}. \tag{3.67}$$

Applying Lemma 2.4.6 with

$$\nu_0 \leq \theta^{-1}C^{-(N+1)}2^{-(2g_1+2)(N+1)^2} = \tilde{C}^{-(N+1)}\theta^N 2^{-(2g_1+2)(N+1)^2} \tag{3.68}$$

completes the proof. We note that $C$ is depending on $\theta^{-1}$ so $\nu_0$ is depending on $\theta^N$ which means $\tilde{C}$ is depending on only data. □

**Remark 3.4.2.** *Whenever we apply Proposition 3.4.5, it is required to have restrictions such as* (3.59) *on a constant $k$ to handle lower order terms properly.*

### Proof of main lemma

Now we are ready to prove Lemma 3.3.1 by applying four propositions in this section.

*Proof.* Without loss of generality, let $(x_0, t_0) := (0, 0)$. Depending on two constants $g_0$ and $g_1$, showing Hölder continuity (finding appropriate connections of the four propositions in this chapter) is rather simple of involves delicate iteration scheme. We will study several cases separately.

**Case I:** When $g_0 = g_1 = 2$ (the differential equation is a uniformly parabolic equation), Hölder continuity of a bounded weak solution $u$ is straightforward. First observe that for any $k > 0$

$$k^2 G\left(\frac{k}{R}\right)^{-1} = \frac{R^2}{G(1)}.$$

For a constant $\eta$ to be chosen later, we begin with the cylinder

$$Q_0 = K_R \times [-16\eta^2 R^2, 0]$$

with the assumption that

$$\text{meas}\,\{Q_0 : u(x,t) < 2M\} \geq \frac{1}{2}|Q_0|.$$

Then Proposition 3.4.1 with constants $\delta_1 = 1/2$ and $\nu = 1/3$ provides that there exist $y \in K_R$, $\eta = \eta(M, \text{data})$, and $\tau \in [\eta^2 R^2, 16\eta^2 R^2]$ such that

$$\text{meas}\left\{K^y_{\eta R} : u(x,-\tau) < M\right\} < (1 - \frac{1}{3})|K^y_{\eta R}|.$$

Then Proposition 3.4.2 with $\epsilon = 2/3$ follows that there exists $j = j(\eta, \text{data})$ such that

$$\text{meas}\left\{K^y_{\eta R} : u(x,t) < 2^{-j}M\right\} < (1 - \frac{1}{3^2})|K^y_{\eta R}|$$

for all $t \in [-\tau, 0]$. By Proposition 3.4.4, for any fixed $\nu \in (0,1)$, there exists $j^* = j^*(\nu, \eta, \text{data})$ such that

$$\text{meas}\left\{K_R \times [-\tau, 0] : u(x,t) < 2^{-j-j^*}M\right\} < \nu\,|K_R \times [-\tau, 0]|\,.$$

Finally, De Giorgi iteration, Proposition 3.4.5 with the restriction

$$b \leq \frac{2^{-j-j^*}M}{R},$$

gives that

$$\operatorname*{ess\,inf}_{Q_1} u(x,t) \geq \frac{2^{-j-j^*}M}{2}$$

where

$$Q_1 = K_{R/2} \times [-\eta^2 (R/2)^2, 0].$$

**Case II:** Suppose that $2 \leq g_0 \leq g_1 < \infty$ (including degenerate types of equations).

We begin with a cylinder

$$Q_0 = K_R \times [-T, 0]$$

where $T$ to be determined later. For a constant $M$, suppose that

$$\text{meas}\,\{(x,t) \in Q_0 : u(x,t) \geq 2M\} \geq \frac{1}{2}|Q_0|.$$

By Proposition 3.4.1 with a fixed constant $\delta_1 = 1/2$, for any constants $\nu_1 \in (0,1)$, there exist a point $y \in K_R$, a time level $\tau \in [T/16, T]$, and a constant $\eta = \eta(M, \nu_1, \text{data})$ such that $K^y_{\eta R} \subset K_R$ and

$$\text{meas}\left\{x \in K^y_{\eta R} : u(x,-\tau) < M\right\} < (1 - \nu_1)\left|K^y_{\eta R}\right|.$$

Then Proposition 3.4.2 provides that for any $\epsilon \in (0,1)$ there exists $j = j(\nu_1, \epsilon, \text{data})$ such that, if

$$\tau \le M^2 G\left(\frac{M}{\eta R}\right)^{-1},$$

then for all $t \in [-\tau, 0]$

$$\text{meas}\left\{x \in K^y_{\eta R} : u(x,t) < 2^{-j}M\right\} < (1-(1-\epsilon)\nu_1)\left|K^y_{\eta R}\right|. \tag{3.69}$$

Now subdivide $K^y_{\eta R}$ into $2^{lN}$ congruent subcylinders, then for any nonnegative integer $l$, there exists $K^{y'}_{2^{-l}\eta R}$ such that

$$\text{meas}\left\{x \in K^{y'}_{2^{-l}\eta R} : u(x,t) < 2^{-j}M\right\} < (1-(1-\epsilon)\nu_1)\left|K^{y'}_{2^{-l}\eta R}\right|. \tag{3.70}$$

We choose a positive integer

$$l \ge \frac{g_1 - 2}{g_0} j,$$

that satisfies

$$2^{(g_1-2)j} 2^{-l g_0} \le 1.$$

Therefore, we have

$$\left(2^{-j}M\right)^2 G\left(\frac{2^{-j}M}{2^{-l}\eta R}\right)^{-1} \le 2^{(g_1-2)j}2^{-lg_0} M^2 G\left(\frac{M}{\eta R}\right)^{-1} \le M^2 G\left(\frac{M}{\eta R}\right)^{-1}.$$

Hence by setting

$$Q_1 = K^{y'}_{2^{-l}\eta R} \times [-\left(2^{-j}M\right)^2 G\left(\frac{2^{-j}M}{2^{-l}\eta R}\right)^{-1}, 0],$$

the equation (3.70) implies that

$$\text{meas}\left\{(x,t) \in Q_1 : u(x,t) < 2^{-j}M\right\} < (1-(1-\epsilon)\nu_1)\left|Q_1\right|. \tag{3.71}$$

Here for a constant $\nu_0 \in (0,1)$ that is from Proposition 3.4.5 with the restriction

$$b \le \frac{2^{-j}M}{2^{-l}\eta R}, \tag{3.72}$$

fix

$$\epsilon = \frac{\nu_0}{1+\nu_0}, \quad \nu_1 = 1 - \nu_0^2$$

which gives

$$(1-\epsilon)\nu_1 = \nu_0.$$

Then with (3.71), we apply Proposition 3.4.5 and conclude

$$\operatorname*{ess\,inf}_{Q_2} u(x,t) \geq \frac{2^{-j}M}{2} \tag{3.73}$$

where

$$Q_2 = K^{y'}_{2^{-l}\eta R/2} \times [-\left(2^{-j}M\right)^2 G\left(\frac{2^{-j}M}{2^{-l}\eta R/2}\right)^{-1}, 0].$$

Let

$$k = 2^{-j}M, \quad \rho = 2^{-l}\eta R/2.$$

Then (3.73) is rewritten that

$$\operatorname*{ess\,inf}_{Q_2} u(x,t) \geq \frac{k}{2} \tag{3.74}$$

where

$$Q_2 = K^{y'}_{\rho} \times [-k^2 G\left(\frac{k}{\rho}\right)^{-1}, 0].$$

For any $\sigma \in (0,1)$, the equation (3.69) is true if a positive constant $M$ is replaced by $\sigma M$. From the equation (3.74), we draw conclusion that for a constant $\sigma \in (0,1)$, it holds that

$$\operatorname*{ess\,inf}_{Q_3} u(x,t) \geq \frac{\sigma k}{2}$$

where

$$Q_3 = K^{y'}_{\rho} \times [-\left(\sigma k\right)^2 G\left(\frac{\sigma k}{\rho}\right)^{-1}, 0].$$

Now let

$$t = \left(2^{-l}\eta\right)^{g_0} \left(\sigma k\right)^2 G\left(\frac{\sigma k}{R}\right)^{-1},$$

then $\sigma$ is a function of $t$. Also let

$$u(x,t) = v(x,t)\sigma(t).$$

Here we apply Proposition 3.4.4 with a linear cutoff function $\zeta$ in the cylinder

$$Q_4 = K_R \times [-t, 0]$$

satisfying

$$|D\zeta| \le \frac{2}{R}, \quad 0 \le \zeta_t \le \frac{2}{t},$$

and the level sets for $v$ that $(v-k_i)_-$ for $k_i = 2^{-i}k$. The local energy estimate by ignoring the first term on the left hand side gives that

$$\begin{aligned}
&\iint_{Q_4} G\left(\sigma|D(v-k_i)_-|\right) G^{r-1}\left(\frac{\zeta\sigma(v-k_i)_-}{R}\right)\sigma^s(v-k_i)_-^s\zeta^q\, dx\, dt \\
&\le \gamma_1 \iint_{Q_4} G^{r-1}\left(\frac{\zeta\sigma(v-k_i)_-}{R}\right)\sigma^{s+2}(v-k_i)_-^{s+2}\zeta^{q-1}\zeta_t\, dx\, dt \\
&\quad + \gamma_2 \iint_{Q_4} G^{r}\left(\frac{\zeta\sigma(v-k_i)_-}{R}\right)\sigma^{s}(v-k_i)_-^{s}\zeta^{q}\, dx\, dt \\
&\quad + \gamma_3 \iint_{Q_4} G(b)G^{r-1}\left(\frac{\zeta\sigma(v-k_i)_-}{R}\right)\sigma^{s}(v-k_i)_-^{s}\zeta^{q-1-2g_1}\, dx\, dt.
\end{aligned} \tag{3.75}$$

With the restriction

$$b \le \frac{\sigma k}{R} \tag{3.76}$$

and $(v-k_i)_- \le k_i$ and increasing functions $w \mapsto G^r(w)w^s$, $w \mapsto G^{r-1}(w)w^{s+2}$, a decreasing function $w \mapsto G^{r-1}(w)w^s$, the inequality (3.75) is simplified to

$$\begin{aligned}
&\iint_{Q_4} G\left(\sigma|D(v-k_i)_-|\right)\chi_{\{k_{j+1}\le v\le k_i\}}\zeta^q\, dx\, dt \\
&\le \iint_{Q_4} \gamma G\left(\frac{\sigma k_i}{R}\right)|K_R\times[-t,0]|\,.
\end{aligned}$$

Then by following lines from the proof of Proposition 3.4.4, we conclude that for any $\nu \in (0,1)$, there exists $j^* = j^*(N,\eta,\nu,\text{data})$ such that

$$\text{meas}\left\{(x,s)\in K_{R/2}\times[-t/2,0] : v(x,s) < 2^{-j^*}k\right\} < \nu\left|K_{R/2}\times[-t/2,0]\right|$$

where for any $\sigma \in (0,1)$

$$\tau^* = \left(2^{-l}\eta\right)^{g_0}(\sigma k)^2\, G\left(\frac{\sigma k}{R}\right)^{-1}. \tag{3.77}$$

By fixing $\sigma$ in (3.77) to be $2^{-j^*}$, we carry Proposition 3.4.5 for $v(x,t)$ using that $u(x,t) = v(x,t)\sigma(t)$. Therefore in the cylinder

$$Q = K_{R/4}\times[-\left(2^{-l}\eta\right)^{g_0}\left(2^{-j^*}k\right)^2 G\left(\frac{2^{-j^*}k}{R/2}\right)^{-1}, 0],$$

it holds that

$$\operatorname*{ess\,inf}_{Q} v(x,t) \ge \frac{2^{-j^*}k}{2}.$$

Hence

$$\operatorname*{ess\,inf}_{Q} u(x,t) \geq 2^{-2j^*-1}k = 2^{-2j^*-j-1}M$$

which leads to our conclusion by choosing

$$T = 16\left(2^{-l}\eta\right)^{g_0}\left(2^{-j^*-j}M\right)^2 G\left(\frac{2^{-j^*-j}M}{R}\right)^{-1}.$$

The equations (3.72) and (3.76) imply that we pick

$$M \geq \max\left\{2^{\frac{g_0+g_1-2}{g_0}j}b\eta R, 2^{j+j^*}bR\right\}.$$

Otherwise, ess osc $w \leq cbR$ for some constant $c > 0$ that leads to Hölder continuity.

