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Optimal networked controllers for networked plants

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DEDICATION

I would like to dedicate this thesis to my parents and my dearest wife. Thank you for all the love and support.

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ABSTRACT

In this thesis, we study networked systems composed of discrete-time systems interacting over discrete-time networks. These systems are emerging in many application areas and require new distributed control and estimation design methodologies. Most existing approaches represent networked system models by structured system models (systems with structured state-space or input-output representations) assuming a complete equivalence between the two models. In this thesis, we carefully analyze the connection between these two models and study the conditions under which networked systems can be viewed as structured systems, and vice versa. Although, networked systems are shown to be equivalent to structured systems in general, we show that modeling the networked systems as systems with structured transfer function matrices is inappropriate for problems which require stabilizability and detectability of the designed networked system. This is due to the lack of constructive proofs in literature to obtain a stabilizable and detectable networked system corresponding to an unstable structured transfer function matrix. This important observation shows that the theory developed for designing distributed controllers using transfer function approaches (where the designed transfer functions can in general be unstable) may not provide a stabilizing networked controller.

We refer to the property of realizing a structured transfer function matrix as a stabilizable and detectable networked system by *network realizability*. Although this problem is mostly open and appears to be difficult, we partially answer this problem by providing a constructive proof to show that stable structured transfer function matrices are always network realizable.

Based on this development, we consider the problem of designing stabilizing networked controllers for a given networked plant. As transfer function approaches are not suitable, we develop a state-space approach using classical Youla-Kučera parameterization techniques to parameterize all internally stabilizing networked controllers for the given networked plant. This formulation allows us to pose the problem of finding stabilizing networked controllers as an unconstrained convex optimization problem, which can be solved using standard techniques. This formulation allows us to solve the optimal networked $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems while ensuring that the solution is a stabilizing networked controller that can be implemented as sub-systems interacting over the given network.

It turns out that the optimal stabilizing networked controllers can have a large order as they trade off complexity for the lack of complete communication graph. The optimal solutions provide performance limitations of the controllers when constrained to be networked. In order to obtain networked controllers with order comparable to that of the networked plant, we provide a methodology to obtain full-order internally stabilizing networked controllers using linear matrix inequalities. This methodology being based on a sufficiency condition, assures only sub-optimal full-order stabilizing networked controllers.

Next, we consider the problem of designing a networked estimator for a given networked plant. We express this problem as a networked control problem for an equivalent plant model and apply our networked controller design approach. We provide the parameterization of all stable networked estimators and the networked estimation problem is expressed as an unconstrained convex optimization problem that can be solved using standard techniques.

Finally, we consider the networked systems over any general delay networks. The results previously developed for systems over zero-delay networks are extended to the case of systems over general delay networks. We conclude the thesis with a look at future research directions - the development of model reduction techniques for networked systems, the development of distributed design methods, and the extension of our design methodology to include network model uncertainties and other distributed performance objectives.

CHAPTER 1. Introduction

With increasing number of applications in the field of networked or spatially interconnected systems, there has been a great surge in research towards design of networked controllers for such systems. One of the main objectives of this research is to find networked controllers that satisfying the desired performance criteria and can also be implemented in a distributed fashion over the same network as that of the plant.

In this thesis, we solve the optimal networked $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems for a class of networked systems composed of heterogeneous sub-systems interacting over a given network. In the networked system model we consider, only local information is passed from a sub-system to it's immediate neighbors over the network in each time instant. The controller is also networked and uses the same interconnection as the networked plant. We restrict our attention to linear time-invariant discrete-time systems.

The literature on decentralized, distributed, and networked control is vast, and it is difficult to provide a thorough review. In the classical decentralized control problem, the plant is generally not interconnected and the controller is made of isolated sub-controllers that use only local measurements and act only on local actuators. These problems are notoriously hard (for example the Witsenhausen problem in [1]) and have motivated the search for controller structures other than just diagonal ones [2–9].

In particular, the availability of communication networks allows controllers to exchange information over the network, and result in structured systems consistent with the available communication network. However, these problems are also usually difficult to solve when the underlying networks for the plant and controller are generic. Important exceptions are obtained for certain networked plant and controller models. Looking for and identifying conveniently searchable structures, in the system state-space or input-output representation, has been the focus of most research in networked or distributed control problems. Examples when network constraints are imposed on the controller transfer function matrix include the cases of spatially invariant systems [4, 10–12], systems with triangular and band structures [5, 6], symmetrically interconnected systems [13], dynamically coupled systems [9], poset-causal systems [14] and in the case of plant and controller structures satisfying quadratic invariance property [8, 15]. These results provide controllers with transfer functions satisfying linear constraints imposed by the underlying network. Examples where network constraints are imposed on the controller state-space matrices include relatively smaller number of cases like networked systems over acyclic networks [16], identical dynamically coupled diagonalizable systems [17] and heterogeneous sub-systems connected over arbitrary undirected graphs considered by [18]. Due to the finite-dimensionality of the state-space approaches, the controllers obtained with network constraints imposed on the state-space matrices are usually sub-optimal.

In the following example, we will show that a large part of the theory developed for distributed controller design does not truly provide a internally stabilizing distributed controller, i.e. a state-space representation of a distributed controller that makes the state-space dynamics of the closed-loop system asymptotically stable.

1.1 Motivational example

Consider the following dynamically coupled system G, based on the model considered in [9], of the form

$$\begin{aligned}
x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + \bar{B}_1 w_1(k) + B_1 u_1(k),\\
x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + \bar{B}_2 w_2(k) + B_2 u_2(k),\\
x_3(k+1) &= A_{32}x_2(k) + A_{33}x_3(k) + \bar{B}_3 w_3(k) + B_3 u_3(k),\\
z_1(k) &= \tilde{C}_{11}x_1(k) + \tilde{D}_1 w_1(k),\\
z_2(k) &= \tilde{C}_{21}x_2(k) + \tilde{D}_2 w_2(k),\\
z_3(k) &= \tilde{C}_{32}x_3(k) + \tilde{D}_3 w_3(k),\\
y_1(k) &= C_{11}x_1(k) + C_{12}x_2(k) + \bar{D}_1 w_1(k),\\
y_2(k) &= C_{21}x_1(k) + C_{22}x_2(k) + \bar{D}_2 w_2(k),\\
y_3(k) &= C_{32}x_2(k) + C_{33}x_3(k) + \bar{D}_3 w_3(k).
\end{aligned} \tag{1.1}$$

where $x_i(k)$, $w_i(k)$, $z_i(k)$, $u_i(k)$ and $y_i(k)$ denote parts of the state vector, exogenous input vector, regulated output vector, control input vector and the measurement output vector for all i. Now, consider the problem of finding finite-dimensional internally stabilizing controllers K of the form

$$\begin{aligned}
x_1^K(k+1) &= A_{11}^K x_1^K(k) + A_{12}^K x_2^K(k) + B_1^K y_1(k),\\
x_2^K(k+1) &= A_{21}^K x_1^K(k) + A_{22}^K x_2^K(k) + B_2^K y_2(k),\\
x_3^K(k+1) &= A_{32}^K x_2^K(k) + A_{33}^K x_3^K(k) + B_3^K y_3(k),\\
u_1(k) &= C_{11}^K x_1^K(k) + C_{12}^K x_2^K(k) + D_1^K y_1(k),\\
u_2(k) &= C_{21}^K x_1^K(k) + C_{22}^K x_2^K(k) + D_2^K y_2(k),\\
u_3(k) &= C_{32}^K x_2^K(k) + C_{33}^K x_3^K(k) + D_3^K y_3(k),
\end{aligned} \tag{1.2}$$

where $x_i^K(k)$ denote parts of the state-vector for controller K for all i. Let $\mathcal{S}$ denote the set of controllers with dynamics given in (1.2). So, the problem can be posed as a search for $K \in \mathcal{S}$ that minimizes an objective function and makes the feedback interconnection of G and K asymptotically stable. In literature, such problems were solved by searching for transfer functions of K which correspond to the state-space equations in (1.2). In this case, the transfer functions corresponding to (1.2) will be of the form

$$K(z): \quad \begin{bmatrix} U_1(z) \\ U_2(z) \\ U_3(z) \end{bmatrix} = \begin{bmatrix} H_{11}(z) & z^{-1}H_{12}(z) & 0 \\ z^{-1}H_{21}(z) & H_{22}(z) & 0 \\ z^{-2}H_{31}(z) & z^{-1}H_{32}(z) & H_{33}(z) \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \\ Y_3(z) \end{bmatrix}, \tag{1.3}$$

where $H_{ij}(z)$ is a real rational proper transfer function matrix for all i and j. Let the set of transfer function matrices of the form (1.3) be represented by $\mathcal{S}_{tf}$. Note that the set $\mathcal{S}_{tf}$ can easily be described in terms of sparsity and delay constraints which are linear constraints. Let the transfer function for G in (1.1) be written in the form

$$G(z): \quad \begin{bmatrix} Z(z) \\ Y(z) \end{bmatrix} = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \begin{bmatrix} W(z) \\ U(z) \end{bmatrix} \tag{1.4}$$

where $Z(z) := [Z_1'(z), Z_2'(z), Z_3'(z)]'$ and similarly for $Y(z)$, $W(z)$ and $U(z)$. So, $G_{22}(z)$ is the mapping from $U(z)$ to $Y(z)$ which can be obtained from (1.1).

1.1.1 Quadratic invariance

Definition 1. *A set $\mathcal{T}$ of transfer function matrices is said to be* quadratically invariant *under $G_{22}(z)$ if $K(z)G_{22}(z)K(z) \in \mathcal{T}$ for every $K(z) \in \mathcal{T}$.*

In [8], the authors showed that the problem of searching for $K(z) \in \mathcal{S}_{tf}$ is convex if $\mathcal{S}_{tf}$ is quadratically invariant under $G_{22}(z)$. In our case, simple algebraic operations show that $\mathcal{S}_{tf}$ is in fact quadratically invariant under $G_{22}(z)$ in (1.4). Then, [8] shows that if there exists a stable stabilizing nominal controller $K_{\text{nom}} \in \mathcal{S}_{tf}$, then Zames' parameterization [19] can be used to parameterize the set of stabilizing controllers in $\mathcal{S}_{tf}$ using a parameter $Q(z) \in \mathcal{S}_{tf}$ which is stable. This parameterization allows them to solve for an optimal stabilizing controller in $\mathcal{S}_{tf}$.

Since our objective is to find an internally stabilizing controller in $\mathcal{S}$ which is described by structural constraints on the state-space matrices of K, one needs to find a stabilizing state-space realization in $\mathcal{S}$ for elements in $\mathcal{S}_{tf}$. We refer to this problem of realizing a structured transfer function as a stabilizable and detectable state-space model with a particular sparsity structure as *structured realization*. This is still an open problem for a general class of systems. Due to the lack of results on structured realization in literature, the transfer function approaches that allow one to find optimal stabilizing controllers in $\mathcal{S}_{tf}$ cannot directly be extended to finding optimal stabilizing controllers in $\mathcal{S}$.

This is the main focus in this thesis. We propose to develop a state-space approach to make the search for stabilizing controllers in $\mathcal{S}$ a convex problem. Instead of re-deriving the results of [8] in a state-space form, which is based on quadratic invariance of transfer function matrices, we found that a state-space formulation of Youla-Kučera parameterization (which is based on linear fractional transformations of state-space representations) is well-suited for our problem. We study *networked systems* and show that they can be expressed as elements of sets of the form $\mathcal{S}$. Then, we study the relationship between $\mathcal{S}$ and $\mathcal{S}_{tf}$. We show that a stable transfer function in $\mathcal{S}_{tf}$ can have a stable state-space realization in $\mathcal{S}$. Using this result, we use a state-space Youla-Kučera parameterization in [20] to parameterize the internally stabilizing controllers in $\mathcal{S}$ in terms of a stable parameter $Q(z) \in \mathcal{S}_{tf}$. This approach allows us to not only search for internally stabilizing controllers in a convex fashion but also assures that the internally stabilizing controller is in $\mathcal{S}$. Also, note that Youla parameterization is based on an observer-based nominal stabilizing controller (which need not be a stable system) while Zames' parameterization used in [8] requires a stable stabilizing nominal controller (which can be more difficult to obtain).

1.2 Organization of thesis

This thesis is organized in the following form. In Chapter 2, we introduce the notation and provide background information related to graph theory, linear algebra and systems theory that will be used in the later parts of the thesis. In Chapter 3, we introduce the networked systems that are considered in the thesis. We describe the dynamics of networked systems using the sub-system dynamics and the network they are interacting on. We first study systems over zero-delay networks and show that such networked systems can be described using structured state-space or structured transfer function matrix representations. We point out the problem of network realizability that has not been addressed thoroughly in literature.

In Chapter 4, we consider the problem of designing a networked controller for a networked plant when both the plant and controller are constrained to be over the same zero-delay network. Using the relationship between networked systems and structured systems, we extend the classical Youla-Kučera parameterization to describe the set of all internally stabilizing networked controllers for a given networked plant using a stable networked parameter Q. Using this parameterization, we show that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ networked control problems are in fact convex optimization problems. In the case of $\mathcal{H}_2$ networked control problem, the constrained convex optimization problem is transformed into an unconstrained convex optimization problem that can be solved easily to get the optimal networked controller.

Since the optimal networked controllers can possibly have a large order, we provide methodologies to design full-order internally stabilizing networked controllers for the given networked plant, in Chapter 5. In Chapter 6, we consider the networked estimation problem where each sub-system of the networked estimatro estimates the states of the corresponding sub-system of the plant be exchanging information with other sub-systems of the estimator. We pose the networked estimation problem as an equivalent networked control problem and solve it using previously developed techniques from Chapter 4.

In Chapter 7, we extend the results for systems over zer-delay networks (given in chapter 3 and Chapter 4) to systems over any general delay networks. The delay shift operator allows us to represent systems over delay networks appropriately and allows us to use the same framework that was developed for systems over zero-delay networks. Some numerical examples are given in Chapter 8 to explain the main results provided in the thesis. Finally, we conclude the thesis and provide directions for future work in Chapter 9.

CHAPTER 2. Preliminaries and Notation

In order to keep this thesis self-contained, we provide most of the notation used in this thesis through this chapter.

The set of natural numbers $\{1,2,\ldots\}$ is represented by $\mathbb{N}$. Including 0, the set $\{0,1,2,\ldots\}$ is represented by $\mathbb{N}_0$. The sets of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$. The open unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$ given by

$$\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

and it's closure and boundary are represented by $\bar{\mathbb{D}}$ and $\partial\mathbb{D}$, respectively, where

$$\bar{\mathbb{D}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \qquad \partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

2.1 Graph Theory

Networked systems are best described using graph-theoretic notation. A *directed graph* or *digraph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of sets where $\mathcal{V}$ is the *vertex-set* whose elements are *vertices* or *nodes*, and $\mathcal{E} \subseteq \mathcal{V}^2$ is the *directed edge-set* whose elements are the *directed edges* or *arcs*. We also use $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the vertex and edge sets of $\mathcal{G}$. $|V(\mathcal{G})|$ and $|E(\mathcal{G})|$ are used to denote the number of vertices and directed edges present in the digraph $\mathcal{G}$, respectively. Let $n_v = |V(\mathcal{G})|$ and $n_e = |E(\mathcal{G})|$. In order to refer to the vertices and directed edges in a digraph $\mathcal{G}$, we assume that the vertices in $\mathcal{V}$ and $\mathcal{E}$ are numbered as $\{v_1, v_2, \ldots, v_n\}$ and $\{e_1, e_2, \ldots, e_n\}$, respectively. Given a digraph, through out this thesis, we assume that the vertices and directed edges are numbered in some fixed order.

An ordered pair $e = (v_i, v_j)$ represents a directed edge from vertex v_i to vertex v_j. The first vertex v_i in the ordered pair (v_i, v_j) is called it's *tail* and the second vertex v_j is it's *head*. A *weighted digraph* is one in which a real value is associated with each edge in the edge-set called *cost* or *weight* of the edge. $W(e_r)$ is used to denote the weight of an edge $e_r \in \mathcal{E}$. We also use a term *unit-weight digraph* to describe a digraph with $W(e_r) = 1$ for all $e_r \in \mathcal{E}$.

A *walk* from vertex v_i to v_j on $\mathcal{G}$ is an alternating sequence of vertices and directed edges, beginning at v_i and ending at v_j, where each edge has the preceding vertex as it's tail and succeeding vertex as it's head. To simplify the notation, a walk from v_i to v_j is represented by only a sequence of vertices $\pi_{ji} = \pi_{ji}(0)\pi_{ji}(1)\ldots\pi_{ji}(r)$ where $\pi_{ji}(0) = v_i$, $\pi_{ji}(r) = v_j$ and $(\pi_{ji}(k), \pi_{ji}(k+1)) \in \mathcal{E}\ \forall\ k \in \{0, 1, \ldots, r-1\}$. A *path* is a walk where all the vertices are distinct. *Length* of a walk is defined as the number of edges in the walk. A *shortest path* from vertex v_i to vertex v_j is defined as a path from v_i to v_j with shortest length. Let the shortest path length from vertex v_i to vertex v_j be denoted by l_{ji}. In the case of weighted digraphs, *weight* of a walk is defined as the sum of the weights of all the edges in the walk. A *minimum-weight path* from vertex v_i to vertex v_j is defined as a path from v_i to v_j with least weight. Let the weight of minimum-weight path from vertex v_i to vertex v_j be denoted by W_{ji}.

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the unique binary matrices (assuming the vertices and edges are numbered in a fixed order) $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_m(\mathcal{G})$ (for all $m \in \mathbb{N}_0$) of size $n_v \times n_v$ are defined as

$$[\mathcal{A}(\mathcal{G})]_{ij} := \begin{cases} 1 & \text{if } i = j \text{ or } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

$$[\mathcal{A}_m(\mathcal{G})]_{ij} := \begin{cases} 1 & \text{if } i = j \text{ or there exists a directed path from vertex } v_j \\ & \text{to vertex } v_i \text{ of length at most } m \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Since the longest path in a digraph with n_v vertices is $n_v - 1$, we note that $\mathcal{A}_k(\mathcal{G}) = \mathcal{A}_{n_v-1}(\mathcal{G})$ for all $k \geq n_v - 1$. From (2.2), it is also easy to see that the shortest path length l_{ij} from vertex v_j to vertex v_i is given by

$$l_{ij} = \begin{cases} 0 & \text{if } i = j \\ \inf\{m \in \mathbb{N}_0 \colon [\mathcal{A}_m(\mathcal{G})]_{ij} \neq 0\} & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that $l_{ij} = \infty$ if there is no path from v_i to v_j, $j \neq i$.

Define directed neighborhood index sets for each vertex v_i given by

$$\begin{aligned} \mathcal{N}_i^- &= \{j | (v_j, v_i) \in \mathcal{E}\} \\ \mathcal{N}_i^+ &= \{j | (v_i, v_j) \in \mathcal{E}\}. \end{aligned} \tag{2.4}$$

Thus, the set of vertices that have directed edges to vertex v_i in $\mathcal{E}$ is given by $\{v_r\}_{r\in\mathcal{N}_i^-}$. Similarly, the set of vertices that have directed edges from vertex v_i in $\mathcal{E}$ is given by $\{v_r\}_{r\in\mathcal{N}_i^+}$.

2.2 Linear algebra and Matrices

We refer to a column-vector as *vector*. To make representations compact, we use the notation $\mathbf{vert}[x_i]_{i\in\mathcal{I}}$ and $\mathbf{hor}[x_i]_{i\in\mathcal{I}}$ for vertical and horizontal concatenation of vectors or matrices $\{x_i\}_{i\in\mathcal{I}}$, of appropriate dimension, where $\mathcal{I}$ is an index set. Let $[x_{ij}]_{i,j\in\mathcal{I}}$ represent a matrix formed by arranging the sub-matrices $\{x_{ij}\}_{i,j}$ as $\mathbf{vert}[\mathbf{hor}[x_{ij}]_{j\in\mathcal{I}}]_{i\in\mathcal{I}}$. Also, let $\mathbf{diag}[x_i]_{i\in\mathcal{I}}$ denote the matrix formed by arranging the vectors or matrices $\{x_i\}_{i\in\mathcal{I}}$ in a block diagonal fashion and the remaining entries being zeros. Sometimes, if the index set $\mathcal{I}$ equals $\{1,\dots,n\}$, then we will not explicitly mention the index set.

Rank of a matrix A is defined as the maximum number of linearly independent columns or rows of A and is represented by $\mathbf{rank}(A)$. A matrix A is said to have *full rank* if a has a rank as large as possible. A square matrix A is said to be *Schur-stable* if all eigenvalues are inside the unit circle, in other words, $(zI - A)$ has full rank for any $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. A' is used to denote the transpose of a matrix A. $\mathbf{Tr}(A)$ denotes the trace of a square matrix A. A^{-1} denotes the inverse of a non-singular square matrix A. A symmetric matrix Q is said to be *positive definite (semi-definite)*, iff $v'Qv > 0 (\geq 0)$ for any non-zero vector v. We write $Q \succ 0$ ($Q \succeq 0$) to denote that Q is positive definite (semi-definite). $Q \succ P$ ($Q \succeq P$) means $Q - P \succ 0$ ($Q - P \succeq 0$).

Lemma 1. *For any square matrix A, if $A + A' \succ 0$, then A is non-singular.*

Proof. We prove this using contradiction. Given $A + A' \succ 0$, assume that A is singular. Then A has an eigenvalue at 0. Let v denote the right eigenvector of A corresponding to eigenvalue 0, i.e. $Av = 0$. First note that A is a non-zero matrix since $A + A' \succ 0$. Thus, v is a non-zero eigenvector. For this non-zero v, we can see that $v'(A + A')v = v'Av + v'A'v = 0$ which contradicts the hypothesis that $A + A' \succ 0$. □

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the *Kronecker product* $A \otimes B \in \mathbb{R}^{mp\times nq}$ is defined as

$$A \otimes B := \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}. \tag{2.5}$$

Given a matrix $A = \left[a_1 \cdots a_n \right] \in \mathbb{C}^{m\times n}$, where $\{a_i\}_i$ denote the columns of A, we associate a vector

$$\mathbf{vec}(A) = \mathbf{vert}[a_i]_i \in \mathbb{C}^{mn} \tag{2.6}$$

which is a vector formed by vertically concatenating the columns of matrix A. Define $\mathbf{vec}^{-1}(\cdot)$ as the inverse operation of the $\mathbf{vec}(\cdot)$ such that $\mathbf{vec}^{-1}(\mathbf{vec}(A)) = A$. When required, we shall use I for an identity matrix and 0 for a zero matrix of appropriate size.