**Case III:** When $1 < g_0 \leq g_1 \leq 2$ (including singular types of equations).

Similar to the proof for Case II, we begin with the cylinder

$$Q_0 = K_R \times [-T, 0]$$

where $T$ to be determined later. For a constant $M$, suppose that

$$\text{meas}\left\{(x,t) \in Q_0 : u(x,t) \geq 2M\right\} \geq \frac{1}{2}\left|Q_0\right|.$$

Then Proposition 3.4.1 with $\delta_1 = 1/2$ provides that for any $\nu_1 \in (0,1)$, there exist a point $y \in K_R$, a time level $\tau \in [T/16, T]$, and $\eta = \eta(M, \delta_1, \nu_1, \text{data}) \in (0,1)$ such that $K_{\eta R}^y \subset K_R$ and

$$\text{meas}\left\{x \in K_{\eta R}^y : u(x,-\tau) < M\right\} < (1-\nu_1)\left|K_{\eta R}^y\right|.$$

By Proposition 3.4.2, for any $\epsilon \in (0,1)$, there exists a positive integer $j = j(\nu_1, \epsilon, \text{data})$ if

$$\tau \leq \left(2^{-j}M\right)^2 G\left(\frac{2^{-j}M}{\eta R}\right)^{-1},$$

then for all $t \in [-\tau, 0]$

$$\text{meas}\left\{x \in K_{\eta R}^y : u(x,t) < 2^{-j}M\right\} < (1-(1-\epsilon)\nu_1)\left|K_{\eta R}^y\right|.$$

By letting

$$\epsilon = \frac{\nu_0}{1+\nu_0}, \quad \nu_1 = 1-\nu_0^2$$

where $\nu_0$ is the constant from Proposition 3.4.5, we apply De Giorgi iteration (Proposition 3.4.5) with the restriction

$$b \leq \frac{2^{-j}M}{\eta R} \tag{3.78}$$

to conclude

$$\operatorname*{ess\,inf}_{Q_1} u(x,t) \geq \frac{2^{-j}M}{2} \tag{3.79}$$

where

$$Q_1 = K^y_{\eta R/2} \times [-\left(2^{-j}M\right)^2 G\left(\frac{2^{-j}M}{\eta R/2}\right)^{-1}, 0].$$

Set

$$k = 2^{-j}M, \quad \rho = \eta R/2.$$

Then (3.79) is saying that

$$\operatorname*{ess\,inf}_{Q_1} u(x,t) \geq \frac{k}{2}$$

where

$$Q_1 = K^y_{\rho} \times [-k^2 G\left(\frac{k}{\rho}\right)^{-1}, 0].$$

Since

$$k^2 G\left(\frac{k}{\rho}\right)^{-1} \geq \eta^{g_1} k^2 G\left(\frac{k}{R}\right)^{-1},$$

by applying Proposition 3.4.4, for any $\nu \in (0,1)$, there exists a positive integer $j^* = j^*(\text{data})$ such that

$$\text{meas}\left\{(x,t) \in K_R \times [-\tau, 0] : u(x,t) < 2^{-j^*}k\right\} < \nu \left|K_R \times [-\tau, 0]\right|. \tag{3.80}$$

Now set

$$u(x,t) = 2^{-j^*}v(x,t).$$

Then we follow the proof of Proposition 3.4.5 with

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \frac{k}{2} + \frac{k}{2^{n+1}}$$

for $n = 0, 1, 2, \ldots$. Then set

$$Q_n = K_{R_n} \times [-k_n^2 G\left(\frac{k_n}{R_n}\right)^{-1}, 0]$$

and set $\zeta_n$ to be a linear cutoff function 1 inside of $Q_{n+1}$ and 0 outside of $Q_n$ satisfying

$$|D\zeta_n| \le \frac{2^{n+2}}{R}, \quad 0 \le (\zeta_n)_t \le 2^n k^{-2} G\left(\frac{k}{R_n}\right).$$

Then the local energy estimate with a level set of $v$ provides that

$$\begin{aligned}
&\sup_t \int_{K_{R_n}} G^{r-1}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s+2}\zeta_n^q \,dx \\
&\quad + \iint_{Q_n} G\left(2^{-j^*}|D(v-k_n)_-|\right) G^{r-1}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s}\zeta_n^q \,dx\,dt \\
&\le \gamma_0 \iint_{Q_n} G^{r-1}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s+2}\zeta_n^{q-1}(\zeta_n)_t \,dx\,dt \qquad (3.81)\\
&\quad + \gamma_1 \iint_{Q_n} G^{r}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s}\zeta_n^{q-1-2g_1} \,dx\,dt \\
&\quad + \gamma_2 \iint_{Q_n} G(b) G^{r-1}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s}\zeta_n^{q} \,dx\,dt.
\end{aligned}$$

Owing to properties of increasing functions $w \mapsto G^r(w)w^s$, $w \mapsto G^{r-1}(w)w^{s+2}$, a decreasing function $w \mapsto G^{r-1}(w)w^s$ and that

$$(v-k_n)_- \le k_n \le k,$$

and by imposing

$$b \le \frac{2^{-j^*}k}{R}, \qquad (3.82)$$

we first note that

$$\begin{aligned}
&G^{r-1}\left(\frac{2^{-j^*}\zeta_n(v-k_n)_-}{R_n}\right)\left(2^{-j^*}(v-k_n)_-\right)^{s+2}\zeta_n^{q-1}(\zeta_n)_t \\
&\le G^{r-1}\left(\frac{2^{-j^*}k_n}{R_n}\right)\left(2^{-j^*}k_n\right)^{s+2} k^{-2} G\left(\frac{k}{R_n}\right) \\
&\le G^{r}\left(\frac{2^{-j^*}k_n}{R_n}\right)\left(2^{-j^*}k_n\right)^{s}
\end{aligned}$$

because

$$2^{-2j^*} G\left(\frac{k}{R_n}\right) \le G\left(\frac{2^{-j^*}k}{R_n}\right)$$

due to $1 < g_0 \le g_1 \le 2$. Hence the right hand side of the inequality (3.81) simplified to

$$RHS \le \gamma G^r\left(\frac{2^{-j^*}k}{R_n}\right)\left(2^{-j^*}k\right)^s \iint_{Q_n} \chi_{\{v<k_n\}} \,dx\,dt.$$

By following the proof of Proposition 3.4.5, we conclude that there exists $\nu_0$ such that for

$$Q_2 = K_R \times [-k^2 G\left(\frac{k}{R}\right)^{-1}, 0]$$

if

$$\operatorname{meas}\{(x,t) \in Q_2 : v(x,t) < k\} < \nu_0 |Q_2|,$$

then

$$\operatorname*{ess\,inf}_{Q_3} v(x,t) \geq \frac{k}{2}$$

where

$$Q_3 = K_{R/2} \times [-k^2 G\left(\frac{k}{R/2}\right)^{-1}, 0].$$

Hence

$$\operatorname*{ess\,inf}_{Q_3} u(x,t) \geq 2^{-j-j^*-1} M.$$

The time length $T$ is now chosen to be

$$T = 16\eta^{g_1} \left(2^{-j} M\right)^2 G\left(\frac{2^{-j} M}{R}\right)^{-1}.$$

Due to the equations (3.78) and (3.82), we restrict

$$M \geq \max\left\{2^j \eta b R, 2^{j+j^*} b R\right\}.$$

$\square$

# CHAPTER 4. Hölder continuity of $Du$

## 4.1 Introduction

Suppose that $u \in W^{1,G}(\Omega_T)$ is a bounded weak solution of

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0 \tag{4.1}$$

under the structure conditions

$$\frac{\partial}{\partial p_j}\mathbf{A}^i(x, t, z, p)\xi_i\xi_j \geq \frac{g(|p|)}{|p|}|\xi|^2, \tag{4.2a}$$

$$\left|\frac{\partial}{\partial p_j}\mathbf{A}^i(x, t, z, p)\right| \leq \Lambda\frac{g(|p|)}{|p|}, \tag{4.2b}$$

$$|\mathbf{A}(x, t, z, p) - \mathbf{A}(y, t, w, p)| \leq \Lambda_1 (1 + g(|p|)) \left[|x - y|^{\alpha_1} + |z - w|^{\alpha_1}\right], \tag{4.2c}$$

$$|\mathbf{A}(x, t, z, p) - \mathbf{A}(x, s, z, p)| \leq \Lambda_1 (1 + g(|p|)) \left[G^{-1}(|t - s|)\right]^{\alpha_2}, \tag{4.2d}$$

for some constants $\Lambda > 0$, $\Lambda_1 \geq 0$, $\alpha_1 \in (0, 1)$, and $\alpha_2 \in (0, 1)$.

For a positive constant $M$ and for an interior point $(x_0, t_0) \in \Omega_T$, we define the cylinder

$$Q_\rho^{(x_0,t_0)} = K_{\rho/M}^{x_0} \times [t_0 - \frac{\rho^2}{G(M)}, 0].$$

We also consider the differential equation

$$\begin{aligned} v_t - \operatorname{div} \tilde{\mathbf{A}}(Dv) = 0 \quad &\text{in } Q_\rho^{(x_0,t_0)}, \\ v = u \quad &\text{on the parabolic boundary of } Q_\rho^{(x_0,t_0)}, \end{aligned} \tag{4.3}$$

where $\tilde{\mathbf{A}}(p) = \mathbf{A}(x_0, t_0, u(x_0, t_0), p)$. Hence (4.3) satisfies the structure conditions,

$$\frac{\partial}{\partial p_j}\tilde{\mathbf{A}}^i(p)\xi_i\xi_j \geq \frac{g(|p|)}{|p|}|\xi|^2, \tag{4.4a}$$

$$\left|\frac{\partial}{\partial p_j}\tilde{\mathbf{A}}^i(p)\right| \leq \Lambda\frac{g(|p|)}{|p|}. \tag{4.4b}$$

We show Hölder continuity of $Du$ following the perturbation argument basically comparing two weak solutions of (4.1) under (4.2) and of (4.3) under (4.4).

To estimate the Hölder continuity of $Dv$, a solution of (4.3) under (4.4), there are three steps. The first step is getting the ess sup $G(|Dv|)$ in terms of the integral form of $G^q(|Dv|)$ using Moser's iteration. Especially here we obtain the integral estimate using a weighted Sobolev inequality. Next step follows by calculating

$$\iint_Q G^q(|Dv|)\,dx\,dt \leq C \iint_Q Dv \cdot \tilde{\mathbf{A}}(Dv)\,dx\,dt.$$

Final step is finding a bound for the integral of $Dv \cdot \tilde{\mathbf{A}}(Dv)$ in terms of known data and the radius of the cylinder. After knowing that $|Dv|$ is locally Hölder continuous, we carry the perturbation argument comparing solutions from (4.1) under (4.2) and from (4.3) under (4.4). At this stage, we require to derive that ess sup $|u-v| \leq$ ess osc $u$ where we adopt the Hölder continuity results from Chapter 3. Also it is important to estimate the integral of $G(|Du-Dv|)$ is bounded by the integral of $1+G(|Du|)$. From these steps, we find the integral estimate of $G(|Du|)$ in terms of some power of $R$ that is the constant defining a cylinder. Based on this estimate we find the bounded mean oscillation estimates which leads to the Campanato space and the Hölder continuity follows by the isomorphic relation.