In this paper, we will come across block matrices that are made up of smaller sub-matrices. These matrices are best described in terms of their sparsity structures. We say a block matrix $A = [A_{ij}]_{i,j\in\{1,\ldots,n\}}$ is *structured according to* an $n\times n$ binary matrix J if the sub-matrices A_{ij} is a zero matrix whenever $J_{ij} = 0$. The dimensions of the sub-matrices $\{A_{ij}\}_{i,j}$ are described using two integer-valued vectors as follows. Let $\mathcal{P}_a = (a_1,\ldots,a_n)$ and $\mathcal{P}_b = (b_1,\ldots,b_n)$ be two $n-$tuples with a_i and b_i being integers for all $i \in \{1,\ldots,n\}$. Then, matrix A is said to be *partitioned according to* $(\mathcal{P}_a,\mathcal{P}_b)$ if the sub-matrix A_{ij} has dimensions $a_i \times b_j$ $\forall i, j$. This definition of partitioning is easily extended to the case of vectors too. A vector x is said to be *partitioned according to* $\mathcal{P}_a$ if it can be written as $\mathbf{vert}[x_i]_{i\in\{1,\ldots,n\}}$ where x_i is a real vector of size a_i for all $i \in \{1,\ldots,n\}$. We say that $\mathcal{P}_a$ is the partition for the vector x.

Definition 2. *Given an $n\times n$ binary matrix J and $n-$tuples $\mathcal{P}_a$, $\mathcal{P}_b$, let $S(J,\mathcal{P}_a,\mathcal{P}_b)$ denote the set of matrices that are partitioned according to $(\mathcal{P}_a,\mathcal{P}_b)$ and structured according to J.*

For example, according to the above definitions, the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 1 & 2 & 1 & 0 & 0 & 0 \\ \hline 3 & 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & 2 & 1 & 2 \end{array}\right] \in S(J,\mathcal{P}_a,\mathcal{P}_b)$$

where $J = \left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, $\mathcal{P}_a = (1,2,1)$ and $\mathcal{P}_b = (1,2,3)$.

The following lemmas are used in the later part of this paper to describe properties of state-space and input-output representations of interconnected systems.

Remark 1. *Given $n-$tuples $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$, let the matrices E and F be partitioned according to $(\mathcal{P}_a,\mathcal{P}_b)$ and $(\mathcal{P}_b,\mathcal{P}_c)$, respectively. Based on block matrix multiplication rules, the product EF is partitioned according to $(\mathcal{P}_a,\mathcal{P}_c)$ and $[EF]_{ij} = \sum_{k=1}^{n} E_{ik}F_{kj}$ where E_{ij} and F_{ij} are the sub-matrices of E and F, respectively.*

Lemma 2. *Let J be an $n\times n$ binary matrix and $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$, $\mathcal{P}_d$ be $n-$tuples. Given matrices $E \in S(I,\mathcal{P}_a,\mathcal{P}_b)$, $F \in S(J,\mathcal{P}_b,\mathcal{P}_c)$ and $G \in S(I,\mathcal{P}_c,\mathcal{P}_d)$, where I is an $n\times n$ identity matrix, the product $EFG \in S(J,\mathcal{P}_a,\mathcal{P}_d)$.*

Proof. From the hypothesis, we see that $E = [E_{ij}]_{i,j}$, $F = [F_{ij}]_{i,j}$ and $G = [G_{ij}]_{i,j}$ where E_{ij} and G_{ij} are zero matrices when $i \neq j$ while $F_{ij} = 0$ when $J_{ij} = 0$. From the Remark 1, it is easy to see that EFG is a block matrix which is partitioned according to $(\mathcal{P}_a,\mathcal{P}_d)$. Thus, we can write $EFG = [H_{ij}]_{i,j}$ in terms of some sub-matrices H_{ij} which have dimensions $\mathcal{P}_a(i)\times\mathcal{P}_d(j)$ and

$$\begin{aligned} H_{ij} &= \sum_{k=1}^{n}\sum_{m=1}^{n} E_{ik}F_{km}G_{mj} \\ &= \sum_{m=1}^{n} E_{ii}F_{im}G_{mj} = E_{ii}F_{ij}G_{jj} \end{aligned} \tag{2.7}$$

since $E_{ik} = 0\ \forall i \neq k$ and $G_{mj} = 0\ \forall m \neq j$. From (2.7), we see that $H_{ij} = 0$ whenever $J_{ij} = 0$ since $F_{ij} = 0$ whenever $J_{ij} = 0$. Thus, EFG is structured according to J and partitioned according to $(\mathcal{P}_a,\mathcal{P}_d)$. □

Lemma 3. *Given an n_v-tuple $\mathcal{P}_a$ and a digraph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with the binary matrices $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_m(\mathcal{G})$ (for all $m \in \mathbb{N}_0$) given by (2.1) and (2.2), let $\{A_i\}_i$ be a sequence of matrices such that $A_i \in S(\mathcal{A}(\mathcal{G}),\mathcal{P}_a,\mathcal{P}_a)$ for all i. Then $B_m = \prod_{k=1}^{m} A_k \in S(\mathcal{A}_m(\mathcal{G}),\mathcal{P}_a,\mathcal{P}_a)$ for all m.*

Proof. From the definition of $\mathcal{A}_m(\mathcal{G})$ in (2.2), we can see that $\mathcal{A}_1(\mathcal{G}) = \mathcal{A}(\mathcal{G})$. Thus, from hypothesis, we know that $B_1 = A_1 \in S(\mathcal{A}_1(\mathcal{G}),\mathcal{P}_a,\mathcal{P}_a)$.

Now, assume that $B_m = \prod_{k=1}^{m} A_k \in S(\mathcal{A}_m(\mathcal{G}),\mathcal{P}_a,\mathcal{P}_a)$ for some $m = p$. From Remark 1, we can see that $B_{p+1} = B_P A_{p+1}$ is partitioned according to $(\mathcal{P}_a,\mathcal{P}_a)$ and the sub-matrices $[B_{p+1}]_{ij}$ are given by

$$[B_{p+1}]_{ij} = \sum_{k=1}^{n} [B_p]_{ik}[A_{p+1}]_{kj}.$$

If there is no path from vertex v_j to vertex v_i of length at most $p+1$, then for all $v_k \in \mathcal{V}$, either there is no path from v_k to v_i of length at most p or there is no directed edge from v_j to v_k. Thus, either $[B_p]_{ik}$ or $[A_{p+1}]_{kj}$ are zero-matrices for all k when $[\mathcal{A}_{p+1}(\mathcal{G})]_{ij} = 0$. Thus, $[B_{p+1}]_{ij}$ is a zero matrix when $[\mathcal{A}_{p+1}(\mathcal{G})]_{ij} = 0$, which implies that $B_{p+1} \in S(\mathcal{A}_{p+1}(\mathcal{G}), \mathcal{P}_a, \mathcal{P}_a)$.

Thus, the given statement is true by mathematical induction. □

2.3 System theory

A system P is represented by a quadruple (A,B,C,D) or

$$P : \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{2.8}$$

in terms of it's state-space matrices A, B, C and D; and state, input and output vectors $x(k)$, $u(k)$ and $y(k)$, respectively. A state-space representation (A,B,C,D) is *asymptotically stable* if A is Schur-stable. (A,B,C,D) is said to *stabilizable* if $\begin{bmatrix} zI-A & B \end{bmatrix}$ has full rank for any $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$. (A,B,C,D) is said to *detectable* if $\begin{bmatrix} zI-A \\ C \end{bmatrix}$ has full rank for any $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$.

Given a state-space representation (A,B,C,D), the unique transfer function matrix corresponding to the system P is given by the $z-$transform of it's impulse response

$$P(z) := \mathbf{tf}(P) := D + \sum_{k=0}^{\infty} CA^k B z^{-k-1} \tag{2.9}$$

which is also concisely represented by

$$P(z) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The Kronecker product in (2.5) can also be extended to transfer function matrices. The delay of a real-rational transfer function $P(z)$ is given by

$$\mathbf{delay}(P(z)) = \inf\{m \in \mathbb{N}_0 : \lim_{z \to \infty} z^m P(z) \neq 0\}. \tag{2.10}$$

Given two systems G and K in terms of their state-space representations

$$G : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \quad K : \begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \end{bmatrix}, \tag{2.11}$$

the *lower linear fractional transformation* (LFT) of G and K is given by the Redheffer star-product

$$\mathbf{lft}(G,K):\begin{bmatrix}x(k+1)\\x_K(k+1)\\z(k)\end{bmatrix}=\begin{bmatrix}A+B_2D_KC_2 & B_2C_K & B_1+B_2D_KD_{21}\\B_KC_2 & A_K & B_KD_{21}\\C_1+D_{12}D_KC_2 & D_{12}C_K & D_{11}+D_{12}D_KD_{21}\end{bmatrix}\begin{bmatrix}x(k)\\x_K(k)\\w(k)\end{bmatrix}. \tag{2.12}$$

In the case when the two systems are given in terms of their transfer function matrices $G(z)$ and $K(z)$ where $G(z)$ is the mapping from $\begin{bmatrix}w(k)\\u(k)\end{bmatrix}$ to $\begin{bmatrix}z(k)\\y(k)\end{bmatrix}$ while $K(z)$ is the mapping from $y(k)$ to $u(k)$, we can partition the transfer function matrix $G(z)$ in terms of $G_{11}(z)$, $G_{12}(z)$, $G_{21}(z)$ and $G_{22}(z)$ as

$$G(z)=\begin{bmatrix}G_{11}(z) & G_{12}(z)\\G_{21}(z) & G_{22}(z)\end{bmatrix},$$

where $G_{22}(z)$ is the mapping from $u(k)$ to $y(k)$. Then the LFT of $G(z)$ and $K(z)$ is given by

$$\mathbf{lft}(G(z),K(z)):=G_{11}(z)+G_{12}(z)K(z)(I-G_{22}(z)K(z))^{-1}G_{21}(z). \tag{2.13}$$

when $G_{22}(\infty)=0$ (i.e. $G_{22}(z)$ is strictly proper).

A discrete-time system is said to be *bounded-input bounded-output (BIBO) stable* if the impulse response of the system is absolutely summable. It is known that a system G is BIBO stable if and only if all the poles of it's transfer function matrix $G(z)$ are inside the unit circle.

A discrete-time system G with a state-space representation (A,B,C,D) is said to be *internally stable* or *asymptotically stable* if A is Schur-stable. It is known that if $G=(A,B,C,D)$ is asymptotically stable, then $\mathbf{tf}(G)$ is BIBO stable but not viceversa.

We say that a system K *stabilizes* a system G (in (2.11)) if $\mathbf{lft}(G,K)$ is BIBO stable and *internally stabilizes* G if $\mathbf{lft}(G,K)$ is asymptotically stable.

Given a discrete-time system G, the $\mathcal{H}_2$ norm of the system is given by

$$\|G(z)\|_2=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{Tr}(G(e^{j\theta})G^*(e^{j\theta})d\theta)} \tag{2.14}$$

where $G(z)$ is the transfer function matrix of G. If a state-space realization of G is given by (A,B,C,D), then the $\mathcal{H}_2$ norm is given by

$$\|G\|_2=\sqrt{\mathbf{Tr}(DD'+CM_cC')}, \tag{2.15}$$

where $M_c\succeq 0$ is the controllability grammian that solves the discrete-time Lyapunov (Stein) equation

$$AM_cA'-M_c+BB'=0. \tag{2.16}$$

The solution of the Stein equation is given by

$$M_c = \sum_{k=0}^{\infty} A^k BB'(A')^k. \tag{2.17}$$

Given a discrete-time system G with a transfer function matrix $G(z)$, the $\mathcal{H}_\infty$ norm of the system is given by

$$\|G(z)\|_\infty = \sup_{\theta\in[0,\pi]} \bar{\sigma}(G(e^{j\theta})) \tag{2.18}$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value function.

Let $\mathcal{R}_p$ denote the set of real-rational proper transfer function matrices, $\mathcal{R}_{sp}$ denote the set of real-rational strictly-proper transfer function matrices and $\mathcal{RH}_\infty$ denote the set of real-rational proper stable transfer function matrices.

CHAPTER 3. Networked systems

In this chapter, we introduce the systems that are considered in this thesis.

Definition 3. *A group of plants or sub-systems interacting over a network is termed as a* networked *or* interconnected *system.*

From Definition 3, it can be seen that a networked system is characterized by the dynamics of the sub-systems and the properties of the network on which they are interacting. In this thesis, we consider only discrete-time sub-systems interacting over discrete-time networks. We model such systems using system theory and graph theory by making further assumptions on the properties of the sub-systems and the interaction network.

3.1 Discrete-time networked system

Definition 4. *A networked system made of n discrete-time finite-dimensional linear time-invariant (DT FDLTI) sub-systems interacting over a discrete-time network is referred to as a* discrete-time networked system.

The dynamics of a discrete-time networked system depends on the dynamics of the sub-systems and the network interconnection.

Assumption 1. *The clock for all the n sub-systems and the network links is assumed to be the same.*

Let $\{P_i\}_{i\in\{1,\ldots,n\}}$ denote the n sub-systems. Let $x_i(k)$ be the local state vector, $u_i(k)$ the local input vector, $y_i(k)$ the local output vector corresponding to the sub-system P_i. Let $\eta_{ri}(k)$ be the message vector transmitted from sub-system P_i to P_r at the time instant k and $\zeta_{ij}(k)$ be the message received by P_i from P_j at time instant k. Let t_{ij} denote the smallest discrete time-delay over the network link from sub-system P_j to P_i. Then the considered state-space representation (assumed to be minimal) for P_i is of the form

$$
\begin{aligned}
x_i(k+1) &= A_{ii}x_i(k) + B_i^u u_i(k) + \sum_{j\in \mathrm{IN}(i)} B_{ij}^{\zeta}\zeta_{ij}(k) \\
y_i(k) &= C_{ii}^{y}x_i(k) + D_i^{yu}u_i(k) + \sum_{j\in \mathrm{IN}(i)} D_{ij}^{y\zeta}\zeta_{ij}(k) \\
\eta_{ri}(k) &= C_{ri}^{\eta}x_i(k)\ \forall\, r \in \mathrm{OUT}(i)
\end{aligned}
\tag{3.1}
$$

and the network dynamics is of the form

$$
\zeta_{ij}(k) = \eta_{ij}(k - t_{ij})\ \forall\, j \in \mathrm{IN}(i), \tag{3.2}
$$

where $\mathrm{IN}(i)$, $\mathrm{OUT}(i)$ denote the index sets for sub-systems that transmit information to P_i and receive from P_i, respectively, i.e. $j \in \mathrm{IN}(i)$ means that there is a network link from P_j to P_i and $j \in \mathrm{OUT}(i)$ means there is a network link from P_i to P_j. Combining (3.1) and (3.2), the collective dynamics of the networked system P made of $\{P_i\}_i$ is given by

$$
\left.
\begin{aligned}
x_i(k+1) &= A_{ii}x_i(k) + B_i^u u_i(k) + \sum_{j\in \mathrm{IN}(i)} A_{ij}x_j(k - t_{ij}) \\
y_i(k) &= C_{ii}^{y}x_i(k) + D_i^{yu}u_i(k) + \sum_{j\in \mathrm{IN}(i)} C_{ij}^{y}x_j(k - t_{ij}),
\end{aligned}
\right.
\quad \forall\, i \in \{1,\dots,n\}
\tag{3.3}
$$

where $A_{ij} := B_{ij}^{\zeta}C_{ij}^{\eta}$ and $C_{ij}^{y} := D_{ij}^{y\zeta}C_{ij}^{\eta}$ for all i, j.

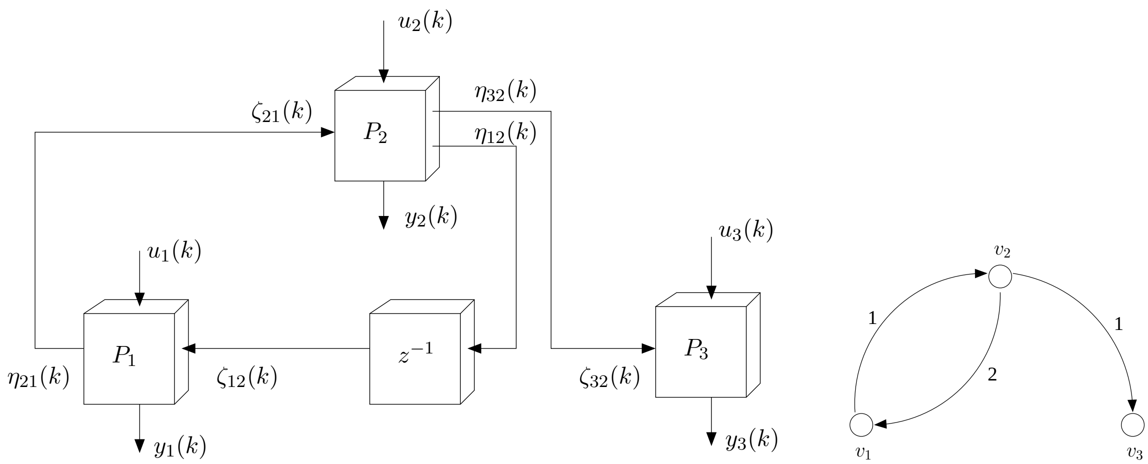
3.1.1 Graphical representation of discrete-time networked systems

The dynamical structure of a discrete-time networked system (3.3) made of n sub-systems (with dynamics given by (3.1)) interacting over a discrete-time network (3.2) can be better represented using a weighted digraph. The graph-theoretic notation makes the equations more concise and makes it easier to understand the structure of the model.

First, we shall see how to identify the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ corresponding to the discrete-time networked system. The vertex-set $\mathcal{V}$ is defined to represent the n sub-systems such that vertex v_i corresponds to the sub-system P_i for all $i \in \{1,\dots,n\}$. Thus, the number of vertices $n_v = n$. The directed edge-set $\mathcal{E}$ is defined based on the interactions between the sub-systems. A directed edge $e_r = (v_j, v_i) \in \mathcal{E}$ if there is a directed network link from sub-system P_j to P_i. Based on the smallest delay

t_{ij} corresponding to the link from P_j to P_i, we assign a weight

$$W((v_j, v_i)) = t_{ij} + 1 \qquad \forall\, (v_j, v_i) \in \mathcal{E}. \tag{3.4}$$



(a) A Networked system

(b) Weighted digraph representing the underlying network

Figure 3.1 A simple example of a discrete-time networked system model made of 3 sub-systems interacting over a discrete-time network.

Note that this representation can replace multiple communication links from sub-system P_j to P_i using just one weighted edge in $\mathcal{E}$. The weight assignment in (3.4) can be better appreciated once we study how an input $u_j(k)$ at sub-system P_j affects the output $y_i(k)$ at P_i in (3.3). By defining the vertices, directed edges and the edge weights, we have a graphical representation of the discrete-time network and the digraph $\mathcal{G}$ is said to *represent* the network interconnection. In terms of the graphical representation $\mathcal{G}$, a discrete-time networked system is given by the state-space equations of sub-systems

$$\begin{aligned} x_i(k+1) &= A_{ii} x_i(k) + B_i^u u_i(k) + \sum_{j \in \mathcal{N}_i^-} B_{ij}^{\zeta} \zeta_{ij}(k) \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yu} u_i(k) + \sum_{j \in \mathcal{N}_i^-} D_{ij}^{y\zeta} \zeta_{ij}(k) \\ \eta_{ri}(k) &= C_{ri}^{\eta} x_i(k)\ \forall\, r \in \mathcal{N}_i^+ \end{aligned} \tag{3.5}$$

and the network interconnection equations given by

$$\zeta_{ij}(k) = \eta_{ij}(k - W((v_j, v_i)) + 1)\ \forall\, (v_j, v_i) \in \mathcal{E}, \tag{3.6}$$

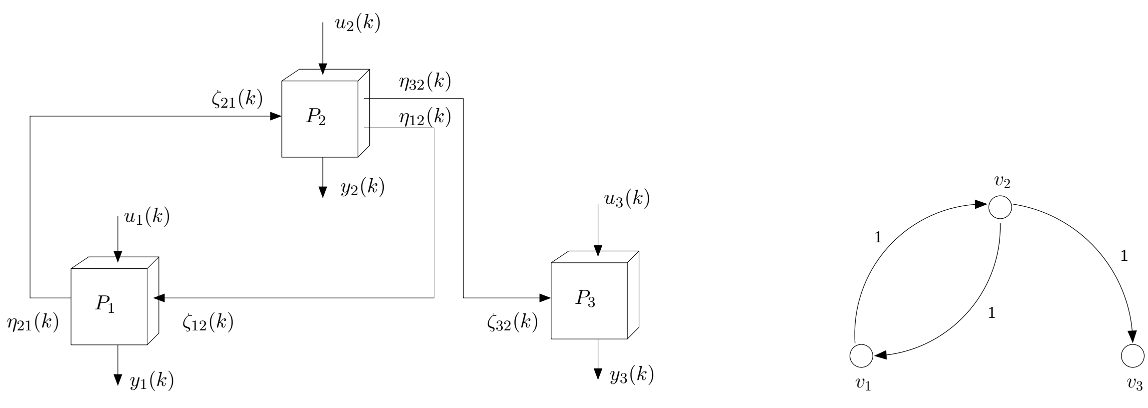
where $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$ are given by (2.4).