By differentiating (4.3) with respect to $x_k$, multiplying the resultant equation by $D_{x_k}v$ and summing on $k$, we notice that $\bar{w} := |Dv|^2$ is a weak solution of the equation

$$\begin{aligned} \bar{w}_t - \operatorname{div}(\tilde{a_{ij}} D\bar{w}) = 0 \quad & \text{in } Q_\rho^{(x_0,t_0)}, \\ v = u \quad & \text{on the parabolic boundary of } Q_\rho^{(x_0,t_0)}, \end{aligned} \tag{4.5}$$

where $a_{ij} = \partial \bar{\mathbf{A}}^i \setminus \partial p_j$ with structure conditions

$$a_{ij}\xi_i\xi_j \geq \frac{g(|p|)}{|p|}|\xi|^2, \tag{4.6a}$$

$$|a_{ij}| \leq \Lambda \frac{g(|p|)}{|p|}, \tag{4.6b}$$

for some positive constant $\Lambda$.

For example, consider the prototype generalized $p-$Laplacian equations

$$v_t - \operatorname{div}\left(\frac{g(|Dv|)}{|Dv|}Dv\right) = 0 \tag{4.7}$$

where $g \in C^1[0,\infty)$. The similar calculations to derive (4.5) from (4.3) generate

$$\bar{w}_t - \operatorname{div}\left(\frac{g(|Dv|)}{|Dv|}D\bar{w}\right) = 0. \tag{4.8}$$

## 4.2 Integral estimates of $Dv$

In this section, we obtain several integral estimates for $Dv$ that will be used in following two sections in this chapter.

**Lemma 4.2.1.** *Suppose that $v$ is a bounded weak solution of (4.3) under the structure conditions (4.4) with a constant $0 < R < 1$. Then there exists a constant $c$ depending on data such that*

$$\iint_{Q_R} G(|Dv|)\,dx\,dt \le c\max\left\{G\left(\frac{\operatorname{ess\,osc}_{Q_{2R}} v}{R}\right)R^{N+2}, \left(\operatorname*{ess\,osc}_{Q_{2R}} v\right)^2 R^N\right\}.$$

*Proof.* Denote the cut-off function $\zeta$ independent of the time variable such that

$$\zeta = \begin{cases} 1 & \text{inside of } K_R \times [t_0, t_1], \\ 0 & \text{on the lateral boundaries of } K_{2R} \times [t_0, t_1] \end{cases}$$

with

$$|D\zeta| \le \frac{1}{R}, \quad \zeta_t = 0.$$

Then

$$\begin{aligned} &\iint_{Q_{2R}} \zeta^{g_1} Dv \cdot \tilde{\mathbf{A}}\,dx\,dt \\ &= \iint_{Q_{2R}} \zeta^{g_1 - 1} D(\zeta v) \cdot \tilde{\mathbf{A}}\,dx\,dt - \iint_{Q_{2R}} \zeta^{g_1-1} v D\zeta \cdot \tilde{\mathbf{A}}\,dx\,dt. \end{aligned}$$

Moreover,

$$\begin{aligned} \iint_{Q_{2R}} \zeta^{g_1-1} D(\zeta v) \cdot \tilde{\mathbf{A}}\,dx\,dt &= -\iint_{Q_{2R}} \zeta^{g_1} v \operatorname{div}\tilde{\mathbf{A}}\,dx\,dt \\ &= -\iint_{Q_{2R}} \zeta^{g_1} v v_t\,dx\,dt \\ &= -\frac{1}{2}\int_{K_{2R}} \zeta^{g_1} v^2\,dx\Big|_{t_0}^{t_1} \end{aligned}$$

because of (4.7), $2vv_t = (v^2)_t$, and $\zeta_t = 0$.

Using the structure conditions (4.4) provides

$$\iint_{Q_{2R}} \zeta^{g_1} G(|Dv|)\,dx\,dt$$
$$\le c \iint_{Q_{2R}} \zeta^{g_1-1}|D\zeta| v g(|Dv|)\,dx\,dt + \sup_{t_0\le t\le t_1} \int_{K_{2R}} \zeta^{g_1} v^2\,dx.$$

By Young's inequality, for any $\epsilon > 0$, note

$$\iint_{Q_{2R}} \zeta^{g_1-1}|D\zeta| v g(|Dv|)\,dx\,dt$$
$$\le \epsilon \iint_{Q_{2R}} \zeta^{g_1} G(|Dv|)\,dx\,dt + c \iint_{Q_{2R}} \zeta^{q} G\left(\frac{v}{\epsilon\zeta R}\right)\,dx\,dt.$$

By choosing $\epsilon = (2c)^{-1}$, we estimate that

$$\iint_{Q_R} \zeta^{g_1} G(|Dv|)\,dx\,dt \le c \sup_{Q_{2R}} G\left(\frac{v}{R}\right)|Q_R| + c \sup_{Q_{2R}} v^2 |K_R|.$$

□

The following lemma is for estimating the $L_\infty$ norm of $G(|Dv|)$ with respect to some power norm of $G(|Dv|)$ using Moser's iteration with the weighted Sobolev embedding theorem, Theorem 2.4.5.

**Lemma 4.2.2.** *Suppose that $v$ is a bounded weak solution of* (4.3) *under the structure conditions* (4.4). *For a positive constant $M > 1$, there exist constants $k$ and $C$ depending on data such that*

$$\sup_{Q_R} G(|Dv|) \le C(M) R^{-N-2} \iint_{Q_{2R}\cap\{|Dv|\ge 2M\}} G(|Dv|)^{2+\frac{k+2}{g_0}+\frac{k}{2}}\,dx\,dt.$$

*Proof.* Assume that $w > 2M$ for some constants $M > 0$ and define $\tau = G(M)$. Let the test function be

$$\varphi(x,t) := w^a G^b(w)\left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2,$$

where nonnegative constants $a, b, c$ are determined later. We also note that

$$\left(1 - \frac{\tau}{G(w)}\right)_+^c = \begin{cases} 1 - \frac{\tau}{G(w)} & \text{if } w > M, \\ 0 & \text{if } w \le M. \end{cases}$$

Our assumption that $w > 2M$ gives

$$(1 - 2^{-g_0}) \le \left(1 - \frac{\tau}{G(w)}\right)_+^c \le 1.$$

Moreover, $\zeta$ is the cutoff function $0 \le \zeta \le 1$ such that

$$\zeta = \begin{cases} 1 & \text{inside } Q_R, \\ 0 & \text{on the parabolic boundary of } Q_{2R}. \end{cases}$$

with

$$|D\zeta| \le \frac{M}{2R}, \quad 0 \le \zeta_t \le \frac{G(M)}{3R^2}.$$

First, we study the integral quantity involved with the derivative with respect to the time variable

$$\iint_{Q_{2R}} \bar{w}_t w^a G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2 \, dx \, dt. \tag{4.9}$$

Owing to $\bar{w} = w^2$, $\bar{w}_t = 2ww_t$. By setting

$$F(w) = \int_0^w 2s^{a+1} G^b(s) \left(1 - \frac{\tau}{G(s)}\right)_+^c ds$$

and taking the integration by parts to (4.9) generate

$$\begin{aligned} \iint_{Q_{2R}} \bar{w}_t \varphi \, dx \, dt &= \iint_{Q_{2R}} [F(w)]_t \, \zeta^2 \, dx \, dt \\ &= \int_{K_{2R/M}} F(w)\zeta^2 \, dx \Big|_{-4R^2/G(M)}^{0} - 2 \iint_{Q_{2R}} F(w)\zeta\zeta_t \, dx \, dt. \end{aligned}$$

Now we estimate $F(w)$ using that

$$F(w) = \int_0^w F'(s) \, ds = wF'(w) - \int_0^w sF''(s) \, ds$$

and

$$sF''(s) \ge [(a+1) + bg_0] \, F'(s)$$
$$sF''(s) \le \left[(a+1) + bg_1 + \frac{cg_1 2^{g_0}}{2^{g_0} - 1}\right] F'(s).$$

Hence

$$F(s) \geq \frac{1}{1+(a+1)+bg_1+\frac{cg_1 2^{g_0}}{2^{g_0}-1}} s^{a+2} G^b(s) \left(1-\frac{\tau}{G(s)}\right)_+^c,$$
$$F(s) \leq \frac{1}{1+(a+1)+bg_1} s^{a+2} G^b(s) \left(1-\frac{\tau}{G(s)}\right)_+^c.$$

We evaluate

$$\begin{aligned} D\varphi &= a w^{a-1} G^b(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta^2 Dv \\ &\quad + b w^a G^{b-1}(w) g(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta^2 Dv \\ &\quad + c w^a G^b(w) \left(1-\frac{\tau}{G(w)}\right)_+^{c-1} \frac{\tau g(w)}{G^2(w)} \zeta^2 Dv \\ &\quad + 2 w^a G^b(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta D\zeta, \end{aligned}$$

to estimate the integral

$$\iint_{Q_{2R}} \tilde{\mathbf{A}} Dw \cdot D\varphi \, dx\, dt = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned} I_1 &\geq 2a \iint_{Q_{2R}} w^a G^b(w) g'(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta^2 |Dv|^2 \, dx\, dt, \\ I_2 &\geq 2b \iint_{Q_{2R}} w^{a+1} G^{b-1}(w) g(w) g'(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta^2 |Dv|^2 \, dx\, dt, \\ I_3 &\geq 2c \iint_{Q_{2R}} w^{a+1} G^b(w) g'(w) \frac{\tau g(w)}{G^2(w)} \left(1-\frac{\tau}{G(w)}\right)_+^{c-1} \zeta^2 |Dv|^2 \, dx\, dt, \\ I_4 &= 4 \iint_{Q_{2R}} \tilde{\mathbf{A}} \zeta Dv \cdot D\zeta w w^a G^b(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \, dx\, dt. \end{aligned}$$

Then because of (2.2), we have

$$I_2 \geq 2bg_0 \iint_{Q_{2R}} w^a G^b(w) g'(w) \left(1-\frac{\tau}{G(w)}\right)_+^c \zeta^2 |Dv|^2 \, dx\, dt.$$

By the Cauchy's inequality, $ab \le \epsilon a^2 + \epsilon^{-1} b^2$, we estimate

$$\begin{aligned}
I_4 &\ge -4 \iint_{Q_{2R}} \zeta |Dv| \cdot |D\zeta| w w^a G^b(w) g'(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c dx\, dt \\
&\ge -4\epsilon \iint_{Q_{2R}} \zeta^2 |Dv|^2 w^a G^b(w) g'(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c dx\, dt \\
&\quad - 4\epsilon^{-1} \iint_{Q_{2R}} w^{a+2} G^b(w) g'(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c |D\zeta|^2 dx\, dt \\
&\ge -4\epsilon \iint_{Q_{2R}} w^a G^b(w) g'(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2 |Dv|^2 dx\, dt \\
&\quad - 4(g_0 - 1) g_0 \epsilon^{-1} \iint_{Q_{2R}} w^a G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c |D\zeta|^2 dx\, dt.
\end{aligned}$$

We choose $\epsilon$ such that $4\epsilon = 2a + abg_0$ which means we cancel out $I_1$ and $I_2$. Therefore all estimates gives that

$$\begin{aligned}
&\int_{K_{2R/M}} w^{a+2} G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2 dx \Big|_{-4R^2/G(M)}^{0} \\
&\quad + \gamma_0 2c \iint_{Q_{2R}} w^{a+1} G^b(w) g'(w) \frac{\tau g(w)}{G^2(w)} \left(1 - \frac{\tau}{G(w)}\right)_+^{c-1} \zeta^2 |Dv|^2 dx\, dt, \\
&\le \gamma_1 \iint_{Q_{2R}} w^{a+2} G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta \zeta_t\, dx\, dt \\
&\quad + \gamma_2 \iint_{Q_{2R}} w^a G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c |D\zeta|^2 dx\, dt.
\end{aligned} \tag{4.10}$$