Definition 5. *A discrete-time networked system P with sub-system dynamics, given by* (3.5)*, satisfying the network interconnection equations given by* (3.6) *is referred to as a* strictly causal interaction of discrete-time FDLTI sub-systems over a discrete-time network represented by a digraph $\mathcal{G}$. *In short, we say "strictly causal interaction over a digraph $\mathcal{G}$".*

3.1.2 Networked systems over zero-delay network

First, we study the case of zero-delay network where $t_{ij} = 0$ for all network links. An extension to a more general delay network will be addressed in Chapter 7. Under the zero-delay condition, the represented digraph $\mathcal{G}$ corresponding to the network is a unit-weight digraph, i.e. $W(e) = 1$ for all $e \in \mathcal{E}$. This case is studied separately because of the emergence of simple sparsity structures (we shall show this in the next section) in both state-space matrices and transfer function matrices of the networked systems. In a general case, the sparsity structures of the state-space matrices are difficult to describe while the transfer function matrices still show some sparsity and delay structures. But the ideas developed for the zero-delay case can be extended to a general case with appropriate modifications. Under the zero-delay network condition, (3.6) becomes

$$\zeta_{ij}(k) = \eta_{ij}(k)\ \forall\, (v_j, v_i) \in \mathcal{E}. \tag{3.7}$$



(a) Networked system

(b) Unit-weight digraph representing the underlying zero-delay network

Figure 3.2 A simple example of a discrete-time networked system model made of 3 sub-systems interacting over a zero-delay network represented by a unit-weight digraph.

Remark 2. *Note that zero-delay network refers to the delay on the network link. It does not refer to the delay of the transfer function from input $u_j(k)$ at sub-system P_j to output $y_i(k)$ at sub-system P_i. We will later see through Remark 4 that the delay from $u_j(k)$ to $y_i(k)$ is equal to l_{ij} given by (2.3).*

By combining the equations (3.5) and (3.7), we can eliminate the network variables $\zeta_{ij}(k)$ and $\eta_{ri}(k)$, and write the state-space equations for the sub-systems as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k) \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yu}u_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^y x_j(k), \end{aligned} \quad \forall\, i \in \{1,\dots,n\} \tag{3.8}$$

where $A_{ij} := B_{ij}^{\zeta}C_{ij}^{\eta}$ and $C_{ij}^y := D_{ij}^{y\zeta}C_{ij}^{\eta}$. Let P denote the discrete-time networked system defined by (3.5) and (3.7). Then the state-space equations of P in (3.8) can also be concisely presented as

$$P: \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_u \\ C_y & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{3.9}$$

where $A := [A_{ij}]_{i,j}$, $B_u := \mathbf{diag}[B_i^u]_i$, $C_y := [C_{ij}^y]_{i,j}$ and $D_{yu} := \mathbf{diag}[D_i^{yu}]_i$ (such that A_{ij} and C_{ij}^y are zero matrices when $(v_j, v_i) \notin \mathcal{E}$ and $i \neq j$) denote the structured state-space matrices; $x(k) := \mathbf{vert}[x_i(k)]_i$, $u(k) := \mathbf{vert}[u_i(k)]_i$ and $y(k) := \mathbf{vert}[y_i(k)]_i$ denote the complete state, input and output vectors corresponding to the networked system P and be partitioned according to $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively. From (3.8), and the structure of $x(k)$, $u(k)$ and $y(k)$, we can see that (using definition 2)

$$\begin{aligned} A &\in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x), \quad B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u), \\ C_y &\in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x), \quad D_{yu} \in S(I, \mathcal{P}_y, \mathcal{P}_u). \end{aligned} \tag{3.10}$$

where $\mathcal{A}(\mathcal{G})$ is given by (2.1) and I is an $n\times n$ identity matrix.

Remark 3. *Note that the discrete-time networked systems considered in this thesis (through (3.8)) are different from the networked systems considered in [16] where the sub-systems were assumed to be instantaneous relays, i.e. any information at a sub-system P_i is assumed to be passed on to any other sub-system P_j that has a directed path to P_i (of any lenght) in next time instant. Our model is based on the networked system considered in [9] where each sub-system can send the local information only to immediate directed neighbors in the next time instant. Thus, our model assures that the network topology exactly describes the information flow from one node to another node with time, unlike the model in [16].*

3.2 Structured systems

In this Section, we look at systems whose state-space and transfer function matrices follow sparsity and delay structures. We will later study how structured and networked systems are related. Based on this relationship, we use structured systems to represent, design and search for networked systems

Definition 6. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n vertices and $n-$tuples $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$; let $\mathcal{A}(\mathcal{G})$ be the unique binary matrix given by (2.1). We define $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ as the set of discrete-time systems with a state-space representation (A, B_u, C_y, D_{yu}) such that $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$, $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ and $D_{yu} \in S(I, \mathcal{P}_y, \mathcal{P}_u)$.*

Also define $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u) = \cup_{\mathcal{P}_x \in \mathbb{N}^n} \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$.

Note that the state-space representations in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ have only structural constraints on the state-space matrices and the state-space representation itself can be non-minimal.

Definition 7. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n vertices and the $n-$tuples $\mathcal{P}_u$ and $\mathcal{P}_y$; let $\mathcal{A}_{n-1}(\mathcal{G})$ be the unique binary matrix given by (2.2) and l_{ij} be defined for all i, j according to (2.3). We define $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as the set of transfer function matrices $P(z) \in S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$ such that the transfer function sub-matrices $P_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i) \times \mathcal{P}_u(j)}$ (where $P(z) = [P_{ij}(z)]_{i,j}$) are such that*

$$\begin{aligned} \mathbf{delay}(P_{ij}(z)) \geq l_{ij} \quad & \text{if } l_{ij} < \infty \\ P_{ij}(z) = 0 \quad & \text{if } l_{ij} = \infty \end{aligned} \tag{3.11}$$

for all i, j.

It is easy to see, from Definitions 6 and 7, that $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ are subspaces. We refer to systems in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, for some $\mathcal{P}_u$ and $\mathcal{P}_y$, as *structured systems over $\mathcal{G}$*.

The sets of asymptotically stable and BIBO stable structured systems over $\mathcal{G}$ with input and output partitions as $\mathcal{P}_u$ and $\mathcal{P}_y$ are denoted by $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, respectively.

Lemma 4. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $n-$tuples $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$. Let P be a structured system with a state-space representation $(A, B_u, C_y, D_{yu}) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ with state vector $x(k)$, output vector $y(k)$ and input vector $u(k)$ partitioned according to $\mathcal{P}_x$, $\mathcal{P}_y$ and $\mathcal{P}_u$, respectively. Then the transfer function matrix of the structured system $\mathbf{tf}(P) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$.*

Proof. Let $P(z)$ be the transfer function of P. From (2.9), we get

$$P(z) = D_{yu} + \sum_{k=0}^{\infty} C_y A^k B_u z^{-k-1}. \tag{3.12}$$

Define $R_0 := D_{yu}$ and $R_{k+1} := C_y A^k B_u$ for all $k \in \mathbb{N}_0$. From Lemmas 2 and 3, and the partitions of state-space matrices from (3.10), we see that

$$\begin{aligned} A^k \in S(\mathcal{A}_k(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x) \Rightarrow \quad & C_y A^k \in S(\mathcal{A}_{k+1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x) \\ \Rightarrow \quad & C_y A^k B_u \in S(\mathcal{A}_{k+1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u) \\ \Rightarrow \quad & R_k \in S(\mathcal{A}_k(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u) \quad \forall\, k \in \mathbb{N}_0. \end{aligned} \tag{3.13}$$

Note that $\mathcal{A}_0(\mathcal{G}) = I$. From (3.12) and definitions of $\{R_k\}_k$, we can write

$$P(z) = \sum_{k=0}^{\infty} R_k z^{-k}. \tag{3.14}$$

Following the facts that

- $\mathcal{A}_k(\mathcal{G}) = \mathcal{A}_{n-1}(\mathcal{G})$ for all $k \geq n-1$,
- $S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$ is a subspace,
- $S(\mathcal{A}_k(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u) \subseteq S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$ for all k,

it is easy to see that $P(z) \in S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$ from (3.14).

Since $P(z)$ is partitioned according to $(\mathcal{P}_y, \mathcal{P}_u)$ we can write $P(z) = [P_{ij}(z)]_{i,j}$, where $P_{ij}(z)$ is the transfer function sub-matrix mapping input vector $u_j(k)$ to output vector $y_i(k)$. From (3.14), we get

$$P_{ij}(z) = \sum_{k=0}^{\infty} [R_k]_{ij} z^{-k}. \tag{3.15}$$

where $[R_k]_{ij}$ is the sub-matrix of R_k, for all k. From (3.15), (2.2) and (2.3); the delay of $P_{ij}(z)$ is given by

$$\begin{aligned} \mathbf{delay}(P_{ij}(z)) &= \inf\{m \in \mathbb{N}_0 : \lim_{z \to \infty} z^m P_{ij}(z) \neq 0\} \\ &= \inf\{m \in \mathbb{N}_0 : \lim_{z \to \infty} z^m \sum_{k=0}^{\infty} [R_k]_{ij} z^{-k} \neq 0\} \\ &\geq \inf\{m \in \mathbb{N}_0 : \lim_{z \to \infty} z^m \sum_{k=0}^{\infty} [\mathcal{A}_k(\mathcal{G})]_{ij} z^{-k} \neq 0\} \\ &= \inf\{m \in \mathbb{N}_0 : \lim_{z \to \infty} z^m \sum_{k=l_{ij}}^{\infty} z^{-k} \neq 0\} = l_{ij}, \end{aligned} \tag{3.16}$$

which implies that $P(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. $\square$

Theorem 1. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $n-$tuples $\mathcal{P}_u$ and $\mathcal{P}_y$.*

1. *Let $P(z)$ be a transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ with input vector $u(k)$ and output vector $y(k)$ partitioned according to $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively. Then there exists a state-space realization (A, B_u, C_y, D_{yu}) of $P(z)$ in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ with state vector $x(k)$ partitioned according to some $n-$tuple $\mathcal{P}_x$.*

2. *If $P(z)$ is also BIBO stable, i.e. $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, then there exists a state-space realization (A, B_u, C_y, D_{yu}) of $P(z)$ in $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ for some $n-$tuple $\mathcal{P}_x$, i.e. A is Schur-stable.*

Proof. A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and transfer function matrix $P(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ are given. So, $P(z)$ is partitioned according to $(\mathcal{P}_y, \mathcal{P}_u)$ and is of the form $P(z) = [P_{ij}(z)]_{i,j}$. Note that $P_{ij}(z)$ is essentially the transfer function matrix mapping $u_j(k)$ to $y_i(k)$, where input $u(k) = \mathbf{vert}[u_r(k)]_r$ and $y(k) = \mathbf{vert}[y_r(k)]_r$ are partitioned according to $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively.

From (3.11), we see that $P_{ij}(z) = 0$ if there is no directed path from v_j to v_i over the digraph $\mathcal{G}$ and $\mathbf{delay}(P_{ij}(z)) \geq l_{ij}$, otherwise. The condition that $P_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i) \times \mathcal{P}_u(j)}$ and $\mathbf{delay}(P_{ij}(z)) \geq l_{ij}$ can equivalently be written as $P_{ij}(z) = z^{-l_{ij}} H_{ij}(z)$ (with possible pole-zero cancellations at origin) where $H_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i) \times \mathcal{P}_u(j)}$. Thus (3.11) can be written as

$$P_{ij}(z) = \begin{cases} z^{-l_{ij}} H_{ij}(z) & \text{if } l_{ij} < \infty \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

where $H_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i) \times \mathcal{P}_u(j)}$ for all i, j. Consider minimal realizations of $P_{ij}(z)$ in the following cases and define local states corresponding to a vertex as shown below.

- When $i = j$, define local states $x_{ii}(k)$ at vertex v_i such that

$$P_{ii}(z): \quad \begin{aligned} x_{ii}(k+1) &= A_{ii} x_{ii}(k) + B_{ii} u_i(k) \\ y_{ii}(k) &= C_{ii} x_{ii}(k) + D_{ii} u_i(k) \end{aligned} \tag{3.18}$$

- When $j \in \mathcal{N}_i^-$, define states $x_{ij}(k)$ at vertex v_j

$$P_{ij}(z): \quad \begin{aligned} x_{ij}(k+1) &= A_{ij} x_{ij}(k) + B_{ij} u_j(k) \\ y_{ij}(k) &= C_{ij} x_{ij}(k) \end{aligned} \tag{3.19}$$

- When $l_{ij} \geq 2$, let a shortest path from vertex v_j to vertex v_i be given by $\pi_{ij} = \pi_{ij}(0)\pi_{ij}(1)\ldots\pi_{ij}(l_{ij})$, where $\pi_{ij}(0) = v_j$ and $\pi_{ij}(l_{ij}) = v_i$. We refer to $\pi_{ij}(p)$ for $p \in \{1,\ldots,l_{ij}-1\}$ as intermediate vertices. In this case, we define states at each vertex on the path as follows.

$$z^{-1}H_{ij}(z): \quad \begin{aligned} x_{ij}^{(0)}(k+1) &= A_{ij}x_{ij}^{(0)}(k) + B_{ij}u_j(k) \\ y_{ij}^{(0)}(k) &= C_{ij}x_{ij}^{(0)}(k) \end{aligned} \tag{3.20}$$

  Note that states $x_{ij}^{(0)}(k)$ are defined at vertex v_j and the outputs $y_{ij}^{(0)}(k)$ are passed to vertex $\pi_{ij}(1)$, i.e. the first vertex in the selected path from v_j to v_i. At vertices $\pi_{ij}(p)$, $p \in \{1,\ldots,l_{ij}-1\}$, we define states $x_{ij}^{(p)}(k)$ corresponding to unit delay systems

$$z^{-1}: \quad \begin{aligned} x_{ij}^{(p)}(k+1) &= y_{ij}^{(p-1)}(k) \\ y_{ij}^{(p)}(k) &= x_{ij}^{(p)}(k). \end{aligned} \tag{3.21}$$

We denote the state vector corresponding to each vertex v_i to be $\tilde{x}_i(k)$, which is formed by appending the states $x_{ii}(k)$, $x_{ri}(k)$ $\forall r \in \mathcal{N}_i^+$ and $x_{ab}^{(p)}(k)$ whenever $\pi_{ab}(p) = v_i$ (for $p \in \{0,\ldots,l_{ab}-1\}$), i.e. when vertex v_i is a vertex on the shortest path from some vertex v_b to some other vertex v_a. A network output vector $\tilde{\eta}_{ri}(k)$, for all $r \in \mathcal{N}_i^+$, is formed by appending $y_{ri}(k)$ and $y_{ab}^{(p)}(k)$ whenever $\pi_{ab}(p) = v_i$ and $\pi_{ab}(p+1) = v_r$ (for $p \in \{0,\ldots,l_{ab}-1\}$). Similarly, a network input vector $\tilde{\zeta}_{ij}(k)$, for all $j \in \mathcal{N}_i^-$, is formed by appending $y_{ij}(k)$ and $y_{ab}^{(p)}(k)$ whenever $\pi_{ab}(p) = v_j$ and $\pi_{ab}(p+1) = v_i$ (for $p \in \{0,\ldots,l_{ab}-1\}$). Note that the network inputs defined at vertex v_i do not affect the network outputs at the same vertex v_i for any time instant k.

At vertex v_i, the output $y_i(k)$ is given by

$$y_i(k) = y_{ii}(k) + \sum_{j \in \mathcal{N}_i^-} y_{ij}(k) + \sum_{j:\ l_{ij} \geq 2} y_{ij}^{(l_{ij}-1)}(k) \tag{3.22}$$

Thus, we can define n sub-systems, $\{\tilde{P}_i\}_i$, each with local states $\tilde{x}_i(k)$, local inputs $u_i(k)$, local outputs $y_i(k)$, network inputs $\tilde{\zeta}_{ij}(k)$ (for all $j \in \mathcal{N}_i^-$) and network outputs $\tilde{\eta}_{ir}(k)$ (for all $r \in \mathcal{N}_i^+$). Following the state-space equations (3.18), (3.19), (3.20), (3.21), (3.22) concerning these states, inputs and outputs at each node, we can see that $\tilde{x}_i(k+1)$ and $y_i(k)$ are linear functions of $\tilde{x}_i(k)$, $u_i(k)$ and $\{\tilde{\zeta}_{ij}(k)\}_{j \in \mathcal{N}_i^-}$; while $\tilde{\eta}_{ri}(k)$ is only a function of $\tilde{x}_i(k)$ (for all $r \in \mathcal{N}_i^+$). Thus, the n sub-systems $\{\tilde{P}_i\}_i$ satisfy the structure given in (3.5) while the network inputs and network outputs satisfy (3.7). Thus the transfer function matrix $P(z)$ is expressed as a networked system $\tilde{P}$ which is a strictly causal interaction of sub-systems $\{\tilde{P}_i\}_i$ over a zero-delay network represented by the given unit-weight digraph $\mathcal{G}$. Following (3.5), (3.7), (3.8) and (3.9), we can get a state-space realization for $\tilde{P}$ as $(A, B_u, C_y, D_{yu}) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ for some $n-$tuple $\mathcal{P}_x$. Following the construction of $\tilde{P}$, we can see that $\tilde{P}$ can be a non-minimal realization of $P(z)$ where $\mathbf{tf}(\tilde{P}) = P(z)$.

In the second case when $P(z)$ is also a BIBO stable transfer function, we show that the construction procedure used in the previous part of the proof also assures asymptotic stability of $\tilde{P}$.

In order to check asymptotic stability of $\tilde{P}$, we consider the zero-input system by assuming $u_i(k) = 0\ \forall i, k$. First, we shall separate the states defined in (3.18), (3.19), (3.20) and (3.21)into two categories. The first category consists of the states corresponding to the transfer function matrices $P_{ij}(z)$, $\forall i \in \{1, \ldots, n\}, j \in \mathcal{N}_i^- \cup \{i\}$ that were defined in (3.18) and (3.19). This set of states can be written as $X_1(k) = \mathbf{vert}[x_{ij}(k)]_{i\in\{1,\ldots,n\}, j\in\mathcal{N}_i^-\cup\{i\}}$. From the state-space equations corresponding to these states, we get

$$X_1(k+1) = \mathbf{diag}[A_{ij}]_{i\in\{1,\ldots,n\}, j\in\mathcal{N}_i^-\cup\{i\}} X_1(k) \tag{3.23}$$

when $u_i(k) = 0$ for all i, k.

The second category consists of the states corresponding to all the $P_{ij}(z)$ when $l_{ij} \geq 2$. For example, assume that a shortest path π_{ij} from vertex v_j to vertex v_i has length greater than 1. Then

$$\pi_{ij} = \pi_{ij}(0)\ \pi_{ij}(1)\ \ldots\ \pi_{ij}(l_{ij})$$

where $l_{ij} \geq 2$, $\pi_{ij}(0) = v_j$ and $\pi_{ij}(l_{ij}) = v_i$. Corresponding to this path, the states earlier defined in (3.20) and (3.21) are $x_{ij}^{(0)}(k)$, $x_{ij}^{(1)}(k)$, $\ldots$, $x_{ij}^{(l_{ij}-1)}(k)$. Let us define

$$X_{ij}(k) = \mathbf{vert}[x_{ij}^{(p)}(k)]_{p\in\{0,\ldots,l_{ij}-1\}}$$

corresponding to the path π_{ij}. From the state-space equations corresponding to these states, we can see that

$$X_{ij}(k+1) = \begin{bmatrix} A_{ij} & & & & \\ C_{ij} & 0 & & & \\ & I & 0 & & \\ & & I & 0 & \\ & & & & \ddots \end{bmatrix} X_{ij}(k). \tag{3.24}$$

Define $X_2(k) = \mathbf{vert}[X_{ij}(k)]_{\{i,j:2\leq l_{ij}<n\}}$ as the set of states corresponding to $P_{ij}(z)$ when $l_{ij} \geq 2$. Note that $X_1(k)$ and $X_2(k)$ constitute all the states defined corresponding to the n sub-systems $\{\tilde{P}_i\}_i$. From (3.23) and (3.24), we can see that the $A-$matrix corresponding to the dynamics of $\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}$ is block lower triangular with $\{A_{ij}\}_{i,j}$ on the diagonal and the rest of the diagonal terms being zero.

By hypothesis, $P(z)$ is BIBO stable which implies that $\{P_{ij}(z)\}_{i,j}$ are all BIBO stable, which in turn implies that $\{H_{ij}(z)\}_{i,j}$ are all BIBO stable. Note that, we assumed minimal realizations of $P_{ij}(z)$ and $H_{ij}(z)$ in (3.18), (3.19) and (3.20) which implies that the matrices $\{A_{ij}\}_{i,j}$ are all Schur-stable. Thus, we can see that the $A-$matrix of the networked realization $\tilde{P}$ is also Schur-stable. This implies that the there exists a state-space realization $\tilde{P} = (A, B_u, C_y, D_{yu}) \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ such that $\mathbf{tf}(\tilde{P}) = P(z)$ when $P(z)$ is BIBO stable. ☐

From Lemma 4 and Theorem 1, we can see that the set of structured systems over a given unit-weight digraph $\mathcal{G}$ can be represented by either $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ or $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, since both the subspaces describe the same set of systems.

3.2.1 Structured realizability

In the case of designing systems for practical use, we need stabilizablity and detectability of the designed systems. For example, stabilizing controller design problems require the designed controllers to be stabilizable and detectable. We refer to the property of realizing a structured transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as a stabilizable and detectable structured state-space representation in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ by *structured realizability*.

Theorem 1 shows that given a digraph $\mathcal{G}$ and any structured transfer function $P(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ there exists a structured system $\tilde{P} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ with the same transfer function. In the case of $P(z)$ being BIBO stable, $\tilde{P}$ was shown to be asymptotically stable (which is stabilizable and detectable). But given a generic unstable structured system in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, the proof of Theorem 1 cannot be used to obtain a stabilizable and detectable structured system $\tilde{P} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ because the construction procedure used in Theorem 1 suggests a non-minimal realization and in general, does not promise stabilizability and detectability of $\tilde{P}$. Even in literature, there is neither a minimal realization technique nor a realization technique that assures stabilizability and detectability for generic structured transfer function matrices while guaranteeing a specific sparsity pattern for the state-space matrices. Thus, structured realizability is still an open problem which needs to be addressed before using transfer function approaches to solve problems which impose sparsity constraints on the state-space representations of the designed systems. This is an important observation and a contribution of this thesis.

3.2.2 Structured systems as Systems over networks

Remark 4. *Following equations* (3.5)*,* (3.7)*,* (3.8)*,* (3.9)*,* (3.10) *and Definition 6, it is easy to note that a discrete-time networked system P that is a strictly causal interaction over a unit-weight digraph $\mathcal{G}$, with state, input and output partitions given by $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$, has a state-space representation in $\mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_u)$. And Lemma 4 shows that the transfer function matrix corresponding to such a networked system belongs to a subspace $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$.*

From Remark 4, we can see that any networked system P with input and output partitions $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively; that is a strictly causal interaction over the given unit-weight digraph $\mathcal{G}$ has a state-space representation in $\mathfrak{S}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ and it's transfer function matrix is in $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$. Since $\mathfrak{S}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ and $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ are subspaces of systems with linear constraints on their state-space representations or transfer function matrices, search for structured systems is relatively easier than searching for systems over networks, and in some cases is also a convex problem. In order to utilize the advantages of structured systems and still design systems over networks, we use the following two results:

- Given a structured system in $\mathfrak{S}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$, we show that there exists a networked system which is a strictly causal interaction over $\mathcal{G}$ with same state-space matrices as that of the given structured system. This result will be shown in Lemma 5.

- Given a stable transfer function matrix in $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$, we show that there exists a stable networked system which is a strictly causal interaction over $\mathcal{G}$ with the same transfer function as the given system. This result will be shown in Corollary 1.

In this section, we address the reverse problem of expressing the elements of $\mathfrak{S}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ or $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ as strictly causal interactions of sub-systems over the given unit-weight digraph $\mathcal{G}$.

Lemma 5. *Given a unit-weight digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $n-$tuples $\mathcal{P}_u$ and $\mathcal{P}_y$, and a structured system $P = (A, B_u, C_y, D_{yu}) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$, there exists a networked system $\tilde{P}$ which is a strictly causal interaction over $\mathcal{G}$ with the same state-space representation (A, B_u, C_y, D_{yu}).*

Proof. Let the $n-$tuple $\mathcal{P}_x$ denote the state partition corresponding to P. Then the state-space matrices A, B_u, C_y and D_{yu} are structured and partitioned as given by (3.10). Thus, we can partition $A = [A_{ij}]_{i,j}$ and $[C^y_{ij}]_{i,j}$ such that A_{ij} and C^y_{ij} are zero matrices when $[\mathcal{A}(\mathcal{G})]_{ij} = 0$.

Define n sub-systems $\{\tilde{P}_i\}_i$ given by

$$\tilde{P}_i: \quad \begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B^u_i u_i(k) + \sum_{j \in \mathcal{N}_i^-} A_{ij}\zeta_{ij}(k) \\ y_i(k) &= C^y_{ii}x_i(k) + D^{yu}_i u_i(k) + \sum_{j \in \mathcal{N}_i^-} C^y_{ij}\zeta_{ij}(k) \\ \eta_{ri}(k) &= x_i(k) \quad \forall\, r \in \mathcal{N}_i^+ \end{aligned} \tag{3.25}$$

for all i, interacting over a network interconnection given by

$$\zeta_{ij}(k) = \eta_{ij}(k) \quad \forall\, (v_j, v_i) \in \mathcal{E} \tag{3.26}$$

where $x(k) := \mathbf{vert}[x_i(k)]_i$, $u(k) := \mathbf{vert}[u_i(k)]_i$ and $y(k) := \mathbf{vert}[y_i(k)]_i$ are partitioned according to $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively.

By combining (3.25), (3.26) and eliminating $\zeta_{ij}(k)$ and $\eta_{ij}(k)$ for all $(v_j, v_i) \in \mathcal{E}$, we get the state-space equations for sub-system P_i as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + \sum_{j \in \mathcal{N}_i^-} A_{ij}x_j(k) + B^u_i u_i(k) \\ y_i(k) &= C^y_{ii}x_i(k) + \sum_{j \in \mathcal{N}_i^-} C^y_{ij}x_j(k) + D^{yu}_i u_i(k) \end{aligned} \quad \forall i \in \{1, \ldots, n\}. \tag{3.27}$$

which implies that that the networked system $\tilde{P}$ obtained by the interaction of sub-systems $\{\tilde{P}_i\}_i$ over the network described by (3.26) has the same state-space representation (A, B_u, C_y, D_{yu}) as the given structured system P. $\square$

Lemma 5 shows that given a unit-weight digraph $\mathcal{G}$ and a structured system P in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, there is a simple way to construct a networked system which is a strictly causal interaction over $\mathcal{G}$, with the same state-space representation. This is mainly possible because there are no bandwidth restrictions on the communication links (i.e. no restriction on the size of messages sent on the links) and also, the communication links do not introduce any noise. Thus, combining equations (3.8), (3.9), (3.10) and Lemma 5, we can treat $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and the set of strictly causal interactions over $\mathcal{G}$ (with input and output partitions $\mathcal{P}_u$ and $\mathcal{P}_y$) as equivalent sets.

Corollary 1. *Given a unit-weight digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $n-$tuples $\mathcal{P}_u$ and $\mathcal{P}_y$, and a stable structured system $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, there exists a stable networked system $\tilde{P}$ which is a strictly causal interaction over $\mathcal{G}$ such that $\mathbf{tf}(\tilde{P}) = P(z)$.*

Proof. The proof follows from Theorem 1. □

Similar to structured realizability, we refer to the property to realizing a structured transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as a stabilizable and detectable networked system which is a strictly causal interaction over $\mathcal{G}$ with the same transfer function as *network realizability*.

Remark 5. *From the proof of Theorem 1 and discussion in Section 3.2.1, we notice that the network realizability problem is also an open problem which needs to be addressed if we want to use transfer function approaches to solve problems that require the designed system to be a networked system that is a strictly causal interaction over $\mathcal{G}$.*

CHAPTER 4. Internal stabilization of networked plants using networked controllers

In this chapter, we consider a family of plants that are networked systems over a given network, and consider the problem of feedback stabilization using a controller which is also a networked system over the same network.

4.1 Networked plant model

A networked plant P is modeled as a strictly causal interaction of sub-systems (as in (3.5)) over a given unit-weight digraph $\mathcal{G}$, but with each sub-system now including local exogenous input vector $w_i(k)$ and local regulated output vector $z_i(k)$. The state-space description of the sub-systems $\{P_i\}_i$ are given by

$$P_i : \begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^w w_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} B_{ij}^{\zeta}\zeta_{ij}(k) \\ z_i(k) &= C_{ii}^z x_i(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{z\zeta}\zeta_{ij}(k) \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yw} w_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{y\zeta}\zeta_{ij}(k) \\ \eta_{ri}(k) &= C_{ri}^{\eta} x_i(k)\ \forall\, r\in\mathcal{N}_i^+ \end{aligned} \quad \forall\, i\in\{1,\ldots,n\} \tag{4.1}$$

where $x_i(k)$ denotes the local state vector, $w_i(k)$ local exogenous input vector, $z_i(k)$ local regulated output vector, $u_i(k)$ local control input vector, $y_i(k)$ the local measurement output vector, η_{ri} (for all $r\in\mathcal{N}_i^+$) the local network outputs and ζ_{ij} (for all $j\in\mathcal{N}_i^-$) the local network inputs corresponding to a sub-system P_i. The discrete-time network corresponding to the unit-weight digraph $\mathcal{G}$ is given by

$$\zeta_{ij}(k) = \eta_{ij}(k)\ \forall\, (v_j, v_i)\in\mathcal{E}. \tag{4.2}$$

Combining (4.1) and (4.2), the network inputs and outputs can be eliminated to give the state-space equations for the sub-systems as

$$
\begin{aligned}
x_i(k+1) &= A_{ii}x_i(k) + B_i^w w_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k),\\
z_i(k) &= C_{ii}^z x_i(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^z x_j(k) \;\forall\, i\in\{1,\dots,n\}\\
y_i(k) &= C_{ii}^y x_i(k) + D_i^{yw} w_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^y x_j(k),
\end{aligned}
\tag{4.3}
$$

where $A_{ij} := B_{ij}^{\zeta} C_{ij}^{\eta}$, $C_{ij}^z := D_{ij}^{z\zeta} C_{ij}^{\eta}$ and $C_{ij}^y := D_{ij}^{y\zeta} C_{ij}^{\eta}$. The state-space equations in (4.3) can also be concisely written as

$$
P : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}
\tag{4.4}
$$

where $A := [A_{ij}]_{i,j}$, $B_w := \mathbf{diag}[B_i^w]_i$, $B_u := \mathbf{diag}[B_i^u]_i$, $C_z := [C_{ij}^z]_{i,j}$, $C_y := [C_{ij}^y]_{i,j}$, $D_{zw} := \mathbf{diag}[D_i^{zw}]_i$, $D_{zu} := \mathbf{diag}[D_i^{zu}]_i$ and $D_{yw} := \mathbf{diag}[D_i^{yw}]_i$ (such that A_{ij}, C_{ij}^z and C_{ij}^y are zero matrices when $(v_j, v_i) \notin \mathcal{E}$ and $i \neq j$) denote the structured state-space matrices; $x(k) := \mathbf{vert}[x_i(k)]_i$, $w(k) := \mathbf{vert}[w_i(k)]_i$, $u(k) := \mathbf{vert}[u_i(k)]_i$, $z(k) := \mathbf{vert}[z_i(k)]_i$ and $y(k) := \mathbf{vert}[y_i(k)]_i$ denote the complete state, exogenous input, control input, regulated output and measurement output vectors corresponding to the networked system P and be partitioned according to $\mathcal{P}_x$, $\mathcal{P}_w$, $\mathcal{P}_u$, $\mathcal{P}_z$ and $\mathcal{P}_y$, respectively. From (4.3), and the partitions of $x(k)$, $w(k)$, $u(k)$, $z(k)$ and $y(k)$, we can see that

$$
\begin{aligned}
&A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x), \quad B_w \in S(I, \mathcal{P}_x, \mathcal{P}_w), \quad B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u),\\
&C_z \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_z, \mathcal{P}_x), \quad D_{zw} \in S(I, \mathcal{P}_z, \mathcal{P}_w), \quad D_{zu} \in S(I, \mathcal{P}_z, \mathcal{P}_u),\\
&C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x), \quad D_{yw} \in S(I, \mathcal{P}_y, \mathcal{P}_w).
\end{aligned}
\tag{4.5}
$$

According to the definition in Section 2.3, a controller K which is a mapping from the measurement outputs $y(k)$ to the control inputs $u(k)$ is said to stabilize the plant P given in (4.4), if $\mathbf{lft}(P,K)$ is BIBO stable and is said to internally stabilize P if $\mathbf{lft}(P,K)$ is asymptotically stable. Given a networked plant P that is a strictly causal interaction over a given digraph $\mathcal{G}$, with dynamics given by (4.3), our main goal is to design internally stabilizing controllers that are also strictly causal interactions over the same digraph $\mathcal{G}$. From the previous chapter, we saw that $\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ are equivalent sets of systems that can be used to represent the networked systems which are strictly causal interactions

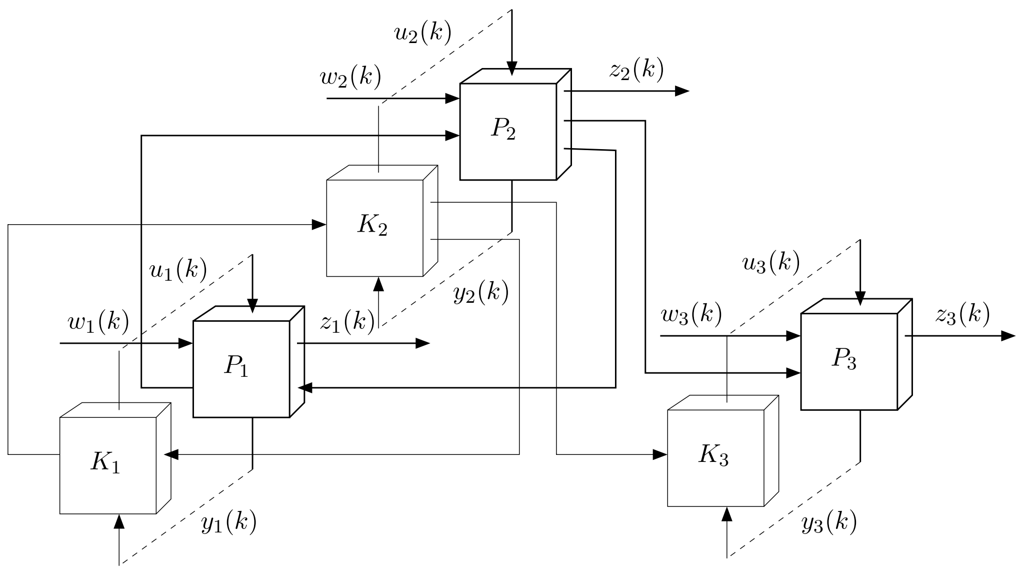


Figure 4.1 A networked controller in feedback with a networked system over the same zero-delay network.

over the given unit-weight digraph $\mathcal{G}$. But Remark 5 points out that designing stabilizing controllers as structured transfer function matrices is not equivalent to designing internally stabilizing controllers that are strictly causal interactions over the given digraph $\mathcal{G}$ unless it is shown how to find a stabilizable and detectable networked system with the same transfer function as an unstable structured transfer function matrix.

Thus, the classical Zames' parameterization [19] and Youla-Kučera parameterization [21,22], which are transfer function approaches for parameterizing all stabilizing controllers, are not suitable for parameterizing all stabilizing networked controllers that are also strictly causal interactions over a given unit-weight digraph $\mathcal{G}$. Instead, we use the state-space approach for Youla-Kučera parameterization based on [23] to parameterize all stabilizing networked controllers, in the next section.

4.2 All internally stabilizing networked controllers

In the standard Youla-Kučera parameterization for internally stabilizing controllers for a general plant [20], the set of all internally stabilizing controllers is constructed from a model based controller and a Youla parameter Q which is a stable system. In our case, the plant P is a networked system. In order to parameterize internally stabilizing networked controllers, first a model based controller J is chosen to be a networked system by finding appropriate F and L. Then Theorem 2 shows that choosing the Youla parameter Q to be a stable networked system will parameterize the stabilizing networked controllers for the given networked plant.

Theorem 2. *Given a unit-weight digraph $\mathcal{G}$ and a stabilizable and detectable networked plant P that is a strictly causal interaction over $\mathcal{G}$ with the sub-system dynamics given by (4.1) and the network interaction given by (4.2). Let the state-space representation for P be given by (4.4) with state-space matrices structured and partitioned according to (4.5). Given there exist matrices $F = [F_{ij}]_{i,j} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$ and $L = \mathbf{diag}[L_i]_i \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ such that $A + B_u F$ and $A + LC_y$ are Schur-stable. Then the set of all internally stabilizing FDLTI controllers for P, which are also strictly causal interactions over $\mathcal{G}$, is parametrized by*

$$K = \mathbf{lft}(J, Q), \tag{4.6}$$

where $J \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u + \mathcal{P}_y, \mathcal{P}_y + \mathcal{P}_u)$ with a state-space representation

$$J : \left[\begin{array}{c} x_J(k+1) \\ \hline u(k) \\ \xi(k) \end{array}\right] = \left[\begin{array}{c|cc} A + B_u F + LC_y & -L & B_u \\ \hline F & 0 & I \\ -C_y & I & 0 \end{array}\right] \left[\begin{array}{c} x_J(k) \\ \hline y(k) \\ \psi(k) \end{array}\right] \tag{4.7}$$

and any FDLTI $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Note that the vectors $x_J(k) := \mathbf{vert}[x_i^J(k)]_i$, $\xi(k) := \mathbf{vert}[\xi_i(k)]_i$ and $\psi(k) := \mathbf{vert}[\psi_i(k)]_i$ are partitioned according to $\mathcal{P}_x$, $\mathcal{P}_y$ and $\mathcal{P}_u$, respectively.

Proof. First, assume that Q is an FDLTI system in $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. It is a well-known result that given J in (4.7) and any stable, causal and FDLTI system Q, the controller given by $K = \mathbf{lft}(J, Q)$ internally stabilizes the given plant P in (4.4). Next, we will show that based on J in (4.7) and a $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, we can get a strictly causal interaction on $\mathcal{G}$ which has the same state-space representation as $\mathbf{lft}(J, Q)$.

Since $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, the state-space representation of Q is given by (A_Q, B_Q, C_Q, D_Q) in the set $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x^Q, \mathcal{P}_u, \mathcal{P}_y)$ for some state partition $\mathcal{P}_x^Q$, which can also be written as

$$\begin{aligned} x_i^Q(k+1) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} A_{ij}^Q x_j^Q(k) + B_i^Q \xi_i(k) \\ \psi_i(k) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} C_{ij}^Q x_j^Q(k) + D_i^Q \xi_i(k) \end{aligned} \qquad \forall i \in \{1, \dots, n\} \tag{4.8}$$

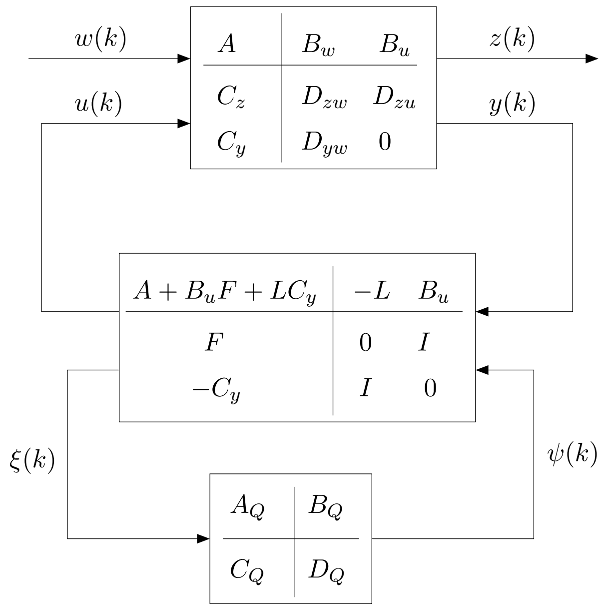


Figure 4.2 Feedback interconnection of the networked plant P and a parametrized controller $K = \mathbf{lft}(J,Q)$.

where $A_Q = [A^Q_{ij}]_{i,j}$, $C_Q = [C^Q_{ij}]_{i,j}$ (with A^Q_{ij} and C^Q_{ij} being zero matrices whenever $[\mathcal{A}(\mathcal{G})]_{ij} = 0$), $B_Q = \mathbf{diag}[B^Q_i]_i$ and $D_Q = \mathbf{diag}[D^Q_i]_i$. Let $x_Q(k) = \mathbf{vert}[x^Q_i(k)]_i$ denotes the state vector of Q. Using the sub-matrices of A, B_u, C_y, F and L; (4.7) can be written as

$$\begin{aligned} x^J_i(k+1) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} (A_{ij} + B^u_i F_{ij} + L_i C^y_{ij}) x^J_j(k) - L_i y_i(k) + B^u_i \psi_i(k), \\ u_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} F_{ij} x^J_j(k) + \psi_i(k), \\ \xi_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} (-C^y_{ij}) x^J_j(k) + y_i(k). \end{aligned} \tag{4.9}$$

for all $i \in \{1,\ldots,n\}$. Combining equations in (4.9) and (4.8), we eliminate the variables $\xi_i(k)$ and $\psi_i(k)$ to write the state-space equations corresponding to $K = \mathbf{lft}(J,Q)$ as

$$\left.\begin{aligned} x^K_i(k+1) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} A^K_{ij} x^K_j(k) + B^K_i y_i(k) \\ u_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} C^K_{ij} x^K_j(k) + D^K_i y_i(k) \end{aligned}\right. \quad \forall i \in \{1,\ldots,n\} \tag{4.10}$$

where $x_i^K(k) = \begin{bmatrix} x_i^J(k) \\ x_i^Q(k) \end{bmatrix}$ and

$$A_{ij}^K := \begin{bmatrix} A_{ij} + B_i^u F_{ij} + L_i C_{ij}^y - B_i^u D_i^Q C_{ij}^y & B_i^u C_{ij}^Q \\ -B_i^Q C_{ij}^y & A_{ij}^Q \end{bmatrix}, \qquad B_i^K := \begin{bmatrix} -L_i + B_i^u D_i^Q \\ B_i^Q \end{bmatrix},$$

$$C_{ij}^K := \begin{bmatrix} F_{ij} - D_i^Q C_{ij}^y & C_{ij}^Q \end{bmatrix}, \qquad D_i^K := D_i^Q.$$

From (4.10), it is easy to see that $K \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x + \mathcal{P}_x^Q, \mathcal{P}_u, \mathcal{P}_y)$. From Lemma 5, we know that (4.10) is equivalent to a strictly causal interaction over $\mathcal{G}$ with the same state-space matrices as in (4.10).

On the otherhand, from the theory of Youla parameterization, we know that given matrices F and L such that $A + B_u F$ and $A + LC_y$ are Schur-stable, any internally stabilizing controller for the plant P is represented by $K = \mathbf{lft}(J, Q)$ where J is given by (4.7) and Q is a stable, causal, FDLTI system. Now, assume that K is a strictly causal interaction over $\mathcal{G}$, which implies that K has a stabilizable and detectable state-space realization in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Then, it is easy to see that K internally stabilizes $\hat{J}$ given by

$$\hat{J} : \begin{bmatrix} x_{\hat{J}}(k+1) \\ \hline \psi(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|cc} A & -L & B_u \\ \hline -F & 0 & I \\ C_y & I & 0 \end{array}\right] \begin{bmatrix} x_{\hat{J}}(k) \\ \hline \xi(k) \\ u(k) \end{bmatrix} \tag{4.11}$$

where $x_{\hat{J}}(k)$ is partitioned according to $\mathcal{P}_x$. Following a similar procedure as before, we see that $Q = \mathbf{lft}(\hat{J}, K) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ and in particular $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. □

Remark 6. *The main result of Theorem 2 is to show that given a networked plant, the set of all internally stabilizing controllers that are also strictly causal interactions over the given $\mathcal{G}$ can be described using the subspace of structured systems given by $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.*

4.2.1 Sufficiency conditions for constructing F and L

Theorem 2 requires matrices matrices $F = [F_{ij}]_{i,j} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$ and $L = \mathbf{diag}[L_i]_i \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ such that $A + B_u F$ and $A + LC_y$ are Schur-stable. The theorem provides a characterization of all internally stabilizing networked controllers over the given network based on the matrices F and L satisfying the above mentioned constraints. In this section, we provide constructive algorithms to obtain such matrices F and L. Note that for stable plants, F and L can always be chosen to be zero matrices. Thus, Theorem 2 and the results of next part of the Chapter provide a networked solution for a stable networked plant. In the case the plant is unstable, we propose the following approach based on relaxed LMI conditions.