We take change of the variables

$$\bar{x} = Mx, \quad \bar{t} = G(M) t$$

which implies

$$Q_\rho = K_{\rho/M} \times [-\frac{\rho^2}{G(M)}, 0] \to \bar{Q}_\rho = K_\rho \times [-\rho^2, 0].$$

Also we note that $D_x w = M D_{\bar{x}} w$. Because $2M < w$, (4.10) becomes below:

$$
\begin{aligned}
& G(M) \sup_{-(2R)^2<\bar{t}<0} \int_{K_{2R}} w^{a+2} G(M) G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2 \, d\bar{x} \\
& \quad + \gamma_0 2cM^2 G(M) \iint_{\bar{Q}_{2R}} w^{a+1} G^b(w) g'(w) \frac{g(w)}{G^2(w)} \left(1 - \frac{\tau}{G(w)}\right)_+^{c-1} \zeta^2 |\bar{D}w|^2 \, d\bar{x}\, d\bar{t}, \\
& \leq \gamma_1 G(M) \iint_{\bar{Q}_{2R}} w^{a+2} G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \bar{\zeta} \bar{\zeta}_t \, d\bar{x}\, d\bar{t} \\
& \quad + \gamma_2 M^2 \iint_{\bar{Q}_{2R}} w^a G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c |\bar{D}\zeta|^2 \, d\bar{x}\, d\bar{t} \\
& \leq \gamma \iint_{\bar{Q}_{2R}} w^{a+2} G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \, d\bar{x}\, d\bar{t}.
\end{aligned}
\tag{4.11}
$$

We then divide (4.11) by $M^2 G(M)$. Moreover, note that

$$
\begin{aligned}
& w^{a+1} G^b(w) g'(w) \frac{g(w)}{G^2(w)} \left(1 - \frac{\tau}{G(w)}\right)_+^{c-1} \zeta^2 |\bar{D}w|^2 \\
& \geq (g_0 - 1) g_0 w^{a-2} G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^{c-1} \zeta^2 |\bar{D}w|^2 \\
& \geq \gamma \left| \bar{D} w^{a/2} G^{b/2} \left(1 - \frac{\tau}{G(w)}\right)_+^{(c-1)/2} \right|^2
\end{aligned}
$$

for some constant $\gamma$. Because $w > 2M$ it follows that

$$
\begin{aligned}
& \sup_{-(2R)^2<\bar{t}<0} \int_{K_{2R}} w^a G(M) G^b(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \zeta^2 \, d\bar{x} \\
& \quad + \gamma_0 \iint_{\bar{Q}_{2R}} \left| \bar{D} w^{a/2} G^{b/2} \left(1 - \frac{\tau}{G(w)}\right)_+^{(c-1)/2} \right|^2 \zeta^2 \, d\bar{x}\, d\bar{t}, \\
& \leq \frac{\gamma}{M^2 G(M)} \iint_{\bar{Q}_{2R}} w^{a+2} G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \, d\bar{x}\, d\bar{t}
\end{aligned}
\tag{4.12}
$$

To handle the left hand side of (4.11), we apply Theorem 2.4.5 with $p = 2$ to the function

$$
\begin{gathered}
w^{a/2} G^{b/2} \left(1 - \frac{\tau}{G(w)}\right)_+^{(c-1)/2}, \\
\lambda^{k/2} = \left(1 - \frac{\tau}{G(w)}\right)_+ \in L^\infty(Q_{2R}).
\end{gathered}
$$

Therefore we have that

$$
\begin{aligned}
& \iint_{\bar{Q}_R} \left| w^{a/2} G^{b/2} \left(1 - \frac{\tau}{G(w)}\right)_+^{(c-1)/2} \right|^{\frac{k+2}{k}} \left(1 - \frac{\tau}{G(w)}\right)_+^{\frac{2}{k}} dx\, dt \\
& \leq C \tau^{N/k} R^{-2(2/k+N/k)} \left( \frac{\gamma}{M^2 G(M)} \iint_{\bar{Q}_{2R}} w^{a+2} G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \, d\bar{x}\, d\bar{t} \right)^{1+2/k}.
\end{aligned}
\tag{4.13}
$$

Let $\theta = 1 + \frac{2}{k}$ and

$$\Sigma = w^{a-k} G^{b-\frac{k}{2}}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c,$$
$$\mu = w^{2+k} G^{1+\frac{k}{2}}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^{-1}.$$

Then (4.13) is written as

$$\left(\iint_{\bar{Q}_R} \Sigma^\theta \mu \, d\bar{x} \, d\bar{t}\right)^{\frac{1}{\theta}} \leq \iint_{\bar{Q}_{2R}} \Sigma \mu \, d\bar{x} \, d\bar{t}.$$

After Moser's type of iteration, we derive that

$$\sup_{Q_R} w^{a-k} G^{b-\frac{k}{2}}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^c \leq C \iint_{Q_{2R}} w^{a+2} G^{b+1}(w) \left(1 - \frac{\tau}{G(w)}\right)_+^{c-1} dx \, dt.$$

By determining $a = k$ , $b = 1 + k/2$ and $c = 1$, we obtain that

$$\sup_{Q_R} G(|Dv|) \leq C(M) \iint_{Q_{2R}} |Dv|^{k+2} G^{2+\frac{k}{2}}(|Dv|) \, dx \, dt.$$

Because $|Dv| \geq 2M$, we make $M > 1$. Therefore

$$|Dv|^{k+2} \leq G(|Dv|)^{\frac{k+2}{g_0}}$$

which leads to the conclusion. □

From Lemma 4.2.2, the maximum of $G(|Dv|)$ is bounded by the integral of $G(|Dv|)^q$ (some constant $q > 1$). To estimate the same quantity with the $L_1$ norm of $G(|Dv|)$, we need to fine out the relationship between $L_q$ norm of $G(|Dv|)$ and $L_1$ norm of $G(|Dv|)$. This step is carried in the following lemma. The proof is following lines of the proof for Lemma 11.22 (pp. 274) from [42] that uses structure conditions to obtain the integral estimate.

**Lemma 4.2.3.** *Suppose that $v$ is a bounded weak solution of* (4.3) *under the structure conditions* (4.4). *For a positive constant $M > 1$ and $q \geq 1$, there exist constant $\beta_1$, $\beta_2$ and $C$ depending on data such that*

$$\iint_{Q_{2R}} G^{q+1}(|Dv|) \, dx \, dt \leq C(M) G^q(M) R^{-2\max\{\beta_1, \beta_2\}} \iint_{Q_{2R}} G(|Dv|) \, dx \, dt$$

*where*

$$\beta_1 = \frac{q g_1}{g_0 q - 1}, \quad \beta_2 = \frac{q g_1}{q(g_1 - g_0) + 1}.$$

*Proof.* To deliver clear proof, here we use the prototype equation (4.7). The similar argument works for for (4.3). We will make note when it requires. Let $\tau = G(M)$ and

$$I = \iint_{Q_{2R}} G^q(|Dv|)\frac{g(|Dv|)}{|Dv|}Dv \cdot Dv\,dx\,dt$$
$$I' = \iint_{Q_{2R}} e^{\lambda\bar{v}}(1+\lambda\bar{v})\zeta^q(G^q(|Dv|)-\tau^q)_+\frac{g(|Dv|)}{|Dv|}Dv \cdot Dv\,dx\,dt,$$

where $\bar{v}$ is a nonnegative bounded weak solution of (1.19) and $\lambda$ is a nonnegative constant to be determined. Here we note that

$$(G^q(|Dv|)-\tau^q)_+ = \begin{cases} G^q(|Dv|)-\tau^q & \text{if } |Dv| > M, \\ 0 & \text{if } |Dv| \le M, \end{cases}$$

and this definition gives that

$$(G^q(|Dv|)-\tau^q)_+ \ge (2^{g_0}-1)\tau^q, \quad \text{if } \ |Dv| \ge 2M.$$

Hence from the settings of $I$ and $I'$, clearly we can derive that

$$I' \ge I - \tau^q \iint_{Q_{2R}} \frac{g(|Dv|)}{|Dv|}Dv \cdot Dv\,dx\,dt$$

which gives

$$I \le I' + \tau^q \iint_{Q_{2R}} \frac{g(|Dv|)}{|Dv|}Dv \cdot Dv\,dx\,dt. \tag{4.14}$$

For (4.3), $\frac{g(|Dv|)}{|Dv|}Dv$ is replaced by $\tilde{\mathbf{A}}$ on the integral quantities $I$ and $I'$ and we need to apply (4.4b) to obtain (4.14).

Now we estimate $I'$ carefully. By reordering $Dv$ in the quantity $I'$, we first notice that

$$e^{\lambda\bar{v}}(1+\lambda\bar{v})Dv = D\left(\bar{v}e^{\lambda\bar{v}}\right).$$

By taking integration by parts knowing that the cutoff function $\zeta$ vanishes on the boundary, we have

$$I' = I_1 + I_2 + I_3$$

where $w = |Dv|$

$$I_1 = -q \iint_{Q_{2R}} \bar{v} e^{\lambda \bar{v}} \zeta^{q-1} (G^q(w) - \tau^q)_+ \frac{g(w)}{w} Dv \cdot D\zeta \, dx\, dt,$$
$$I_2 = -q \iint_{Q_{2R}} \bar{v} e^{\lambda \bar{v}} \zeta^{q} G^{q-1}(w) g(w) \frac{g(w)}{w} Dv \cdot Dv \, dx\, dt,$$
$$I_3 = - \iint_{Q_{2R}} \bar{v} e^{\lambda \bar{v}} \zeta^{q} (G^q(w) - \tau^q)_+ \operatorname{div} \left( \frac{g(w)}{w} Dv \right) \, dx\, dt.$$

For simpler notation let $E = \sup \bar{v} e^{\lambda \bar{v}}$. Then

$$I_1 \leq \frac{qEM}{2R} \iint_{Q_{2R}} (\zeta G(w))^{q-1} \, G(w) g(w) \, dx\, dt.$$

Also

$$I_2 \leq qE \iint_{Q_{2R}} \left( \zeta^q G^q(w) \frac{g(w)}{w} Dv \right)^{1/2} \left( \zeta^q G^{q-2}(w) g^2(w) \frac{g(w)}{w} Dv |Dv|^2 \right)^{1/2} \, dx\, dt,$$
$$\leq qE\epsilon I + \frac{qE}{\epsilon} \iint_{Q_{2R}} \zeta^q G^{q-2}(w) g^2(w) \frac{g(w)}{w} w |Dv|^2 \, dx\, dt,$$
$$\leq \frac{1}{4} I + (qE)^2 \iint_{Q_{2R}} \zeta^q G^{q}(w) g(w) \frac{|Dv|^2}{w^2} \, dx\, dt,$$

by Cauchy's inequality and choosing $\epsilon$ such that $qE\epsilon = 1/4$ (that is, $\epsilon^{-1} = 4qE$). Before estimating $I_3$, we make an observation that

$$\operatorname{div} \left( \frac{g(w)}{w} Dv \right) = \left( \frac{vg'(w) - g(w)}{w^2} \right) Dw \cdot Dv + \sum_k \frac{g(w)}{w} D_k Dv$$

and

$$vg'(w) - g(w) \leq (g_1 - 2) g(w).$$

If we play with (4.3), then

$$\operatorname{div} \mathbf{A}(\tilde{D}v) = a_{ii} + |Dv|^{-1} a_{ij} D_{ij} v.$$