The stability test for discrete-time systems is given by a discrete-time Lyapunov equation or a Stein equation. In [24], the stability test has been expressed as a feasibility problem as shown in the following lemma. This formulation is best suited for imposing sparsity constraints on F and L.

Lemma 6. *A matrix A is Schur stable if, and only if, there exist a symmetric matrix $M = M'$ and a matrix G such that the LMI*

$$\begin{bmatrix} M & AG \\ G'A' & G+G'-M \end{bmatrix} \succ 0 \tag{4.12}$$

is feasible.

Note that, in Lemma 6, there are no constraints on the matrix G, which is a free parameter. We extend Lemma 6 to construct matrices F and L with the required sparsity constraints by imposing constraints on the free parameter G and solving the following convex feasibility problems.

Lemma 7. *Given matrices A and B_u that are partitioned according to $(\mathcal{P}_x, \mathcal{P}_x)$ and $(\mathcal{P}_x, \mathcal{P}_u)$, respectively, there exists a matrix $F \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$ such that $A + B_uF$ is Schur-stable if the following feasibility problem has a solution*

$$\begin{aligned} \min \quad & 1 \\ \textit{subject to} \quad & \begin{bmatrix} M & AG+B_uR \\ (AG+B_uR)' & G+G'-M \end{bmatrix} \succ 0, \\ & G \in S(I, \mathcal{P}_x, \mathcal{P}_x), \\ & R \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x). \end{aligned} \tag{4.13}$$

Proof. If (4.13) has a solution, then $G + G' \succ M \succ 0$ which implies that G is non-singular (from Lemma 1) and thus G^{-1} exists. Combining (4.13) with Lemma 6, we note that $A + B_uRG^{-1}$ is Schur-stable. Due to the structure of R and G in (4.13), it is easy to see (using Lemma 2) that $F = RG^{-1} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$ and $A + B_uF$ is Schur-stable. □

Lemma 8. *Given matrices A and C_y that are partitioned according to $(\mathcal{P}_x, \mathcal{P}_x)$ and $(\mathcal{P}_y, \mathcal{P}_x)$, respectively, there exists a matrix $L \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ such that $A + LC_y$ is Schur-stable if the following feasibility* *problem has a solution*

$$
\begin{aligned}
\min \quad & 1 \\
\textit{subject to} \quad & \begin{bmatrix} M & A'G + C_y'R \\ (A'G + C_y'R)' & G + G' - M \end{bmatrix} \succ 0, \\
& G \in S(I, \mathcal{P}_x, \mathcal{P}_x), \\
& R \in S(I, \mathcal{P}_y, \mathcal{P}_x).
\end{aligned} \tag{4.14}
$$

Proof. The proof is similar to that of Lemma 7. If (4.14) has a solution, then $G + G' \succ M \succ 0$ which implies that G is non-singular (from Lemma 1) and thus G^{-1} exists. Combining (4.14) with Lemma 6, we note that $A' + C_y' RG^{-1}$ is Schur-stable. Due to the structure of R and G in (4.14), it is easy to see that $L = (RG^{-1})' \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ and $A' + C_y' L'$ is Schur-stable, which implies $A + LC_y$ is Schur-stable. □

In this section, we only provide sufficiency conditions for constructing the matrices F and L with the required properties. Necessary conditions for the existence of such matrices is a more involved topic and is left for future work.

4.3 Optimal solution for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ networked controller design problems

Let $\mathcal{G}$ denote the unit-weight digraph representing a zero-delay network interaction. Given a networked plant P with sub-system dynamics following (4.1) that are interacting over a network specified by (4.2). Then the problem of finding an internally stabilizing networked controller, that is also a strictly causal interaction over $\mathcal{G}$, while minimizing an objective function is referred to as *Networked controller design problem* or *Networked control problem*. In this section, we show how to solve the following norm-minimizing networked control problems

$$
\begin{aligned}
\min \quad & \|T_{zw}\|_\alpha \\
\text{subject to} \quad & K \quad \text{is a strictly causal interaction over } \mathcal{G}, \\
& T_{zw} \quad \text{is asymptotically stable}
\end{aligned} \tag{4.15}
$$

where $T_{zw} = \mathbf{lft}(P, K)$ denotes the closed-loop mapping from $w(k)$ to $z(k)$, and $\alpha = 2$ or ∞. In the case when $\alpha = 2$, the solution for (4.15) is referred to as $\mathcal{H}_2$ *networked controller* and in the case when $\alpha = \infty$, it is called H_∞ *networked controller*.

Theorem 2 provides the parameterization of internally stabilizing networked controllers that are strictly causal interactions over $\mathcal{G}$ as $K = \textbf{lft}(J,Q)$ where J is given by (4.7) and $Q \in \mathfrak{S}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$ is a parameter. If there exists matrices $F \in S(\mathcal{A}(\mathcal{G}),\mathcal{P}_u,\mathcal{P}_x)$ and $L \in S(I,\mathcal{P}_x,\mathcal{P}_y)$ such that $A+B_uF$ and $A+LC_y$ are Schur-stable, then the set of all closed-loop transfer matrices from $w(k)$ to $z(k)$ for an internally stabilizing networked controller (which is a strictly causal interaction over $\mathcal{G}$) can be obtained using Theorem 2 and the results from ([20]) as

$$\mathfrak{C}_{zw} := \{T_{11}(z)+T_{12}(z)Q(z)T_{21}(z):\ Q(z) = \textbf{tf}(Q),\ Q \in \mathfrak{S}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)\} \tag{4.16}$$

where

$$\begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} = \left[\begin{array}{cc|cc} A+B_uF & -B_uF & B_w & B_u \\ 0 & A+LC_y & B_w+LD_{yw} & 0 \\ \hline C_z+D_{zu}F & -D_{zu}F & D_{zw} & D_{zu} \\ 0 & C_y & D_{yw} & 0 \end{array}\right]. \tag{4.17}$$

Thus, the norm-minimization networked control problem in (4.15) can be written as

$$\begin{array}{ll} \min & \|T_{zw}(z)\|_\alpha \\ \text{subject to} & T_{zw}(z) \in \mathfrak{C}_{zw} \end{array} \qquad \text{for } \alpha = 2 \text{ or } \infty. \tag{4.18}$$

Since the closed-loop transfer function matrix is simply an affine function of the Youla parameter Q, we can rewrite the problem in (4.18) as a convex optimization problem

$$\begin{array}{ll} \min & \|T_{11}(z)+T_{12}(z)Q(z)T_{21}(z)\|_\alpha \\ \text{subject to} & Q(z) = \textbf{tf}(Q), \\ & Q \in \mathfrak{S}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y) \end{array} \qquad \text{for } \alpha = 2 \text{ or } \infty. \tag{4.19}$$

Following the results of Lemma 4 and Theorem 1, we note that the condition $Q \in \mathfrak{S}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$ is equivalent to $\textbf{tf}(Q) \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$. Since it is convenient to solve the problem (4.19) in the frequency domain, we write (4.19) as

$$\begin{array}{ll} \min & \|T_{11}(z)+T_{12}(z)Q(z)T_{21}(z)\|_\alpha \\ \text{subject to} & Q(z) \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y) \end{array} \qquad \text{for } \alpha = 2 \text{ or } \infty. \tag{4.20}$$

The problem is now reduced to a standard convex optimization form which can be colved using convex programming [6]. In the case of $\alpha = 2$, the optimization problem in (4.20) can equivalently be expressed as an unconstrained optimization problem by following the methodology used in [8] that has a similar problem setting. By extending the vectorization idea for complex matrices (2.6) to transfer function matrices, given $G(z) \in \mathcal{R}_p^{a\times b}$, we write

$$\mathbf{vec}(G(z)) = \mathbf{vert}\left[\mathbf{vert}[G_{ij}(z)]_{i\in\{1,\ldots,a\}}\right]_{j\in\{1,\ldots,b\}} \in \mathcal{R}_p^{ab\times 1} \tag{4.21}$$

which is nothing but arranging the columns of the matrix $G(z)$ to form a vector. It is also easy to see that inverse operation from vector to a matrix form is well-defined. It is represented by $\mathbf{vec}^{-1}(\cdot)$.

Let $\mathbf{vec}(\mathfrak{T}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)) = \{\mathbf{vec}(Q(z))|Q(z) \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)\}$ denote the set of vectorized elements of $\mathfrak{T}^s(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$. If $\mathcal{P}_u = (\mathcal{P}_u(1),\ldots,\mathcal{P}_u(n))$ denotes the output partition, then denote $n_u := \sum_i \mathcal{P}_u(i)$ to represent the total number of outputs. Similarly, denote n_y to represent the total number of inputs. It can be seen that $\mathbf{vec}(\mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)) \in \mathcal{RH}_\infty^{n_u n_y\times 1}$ is a sub-space due to the delay and sparsity constraints imposed by the set $\mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$. Let a denote the total number of elements of $Q(z) \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$ that are not constrained to be zero. From (3.11), we can infer that $Q_{ij}(z)$ is of the form $z^{-t(i,j)}H_{ij}(z)$ (with possible pole-zero cancellations at origin) where $H_{ij}(z) \in \mathcal{RH}_\infty$ and $t(i,j)$ is based on $\mathcal{G}$ and partitions $\mathcal{P}_u$ and $\mathcal{P}_y$. Thus, we can separate the sparsity and delay terms of the form $z^{-t(i,j)}$ into a matrix $S(z) \in \mathcal{R}_p^{n_u n_y\times a}$ and say

$$Q(z) \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u) \Longleftrightarrow \mathbf{vec}(Q(z)) = S(z)H(z) \quad \text{for some } H(z) \in \mathcal{RH}_\infty^{a\times 1}. \tag{4.22}$$

For example, consider the following $Q(z)$ and the decomposition of its vectorization.

$$Q(z) = \begin{bmatrix} \frac{z+1}{z-0.5} & \frac{0.5}{z-0.8} & 0 \\ \frac{-0.1}{z-0.5} & \frac{z+0.1}{z-0.1} & 0 \\ \frac{1}{(z-0.1)(z-0.8)} & \frac{0.3}{z-0.8} & \frac{z-0.2}{z-0.5} \end{bmatrix} \tag{4.23}$$

$$\Rightarrow \mathbf{vec}(Q(z)) = \begin{bmatrix} \frac{z+1}{z-0.5} \\ \frac{-0.1}{z-0.5} \\ \frac{1}{(z-0.1)(z-0.8)} \\ \frac{0.5}{z-0.8} \\ \frac{z+0.1}{z-0.1} \\ \frac{0.3}{z-0.8} \\ 0 \\ 0 \\ \frac{z-0.2}{z-0.5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z+1}{z-0.5} \\ \frac{-0.1z}{z-0.5} \\ \frac{z^2}{(z-0.1)(z-0.8)} \\ \frac{0.5z}{z-0.8} \\ \frac{z+0.1}{z-0.1} \\ \frac{0.3z}{z-0.8} \\ \frac{z-0.2}{z-0.5} \end{bmatrix} \tag{4.24}$$

$$=: S(z)H(z)$$

Note that $S(z)$ contains both the delay and sparsity constraints imposed by the set $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. Using the results of vectorization, we get

$$\begin{aligned} \|T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)\|_2 &= \|\mathbf{vec}(T_{11}(z) + T_{12}(z)Q(z)T_{21}(z))\|_2 \\ &= \left\|\mathbf{vec}(T_{11}(z)) + (T_{21}(z)' \otimes T_{12}(z))\,\mathbf{vec}(Q(z))\right\|_2 \\ &= \left\|\mathbf{vec}(T_{11}(z)) + (T_{21}(z)' \otimes T_{12}(z))S(z)H(z)\right\|_2 \end{aligned}$$

Thus, we can pose the problem (4.20) (when $\alpha = 2$) as an unconstrained $\mathcal{H}_2$ problem

$$\begin{aligned} \min \quad & \left\|\mathbf{vec}(T_{11}(z)) + (T_{21}(z)' \otimes T_{12}(z))S(z)H(z)\right\|_2 \\ \text{subject to} \quad & H(z) \in \mathcal{RH}_\infty^{\alpha \times 1}, \end{aligned} \tag{4.25}$$

which can be solved using standard techniques. Let $H^\star(z)$ denote the solution of the unconstrained convex optimization problem (4.25). Then the corresponding optimal $Q^\star(z)$ is given by $Q^\star(z) = \mathbf{vec}^{-1}(S(z)H^\star(z))$. Since $Q^\star(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, we can obtain a realization $\tilde{Q} = (\tilde{A}_Q, \tilde{B}_Q, \tilde{C}_Q, \tilde{D}_Q) \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, using Theorem 1, such that $Q^\star(z) = \mathbf{tf}(\tilde{Q})$ and $\tilde{A}_Q$ is Schur-stable. The corresponding controller is given by $K^\star = \mathbf{lft}(J, \tilde{Q})$, where J is given by (4.7). Using Lemma 5, one can obtain a strictly causal interaction over given $\mathcal{G}$ with the same state-space representation as $K^\star$. From Theorem 2 and the problem formulation in (4.15), we can see that $K^\star$ thus designed is the optimal internally stabilizing networked controller that is a strictly causal interaction over $\mathcal{G}$ for the given networked plant P.

CHAPTER 5. Full-order networked controllers

In the previous chapter, we showed how to find an optimal internally stabilizing networked controller given a networked plant. In order to obtain the optimal controller, we converted the networked control problem in (4.15) into an infinite dimensional unconstrained optimization problem in (4.20). Note that the solution of (4.20) gives a transfer function matrix for the Q parameter and the use of Theorem 1 leads to a structured state-space representation for Q which can have very large order. In the case of centralized plants (where there are no structural constraints on state-space or transfer functions), classical theory says that a *full-order controller* (a controller with the same number of states as that of the plant) can be an optimal solution to the centralized control problem. As we mentioned, in the case of networked control problem, the optimal controller might necessarily be of higher-order than the given plant, which may not be a good option for practical purposes. Model reduction is one option to reduce the order of the optimal networked controller but the available techniques for model reduction do not promise any required sparsity structures for the state-space matrices (as we require) of the reduced-order models. In this chapter we look at design of full-order networked controllers, which is an alternative option for model reduction.

5.1 Full-order $\mathcal{H}_2$ networked controller design

Let $\mathcal{G}$ denote the unit-weight digraph representing a zero-delay network interaction. Given a networked plant P with sub-system dynamics following (4.1) that are interacting over a network specified by (4.2). Then P has a state-space realization of the form (4.4) with state-space matrices structured and partitioned according to (4.5). A controller K is said to be a *full-order networked controller for P* if it is a strictly causal interaction over $\mathcal{G}$ with a state-space realization in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ that internally stabilizes P. Thus, a full-order $\mathcal{H}_2$ networked control problem can be posed as

$$\begin{aligned} \min \quad & \|T_{zw}\|_2 \\ \text{subject to} \quad & K \quad \text{is a full-order strictly causal interaction over } \mathcal{G}, \\ & T_{zw} \quad \text{is asymptotically stable}, \end{aligned} \tag{5.1}$$

where $T_{zw} = \mathbf{lft}(P,K)$ denotes the closed-loop mapping from $w(k)$ to $z(k)$.

In order to solve the problem in (5.1), we need to search for a controller K in $\mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_u,\mathcal{P}_y)$ that also makes the closed-loop transfer function $T_{zw} = \mathbf{lft}(P,K)$ internally stable. Let the state-space representation of K be $(A_K,B_K,C_K,D_K) \in \mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_u,\mathcal{P}_y)$. Let $x_K(k)$ denote the states of the controller which is partitioned according to $\mathcal{P}_x$. By connecting the controller K in feedback with the plant P, we get a state-space representation for the closed-loop system T_{zw} (using (2.12)) as

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ x_K(k+1) \end{bmatrix} &= \tilde{A} \begin{bmatrix} x(k) \\ x_K(k) \end{bmatrix} + \tilde{B}w(k) \\ z(k) &= \tilde{C} \begin{bmatrix} x(k) \\ x_K(k) \end{bmatrix} + \tilde{D}w(k) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \tilde{A} :&= \begin{bmatrix} A + B_u D_K C_y & B_u C_K \\ B_K C_y & A_K \end{bmatrix}, \ \tilde{B} := \begin{bmatrix} B_w + B_u D_K D_{yw} \\ B_K D_{yw} \end{bmatrix} \\ \tilde{C} :&= \begin{bmatrix} C_z + D_{zu} D_K C_y & D_{zu} C_K \end{bmatrix}, \ \tilde{D} := \begin{bmatrix} D_{zw} + D_{zu} D_K D_{yw} \end{bmatrix} \end{aligned} \tag{5.3}$$

The following lemma gives a characterization of $\mathcal{H}_2$ norm constraint for a discrete-time FDLTI system in terms of linear matrix inequalities (LMIs).

Lemma 9. *Given a system P with a state-space realization (A,B,C,D), A is Schur-stable and $\|P\|_2^2 < \mu$ if and only if, there exists symmetric matrices M and W such that $\mathbf{Tr}(W) < \mu$ and*

$$\begin{bmatrix} W & CM & D \\ (\cdot)' & M & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} M & AM & B \\ (\cdot)' & M & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0 \tag{5.4}$$

are feasible.

Proof. This proof is a discrete-time version of the proposition in [25].

To prove (only if) part, assume that A is Schur-stable and $\|P\|_2^2 < \mu$ for some $\mu \geq 0$. From (2.15), we see that

$$\begin{aligned} &\mathbf{Tr}(DD' + CM_cC') < \mu \\ \Rightarrow \exists W \succ 0 \ni DD' + CM_cC' \prec W, \quad &\mathbf{Tr}(W) < \mu. \end{aligned} \tag{5.5}$$

where M_c is given by (2.17). For $\varepsilon \geq 0$, define

$$M(\varepsilon) = \sum_{k=0}^{\infty} A^k (BB' + \varepsilon I)(A')^k. \tag{5.6}$$

We can see that $M(\varepsilon)$ is continuous in ε and equals M_c when $\varepsilon = 0$. From [25], we know that $M(\varepsilon) \succ M_c$ for any $\varepsilon > 0$. Using these properties of $M(\varepsilon)$ and combining with (5.5), we can say that $\exists \varepsilon > 0$ such that

$$DD' + CM_cC' \prec DD' + CM(\varepsilon)C' \prec W. \tag{5.7}$$

For this ε, we note that

$$AM(\varepsilon)A' - M(\varepsilon) + BB' + \varepsilon I = 0. \tag{5.8}$$

Combining equations (5.5), (5.7) and (5.8), we can say that the LMIs in (5.4) are satisfied for some $M(\varepsilon) \succ 0$ and $W \succ 0$.

To prove (if) part, assume that the LMIs in (5.4) are satisfied for some $M \succ 0$ and $W \succ 0$. From (5.4), we note that

$$\begin{bmatrix} M & AM \\ MA' & M \end{bmatrix} \succ 0 \tag{5.9}$$

which implies that A is Schur-stable. Using Schur complements, (5.4) also imply that

$$\begin{aligned} & M - AMA' - BB' \succ 0, \quad W \succ DD' + CMC', \\ \Rightarrow \quad & M \succ M_c \quad \Rightarrow \quad CMC' \succ CM_cC', \\ \Rightarrow \quad & W \succ DD' + CMC' \succ DD' + CM_cC', \\ \Rightarrow \quad & \mu > \mathbf{Tr}(W) > \mathbf{Tr}(DD' + CMC') > \mathbf{Tr}(DD' + CM_cC'), \\ \Rightarrow \quad & \|P\|_2^2 < \mu. \end{aligned}$$

Thus the LMIs in (5.4) imply that A is Schur-stable and $\|P\|_2^2 < \mu$. $\square$

As shown in [24], the LMIs in Lemma 9 can be extended by introducing an additional matrix variable so that the product of A and M does not appear.

Lemma 10. *Given a system P with a state-space realization (A,B,C,D), A is Schur-stable and $\|P\|_2^2 < \mu$ if and only if, there exists a matrix G and symmetric matrices M and W such that $\mathbf{Tr}(W) < \mu$ and*

$$\begin{bmatrix} W & CG & D \\ (\cdot)' & G+G'-M & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} M & AG & B \\ (\cdot)' & G+G'-M & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0 \tag{5.10}$$

is feasible.

Proof. The proof is very much similar to the one in [24] with a minor difference due to a non-zero D in (5.10).

First assume that A is Schur-stable and $\|P\|_2^2 < \mu$. From Lemma 9, we know that there exists symmetric matrices M and W such that the LMIs in (5.4) are satisfied. It is easy to note that a choice of $G = M$ would ensure that the LMIs in (5.10) are also satisfied.

Next, assume that there exists a matrix G and symmetric matrices M and W such that the LMIs in (5.10) are satisfied. Note that $G+G'-M \succ 0$ implies that G is non-singular (from Lemma 1) and G^{-1} exists. Since $G+G' \succ M \succ 0$, we get $(I-G^{-1}M)'M(I-G^{-1}M) \succ 0$, which implies that $G+G'-M \succ G'M^{-1}G$. Combining this observation with the LMIs in (5.10), we can write

$$\begin{bmatrix} W & CG & D \\ (\cdot)' & G'M^{-1}G & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} M & AG & B \\ (\cdot)' & G'M^{-1}G & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0 \tag{5.11}$$

Define a block diagonal matrix $T := \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & G^{-1}M & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$. Multiplying T from the right and T' from the left, the LMIs in (5.11) transform into the LMIs in (5.4). Since symmetric matrices M and W satisfy the LMIs in (5.4), Lemma 9 shows that $\|P\|_2^2 < \mu$ and A is Schur-stable. $\square$

Note that G can be any matrix and does not have any structural constraints like symmetry. This property of decoupling A and C from M allows us to parameterize the controllers belonging to $\mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_u)$ in a flexible form and write the $\mathcal{H}_2$ networked control problem in (5.1) as a *semi-definite program* (SDP) which can be efficiently solved.