Therefore

$$I_3 = I_4 + I_5$$

where

$$I_4 \leq E \iint_{Q_{2R}} \zeta^q (G^q(w) - \tau^q)_+ \left( \frac{vg'(w) - g(w)}{w^2} \right) Dw \cdot Dv \, dx\, dt$$
$$I_5 \leq E \iint_{Q_{2R}} \zeta^q (G^q(w) - \tau^q)_+ \frac{g(w)}{w} |Dw| \, dx\, dt.$$

Then we have that

$$\begin{aligned} I_4 &\le E g_1 \iint_{Q_{2R}} \zeta^q G^q(w) \frac{g(w)}{w^2} Dw \cdot Dv\, dx\, dt \\ &\le E g_1 \epsilon I + \frac{E g_1}{\epsilon} \iint_{Q_{2R}} \zeta^q G^q(w) \frac{g(w)}{w^3} |Dv||Dw|^2\, dx\, dt \\ &\le \frac{1}{4} I + (E g_1)^2 \iint_{Q_{2R}} \zeta^q G^q(w) g(w) \frac{|Dw|^2}{w^2}\, dx\, dt \end{aligned}$$

by Cauchy's inequality and setting $\epsilon$ such that $Eg_1\epsilon = 1/4$ (that is, $\epsilon^{-1} = 4Eg_1$). For any $\delta > 0$, Cauchy's inequality to $I_5$ implies that

$$I_5 \le E \iint_{Q_{2R}} \zeta^q (G^q(w) - \tau^q)_+ g(w)\, dx\, dt + E \iint_{Q_{2R}} \zeta^q G^q(w) g(w) \frac{|Dw|^2}{w^2}\, dx\, dt.$$

It says that

$$\begin{aligned} \frac{1}{2} I \le \tau^q & \iint_{Q_{2R}} \frac{g(|Dv|)}{|Dv|} Dv \cdot Dv\, dx\, dt \\ &+ \iint_{Q_{2R}} \zeta^q (G^q(w) - \tau^q)_+ g(w)\, dx\, dt \\ &+ \frac{qEM}{2R} \iint_{Q_{2R}} (\zeta G(w))^{q-1} G(w) g(w)\, dx\, dt \\ &+ \left[(qE)^2 + (Eg_1)^2 + E\right] \iint_{Q_{2R}} \zeta^q G^q(w) g(w) \frac{|Dw|^2}{w^2}\, dx\, dt. \end{aligned}$$

To estimate the integral quantity

$$\iint_{Q_{2R}} \zeta^q G^q(w) g(w) \frac{|Dw|^2}{w^2}\, dx\, dt,$$

we go back to (4.8) with the test function

$$\varphi = G^{q-1}(w) g(w) \zeta^q.$$

First note that

$$D\varphi = [(q-1)g^2(w) + g'(w)G(w)] G^{q-2}(w) \zeta^q Dw + q G^{q-1}(w) g(w) \zeta^{q-1} D\zeta.$$

Then through similar calculation appeared in Lemma 4.2.2, we are able to obtain that

$$\begin{aligned} &\iint_{Q_{2R}} \zeta^q G^q(w) g(w) \frac{|Dw|^2}{w^2}\, dx\, dt \\ &\le \gamma \iint_{Q_{2R}} \left[v G^q(w) \zeta^{q-1} \zeta_t + G^q(w) g(w) \zeta^{q-2} |D\zeta|^2\right] dx\, dt \\ &\le \gamma \frac{G(M)}{R^2} \iint_{Q_{2R}} v G^q(w)\, dx\, dt + \gamma \frac{M^2}{R^2} \iint_{Q_{2R}} G^q(w) g(w) \zeta^{q-2}\, dx\, dt. \end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2}I \le \tau^q &\iint_{Q_{2R}} \frac{g(|Dv|)}{|Dv|} Dv \cdot Dv \, dx \, dt \\
&+ \left[1 + \frac{qEM}{2R} + \frac{E'\gamma M^2}{R^2}\right] \iint_{Q_{2R}} \zeta^{q-2} G^q(w) g(w) \, dx \, dt \\
&+ \frac{E'G(M)}{R^2} \iint_{Q_{2R}} \zeta^{q-1} w G^q(w) \, dx \, dt.
\end{aligned}$$

where $E' = (qE)^2 + (Eg_1)^2 + E$.

Finally we note that

$$\begin{aligned}
CG^q(w)g(w) &\le Cg_1 \frac{G^q(w)}{w} G(w) \\
&\le \epsilon_1 \left(\frac{G^q(w)}{w}\right)^{p_1} G(w) + \epsilon_1^{-1} C^{p_1'} G(w)
\end{aligned}$$

with

$$p_1 = \frac{qg_1}{g_0 q - 1}, \quad p_1' = \frac{qg_1}{q(g_1 - g_0) + 1}.$$

Also similarly

$$\begin{aligned}
CvG^q(w) &= CwG^{q-1}(w)G(w) \\
&\le \epsilon_2 \left(wG^{q-1}(w)\right)^{p_2} G(w) + \epsilon_2^{-1} C^{p_2'} G(w)
\end{aligned}$$

with

$$p_2 = \frac{qg_1}{1 + g_0(q-1)}, \quad p_2' = \frac{qg_1}{q(g_1 - g_0) + (g_0 - 1)}.$$

By fixing $\epsilon_1 = \epsilon_2 = 1/8$, we derive that

$$\frac{1}{4}I \le \left[\tau^q + R^{-2}C_1^{p_1'} + R^{-2}C_2^{p_2'}\right] \iint_{Q_{2R}} G(|Dv|) \, dx \, dt$$

where

$$C_1 = (1 + qE + E'\gamma)M^2, \quad C_2 = E'G(M).$$

□

## 4.3 The $p-$energy estimates

The perturbation theory is based on comparing behavior of solutions from (4.1) under (4.2) and (4.3) under (4.4). Below approximation argument in Section 4.2 and the Hölder continuity of a bounded weak solution of (4.1) under (4.2) from Chapter 3 will make us to estimate of the supreme of $|u - v|$ in terms of some power of the side length of a cylinder.

**Lemma 4.3.1.** *Let $u$ be a bounded solution of* (4.1) *under* (4.2) *and $v$ be a solution of* (4.3) *under* (4.4). *Then*

$$\operatorname*{ess\,sup}_{Q_R} |u - v| \leq \operatorname*{ess\,osc}_{Q_R} u.$$

*Proof.* We define $F$ by $F(t) = g(t)/t$ for $t > 0$, and for $\epsilon \in (0, 1), \eta \in C^{0,1}(\mathbb{R}_+)$, then define

$$F_\epsilon(t) = F\left(\min\left\{\frac{1}{\epsilon}, t + \epsilon\right\}\right),$$

$$A_\epsilon(p) = (1 - \eta(|p|))\, \tilde{A}(p) + \eta(|p|) F_\epsilon(|p|) p.$$

Then for $|p| < \frac{1}{\epsilon} - \epsilon$, we have

$$a_\epsilon^{ij}(p) = (1 - \eta)\tilde{a}_{ij} + \eta\left[\frac{F'(|p| + \epsilon)}{|p| + \epsilon} p_j p_j + F(|p| + \epsilon)\right] + \eta'\left[-A^i + F_\epsilon p_i\right]\frac{p_j}{|p|}.$$

Now suppose $\eta(t) = 1$ for $t < \epsilon$. Then

$$\begin{aligned}
&\frac{F'(|p| + \epsilon)}{|p| + \epsilon}(p \cdot \xi)^2 + F_\epsilon(|p|)|\xi|^2 \\
&\geq \left\{(\delta - 1)\frac{(p \cdot \xi)^2}{|p| + \epsilon} + |\xi|^2\right\} F_\epsilon(|p|) \\
&\geq \delta F_\epsilon(|p|)|\xi|^2,
\end{aligned}$$

and hence $a_\epsilon^{ij}\xi_i\xi_j \geq \delta F_\epsilon |\xi|^2$ for $|p| < \epsilon$. For $|p| \geq \epsilon$, we have $|p| \leq |p| + \epsilon \leq 2|p|$ and hence $F_\epsilon(|p|) \geq 2^{\delta - 1} F(|p|)$. It follows that, for $|p| \in (\epsilon, \frac{1}{\epsilon} - \epsilon)$,

$$\begin{aligned}
&a_\epsilon^{ij}\xi_i\xi_j \\
&\geq (1 - \eta)F|\xi|^2 + \eta\delta F_\epsilon|\xi|^2 + \epsilon|\xi|^2 - |\eta'||p|\{\Lambda F + F_\epsilon\}|\xi|^2 \\
&\geq \left\{\frac{\delta}{2} - (|\eta'||p|)(\Lambda + 1)\right\} F_\epsilon|\xi|^2.
\end{aligned}$$

Finally if $\eta(t) = 1$ for $t > \frac{1}{\epsilon} - \epsilon$, we have $a_\epsilon^{ij}\xi_i\xi_j = F_\epsilon|\xi|^2$. Now set $K = \delta/8(1 + \Lambda)$ and suppose $\epsilon < e^{2/K}/2$. Then the choice

$$\eta(t) = \begin{cases} 1 & 0 \le t < \epsilon, \\ 1 - K\ln(t/\epsilon) & \epsilon \le t < e^{1/K}\epsilon \\ 0 & e^{1/K}\epsilon \le t < \frac{1}{2\epsilon}e^{1/K} \\ K\ln(t/\epsilon) & \frac{1}{2\epsilon}e^{1/K} \le t < \frac{1}{2\epsilon} \\ 1 & \frac{1}{2\epsilon} \le t \end{cases}$$

gives $a_\epsilon^{ij}\xi_i\xi_j \ge \frac{1}{4}\delta F_\epsilon|\xi|^2$. Similarly

$$|\tilde{a}_\epsilon^{ij}| \le 2(\Lambda + g_0 + 1)F_\epsilon.$$

Since $F$ is uniformly bounded on compact subsets of $(0,\infty)$, we can use an additional approximation argument to conclude that there is a $C^{1,\beta}$ solution of

$$(v_j)_t - \operatorname{div} A_{1/j}(Dv_j) = 0$$

for sufficiently large integers $j$.

Now $A_{1/j}$ converges uniformly to $\tilde{A}$ and $g_j$ converges uniformly to $g$ on compact subsets of $(0,\infty)$. The uniform estimates on the $v_j$'s guarantee that $v_j$ converges to $v$ locally in $C^1$ (after taking a uniform convergent sequence). □

**Remark 4.3.1.** *Then the Hölder continuity from Chapter 3 gives that*

$$\operatorname*{ess\,osc}_{Q_R} u \le \theta(R)$$

*which means*

$$\operatorname*{ess\,sup}_{Q_R} |u - v| \le \theta(R).$$

*Moreover,*

$$\operatorname*{ess\,osc}_{Q_R} v \le c\theta(R)$$

*because*

$$|v| \le |v - u| + |u|.$$

Based on a uniform estimate of $\sup |u-v|$ from Lemma 4.3.1 and Remark 4.3.1, we estimate the integral of $G(|Du-Dv|)$ in terms of the integral of $1+G(|Du|)$. Because of $G$ is not the power function, we need to split two sets depending on the largeness of $|Du-Dv|$ compared to $2|Du|$. Then considering each case separately and putting them together later will leads to our conclusion. Here we mimic the steps on pp. 342 - 345 from [41].