Theorem 3. *Given a unit-weight digraph $\mathcal{G}$ and a networked plant P with sub-system dynamics given by* (4.1) *interacting over a network defined by* (4.2)*. Let the plant dynamics be given by a state-space representation in* (4.4) *with state-space matrices of the form* (3.10)*.*

If there exist matrices X, Y, S in $S(I, \mathcal{P}_x, \mathcal{P}_x)$ (with $S - YX$ being non-singular); $Q \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $L \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$, $F \in S(I, \mathcal{P}_x, \mathcal{P}_y)$, $R \in S(I, \mathcal{P}_u, \mathcal{P}_y)$; symmetric matrices M, H, W and any general matrix J of dimensions $n_x \times n_x$ (where $n_x = \sum_i \mathcal{P}_x(i)$) such that

$$\mathbf{Tr}(W) < \mu, \tag{5.12}$$

$$\begin{bmatrix} W & C_zX + D_{zu}L & C_z + D_{zu}RC_y & D_{zw} + D_{zu}RD_{yw} \\ (\cdot)' & X + X' - M & I + S' - J & 0 \\ (\cdot)' & (\cdot)' & Y + Y' - H & 0 \\ (\cdot)' & (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \tag{5.13}$$

$$\begin{bmatrix} M & J & AX + B_uL & A + B_uRC_y & B_w + B_uRD_{yw} \\ (\cdot)' & H & Q & YA + FC_y & YB_w + FD_{yw} \\ (\cdot)' & (\cdot)' & X + X' - M & I + S' - J & 0 \\ (\cdot)' & (\cdot)' & (\cdot)' & Y + Y' - H & 0 \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \tag{5.14}$$

then there exists $K \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ such that $\|\mathbf{lft}(P,K)\|_2^2 < \mu$ and $\mathbf{lft}(P,K)$ is asymptotically stable.

Proof. Given the block-diagonal matrices X, Y and S , choose matrices U and V in $S(I, \mathcal{P}_x, \mathcal{P}_x)$ such that $VU = S - YX$. This is possible because $S - YX$ is assumed to be non-singular. One simple way is to choose $V = I$ and $U = S - YX$. Then construct the matrices A_K, B_K, C_K and D_K in the following order

$$\begin{aligned} D_K &:= R, \\ C_K &:= (L - RC_yX)U^{-1}, \\ B_K &:= V^{-1}(F - YB_uR), \\ A_K &:= V^{-1}[Q - Y(A + B_uRC_y)X - VB_KC_yX]U^{-1} - V^{-1}YB_uC_K \end{aligned} \tag{5.15}$$

Based on the structure and partitions of A, B_u, C_y, Q, L, F, R (from hypothesis) and U, V from construction, we can see that $(A_K, B_K, C_k, D_K) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$. Let us denote (A_K, B_K, C_k, D_K) by a system

K. The state-space equations of the closed-loop system $T_{zw} = \mathbf{lft}(P,K)$ is given by (5.2) and the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are given by (5.3).

Let $G = \left[\begin{smallmatrix} X & \Gamma \\ U & \Lambda \end{smallmatrix}\right]$, where $\Gamma = (I - XY')(V')^{-1}$ and $\Lambda = -(UY')(V')^{-1}$. Then consider a transformation matrix $T = \left[\begin{smallmatrix} I & Y' \\ 0 & V' \end{smallmatrix}\right]$. Note that T^{-1} exists and is equal to $\left[\begin{smallmatrix} I & Y'(V')^{-1} \\ 0 & (V')^{-1} \end{smallmatrix}\right]$, since V is non-singular. Also,

$$GT = \begin{bmatrix} X & \Gamma \\ U & \Lambda \end{bmatrix} \begin{bmatrix} I & Y' \\ 0 & V' \end{bmatrix} = \begin{bmatrix} X & I \\ U & 0 \end{bmatrix} \tag{5.16}$$

Combining equation (5.15), (5.3) and (5.16), we get the following identities

$$\begin{aligned} T'\tilde{A}GT &= \begin{bmatrix} AX + B_u L & A + B_u R C_y \\ Q & YA + FC_y \end{bmatrix}, & T'\tilde{B} &= \begin{bmatrix} B_w + B_u R D_{yw} \\ YB_w + FD_{yw} \end{bmatrix}, \\ \tilde{C}GT &= \begin{bmatrix} C_z X + D_{zu} L & C_z + D_{zu} R C_y \end{bmatrix}, & T'(G+G')T &= \begin{bmatrix} X + X' & I + S' \\ (\cdot)' & Y + Y' \end{bmatrix}. \end{aligned} \tag{5.17}$$

Substituting (5.17) in (5.13) and (5.14) give us the following inequalities

$$\begin{bmatrix} W & \tilde{C}GT & \tilde{D} \\ (\cdot)' & T'(G+G')T - \bar{M} & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \tag{5.18}$$

$$\begin{bmatrix} \bar{M} & \tilde{A}GT & \tilde{B} \\ (\cdot)' & T'(G+G')T - \bar{M} & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0 \tag{5.19}$$

where $\bar{M} :\ = \left[\begin{smallmatrix} M & J \\ J' & H \end{smallmatrix}\right]$ is a positive definite matrix. Let $\tilde{T} := \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$ and $\hat{T} := \left[\begin{smallmatrix} T^{-1} & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$. Multiplying (5.18) with $\tilde{T}'$ on the left and $\tilde{T}$ on the right; and (5.19) with $\hat{T}'$ on the left and $\hat{T}$ on the right gives us

$$\begin{bmatrix} W & \tilde{C}G & \tilde{D} \\ (\cdot)' & G + G' - \tilde{M} & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} \tilde{M} & \tilde{A}G & \tilde{B} \\ (\cdot)' & G + G' - \tilde{M} & 0 \\ (\cdot)' & (\cdot)' & I \end{bmatrix} \succ 0 \tag{5.20}$$

where $\tilde{M} :\ = (T')^{-1}\bar{M}T^{-1}$, is positive definite since $\bar{M} \succ 0$. From (5.20), Lemma 10 can be used to show that $\|\mathbf{lft}(P,K)\|_2^2 < \mu$ (where $K = (A_K, B_K, C_K, D_K)$ while A_K, B_K, C_K and D_K are given by (5.15)) and $\tilde{A}$ is Schur-stable which means that the closed-loop system is internally stable.

Thus $K = (A_K, B_K, C_K, D_K) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ internally stabilizes P and $\|\mathbf{lft}(P,K)\|_2^2 < \mu$. $\square$

Consider a semi-definite program (SDP) given by

$$\begin{aligned} \min \quad & \mu \\ \text{subject to} \quad & 0 < \operatorname{trace}(W) < \mu \\ & \text{LMIs (5.13) and (5.14) are satisfied} \end{aligned} \tag{5.21}$$

Based on Theorem 3, it is easy to show that if we find a solution to the optimization problem given in (5.21) then we can obtain a full-order stabilizing networked controller (that is a strictly causal interaction over $\mathcal{G}$) for a networked plant P described by (4.4) using equations in (5.15).

Remark 7. *Note that Theorem 3 only provides us a sufficiency condition to find a full-order stabilizing networked controller. Since we do not have a necessary condition, the controller obtained from* (5.21) *and Theorem 3 is only a sub-optimal solution to the full-order $\mathcal{H}_2$ networked control problem given in* (5.1).

Note that the solution of the SDP (5.21) might give matrices X, Y and S such that $S - YX$ is singular. Under that situation, one can perturb the matrix by εI, for some small ε, to calculate non-singular block-diagonal matrices U and V such that $VU = S - YX + \varepsilon I$. This might disrupt the performance of the synthesized controller slightly but is not a big problem.

5.2 Full-order $\mathcal{H}_\infty$ networked controller design

Let $\mathcal{G}$ denote the unit-weight digraph representing a zero-delay network interaction. Let P be a networked plant with a state-space realization of the form (4.4) while the state-space matrices are structured and partitioned according to (4.5). Similar to the $\mathcal{H}_2$ networked control problem in (5.1), a full-order $\mathcal{H}_\infty$ networked control problem can be posed as

$$\begin{aligned} \min \quad & \|T_{zw}\|_\infty \\ \text{subject to} \quad & K \quad \text{is a full-order strictly causal interaction over } \mathcal{G}, \\ & T_{zw} \quad \text{is asymptotically stable}, \end{aligned} \tag{5.22}$$

where $T_{zw} = \mathbf{lft}(P, K)$ denotes the closed-loop mapping from $w(k)$ to $z(k)$.

The procedure for solving the $\mathcal{H}_\infty$ networked control problem given in (5.22) is very much similar to the procedure followed for the $\mathcal{H}_2$ counter-part. In order to solve the problem in (5.22), we need to search for a stabilizing controller K in $\mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_u,\mathcal{P}_y)$ that also minimizes the $\mathcal{H}_\infty$ norm of the closed-loop transfer function T_{zw}. Let the state-space representation of K be $(A_K,B_K,C_K,D_K) \in \mathfrak{S}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_u,\mathcal{P}_y)$. Let $x_K(k)$ denote the state vector of the controller which is partitioned according to $\mathcal{P}_x$. The feedback interconnection of the plant P and a networked controller K gives a closed-loop state-space representation as given in (5.2).

The following lemma gives a characterization of $\mathcal{H}_\infty$ norm constraint for a discrete-time linear time-invariant system in terms of LMIs.

Lemma 11. *Let P be a system with (A,B,C,D) as it's state-space realization. Then A is Schur-stable and $\|P\|_\infty^2 < \mu$ if and only if there exists symmetric matrix $M \succ 0$ such that*

$$\begin{bmatrix} M & AM & B & 0 \\ (\cdot)' & M & 0 & MC' \\ (\cdot)' & (\cdot)' & I & D' \\ (\cdot)' & (\cdot)' & (\cdot)' & \mu I \end{bmatrix} \succeq 0 \tag{5.23}$$

is feasible.

Proof. The proof for this lemma can be obtained from a scaled version of the bounded-real lemma for discrete-time systems. The following statement can be obtained by slightly modifying the derivations in [26]. P is asymptotically stable and $\|P\|_\infty^2 < \mu$ if and only if there exist $\tilde{M} \succ 0$ and matrices L, W such that

$$\begin{aligned} A'MA + C'C + L'L &= \tilde{M}, \\ B'MB + D'D + W'W &= \mu I, \\ A'MB + C'D + L'W &= 0. \end{aligned} \tag{5.24}$$

This scaled version of bounded-real lemma (5.24) can be written in terms of LMIs as follows

$$\begin{aligned}
& \begin{bmatrix} L' \\ W' \end{bmatrix} \begin{bmatrix} L & W \end{bmatrix} \succeq 0 \\
\Leftrightarrow \quad & \begin{bmatrix} \tilde{M} - A'\tilde{M}A - C'C & A'\tilde{M}B - C'D \\ (\cdot)' & \mu I - B'\tilde{M}B - D'D \end{bmatrix} \succeq 0 \\
\Leftrightarrow \quad & \begin{bmatrix} \tilde{M} & 0 \\ 0 & \mu I \end{bmatrix} - \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix} \begin{bmatrix} \tilde{M} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \succeq 0 \\
\Leftrightarrow \quad & \begin{bmatrix} \bar{M} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix} \begin{bmatrix} \bar{M} & 0 \\ 0 & \mu^{-1} I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \succeq 0 \\
\Leftrightarrow \quad & \begin{bmatrix} \bar{M}^{-1} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \bar{M}^{-1}A' & \bar{M}^{-1}C' \\ B' & D' \end{bmatrix} \begin{bmatrix} \bar{M} & 0 \\ 0 & \mu^{-1} I \end{bmatrix} \begin{bmatrix} A\bar{M}^{-1} & B \\ C\bar{M}^{-1} & D \end{bmatrix} \succeq 0 \\
\Leftrightarrow \quad & \begin{bmatrix} M & 0 & MA' & MC' \\ 0 & I & B' & D' \\ AM & B & M & 0 \\ CM & D & 0 & \mu I \end{bmatrix} \succeq 0 \quad \Leftrightarrow \quad \begin{bmatrix} M & AM & B & 0 \\ MA' & M & 0 & MC' \\ B' & 0 & I & D' \\ 0 & CM & D & \mu I \end{bmatrix} \succeq 0
\end{aligned}$$

where $\bar{M} :\ = \mu^{-1}\tilde{M}$ and $\tilde{M} = \bar{M}^{-1}$. Both of them are positive definite because $\tilde{M}$ is positive definite and $\mu > 0$. Since $\left[\begin{smallmatrix} L' \\ W' \end{smallmatrix}\right] \left[\begin{smallmatrix} L & W \end{smallmatrix}\right] \succeq 0$ for all L and W, we get that A is Schur-stable and $\|P\|_\infty^2 < \mu$ if and only if there exist $M \succ 0$ such that (5.23) is satisfied. □

As shown in [24], an LMI characterization of the H_∞ norm constraint for a discrete-time linear time-invariant system can be expressed as shown in the following lemma.

Lemma 12. *Given a system P with a state-space realization (A, B, C, D), A is Schur-stable and $\|P\|_\infty^2 < \mu$ if and only if there exists a matrix G and a symmetric matrix M such that*

$$\begin{bmatrix} M & AG & B & 0 \\ (\cdot)' & G + G' - M & 0 & G'C' \\ (\cdot)' & (\cdot)' & I & D' \\ (\cdot)' & (\cdot)' & (\cdot)' & \mu I \end{bmatrix} \succeq 0 \tag{5.25}$$

is feasible.

Proof. Using a similar argument as in the proof for Lemma 10, (only if) part can be proved by choosing $G = M$ and using Lemma 11.

(if) part is proved by using $G + G' - M \succ G'M^{-1}G$, which means that the LMI in (5.25) implies

$$\begin{bmatrix} M & AG & B & 0 \\ (\cdot)' & G'M^{-1}G & 0 & G'C' \\ (\cdot)' & (\cdot)' & I & D' \\ (\cdot)' & (\cdot)' & (\cdot)' & \mu I \end{bmatrix} \succeq 0 \tag{5.26}$$

Define a block diagonal matrix $T := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & G^{-1}M & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$. Multiplying T from the right and T' from the left, the LMI in (5.26) transforms into the LMI in (5.23). Since symmetric matrices M and W satisfy (5.23), Lemma 11 shows that $\|P\|_\infty^2 < \mu$ and A is Schur-stable. □

Theorem 4. *Given a unit-weight digraph $\mathcal{G}$ and a networked plant P with sub-system dynamics given by* (4.1) *interacting over a network defined by* (4.2)*. Let the plant dynamics be given by a state-space representation in* (4.4) *with state-space matrices of the form* (3.10)*.*

If there exist matrices X, Y, S in $S(I, \mathcal{P}_x, \mathcal{P}_x)$ (with $S - YX$ being non-singular); $Q \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $L \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_u, \mathcal{P}_x)$, $F \in S(I, \mathcal{P}_x, \mathcal{P}_y)$, $R \in S(I, \mathcal{P}_u, \mathcal{P}_y)$; symmetric matrices M, H, W and any general matrix J of dimensions $n_x \times n_x$ (where $n_x = \sum_i \mathcal{P}_x(i)$) such that

$$\begin{bmatrix} M & J & AX + B_u L & A + B_u R C_y & B_w + B_u R D_{yw} & 0 \\ (\cdot)' & H & Q & YA + FC_y & YB_w + FD_{yw} & 0 \\ (\cdot)' & (\cdot)' & X + X' - M & I + S' - J & 0 & X'C_z' + L'D_{zu}' \\ (\cdot)' & (\cdot)' & (\cdot)' & Y + Y' - H & 0 & C_z' + C_y'R'D_{zu}' \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & I & D_{zw}' + D_{yw}'R'D_{zu}' \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & \mu I \end{bmatrix} \succeq 0, \tag{5.27}$$

then there exists $K \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ such that $\|\mathbf{lft}(P,K)\|_\infty^2 < \mu$ and $\mathbf{lft}(P,K)$ is asymptotically stable.

Proof. The proof for this theorem is very much similar to the proof for Theorem 3. So, we shall skip the details. □

Consider a semi-definite program (SDP) given by

$$\begin{aligned} \min \quad & \mu \\ \text{subject to} \quad & \text{LMI (5.27) is satisfied} \end{aligned} \tag{5.28}$$

Based on Theorem 4, it is easy to show that if we find a solution to the optimization problem given in (5.28) then we can obtain a full-order stabilizing networked controller (that is a strictly causal interaction over $\mathcal{G}$) for a networked plant P described by (4.4) using equations in (5.15).

Remark 8. *Note that Theorem 4 only provides us a sufficiency condition to find a full-order stabilizing networked controller. Since we do not have a necessary condition, the controller obtained from* (5.28) *and Theorem 4 is only a sub-optimal solution to the full-order* $\mathcal{H}_\infty$ *networked control problem given in* (5.22).

CHAPTER 6. Networked estimation

In the previous chapters, we considered the networked control problem where a networked controller is designed to internally stabilized a given networked plant while minimizing the norm of the closed-loop system. In this chapter, we consider the problem of *networked estimation* or *filtering* which is the design of networked estimators that are strictly causal interactions over a given unit-weight digraph $\mathcal{G}$. The objective of this problem is to make each sub-system of the networked estimator asymptotically track the states of the corresponding sub-system of the networked plant by exchanging information with other estimator sub-systems.

In the following sections, the above mentioned networked estimation problem is formulated and analyzed to estimate the states of a given plant by minimizing the effect of external disturbances and measurement noise. We shall make some assumptions about detectability of the plant dynamics to assure the existence of a networked estimator.

6.1 Networked filtering for networked systems

Let a unit-weight digraph $\mathcal{G}$ be the representation of a given zero-delay network interconnection. Given a networked system P made of discrete-time FDLTI sub-systems $\{P_i\}_i$ interacting over the network represented by $\mathcal{G}$. Let $x_i(k)$ be the state vector and $w_i(k)$ denote the disturbance and measurement noise vector corresponding to P_i at time instant k. The dynamics of sub-system P_i is be given by the following state equations

$$
\begin{aligned}
x_i(k+1) &= A_{ii}x_i(k) + B_i w_i(k) + \sum_{j\in\mathcal{N}_i^-} B_{ij}^{\zeta}\zeta_{ij}(k), \\
y_i(k) &= C_{ii}x_i(k) + D_i w_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{\zeta}\zeta_{ij}(k), \qquad \forall\, i \in \{1,\dots,n\} \\
\eta_{ri}(k) &= C_{ri}^{\eta}x_i(k),\ \forall r \in \mathcal{N}_i^+
\end{aligned}
\tag{6.1}
$$

where $\eta_{ri}(k)$ denotes the message vector transmitted from sub-system P_i to sub-system P_r, while $\zeta_{ij}(k)$ denotes the vector received by sub-system P_i from sub-system P_j at time instant k. The zero-delay network interaction is written as

$$\zeta_{ij}(k) = \eta_{ij}(k) \quad \forall\, (v_j, v_i) \in \mathcal{E} \tag{6.2}$$

Combining the equations (6.1) and (6.2), we get state-space equations corresponding to the networked system P as follows

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i w_i(k) + \sum_{j \in \mathcal{N}_i^-} A_{ij}x_j(k), \\ y_i(k) &= C_{ii}x_i(k) + D_i w_i(k) + \sum_{j \in \mathcal{N}_i^-} C_{ij}\zeta_{ij}(k), \end{aligned} \tag{6.3}$$

where $A_{ij} = B_{ij}^{\zeta} C_{ij}^{\eta}$ and $C_{ij} = D_{ij}^{\zeta} C_{ij}^{\eta}$. The equations in (6.3) can be written in a simpler form as

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k) \end{aligned} \tag{6.4}$$

where $x(k) = \mathbf{vert}[x_i(k)]_i$, $y(k) = \mathbf{vert}[y_i(k)]_i$ and $w(k) = \mathbf{vert}[w_i(k)]_i$ are the state, measurement and disturbance vectors (partitioned according to $\mathcal{P}_x$, $\mathcal{P}_y$ and $\mathcal{P}_w$, respectively); while $A := [A_{ij}]_{i,j} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B := \mathbf{diag}[B_i]_i \in S(I, \mathcal{P}_x, \mathcal{P}_w)$, $C := [C_{ij}]_{i,j} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ and $D := \mathbf{diag}[D_i]_i \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_w)$.

Corresponding to this networked system P, we design a networked estimator E (as shown in Fig. 6.1) such that each sub-unit E_i estimates the states of the sub-system P_i by exchanging messages over the same causal network $\mathcal{G}$. In a norm-minimizing networked filtering problem, our objective is to minimize $\|x(k) - \hat{x}(k)\|_\alpha$ (for $\alpha = 2$ or ∞) where $\hat{x}(k) :\ = \mathbf{vert}[\hat{x}_i(k)]_i$ and $\hat{x}_i(k)$ denotes the estimated state vector corresponding to each sub-system E_i.

This problem can easily be converted into a networked control problem, discussed in previous chapters, by treating estimates as control inputs and writing an equivalent generalized plant's sub-systems G_i as follows

$$\begin{aligned} x_i(k+1) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} A_{ij}x_j(k) + B_i w_i(k), \\ z_i(k) &= x_i(k) - u_i(k), \qquad\qquad \forall\, i \\ y_i(k) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} C_{ij}x_j(k) + D_i w_i(k) \end{aligned} \tag{6.5}$$

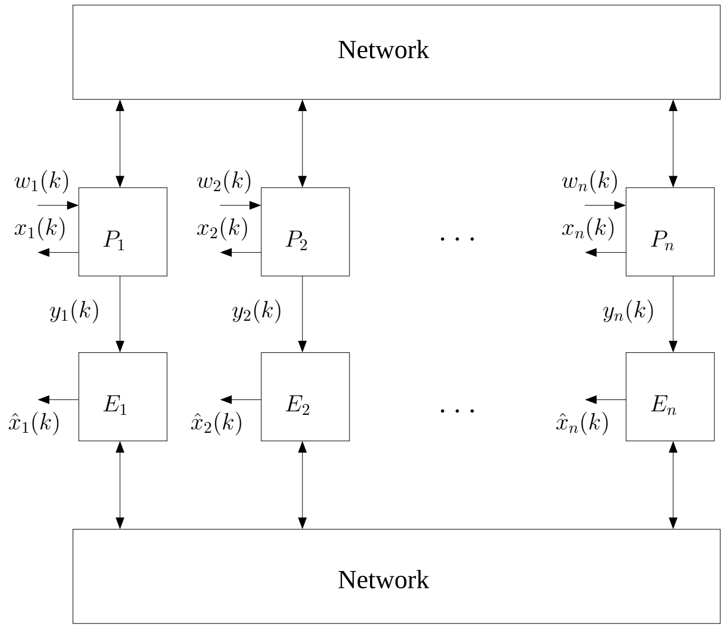


Figure 6.1 Networked plant P and a networked estimator E in terms of their sub-systems $\{P_i\}_i$ and $\{E_i\}_i$.

where $u_i(k) = \hat{x}_i(k)$ is the state estimate and $z_i(k)$ represents the estimation error corresponding to P_i at time instant k, for all i. Since the dimension of $x_i(k)$ and $\hat{x}_i(k)$ are the same, we know that $u(k) := \mathbf{vert}[u_i(k)]_i$ and $z(k) := \mathbf{vert}[z_i(k)]_i$ are partitioned according to $\mathcal{P}_x$. Pictorially, we can view the problem as Fig. 6.2 where G is the generalized plant, corresponding to the networked system P, with a state-space representation given by (based on the state-space matrices of P)

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ I & 0 & -I \\ C & D & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{6.6}$$

and E is stable networked estimator that is a strictly causal interaction over $\mathcal{G}$. Note that the generalized plant in (6.6) is similar to the networked plant in (4.4).