**Lemma 4.3.2.** *Let $u$ be a bounded solution of* (4.1) *under* (4.2) *and $v$ be a solution of* (4.3) *under* (4.4). *Then there exists a constant $C$ depending on data such that*

$$\iint_{Q_R} G\left(|Du-Dv|\right)\,dx\,dt \le C\theta(R)^{\alpha_1}\iint_{Q_R}\left(1+G(|Du|)\right)\,dx\,dt.$$

*Proof.* Let

$$I=\iint_{Q_R}\left[\tilde{\mathbf{A}}(Du)-\tilde{\mathbf{A}}(Dv)\right]\cdot[Du-Dv]\,dx\,dt,$$

and

$$S_1=\left\{(x,t)\in Q_R: |Du-Dv|\le 2|Du|\right\},$$
$$S_2=\left\{(x,t)\in Q_R: |Du-Dv|> 2|Du|\right\}.$$

Owing to the structure conditions (4.2b) and (4.2c), it follows that

$$\begin{aligned} I &\le \iint_{Q_R}\left(\Lambda_1|u-v|^{\alpha_1}g(1+|Du|)|Du-Dv|+\Lambda g(1+|Du|)|Du-Dv|\right)\,dx\,dt \\ &\le C\theta(R)^{\alpha_1}\iint_{Q_R}G(1+|Du|)\,dx\,dt+\epsilon\iint_{Q_R}G(|Du-Dv|)\,dx\,dt \end{aligned} \tag{4.15}$$

for any positive constant $\epsilon$ with the aid of Lemma and Young's inequality.

Now to estimate a lower bound of the quantity $I$, we apply (4.2a) and the mean value theorem

$$I\ge\iint_{Q_R}\int_0^1 g'\left(|Du+(1-t)(Du-Dv)|\right)|Du-Dv|^2dt\,dx\,dt.$$

Then we make observation using the triangle inequality that

$$\frac{1}{2}|Du|\le|Du+(1-t)(Du-Dv)|\le 3|Du|\quad\text{on } S_1 \text{ for } t\ge\frac{3}{4},$$
$$\frac{1}{4}|Du-Dv|\le|Du+(1-t)(Du-Dv)|\le 3|Du-Dv|\quad\text{on } S_2 \text{ for } t\le\frac{1}{4}.$$

Therefore

$$I \geq C \iint_{S_1} g'(|Du|)|Du - Dv|^2 \, dx \, dt + C \iint_{S_2} G(|Du - Dv|) \, dx \, dt. \tag{4.16}$$

Then two inequalities (4.15) and (4.16) deliver that

$$\begin{aligned} &\iint_{S_2} G(|Du - Dv|) \, dx \, dt \\ &\leq C\theta(R)^{\alpha_1} \iint_{Q_R} G(1 + |Du|) \, dx \, dt + \epsilon \iint_{Q_R} G(|Du - Dv|) \, dx \, dt \\ &\leq C\theta(R)^{\alpha_1} \iint_{Q_R} G(1 + |Du|) \, dx \, dt + \epsilon \iint_{S_1} G(1 + |Du|) \, dx \, dt \\ &\quad + \epsilon \iint_{S_2} G(|Du - Dv|) \, dx \, dt. \end{aligned} \tag{4.17}$$

By choosing $\epsilon = C/2$, we obtain that

$$\iint_{S_2} G(|Du - Dv|) \, dx \, dt \leq C\theta(R)^{\alpha_1} \iint_{Q_R} G(1 + |Du|) \, dx \, dt. \tag{4.18}$$

To estimate the integral over $S_1$, we calculate

$$\begin{aligned} &\iint_{S_1} G(|Du - Dv|) \, dx \, dt \\ &\leq C \iint_{S_1} \left[g'(|Du - Dv|)|Du - Dv|\right]^{1/2} \left[|Du - Dv|^3 g(|Du - Dv|)\right]^{1/2} \, dx \, dt \\ &\leq C \iint_{S_1} \left[g'(|Du|)|Du|\right]^{1/2} \left[|Du - Dv|^3 g(|Du - Dv|)\right]^{1/2} \, dx \, dt \\ &\leq C \iint_{S_1} \left[g'(|Du|)|Du - Dv|^2\right]^{1/2} \left[g(|Du - Dv|)|Du||Du - Dv|\right]^{1/2} \, dx \, dt, \end{aligned}$$

using various inequalities for $g'$ and $g$, and the setting on the set $S_1$. By Cauchy's inequality we derive that

$$\begin{aligned} &\iint_{S_1} G(|Du - Dv|) \, dx \, dt \\ &\leq C \iint_{S_1} g'(|Du|)|Du - Dv|^2 \, dx \, dt + C \iint_{S_1} g(|Du - Dv|)|Du||Du - Dv| \, dx \, dt \\ &\leq CI + C \iint_{S_1} g(|Du - Dv|)|Du||Du - Dv| \, dx \, dt \\ &\leq C\theta(R)^{\alpha_1} \iint_{Q_R} G(1 + |Du|) \, dx \, dt \\ &\quad + \epsilon \iint_{S_1} G(|Du - Dv|) \, dx \, dt + C \iint_{S_1} G(1 + |Du|) \, dx \, dt \end{aligned}$$

because of (4.15), (4.16), and (4.18). Fixing $\epsilon = 1/2$ yields

$$\iint_{S_1} G(|Du - Dv|)\,dx\,dt \le C\theta(R)^{\alpha_1} \iint_{Q_R} G(1 + |Du|)\,dx\,dt.$$

□

To obtain the conclusion of this section, we modify Lemma 5 (pp. 392 - 293) from [46]. Once we obtain the integral estimates of $G(1 + |Du|)$ in terms of the mixture of the integral estimate $G(1 + |Du|)$ in a bigger cylinder with some power of $R$, an iteration scheme yields our conclusion.

**Theorem 4.3.3.** *Suppose that $u$ is a bounded weak solution of* (4.1) *under* (4.2). *For a constant $0 < R < 1$ and any $\rho$ such that $0 < \rho < R/2$, there exist some constants $\delta \in (0, 1)$ and $C = C(\mathrm{data}, \|G(|Du|)\|_{1,Q_1})$ satisfying*

$$\iint_{Q_\rho} (1 + G(|Du|))\,dx\,dt \le C\rho^{\delta}.$$

*Proof.* Because of the triangle inequality and Lemma 4.3.2, we obtain

$$\begin{aligned}
&\iint_{Q_R} [1 + G(|Dv|)]\,dx\,dt \\
&\le c \iint_{Q_R} G(|Dv - Du|)\,dx\,dt + c \iint_{Q_R} [1 + G(|Du|)]\,dx\,dt \\
&\le \{c\theta(R)^{\alpha_1} + c\} \iint_{Q_R} [1 + G(|Du|)]\,dx\,dt \\
&\le c \iint_{Q_R} [1 + G(|Du|)]\,dx\,dt
\end{aligned} \tag{4.19}$$

due to $R < 1$ and $\theta(R) < 1$. Also by applying Lemma 4.2.2 and Lemma 4.2.3, note

$$\begin{aligned}
\iint_{Q_\rho} [1 + G(|Dv|)]\,dx\,dt &\le c\rho^{N+2} \sup_{Q_{R/2}} G(|Dv|) \\
&\le c\left(\frac{\rho}{R}\right)^{N+2} R^{-\beta} \iint_{Q_R} [1 + G(|Dv|)]\,dx\,dt \\
&\le c\left(\frac{\rho}{R}\right)^{N+2} R^{-\beta} \iint_{Q_R} [1 + G(|Du|)]\,dx\,dt
\end{aligned} \tag{4.20}$$

owing to (4.19), for a positive constant $\beta = \beta(\mathrm{data}) \in (0, 1)$.

By applying the triangle inequality, we derive below by Lemma 4.3.2 and (4.20)

$$\begin{aligned}
&\iint_{Q_\rho} [1+G(|Du|)]\, dx\, dt \\
&\le c\iint_{Q_\rho} G(|Du-Dv|)\, dx\, dt + c\iint_{Q_\rho} [1+G(|Dv|)]\, dx\, dt \\
&\le c\theta(\rho)^{\alpha_1}\iint_{Q_\rho} [1+G(|Du|)]\, dx\, dt + c\left(\frac{\rho}{R}\right)^{N+2} R^{-\beta}\iint_{Q_R} [1+G(|Du|)]\, dx\, dt.
\end{aligned} \tag{4.21}$$

Because of (4.19) interchanging $Du$ and $Dv$ and Lemma 4.2.1, we observe that

$$\begin{aligned}
\iint_{Q_\rho} [1+G(|Du|)]\, dx\, dt &\le c\iint_{Q_{R/2}} [1+G(|Dv|)]\, dx\, dt \\
&\le cG\left(\frac{\theta(R)}{R}\right) R^{N+2} + c\theta R^2 R^N \\
&\le c\theta(R)^{g_0} R^{N+2-g_1} + c\theta R^2 R^N
\end{aligned} \tag{4.22}$$

and so

$$\iint_{Q_\rho} [1+G(|Du|)]\, dx\, dt \le \begin{cases} cR^{N+2-g_1} & \text{if } g_1 > 2, \\ c\theta R^2 R^N & \text{if } g_1 \le 2. \end{cases}$$

For some constant $0<\gamma<2$, we have

$$\iint_{Q_\rho} [1+G(|Du|)]\, dx\, dt \le cR^{N+\gamma-\sigma\kappa}.$$

For a constant $0<\gamma<2$, we observe that

$$\left(\frac{\rho}{R}\right)^{N+2} R^{-\beta} = \left(\frac{\rho}{R}\right)^{N+2-\gamma} \rho^\gamma R^{-\beta-\gamma} \le \left(\frac{\rho}{R}\right)^{N+2-\gamma}$$

by setting

$$\rho \le R^{\frac{\beta+\gamma}{\gamma}}.$$

Let

$$F(r) = \iint_{Q_r} [1+G(|Du|)]\, dx\, dt,$$

a nonnegative and nondecreasing function. Then (4.21) and (4.22) yields

$$F(\rho) \le c\left(\frac{\rho}{R}\right)^{N+2-\gamma} F(R) + ccR^{N+\gamma-\sigma\kappa}.$$

By Lemma 2.4.8, there exist a constant $\delta' \in (0,1)$ such that

$$F(\rho) \le c\rho^{\delta} \le c\rho^{\delta'}\, [F(1)+1]$$

that leads to our conclusion. □

## 4.4 The bounded mean oscillation estimates

Denote

$$\{f\}_\rho = \frac{1}{|Q_\rho|}\iint_{Q_\rho} f(x,t)\,dx\,dt$$

for $f \in L_{1,Q_\rho}$. Let $G(|f|) \in L_{1,Q_\rho}$. We then observe that for any $L \in \mathbb{R}$

$$G\left(|f - \{f\}_\rho|\right) \le cG\left(|f-L|\right) + cG\left(|L - \{f\}_\rho|\right),$$

because $G$ is a convex function. Using the fact that $G\left(|L-\{f\}_\rho|\right)$ is a constant, we derive

$$\begin{aligned}\frac{1}{|Q_\rho|}\iint_{Q_\rho} G\left(|L-\{f\}_\rho|\right)\,dx\,dt &= G\left(|L-\{f\}_\rho|\right)\\ &= G\left(\frac{1}{|Q_\rho|}\iint_{Q_\rho} |L-\{f\}_\rho|\,dx\,dt\right)\\ &\le \frac{1}{|Q_\rho|}\iint_{Q_\rho} G\left(|L-\{f\}_\rho|\right)\,dx\,dt\end{aligned}$$

using the Jensen's inequality for the last inequality. Therefore we conclude

$$\frac{1}{|Q_\rho|}\iint_{Q_\rho} G\left(|f-\{f\}_\rho|\right)\,dx\,dt \le C\frac{1}{|Q_\rho|}\iint_{Q_\rho} G\left(|f-L|\right)\,dx\,dt \tag{4.23}$$

for any constant $L$.