Our objective to design a networked estimator for a networked plant P can be interpreted as design of a stable and networked controller E for the generalized plant G that minimizes the closed-loop system norm $\|T_{zw}\|_\alpha = \|\mathbf{lft}(G,E)\|_\alpha$ for $\alpha = 2$ or ∞. Based on previous chapters (Lemma 4, Theorem 1 and Lemma 5), we notice that stable networked estimators which are strictly causal interactions over unit-weight digraph $\mathcal{G}$ can equivalently be treated as elements of $\mathfrak{S}^s(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y)$ or $\mathfrak{T}^s(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y)$. Thus the

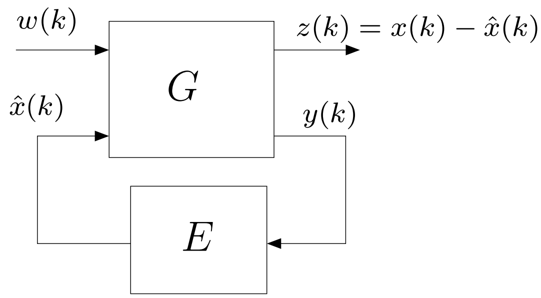


Figure 6.2 An equivalent model using a generalized plant G in a feedback interconnection with the networked estimator E.

networked filtering problem can be written as

$$\begin{aligned} &\min \quad \|T_{zw}\|_{\alpha} \\ &\text{subject to } E \in \mathfrak{S}^{s}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y) \text{ or } \mathfrak{T}^{s}(\mathcal{G},\mathcal{P}_x,\mathcal{P}_y), \\ &\qquad T_{zw} \text{ is BIBO stable.} \end{aligned} \tag{6.7}$$

For estimation, we only require the closed-loop transfer function T_{zw} to be BIBO stable and do not require internal stabilization of the given plant. Thus, the problem in (6.7) is much simpler than the corresponding networked controller design problem in (4.15). Since we only require BIBO stability of T_{zw}, one can also use the results of [8] to solve the problem in (6.7).

6.1.1 Parametrization of all stable networked estimators

Using the methodology given in the previous chapter, we parameterize the set of all possible stable networked estimators that are strictly causal interactions over $\mathcal{G}$ for a given networked plant P over $\mathcal{G}$, using the following theorem.

Theorem 5. *Given a unit-weight digraph $\mathcal{G}$ and a networked system P which is a strictly causal interaction over $\mathcal{G}$ with a state-space representation given by* (6.4)*. Given a matrix $L \in S(I,\mathcal{P}_x,\mathcal{P}_y)$ such that $A+LC$ is stable. Then the set of all stable networked estimators (that are strictly causal interactions over $\mathcal{G}$) that drive the estimates $\hat{x}(k)$ asymptotically to $x(k)$ is given by*

$$E = \textbf{lft}(J,Q)$$

where $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$ *and*

$$J: \quad \left[\begin{array}{c} x_J(k+1) \\ \hline \hat{x}(k) \\ \xi(k) \end{array}\right] \left[\begin{array}{c|cc} A+LC & -L & 0 \\ \hline I & 0 & I \\ -C & I & 0 \end{array}\right] \left[\begin{array}{c} x_J(k) \\ \hline y(k) \\ \psi(k) \end{array}\right]. \tag{6.8}$$

Note that the vectors $x_J(k) := \mathbf{vert}[x_i^J(k)]_i$, $\hat{x}(k) := \mathbf{vert}[\hat{x}_i(k)]_i$, $\xi(k) := \mathbf{vert}[\xi_i(k)]_i$ *and* $\psi(k) := \mathbf{vert}[\psi_i(k)]_i$ *are partitioned according to* $\mathcal{P}_x$, $\mathcal{P}_x$, $\mathcal{P}_y$ *and* $\mathcal{P}_x$, *respectively.*

Proof. We prove this Theorem as a special case of Theorem 2 .

Given the networked plant P in (6.4), define a generalized plant G in (6.6) such that $u_i(k) = \hat{x}_i(k)$ for all i.

First, assume that $Q = (A_Q, B_Q, C_Q, D_Q) \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$. So, A_Q is Schur-stable. Let $x_Q(k)$ denote the state vector of Q which is partitioned according to $\mathcal{P}_x^Q$. Then, $E = \mathbf{lft}(J, Q)$ where J is given by (6.8). Using (2.12), we get a state-space representation for E to be

$$\left[\begin{array}{c} x_J(k+1) \\ x_Q(k+1) \\ \hline u(k) \end{array}\right] \left[\begin{array}{cc|c} A+LC & 0 & -L \\ -B_Q C & A_Q & B_Q \\ \hline I - D_Q C & C_Q & D_Q \end{array}\right] \left[\begin{array}{c} x_J(k) \\ x_Q(k) \\ \hline y(k) \end{array}\right]. \tag{6.9}$$

Since $A+LC$ and A_Q are Schur-stable, we can see that E in (6.9) is asymptotically stable.

The state vector $x_J(k) = \mathbf{vert}[x_i^J(k)]_i$ is partitioned according to $\mathcal{P}_x$ and $x_Q(k) = \mathbf{vert}[x_i^Q(k)]_i$ is partitioned according to $\mathcal{P}_x^Q$. Let the state-vector for E be expressed as $x_E(k) := \mathbf{vert}[x_i^E(k)]_i$ where $x_i^E(k) = \left[\begin{smallmatrix} x_i^J(k) \\ x_i^Q(k) \end{smallmatrix}\right]$. Thus $x_E(k)$ is partitioned according to $\mathcal{P}_x + \mathcal{P}_x^Q$. Then the dynamics of E in (6.9) can equivalently be written as

$$\begin{aligned} x_i^E(k+1) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} A_{ij}^E x_j(k) + B_i^E y_i(k), \\ u_i(k) &= \sum_{j \in \mathcal{N}_i^- \cup \{i\}} C_{ij}^E x_j(k) + D_i^E y_i(k), \end{aligned} \qquad \forall\, i \tag{6.10}$$

where

$$\begin{aligned} A_{ij}^E &= \begin{bmatrix} A_{ij} + L_i C_{ij} & 0 \\ -B_i^Q C_{ij} & A_{ij}^Q \end{bmatrix}, \quad B_i^E = \begin{bmatrix} -L_i \\ B_i^Q \end{bmatrix}, \\ C_{ij}^E &= \begin{bmatrix} I - D_i^Q C_{ij} & C_{ij}^Q \end{bmatrix}, \quad D_i^E = D_i^Q. \end{aligned} \qquad \forall i, j \tag{6.11}$$

Based on the state-space dynamics in(6.10) and the fact that E is asymptotically stable, we can say that $E \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x + \mathcal{P}_x^Q, \mathcal{P}_u, \mathcal{P}_y)$. Lemma 5 shows that E can be viewed as a strictly causal interaction over $\mathcal{G}$.

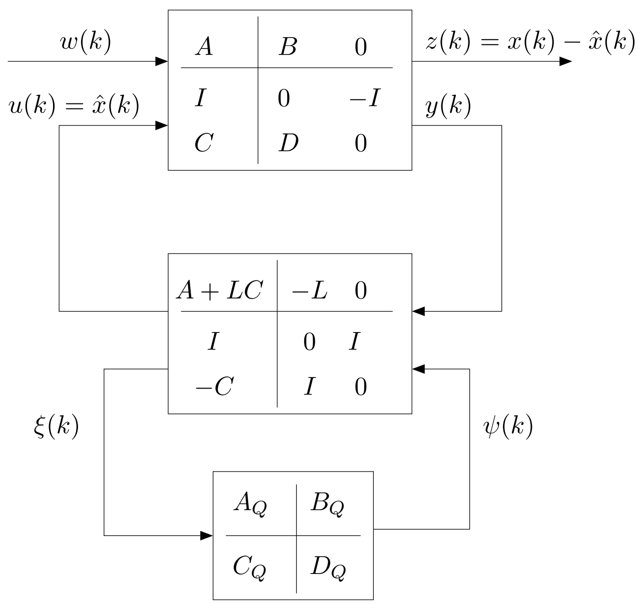


Figure 6.3 Representing an estimation problem as a feedback interconnection of generalized plant G and a parametrized estimator $E = \mathbf{lft}(J, Q)$.

To show that E given by (6.9) estimates the states of the networked system P, we also need to show that $\hat{x}(k) \to x(k)$ as $k \to \infty$ where $\hat{x}(k)$ denotes the estimated vector from E and $x(k)$ denotes the state vector of P at time instant k. By defining $\overline{x}(k) := x(k) - x_J(k)$ and following the equations (6.4) and (6.9), we get

$$\begin{aligned} \overline{x}(k+1) &= x(k+1) - x_J(k+1) \\ &= Ax(k) + Bw(k) - (A+LC)x_J(k) + Ly(k) \\ &= (A+LC)(x(k) - x_J(k)) + (B+LD)w(k) \\ &= (A+LC)\overline{x}(k) + (B+LD)w(k), \end{aligned} \tag{6.12}$$

$$\begin{aligned} x_Q(k+1) &= A_Q x_Q(k) + B_Q(-Cx_J(k) + y(k)) \\ &= A_Q x_Q(k) + B_Q C(x(k) - x_J(k)) + B_Q D w(k) \\ &= A_Q x_Q(k) + B_Q C\overline{x}(k) + B_Q D w(k), \end{aligned} \tag{6.13}$$

$$\begin{aligned} z(k) = x(k) - \hat{x}(k) &= x(k) - u(k) \\ &= x(k) - (I - D_Q C) x_J(k) - C_Q x_Q(k) - D_Q y(k) \\ &= (I - D_Q C)(x(k) - x_J(k)) - C_Q x_Q(k) - D_Q D w(k) \\ &= (I - D_Q C)\overline{x}(k) - C_Q x_Q(k) - D_Q D w(k) \end{aligned} \tag{6.14}$$

From (6.12), (6.13) and (6.14), the dynamics connecting the estimation error $z(k)$ and the input disturbance $w(k)$ can be written in terms of the states $\overline{x}(k)$ and $x_Q(k)$ as follows

$$\begin{aligned} \begin{bmatrix} \overline{x}(k+1) \\ x_Q(k+1) \end{bmatrix} &= \begin{bmatrix} A+LC & 0 \\ B_Q C & A_Q \end{bmatrix} \begin{bmatrix} \overline{x}(k) \\ x_Q(k) \end{bmatrix} x(k) + \begin{bmatrix} B+LD \\ B_Q D \end{bmatrix} w(k), \\ z(k) &= \begin{bmatrix} I - D_Q C & -C_Q \end{bmatrix} \begin{bmatrix} \overline{x}(k) \\ x_Q(k) \end{bmatrix} - D_Q D w(k). \end{aligned} \tag{6.15}$$

Since $A+LC$ and A_Q are Schur-stable, so is $\left[\begin{smallmatrix} A+LC & 0 \\ B_Q C & A_Q \end{smallmatrix}\right]$ because of its block-diagonal structure. Thus, the estimation error asymptotically goes to zero using $E = \mathbf{lft}(J,Q)$.

On the otherhand, from the theory of Youla parameterization, we know that given L such that $A + LC_y$ is Schur-stable, any stabilizing estimator for G given by (6.6) is represented by $E = \mathbf{lft}(J,Q)$ where J is given by (6.8) and Q is a stable, causal, FDLTI system. If we also assume that E is a stable strictly causal interaction over $\mathcal{G}$, then E has a state-space realization in $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$. Then, it is easy to see that E stabilizes $\hat{J}$ given by

$$\hat{J} : \left[\begin{array}{c} x_{\hat{J}}(k+1) \\ \hline \psi(k) \\ y(k) \end{array}\right] = \left[\begin{array}{c|cc} A & -L & 0 \\ \hline -I & 0 & I \\ C & I & 0 \end{array}\right] \left[\begin{array}{c} x_{\hat{J}}(k) \\ \hline \xi(k) \\ u(k) \end{array}\right] \tag{6.16}$$

where $x_{\hat{J}}(k)$ is partitioned according to $\mathcal{P}_x$. Following a similar procedure as before, we see that $Q = \mathbf{lft}(\hat{J}, E) \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$. □

6.1.2 Optimal networked estimator

Using the sufficiency condition given in Lemma 8, we know that if the feasibility problem in (4.14) has a solution, then there exists $L \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ such that $A + LC_y$ is Schur-stable. If such an L exists, the set of all closed-loop transfer matrices from $w(k)$ to $z(k)$ can be obtained using Theorem 5 and following (4.16), (4.17) as

$$\mathfrak{C}_{zw} = \{T_{11}(z) + T_{12}(z)Q(z)T_{21}(z) : \; Q(z) = \mathbf{tf}(Q), \; Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)\} \tag{6.17}$$

where

$$\begin{aligned} \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} &= \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A+LC & B+LD & 0 \\ \hline 0 & I & 0 & -I \\ 0 & C & D & 0 \end{array}\right] \\ &= \left[\begin{array}{c|cc} A+LC & B+LD & 0 \\ \hline I & 0 & -I \\ C & D & 0 \end{array}\right]. \end{aligned} \tag{6.18}$$

Note that (6.18) corresponds to

$$T_{11}(z) = \left[\begin{array}{c|c} A+LC & B+LD \\ \hline I & 0 \end{array}\right], \; T_{12}(z) = -I,$$

$$T_{21}(z) = \left[\begin{array}{c|c} A+LC & B+LD \\ \hline C & D \end{array}\right], \; T_{22}(z) = 0.$$

Since the closed-loop transfer matrix is simply an affine function of the Youla parameter Q, we can rewrite the networked estimation problem in (6.7) as a convex optimization problem

$$\begin{aligned} \min \quad & \|T_{11}(z) - Q(z)T_{21}(z)\|_\alpha \\ \text{subject to} \quad & Q(z) = \mathbf{tf}(Q), \qquad\qquad \text{for } \alpha = 2 \text{ or } \infty \\ & Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y), \end{aligned} \tag{6.19}$$

which is similar to the problem (4.19). Following the results of Lemma 4 and Theorem 1, we note that the condition $Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$ is equivalent to $\mathbf{tf}(Q) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$. Since it is convenient to solve the problem (6.19) in the frequency domain, we write (6.19) as

$$\begin{aligned} \min \quad & \|T_{11}(z) - Q(z)T_{21}(z)\|_\alpha \qquad \text{for } \alpha = 2 \text{ or } \infty \\ \text{subject to} \quad & Q(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y). \end{aligned} \tag{6.20}$$

Using the same vectorization ideas as in Chapter 4, we can pose the problem (6.20) (when $\alpha = 2$) as an unconstrained $\mathcal{H}_2$ problem

$$\begin{aligned} \min \quad & \left\| \mathbf{vec}(T_{11}(z)) - (T_{21}(z)' \otimes I)S(z)H(z) \right\|_2 \\ \text{subject to} \quad & H(z) \in \mathcal{RH}_\infty^{a\times 1} \end{aligned} \tag{6.21}$$

where a denotes the total number of elements of $Q(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$ that are not constrained to be zero and $S(z)$ is given by (4.22). The unconstrained convex optimization problem in (6.21) can be solved using standard techniques. Let $H^\star(z)$ denote the solution of the optimization problem (6.21). Then the corresponding optimal $Q^\star(z)$ is given by $Q^\star(z) = \mathbf{vec}^{-1}(S(z)H^\star(z))$. Since $Q^\star(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$, we can obtain a state-space realization $\tilde{Q} = (\tilde{A}_Q, \tilde{B}_Q, \tilde{C}_Q, \tilde{D}_Q) \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y)$, using Theorem 1, such that $Q^\star(z) = \mathbf{tf}(\tilde{Q})$ and $\tilde{A}_Q$ is Schur-stable. The corresponding estimator is given by $E^\star = \mathbf{lft}(J, \tilde{Q})$, where J is given by (6.8). Using Lemma 5, one can obtain a strictly causal interaction over given $\mathcal{G}$ with the same state-space representation as $E^\star$. From Theorem 5 and the problem formulation in (6.7), we can see that $E^\star$ thus designed is the optimal stable networked estimator that is a strictly causal interaction over $\mathcal{G}$ for the given networked plant P in (6.3).

CHAPTER 7. Networked systems over delay networks

In previous chapters, we studied networked systems that are strictly causal interactions over zero-delay networks. We saw that the state-space and input-output representations of a strictly causal interaction of sub-systems over a zero-delay network could be described using a unit-weight digraph $\mathcal{G}$ corresponding to the zero-delay network. Based on these connections with $\mathcal{G}$, we derived networked controllers and estimators for networked plants when the plants, controllers and estimators are all strictly causal interactions over the same digraph $\mathcal{G}$. Now, we look at possible extensions of the theory developed for zero-delay network case to a general delay network case.

Let the discrete-time networked system be represented by a weighted digraph $\mathcal{G}$ as described in Section 3.1.1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted digraph where $W((v_i, v_j)) \in \mathbb{N}$ denotes the weight of the edge $(v_i, v_j) \in \mathcal{E}$. Equation (3.4) shows that the network delay t_{ij} (on the communication link from sub-system P_j to P_i) and $W((v_j, v_i))$ are related by

$$t_{ij} = W((v_j, v_i)) - 1 \qquad \forall\, (v_j, v_i) \in \mathcal{E}. \tag{7.1}$$

We also defined W_{ij} as the weight of a minimum-weight path from vertex v_j to vertex v_i. If π is a directed path, we denote the weight of π by $W(\pi)$, which is the sum of weights of all the edges in the path. Thus, we can write

$$W_{ij} = \inf\{W(\pi) : \pi \text{ is a directed path from vertex } v_j \text{ to } v_i\}. \tag{7.2}$$

Note that $\pi = v_i$ is treated as a directed path from vertex v_i to v_i and weight of such a path is equal to zero. So, $W_{ii} = 0$ for all i since there are no edges in the path $\pi = v_i$. If there is no directed path from vertex v_j to v_i, $j \neq i$, then $W_{ij} = \infty$ since infimum (in (7.2)) of an empty set is treated as ∞.

Given a weighted digraph $\mathcal{G}$, let a networked system P be described in terms of it's sub-system dynamics given by (3.5) and the network interaction given by (3.6). By combining (3.5), (3.6) and

(7.1), we can eliminate the network variables $\zeta_{ij}(k)$ and $\eta_{ir}(k)$, and write the dynamics of the networked system P as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k-t_{ij}) \\ y_i(k) &= C_{ii}^y x_i(k) + D_{ii}^{yu}u_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^y x_j(k-t_{ij}), \end{aligned} \qquad \forall\, i\in\{1,\dots,n\} \tag{7.3}$$

where $A_{ij} := B_{ij}^{\zeta}C_{ji}^{\eta}$ and $C_{ij}^y := D_{ij}^{y\zeta}C_{ji}^{\eta}$.

Lemma 13. *Given a weighted digraph $\mathcal{G}$, a networked system P that is a strictly causal interaction over $\mathcal{G}$ with dynamics given by (7.3) is asymptotically stable if and only if $(zI - A(z))$ has full rank for any $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$ where*

$$[A(z)]_{ij} := \begin{cases} A_{ii} & \text{if } i = j, \\ z^{-t_{ij}}A_{ij} & \text{if } (v_j, v_i)\in\mathcal{E}, \\ 0 & \text{otherwise} \end{cases} \tag{7.4}$$

where t_{ij} is given by (7.1).

Proof. The dynamics of P are given by (7.3). In order to check the stability of the system, we can assume the inputs to be zero and disregard the outputs and just consider the autonomous part of P given by

$$x_i(k+1) = A_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k-t_{ij}) \quad \forall i \tag{7.5}$$

Let us define $x_{ij}^{(0)}(k) = x_j(k)$ for all $i\in\{1,\dots,n\}$ and $j\in\mathcal{N}_i^-$. Corresponding to the non-zero delays in the communication links (given by (7.1)), define the following *network states* $\{x_{ij}^{(r)}(k)\}_{i,j,r}$ for all $i\in\{1,\dots,n\}$, $j\in\mathcal{N}_i^-$ and $r\in\{1,\dots,t_{ij}\}$ (when $t_{ij}\neq 0$)

$$x_{ij}^{(r)}(k) := x_{ij}^{(r-1)}(k-1) \tag{7.6}$$

Thus, the dynamics in (7.5) can be written as

$$x_i(k+1) = A_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_{ij}^{(t_{ij})}(k) \quad \forall i. \tag{7.7}$$

By defining a state-vector $\overline{x}(k)$ of the form

$$\overline{x}(k) = \begin{bmatrix} \mathbf{vert}[x_i(k)]_{i\in\{1,\dots,n\}} \\ \mathbf{vert}[x_{ij}^{(r)}(k)]_{i\in\{1,\dots,n\},j\in\mathcal{N}_i^-,r\in\{1,\dots,t_{ij}\}} \end{bmatrix}, \tag{7.8}$$

we can write the state equations corresponding to (7.5) as

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) \tag{7.9}$$

which is a collection of the equations (7.6) and (7.7). From the formulation of (7.9), we can see that the given networked system P is asymptotically stable iff $\bar{A}$ is Schur-stable, i.e. $(zI - \bar{A})$ has full rank for all $z \in \mathbb{C}\backslash\bar{\mathbb{D}}$.

We now show that for any $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$, $(\lambda I - \bar{A})$ does not have full rank iff $(\lambda I - A(\lambda))$ does not have full rank, which will prove the hypothesis.