To obtain Hölder continuity of $Du$ using theorems isomorphic relation between Campanato spaces and Hölder space, we estimate the bounded mean oscillation using Theorem 4.3.3 from the previous section. Due to Theorem 4.3.3, it is possible to estimate the essential oscillation of $|Dv|$ in terms of some power of the side length of a cylinder. Depending on the size of $|Dv|$, we have two cases to consider. First, if $|Dv|$ is somewhat small quantity overall the cylinder, then there exists a subcylinder where $|Dv|$ is bounded almost everywhere. Otherwise (in case $|Dv|$ is greater than some positive constants overall the cylinder), we obtain integral estimates of the bounded mean oscillation. Finally an iteration step will leads to the conclusion that yields the Hölder continuity eventually.

**Lemma 4.4.1.** *For $v$, a solution of* (4.3) *under* (4.4)*, there exist constants $C$ and $\gamma$ such that*

$$\sup_{Q_{R/2}} G(|Dv|) \le CR^{-\gamma}.$$

*Proof.* Rewrite the conclusion in Theorem 4.3.3 with a constant $\eta = \delta - (N+2)$ such that

$$\frac{1}{|Q_R|} \iint_{Q_R} [1 + G(|Du|)] \, dx \, dt \leq cR^{-\eta}. \tag{4.24}$$

By Lemma 4.2.2 and Lemma 4.2.3, there exists a constant $\beta > 0$ such that

$$\sup_{Q_{R/2}} G(|Dv|) \leq cR^{-N-2} R^{-2\beta} \iint_{Q_R} G(|Dv|) \, dx \, dt.$$

But then by the triangle inequality, Lemma 4.3.2, and $0 < R < 1$, we have

$$\begin{aligned} \sup_{Q_{R/2}} u(x,t) &\leq cR^{-N-2} R^{-2\beta} \iint_{Q_R} [1 + G(|Du|)] \, dx \, dt \\ &\leq cR^{-2\beta-\eta} \end{aligned}$$

because of (4.24). Hence the conclusion follows by letting $\gamma = 2\beta + \eta$. □

Two Propositions below are equivalent to Proposition 1.1 and Proposition 1.2 on Section IX.1-(i) of [11]. Denote

$$Q_R(\mu) = K_R \times [-\frac{\mu^2}{G(\mu)} R^2, 0]$$

for some constant $\mu > 0$ which is intrinsically scaled cylinder.

**Proposition 4.4.2.** *There exist constants $\nu, c, \delta$ in $(0,1)$ determined by data such that if*

$$\text{meas}\,\{Q_R : |Dv| < (1-\nu)\mu\} < \nu |Q_R|$$

*where $\mu = \text{ess sup}\,|Dv|$, then*

$$\iint_{Q_{\delta^{n+1}R}(\mu)} |Du - \{Du\}_{n+1}|^2 \, dx \, dt \leq c\delta^{N+2} \iint_{Q_{\delta^n R}(\mu)} |Du - \{Du\}_n|^2 \, dx \, dt$$

*for all $n = 1, 2, \ldots,$ where*

$$\{Du\}_n = \frac{1}{|Q_{\delta^n R}(\mu)|} \iint_{Q_{\delta^n R}(\mu)} Du \, dx \, dt.$$

**Proposition 4.4.3.** *There exist constants $0 < \sigma < \nu < 1$, if*

$$\text{meas}\,\{Q_R : |Dv| < (1-\nu)\mu\} \geq \nu |Q_R|$$

*where $\mu = \text{ess sup}\,|Dv|$, then*

$$|Dv|(x,t) \leq \eta\mu$$

*for all $(x,t) \in Q_{\sigma R}(\mu)$.*

*Proof.* To prove Proposition 4.4.2 and Proposition 4.4.3, we quote argument from [11] on Chapter IX (pp. 246) about Hölder continuity of $Du$ that is parallel to the proof we carried in Chapter 3 to show Hölder continuity of a bounded weak solution. □

**Lemma 4.4.4.** *For $v$, a solution of* (4.3) *under* (4.4)*, there exist constants $C$ and $\gamma_1$ such that*

$$\operatorname*{ess\,osc}_{Q_{R/2}} |Dv| \le CR^{\alpha}.$$

*Proof.* Apply Proposition 4.4.2 and Proposition 4.4.3 to estimate the oscillation of $|Dv|$. Then the places wherever $\|Dv\|_\infty$ is used are replaced by $R^\alpha$ due to Lemma 4.4.1.

From Proposition 4.4.2 and Proposition 4.4.3, we obtain parallel version of Theorem 3.3.4 and Corollary 3.3.5 replacing $v$ by $Dv$ and $\operatorname{dist}(t_1, t_2) = \sqrt{|t_1 - t_2|}$. Owing to Lemma 4.4.1, $\|Dv\|_\infty$ is replaced by $CR^{-\gamma/g_0}$. Then there exists $\alpha$ such that

$$\operatorname*{ess\,osc}_{Q_{R/2}} G\left(|Dv|\right) \le C(\mu)R^{-\alpha}.$$

□

**Theorem 4.4.5.** *For a bounded solution $u$ of* (4.1) *under* (4.2)*, there exist $C$ and $\alpha$ such that, for any $0 < \rho < R < 1$,*

$$\iint_{Q_\rho} G\left(|Du - \{Du\}_\rho|\right)\, dx\, dt \le \left(\frac{\rho}{R}\right)^{\eta} \iint_{Q_R} G\left(|Du - \{Du\}_\rho|\right)\, dx\, dt$$

*Proof.* Let $0 < R < 1$. For $R/2 \le \rho < R$, we have that

$$\begin{aligned}&\iint_{Q_\rho} G\left(|Dv - \{Dv\}_\rho|\right)\, dx\, dt \\ &\le C2^{N+2}\left(\frac{\rho}{R}\right)^{N+2} \iint_{Q_R} G\left(|Dv - \{Dv\}_R|\right)\, dx\, dt + c\rho^{N+2-\alpha}.\end{aligned}$$

For $\rho < R/2$, we use (4.23) and Lemma 4.4.4 to obtain

$$\begin{aligned}\iint_{Q_\rho} G\left(|Dv - \{Dv\}_\rho|\right)\, dx\, dt &\le c\iint_{Q_\rho} G\left(|Dv - L|\right)\, dx\, dt \\ &\le c\rho^{N+2} \operatorname*{ess\,osc}_{Q_\rho} G(|Dv|) \\ &\le c\rho^{N+2-\alpha},\end{aligned}$$

for any constant $L$. Therefore, for all $0 < \rho < R$,

$$\begin{aligned}&\iint_{Q_\rho} G\left(|Dv - \{Dv\}_\rho|\right)\,dx\,dt\\ &\leq C2^{N+2}\left(\frac{\rho}{R}\right)^{N+2}\iint_{Q_R} G\left(|Dv - \{Dv\}_R|\right)\,dx\,dt + C\rho^{N+2-\alpha}.\end{aligned}$$

Because of (4.23) and the triangle inequality, we derive for any constant $L$

$$\begin{aligned}&\iint_{Q_\rho} G\left(|Du - \{Du\}_\rho|\right)\,dx\,dt\\ &\leq C\iint_{Q_\rho} G\left(|Du - L|\right)\,dx\,dt\\ &\leq C\iint_{Q_R} G\left(|Dv - L|\right)\,dx\,dt + c\iint_{Q_R} G\left(|Dv|\right)\,dx\,dt\\ &\leq C2^{N+2}\left(\frac{\rho}{R}\right)^{N+2}\iint_{Q_R} G\left(|Dv - L|\right)\,dx\,dt + c\rho^{N+2+\alpha g_0}.\end{aligned}$$

Also we note that

$$\begin{aligned}&\iint_{Q_R} G\left(|Dv - \{Dv\}_R|\right)\,dx\,dt\\ &\leq c\iint_{Q_R} G\left(|Du - L|\right)\,dx\,dt + \iint_{Q_R} G\left(|Du - Dv|\right)\,dx\,dt\\ &\leq c\iint_{Q_R} G\left(|Du - L|\right)\,dx\,dt + C\theta(R)^{\alpha_1}\iint_{Q_R}\left[1 + G(|Du|)\right]\,dx\,dt\\ &\leq c\iint_{Q_R} G\left(|Du - L|\right)\,dx\,dt.\end{aligned}$$

Therefore it follows

$$\begin{aligned}&\iint_{Q_\rho} G\left(|Du - \{Du\}_\rho|\right)\,dx\,dt\\ &\leq C\left(\frac{\rho}{R}\right)^{N+2}\iint_{Q_R} G\left(|Du - \{Du\}_R|\right)\,dx\,dt + c\rho^{N+2-\alpha}.\end{aligned} \tag{4.25}$$

Let

$$F(r) = \iint_{Q_r} G\left(|Du - \{Du\}_\rho|\right)\,dx\,dt$$

then we are able to apply Lemma 2.4.7 due to the equation (4.25). Hence there exists $\eta \in (0,1)$ such that

$$F(\rho) \leq c\left(\frac{\rho}{R}\right)^{\eta} F(R) + \rho^{N+2-\alpha},$$

which leads to the conclusion. □

## 4.5 Hölder continuity of $Du$

We introduce two theorems below will be used in Chapter 4 to make the final conclusion to obtain the Hölder continuity from the bounded mean oscillation estimate. The notation $\mathcal{L}^{p,\lambda}(\Omega)$ refer the Morrey spaces that for $u \in L_{p,\Omega}$ define

$$L^{p,\lambda}(\Omega) = \Big\{ u \in L^p(\Omega) | \|u\|_{L^{p,\lambda}(\Omega)} < \infty \Big\}$$

where

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{x\in\Omega, 0<\rho<\mathrm{diam}\Omega} \rho^{-\lambda} \int_{\Omega_{x,\rho}} |u|^p \, dx \right\}^{\frac{1}{p}}.$$

We also define the Campanato spaces for $p \geq 1$ and $\lambda \geq 0$ that

$$\mathcal{L}^{p,\lambda} = \Big\{ u \in L^p(\Omega) | \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} < \infty \Big\}$$

where

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{\mathcal{L}^{p}(\Omega)} + [u]_{p,\lambda},$$

$$[u]_{p,\lambda} = \left\{ \sup_{x\in\Omega, 0<\rho<\mathrm{diam}\Omega} \rho^{-\lambda} \int_{\Omega_{x,\rho}} |u(x) - u_{x_0,\rho}|^p \, dx \right\}^{\frac{1}{p}}.$$

**Theorem 4.5.1.** *(An integral characterization of Hölder continuous functions) Let $\Omega$ be any Lipschitz domain and $n < \lambda \leq n + p$. Then $\mathcal{L}^{p,\lambda}(\Omega)$ is isomorphic to the space $C^{0,\alpha}(\Omega)$ with $\alpha = \frac{\lambda-n}{p}$. Moreover if $u \in \mathcal{L}^{p,\lambda}(\Omega)$ with $\lambda > n + p$, then $u$ is constant in $\Omega$.*

This theorem is quoted from [24] Theorem 1.2 pp. 70.

**Theorem 4.5.2.** *If*

$$\int_{B_\rho(x)} |u - \{u\}_\rho|^p \, dx \leq c\rho^{n+p\alpha}, \quad \alpha \in (0, 1]$$

*for $x$ in an open set $\Omega$ and for all $p < \min(R_0, \mathrm{dist}(x, \partial\Omega))$ for some $R_0$, then $u$ is locally Hölder continuous with exponent $\alpha$ in $\Omega$.*

This theorem is quoted from [24] Theorem 1.3 pp. 72.

From Theorem 4.4.5, we apply the isomorphism theorems given by Theorem 4.5.1 and Theorem 4.5.2. Basically the conclusion from Theorem 4.4.5 implies the membership of $Du$

in the Campanato space $\mathcal{L}^{1,\alpha}$ and hence $Du$ is Hölder continuous. Here we actually have two different size of cylinders, one intrinsically scaled cylinder for estimating the essential oscillation of $|Dv|$ and another one is regular sized cylinder. Therefore we need to link those two differently sized cylinders to obtain complete story.

# CHAPTER 5. Summary and Further Discussions

## 5.1 Summary

## 5.2 Hölder continuity of $u$

In Chapter 3, we study that a weak bounded solution of (1.19) under the structure conditions (1.20) that carry constants for lower order terms. Let us consider the same partial differential equation under the structure conditions

$$\mathbf{A}(x,t,u,Du)\cdot Du \geq G(|Du|) - G(\varphi_0(x,t)), \tag{5.1a}$$

$$|\mathbf{A}(x,t,u,Du)|\, a_1 \leq a_1 g(|Du|) + g(\varphi_1(x,t)), \tag{5.1b}$$

$$|B(x,t,u,Du)| \leq a_2 G(|Du|) + G(\varphi_2(x,t)), \tag{5.1c}$$

with positive measurable functions $\varphi_0$, $\varphi_1$, and $\varphi_2$ in an appropriate function space. Because of the usage of the generalized function, we are no longer allowed for applying Hölder inequality which makes difficulties deriving appropriate energy estimate. Also using various inequality such as Lemma 2.1.1 has limitation because we have to divide cases when the function value is greater or less than 1. To obtain a proper (possibly broadest) spaces for such lower order functions depending on $x$ and $t$, we need to use different method such as perturbation argument (somewhat Hölder continuous).

## 5.3 Existence theory

We first report known results in existence theory the Cauchy problems for the heat equation and the $p-$Laplacian equation. A classical result of Tychonov [57] states that the Cauchy problem for the heat equation

$$
\begin{aligned}
u_t - \Delta u = 0, \qquad & \text{in } \mathbb{R}^N \times (0,T),\ 0 < T < \infty, \\
u(\cdot,0) = u_0(\cdot), &
\end{aligned}
$$

is uniquely solvable for continuous initial data $x \mapsto u_0(x)$ satisfying the growth condition

$$
|u_0(x)| \le C \exp\left(a|x|^2\right), \quad \text{as } |x| \to \infty,
$$

for some positive constants $C$ and $a$. In such a case, the solution $u$ exists in the strip $\mathbb{R}^N \times (0, a/4)$. Roughly for the Cauchy problem for $p-$Laplacian equation (nonlinear version of the heat equation),

$$
\begin{aligned}
u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \qquad & \text{in } \mathbb{R}^N \times (0,T),\ 0 < T < \infty, \\
u(\cdot,0) = u_0(\cdot), &
\end{aligned}
\tag{5.2}
$$

the issue of growth conditions on the initial data $x \mapsto u_0(x)$ should be addressed. Here the range of the number $p$ whether $p > 2$ or $p^* < p < 2$ or $1 < p \le p^*$ plays importantly because of quite different behavior of solutions in each case.

When $p > 2$ (refer [19]), if $u_0 \in L^\infty_{\text{local}}(\mathbb{R}^N)$ provided

$$
|u_0(x)| \le c_0 |x|^{\frac{p}{p-2}} \quad \text{as } |x| \to \infty
$$

for some positive constant $c_0$, then the Cauchy problem (5.2) with $p > 2$ is uniquely solvable in a weak sense in the strip

$$
S_T = \mathbb{R}^N \times (0,T), \quad T = \frac{C(N,p)}{c_0^{p-2}}
$$

for some positive constant $C$ depending only by $N$ and $p$. (From the quantity of $T$, we can observe that the method of intrinsic scaling is used.) In fact, $u_0$ can be lie in $L^1_{\text{loc}}(\mathbb{R}^N)$ by rephrasing the growth condition in terms of suitable integral averages. We note the Barenblatt solution

$$\begin{aligned}
&\mathcal{B}(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left( \frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}, \quad t > 0, \\
&\gamma_p = \left( \frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}, \quad p > 2, \\
&\lambda = N(p-2) + p,
\end{aligned} \tag{5.3}$$

of the Cauchy problem (5.2) with the initial data $\mathcal{B}(x,0) = M\delta_0$ where $\delta_0$ is the Dirac delta function concentrated at the origin and a positive constant $M$ determined by $L^1(\mathbb{R}^N)$ norm of the solution. Note that for $t > 0$

$$\begin{aligned}
\|\mathcal{B}(\cdot,t)\|_{\mathbb{R}^N}^{\infty} &\leq t^{-\frac{N}{\lambda}}, \\
\|D\mathcal{B}(\cdot,t)\|_{\mathbb{R}^N}^{\infty} &\leq t^{-\frac{N+1}{\lambda}} \frac{p}{p-1} \gamma_p^{\frac{p-1}{p}}
\end{aligned}$$

which is very close to desired estimates for $\|u(\cdot,t)\|_{\Omega}^{\infty}$ and $\|Du(\cdot,t)\|_{\Omega}^{\infty}$ for the Cauchy problem (5.2) with the initial data carrying suitable growth condition. The $L^\infty$ estimates are followed from $L_q$ norm of $|Du|$ which involved delicate work for estimating $\iint |Du|^{p-1}\,dx\,dt$.

The Cauchy problem (5.2) in case $1 < p < 2$ is solvable as long as the initial data satisfy $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$, regardless of the growth condition of the initial data (Refer [20]). However the approach is quite different and deep compared to degenerate case. Unlike solutions in case $p > 2$, in general solutions in case $1 < p < 2$ are not locally bounded. Specifically, if

$$u_0 \in L^r_{\text{loc}}(\mathbb{R}^N), \quad \text{for} \ \ r \geq 1,$$

and

$$p > \frac{2N}{N+r},$$

then the solution $u$ of (5.2) belongs to $L^\infty_{\text{loc}}(S_T)$. Otherwise one of the above conditions fails or $u$ is not locally bounded. Here $C^{1,\alpha}$ type of estimate does not work and a different approach is used that a truncated solution is somewhat regular. Let

$$u_k = \min\{u, k\}, \quad \text{for any} \ \ k > 0.$$

Then

$$|Du_k| \in L^p_{\text{loc}}(S_T), \quad \frac{\partial}{\partial t} u_k \in L^1_{\text{loc}}(S_T).$$

By investigating

$$\left| Du^{\frac{p-1-\alpha}{p}} \right| \in L^p_{\text{loc}}(S_T), \quad \text{for any } \alpha \in (0, p-1), \tag{5.4}$$

deriving the the integral of $|Du|^{p-1}$ is made in terms of the integral quantity depending on a cylinder and some power norm of $u$. One of step is carrying Galerkin type of approximation with the initial data belong to$C_0^\infty(\mathbb{R}^N)$ for a solution $u_k$.

Now consider the Cauchy problem for generalized $p-$Laplacian equation

$$u_t - \text{div}\left(g'(|Du|)Du\right) = 0, \qquad \text{in } S_T = \mathbb{R}^N \times (0,T),\ 0 < T < \infty, \tag{5.5}$$

$$u(\cdot, 0) = u_0(\cdot), \qquad u_0(\cdot) \in L^1_{\text{loc}}(S_T).$$

Then Hölder regularity for $Du$ from Chapter 4 can be used for existence theory when $g_0 \geq 2$ (which includes degenerate case) although we may lose sharpness of the original existence theory. In case $g_1 \leq 2$, we choose test functions that actually vanish places where $u$ is somewhat large. Note that showing (5.4) is equivalent to showing that

$$\left\| u^{-1} G(|Du|) \right\|^1_{S_T} < \infty.$$

Many estimates can be rewritten in terms of functions $g'$, $g$, or $G$. One of goals is developing methods that gives sharp estimate even for generalized equation and capturing the effect of the difference of two constants $g_0$, and $g_1$.

## 5.4 Harnack inequality

For the heat equation, we set up two subcylinders about a point $(x_0, t_0)$ with a fixed constant $\sigma \in (0, 1)$

$$Q^+ = K_{\sigma R}(x_0) \times (t_0 + (\sigma R)^2, t_0 + R^2],$$

$$Q^- = K_{\sigma R}(x_0) \times (t_0 - R^2, t_0 - (\sigma R)^2].$$

Then the Harnack estimate is

$$\operatorname*{ess\,sup}_{Q^-} u(x,t) \le \gamma(\sigma) \operatorname*{ess\,inf}_{Q^+} u(x,t).$$

We notice that $Q^-$ and $Q^+$ are away with a positive time gap $2(\sigma R)^2$. By sending $\sigma \to 1$, the constant $\gamma(\sigma)$ is stable and we have

$$\gamma^{-1} \operatorname*{ess\,sup}_{K_R(x_0)} u(\cdot, t_0 - R^2) \le u(x_0, t_0) \le \gamma \operatorname*{ess\,sup}_{K_R(x_0)} u(\cdot, t_0 + R^2)$$

for a constant $\gamma$ depending on data.

# CHAPTER 6. Appendix

## 6.1 Appendix A. Notation

1. $Du = \nabla u$, the gradient vector of $u$.

2. $|Du| = \left(\sum_{k=1}^{N} |D_k u|^2\right)^{1/2}$

3. $\partial_p \Omega_T$: parabolic boundary of $\Omega_T$

## 6.2 Appendix B. Function spaces

1. $C(\Omega) = \{u : \Omega \to \mathbb{R} | \quad u \text{ continuous}\}$.

2. $C^k(\Omega) = \{u : \Omega \to \mathbb{R} | \quad u \text{ is } k\text{−times continuously differentiable}\}$.

3. $L^p(\Omega) = \{u : \Omega \to \mathbb{R} | \quad u \text{ is Lebesgue measurable, } \|u\|_{p,\Omega} < \infty\}$

   where

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}.$$

4. $L^\infty(\Omega) == \{u : \Omega \to \mathbb{R} | \quad u \text{ is Lebesgue measurable, } \|u\|_{\infty,\Omega} < \infty\}$

   where

$$\|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_\Omega |u|.$$

5. The Sobolev spaces

$$W^{k,p}(\Omega) = \left\{u : \Omega \to \mathbb{R} | \quad \text{upto } k\text{-th derivatives are exist, } \|u\|_{W^{k,p},\Omega} < \infty\right\}$$

where

$$\|u\|_{W^{k,p},\Omega} = \left(\|u\|_{p,\Omega} + \sum_{i=1}^{i=k} \|D^i u\|_{p,\Omega}\right)^{\frac{1}{p}}.$$

6. The $\alpha-$th Hölder seminorm: for distinct points $X, Y$ in $\Omega$

$$[u]_{\alpha,\Omega} = \sup_{X,Y} \frac{|u(X) - u(Y)|}{|X - Y|^{\alpha}}.$$

7. The $\alpha-$th Hölder norm

$$|u|_{\alpha,\Omega} = \|u\|_{\infty,\Omega} + [u]_{\alpha,\Omega}.$$

8. The $k-$the derivative Hölder space

$$C^{k,\alpha}(\Omega) = \{u : \Omega \to \mathbb{R}| \quad \text{upto } k\text{−th derivatives are exist, } |u|_{\alpha,\Omega} < \infty\}.$$

9. $V^0(\Omega_T) = L^{\infty}\left(0, T; L^2(\Omega)\right) \cap C(0, T; W^{1,2}(\Omega))$ where

$$L^{\infty}(0, T; L^2(\Omega)) = \left\{u : \int_0^T \left(\int_{\Omega} u^2\, dx\right)^{\frac{1}{2}} dt < \infty\right\}.$$

10. $V^{1,2}(\Omega_T) = L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$ where

$$L^2(0, T; L^2(\Omega)) = \left\{u : \left(\int_0^T \int_{\Omega} u^2\, dx\, dt\right)^{\frac{1}{2}} < \infty\right\}.$$

11. $V^{1,p}(\Omega_T) = L^{\infty}(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,2}(\Omega))$ where

$$L^p(0, T; W^{1,2}(\Omega)) = \left\{u : \left(\int_0^T \left(\int_{\Omega} u^2\, dx\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} < \infty\right\}.$$

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