($\Rightarrow$) Assume that $(\lambda I - \bar{A})$ does not have full rank for some $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$. Then there exists a vector $\bar{V}$ of the form

$$\bar{V} = \begin{bmatrix} \mathbf{vert}[V_i]_{i\in\{1,\ldots,n\}} \\ \mathbf{vert}[V_{ij}^{(r)}]_{i\in\{1,\ldots,n\},j\in\mathcal{N}_i^-,r\in\{1,\ldots,t_{ij}\}} \end{bmatrix}, \tag{7.10}$$

for some $\{V_i\}_{i\in\{1,\ldots,n\}}$ (dimension of the vector V_i is $\mathcal{P}_x(i)\times 1$) and $\{V_{ij}^{(r)}\}_{i\in\{1,\ldots,n\},j\in\mathcal{N}_i^-,r\in\{1,\ldots,t_{ij}\}}$ such that $(\lambda I - \bar{A})\bar{V} = 0$ or

$$\begin{aligned} \lambda V_i &= A_{ii}V_i + \sum_{j\in\mathcal{N}_i^-} A_{ij}V_{ij}^{(t_{ij})} \quad \forall\, i \in \{1,\ldots,n\} \\ \lambda V_{ij}^{(r)} &= V_{ij}^{(r-1)} \quad \forall\, i \in \{1,\ldots,n\}, j \in \mathcal{N}_i^-, r \in \{1,\ldots,t_{ij}\} \end{aligned} \tag{7.11}$$

where $V_{ij}^{(0)} = V_j$. From (7.11), we note that

$$\begin{aligned} \lambda V_i &= A_{ii}V_i + \sum_{j\in\mathcal{N}_i^-} \lambda^{-t_{ij}} A_{ij}V_j \quad \forall\, i \in \{1,\ldots,n\} \\ \Rightarrow \quad & (\lambda I - A(\lambda))V = 0 \end{aligned} \tag{7.12}$$

where $V = \mathbf{vert}[V_i]_i$ (partitioned according to $\mathcal{P}_x$) and $A(\lambda)$ is given by (7.4). Thus, (7.12) shows that $(\lambda I - A(\lambda))$ does not have full rank if $(\lambda I - \bar{A})$ does not have full rank.

($\Leftarrow$) Assume that $(\lambda I - A(\lambda))$ does not have full rank for some $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$. Then there exists a vector $V = \mathbf{vert}[V_i]_i$, partitioned according to $\mathcal{P}_x$, such that $(\lambda I - A(\lambda))V = 0$.

By defining $\{V_{ij}^{(r)}\}_{i\in\{1,\ldots,n\},j\in\mathcal{N}_i^-,r\in\{1,\ldots,t_{ij}\}}$ such that

$$V_{ij}^{(r)} = \lambda^{-1}V_{ij}^{(r-1)} \quad \forall\, i \in \{1,\ldots,n\}, j \in \mathcal{N}_i^-, r \in \{1,\ldots,t_{ij}\} \tag{7.13}$$

where $V_{ij}^{(0)} = V_j$ for all $i, j \in \mathcal{N}_i^-$. Following the same procedure as before, it is easy to show that $(\lambda I - \bar{A})\bar{V} = 0$ for $\bar{V}$ formed from (7.10). Thus $(\lambda I - \bar{A})$ does not have full rank if $(\lambda I - A(\lambda))$ does not have full rank. □

In order to describe networked systems over delay networks in (7.3) in a simpler fashion, we introduce a delay shift operator denoted by $\mathbf{q}$ such that $x(k-1) = \mathbf{q}x(k)$ where $x(k)$ is any discrete-time signal. The delay shift operator was also used in [9] to describe systems over delay networks. From the definition of the shift operator, it is easy to see that the transfer function corresponding to the operator is z^{-1}. Based on the shift operator, we call a matrix $J(\mathbf{q})$ *sparsity and delay pattern matrix* if it's entries $[J(\mathbf{q})]_{ij}$ are either 0 or $\mathbf{q}^r$ for some $r \in \mathbb{N}_0$. Note that, such a sparsity and delay pattern matrix can be used to describe not just the sparsity pattern in state-space or transfer function matrices but also the delay terms.

We say that a matrix $A(\mathbf{q})$ is *structured according to a sparsity and delay structure* $J(\mathbf{q})$ if $[A(\mathbf{q})]_{ij} = [J(\mathbf{q})]_{ij} A_{ij}(\mathbf{q})$ (for all $i,\ j$) where $\{A_{ij}(\mathbf{q})\}_{i,j}$ are all matrices of appropriate dimensions containing polynomials of $\mathbf{q}$.

Definition 8. *Given a sparsity and delay pattern matrix $J(\mathbf{q})$ and $n-$tuples $\mathcal{P}_a$, $\mathcal{P}_b$, let $S(J(\mathbf{q}), \mathcal{P}_a, \mathcal{P}_b)$ denote the set of matrices that are partitioned according to $(\mathcal{P}_a, \mathcal{P}_b)$ and structured according to $J(\mathbf{q})$.*

Given a weighted digraph $\mathcal{G}$ with n vertices, using the delay shift operator $\mathbf{q}$, we shall define sparsity and delay structures on $\mathcal{G}$ by extending the definition of $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_m(\mathcal{G})$ to $\mathcal{A}(\mathcal{G}, \mathbf{q})$ and $\mathcal{A}_m(\mathcal{G}, \mathbf{q})$ (of dimension $n \times n$) for $m \in \mathbb{N}_0$ given by

$$[\mathcal{A}(\mathcal{G}, \mathbf{q})]_{ij} := \begin{cases} 1 & \text{if } i = j, \\ \mathbf{q}^{t_{ij}} & \text{if } (v_j, v_i) \in \mathcal{E}, \\ 0 & \text{otherwise} \end{cases} \tag{7.14}$$

$$[\mathcal{A}_m(\mathcal{G}, \mathbf{q})]_{ij} := \begin{cases} 1 & \text{if } i = j, \\ \mathbf{q}^{W(\pi_{ij}) - l(\pi_{ij})} & \text{if } \pi_{ij} \text{ is a directed path from vertex } v_j \text{ to } v_i \\ & \text{of length at most } m \text{ and with smallest weight} \\ 0 & \text{otherwise.} \end{cases} \tag{7.15}$$

where t_{ij} is given by (7.1) and $l(\pi_{ij})$ denotes the length of path π_{ij}.

Based on the sparsity and delay pattern matrices in (7.14) and (7.15), we can extend Lemma 3 in the following way.

Lemma 14. *Given an $n-$tuple $\mathcal{P}_a$ and a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (with n vertices) with the sparsity and delay structures $\mathcal{A}(\mathcal{G}, \mathbf{q})$ and $\mathcal{A}_m(\mathcal{G}, \mathbf{q})$ (for all $m \in \mathbb{N}_0$) given by (7.14) and (7.15), let $\{A_i(\mathbf{q})\}_i$ be a sequence of matrices such that $A_i(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G}, \mathbf{q}), \mathcal{P}_a, \mathcal{P}_a)$ for all i. Then $B_m(\mathbf{q}) = \prod_{k=1}^{m} A_k(\mathbf{q}) \in S(\mathcal{A}_m(\mathcal{G}, \mathbf{q}), \mathcal{P}_a, \mathcal{P}_a)$ for all m.*

Proof. From the definition of $\mathcal{A}_m(\mathcal{G}, \mathbf{q})$ in (7.15), we can see that $\mathcal{A}_1(\mathcal{G}, \mathbf{q}) = \mathcal{A}(\mathcal{G}, \mathbf{q})$. Thus, from hypothesis, we know that $B_1(\mathbf{q}) = A_1(\mathbf{q}) \in S(\mathcal{A}_1(\mathcal{G}, \mathbf{q}), \mathcal{P}_a, \mathcal{P}_a)$.

Now, assume that $B_m(\mathbf{q}) = \prod_{k=1}^{m} A_k(\mathbf{q}) \in S(\mathcal{A}_m(\mathcal{G}, \mathbf{q}), \mathcal{P}_a, \mathcal{P}_a)$ for some $m = p$. From Remark 1, we can see that $B_{p+1}(\mathbf{q}) = B_P(\mathbf{q})A_{p+1}(\mathbf{q})$ is partitioned according to $(\mathcal{P}_a, \mathcal{P}_a)$ and the sub-matrices $[B_{p+1}(\mathbf{q})]_{ij} = \sum_{k=1}^{n} [B_p(\mathbf{q})]_{ik} [A_{p+1}(\mathbf{q})]_{kj}$. We see that

$$[A_{p+1}(\mathbf{q})]_{kj} = \begin{cases} H_{kk}(\mathbf{q}) & \text{if } k = j, \\ \mathbf{q}^{W((v_j, v_k))-1} H_{kj}(\mathbf{q}) & \text{if } (v_j, v_k) \in \mathcal{E}, \\ 0 & \text{otherwise} \end{cases} \tag{7.16}$$

$$[B_p(\mathbf{q})]_{ik} := \begin{cases} R_{ii}(\mathbf{q}) & \text{if } i = k, \\ \mathbf{q}^{W(\pi_{ik})-l(\pi_{ik})} R_{ik}(\mathbf{q}) & \text{if } \pi_{ik} \text{ is a directed path from vertex } v_k \text{ to } v_i \\ & \text{of length at most } p \text{ and with smallest weight} \\ 0 & \text{otherwise} \end{cases} \tag{7.17}$$

where $\{H_{kj}(\mathbf{q})\}_{kj}$ and $\{R_{ik}(\mathbf{q})\}_{ik}$ are matrices with elements as polynomials in $\mathbf{q}$, for all i, j and k.

If there is no path from vertex v_j to vertex v_i of length at most $p+1$, then for all $v_k \in \mathcal{V}$, either there is no path from v_k to v_i of length at most p or there is no directed edge from v_j to v_k. Thus, either $[B_p(\mathbf{q})]_{ik}$ or $[A_{p+1}(\mathbf{q})]_{kj}$ are zero-matrices for all k when $[\mathcal{A}_{p+1}(\mathcal{G}, \mathbf{q})]_{ij} = 0$. Thus, $[B_{p+1}(\mathbf{q})]_{ij}$ is a zero matrix when $[\mathcal{A}_{p+1}(\mathcal{G}, \mathbf{q})]_{ij} = 0$.

Looking at all the paths from vertex v_j to v_i, we can also note that

$$[B_{p+1}(\mathbf{q})]_{ij} := \begin{cases} T_{ii}(\mathbf{q}) & \text{if } i = j, \\ \mathbf{q}^{W(\pi_{ij})-l(\pi_{ij})}\, T_{ij}(\mathbf{q}) & \text{if } \pi_{ij} \text{ is a directed path from vertex } v_j \text{ to } v_i \\ & \text{of length at most } p+1 \text{ and with smallest weight} \\ 0 & \text{otherwise} \end{cases} \tag{7.18}$$

for some $T_{ij}(\mathbf{q})$, which implies that $B_{p+1} \in S(\mathcal{A}_{p+1}(\mathcal{G},\mathbf{q}),\mathcal{P}_a,\mathcal{P}_a)$.

Thus, the given statement is true by mathematical induction. □

Using the delay shift operator $\mathbf{q}$ and Definition 8, we can write the dynamics of the networked system in (7.3) using a concise form

$$P : \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\mathbf{q}) & B_u \\ C_y(\mathbf{q}) & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{7.19}$$

where

$$[A(\mathbf{q})]_{ij} = \begin{cases} A_{ii} & \text{if } i = j \\ \mathbf{q}^{t_{ij}} A_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}, \qquad [C_y(\mathbf{q})]_{ij} = \begin{cases} C^y_{ii} & \text{if } i = j \\ \mathbf{q}^{t_{ij}} C^y_{ij} & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}, \tag{7.20}$$

$$B_u = \mathbf{diag}[B^u_i]_i \quad , \qquad D_{yu} = \mathbf{diag}[D^{yu}_i]_i$$

and t_{ij} is given by (7.1). Thus $A(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_x,\mathcal{P}_x)$, $B_u \in S(I,\mathcal{P}_x,\mathcal{P}_u)$, $C_y(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_y,\mathcal{P}_x)$ and $D_{yu} \in S(I,\mathcal{P}_y,\mathcal{P}_u)$.

Using the notation introduced in this section to describe networked systems over delay networks, we can extend almost all the definitions and results for strictly causal interactions over zero-delay networks to strictly causal interactions over delay networks.

7.1 Structured systems

In Chapter 3, we saw that networked systems that are strictly causal interactions over a zero-delay network can be described using structured systems over a unit-weight digraph. In this section, we extend the results and show that networked systems that are strictly causal interactions over a delay network can be described using structured systems over a weighted digraph.

Definition 9. *Given a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n vertices and the n−tuples $\mathcal{P}_u$ and $\mathcal{P}_y$; let $\mathcal{A}_{n-1}(\mathcal{G})$ be the unique binary matrix given by (2.2) and W_{ij} be defined for all i, j according to (7.2). We define $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as the set of transfer function matrices $P(z) \in S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$ such that the transfer function sub-matrices $P_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i)\times\mathcal{P}_u(j)}$ (where $P(z) = [P_{ij}(z)]_{i,j}$) are such that*

$$\begin{aligned} \mathbf{delay}(P_{ij}(z)) \geq W_{ij} \quad &\text{if } W_{ij} < \infty \\ P_{ij}(z) = 0 \quad &\text{if } W_{ij} = \infty \end{aligned} \tag{7.21}$$

for all i, j.

Lemma 15. *Given a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and n−tuples $\mathcal{P}_x$, $\mathcal{P}_u$, $\mathcal{P}_y$, $\mathcal{P}_\eta$ and $\mathcal{P}_\zeta$, let P be a networked system with sub-system dynamics given by (3.5) interacting over network interconnection (3.6) where $x(k) = \mathbf{vert}[x_i(k)]_i$, $u(k) = \mathbf{vert}[u_i(k)]_i$, $y(k) = \mathbf{vert}[y_i(k)]_i$, $\eta(k) = \mathbf{vert}[\eta_{ri}(k)]_{i,r\in\mathcal{N}_i^+}$ and $\zeta(k) = \mathbf{vert}[\zeta_{ij}(k)]_{i,j\in\mathcal{N}_i^-}$ are partitioned according to $\mathcal{P}_x$, $\mathcal{P}_u$, $\mathcal{P}_y$, $\mathcal{P}_\eta$ and $\mathcal{P}_\zeta$, respectively. Then $\mathbf{tf}\,P \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$.*

Proof. Based on the hypothesis, we can see that the dynamics of P can be written using (7.19) where $A(\mathbf{q})$, B_u, $C_y(\mathbf{q})$ and D_{yu} are given by (7.20). Note that $A(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G}, \mathbf{q}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$, $C_y(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G}, \mathbf{q}), \mathcal{P}_y, \mathcal{P}_x)$ and $D_{yu} \in S(I, \mathcal{P}_y, \mathcal{P}_u)$.

Let $P(z)$ be the transfer function of P. Using the fact that the transfer function of delay shift operator $\mathbf{q}$ is z^{-1} and from (2.9), we get

$$P(z) = D_{yu} + \sum_{k=0}^{\infty} C_y(z^{-1})(A(z^{-1}))^k B_u z^{-k-1}. \tag{7.22}$$

Define $R_0(z) := D_{yu}$ and $R_{k+1}(z) := C_y(z^{-1})(A(z^{-1}))^k B_u$ for all $k \in \mathbb{N}_0$. From Lemmas 2 and 14, and (7.20), we see that

$$\begin{aligned} & (A(z^{-1}))^k \in S(\mathcal{A}_k(\mathcal{G}, z^{-1}), \mathcal{P}_x, \mathcal{P}_x) \\ \Rightarrow \quad & C_y(z^{-1})(A(z^{-1}))^k \in S(\mathcal{A}_{k+1}(\mathcal{G}, z^{-1}), \mathcal{P}_y, \mathcal{P}_x) \\ \Rightarrow \quad & C_y(z^{-1})A^k B_u \in S(\mathcal{A}_{k+1}(\mathcal{G}, z^{-1}), \mathcal{P}_y, \mathcal{P}_u) \\ \Rightarrow \quad & R_k(z) \in S(\mathcal{A}_k(\mathcal{G}, z^{-1}), \mathcal{P}_y, \mathcal{P}_u) \quad \forall\, k \in \mathbb{N}_0. \end{aligned}$$

Note that $\mathcal{A}_0(\mathcal{G}, z^{-1}) = I$. From (7.22) and definitions of $\{R_k(z)\}_k$, we can write

$$P(z) = \sum_{k=0}^{\infty} R_k(z) z^{-k}. \tag{7.23}$$

Following the proof of Lemma 4, it is easy to see that $P(z) \in S(\mathcal{A}_{n-1}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_u)$. Since $P(z)$ is partitioned according to $(\mathcal{P}_y, \mathcal{P}_u)$ we can write $P(z) = [P_{ij}(z)]_{i,j}$, where $P_{ij}(z)$ is the transfer function sub-matrix mapping input vector $u_j(k)$ to output vector $y_i(k)$. From (7.23), we get

$$P_{ij}(z) = \sum_{k=0}^{\infty} [R_k(z)]_{ij} z^{-k}. \tag{7.24}$$

where $[R_k(z)]_{ij}$ is the sub-matrix of $R_k(z)$, for all k. From (7.24), (7.15) and (7.1); the delay of $P_{ij}(z)$ is given by

$$\begin{aligned}
\mathbf{delay}(P_{ij}(z)) &= \inf\{m \in \mathbb{N}_0 : \lim_{z\to\infty} z^m P_{ij}(z) \neq 0\} \\
&= \inf\{m \in \mathbb{N}_0 : \lim_{z\to\infty} z^m \sum_{k=0}^{\infty} [R_k(z)]_{ij} z^{-k} \neq 0\} \\
&\geq \inf\{m \in \mathbb{N}_0 : \lim_{z\to\infty} z^m \sum_{k=0}^{\infty} [\mathcal{A}_k(\mathcal{G}, z^{-1})]_{ij} z^{-k} \neq 0\} \\
&= \inf\{m \in \mathbb{N}_0 : \lim_{z\to\infty} z^m \sum_{k=l(\pi)}^{\infty} z^{(l(\pi)-W(\pi))} z^{-k} \neq 0,\ \pi \text{ is a path from } v_j \text{ to } v_i\} \\
&= \inf\{W(\pi) : \pi \text{ is a path from } v_j \text{ to } v_i\} = W_{ij}
\end{aligned} \tag{7.25}$$

which implies that $P(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. □

Theorem 6. *Given a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $n-$tuples $\mathcal{P}_u$ and $\mathcal{P}_y$.*

1. *Let $P(z)$ be a transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ with input vector $u(k)$ and output vector $y(k)$ partitioned according to $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively. Then there exists a networked system $\tilde{P}$ with sub-system dynamics given by* (3.5) *interacting over a network interconnection* (3.6) *such that* $\mathbf{tf}(\tilde{P}) = P(z)$.

2. *If $P(z)$ is also BIBO stable, then there exists a stable networked system $\tilde{P}$ which is a strictly causal interaction over $\mathcal{G}$ such that* $\mathbf{tf}(\tilde{P}) = P(z)$.

Proof. The proof of this Theorem is very similar to that of the proof of Theorem 1.

A weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and transfer function matrix $P(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ are given. So, $P(z)$ is partitioned according to $(\mathcal{P}_y, \mathcal{P}_u)$ and is of the form $P(z) = [P_{ij}(z)]_{i,j}$. Note that $P_{ij}(z)$ is essentially the transfer function matrix mapping $u_j(k)$ to $y_i(k)$, where input $u(k) = \mathbf{vert}[u_r(k)]_r$ and $y(k) = \mathbf{vert}[y_r(k)]_r$ are partitioned according to $\mathcal{P}_u$ and $\mathcal{P}_y$, respectively.

From (7.21), we see that $P_{ij}(z) = 0$ if there is no directed path from v_j to v_i over the digraph $\mathcal{G}$ and $\mathbf{delay}(P_{ij}(z)) \geq W_{ij}$ (where W_{ij} is the weight of minimum weight path from v_j to v_i given by (7.2)), otherwise. The condition that $P_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i)\times\mathcal{P}_u(j)}$ and $\mathbf{delay}(P_{ij}(z)) \geq W_{ij}$ can equivalently be written as $P_{ij}(z) = z^{-W_{ij}} H_{ij}(z)$ (with possible pole-zero cancellations at origin) where $H_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i)\times\mathcal{P}_u(j)}$. Thus (7.21) can be written as

$$P_{ij}(z) = \begin{cases} z^{-W_{ij}} H_{ij}(z) & \text{if } W_{ij} < \infty \\ 0 & \text{otherwise} \end{cases} \tag{7.26}$$

where $H_{ij}(z) \in \mathcal{R}_p^{\mathcal{P}_y(i)\times\mathcal{P}_u(j)}$ for all i, j.

When $i \neq j$, let a minimum-weight path from vertex v_j to vertex v_i be given by

$$\pi_{ij} = \pi_{ij}(0)\pi_{ij}(1)\ldots\pi_{ij}(m_{ij}),$$

where $\pi_{ij}(0) = v_j$ and $\pi_{ij}(m_{ij}) = v_i$, i.e. m_{ij} is the length of the minimum-weight path. Note that a minimum-weight path need not have the shortest-length, i.e. $m_{ij} \geq l_{ij}$. We refer to $\pi_{ij}(p)$, for $p \in \{1, \ldots, m_{ij} - 1\}$, as intermediate vertices. Let $W_{ij}(p)$ denote the weight of the directed edge $(\pi_{ij}(p), \pi_{ij}(p+1))$ for $p \in \{0, \ldots, m_{ij} - 1\}$. Thus, $W_{ij} = \sum_{p=0}^{m_{ij}-1} W_{ij}(p)$. We also denote the delay corresponding to the network link from $\pi_{ij}(p)$ to $\pi_{ij}(p+1)$ by $t_{ij}(p)$, for $p \in \{0, \ldots, m_{ij} - 1\}$. By (7.1), we get that $t_{ij}(p) = W_{ij}(p) - 1$ for all p.

Consider minimal realizations of $P_{ij}(z)$ in the following cases and define local states corresponding to a vertex as shown below.

- When $i = j$, define local states $x_{ii}(k)$ at vertex v_i such that

$$P_{ii}(z): \quad \begin{aligned} x_{ii}(k+1) &= A_{ii} x_{ii}(k) + B_{ii} u_i(k) \\ y_{ii}(k) &= C_{ii} x_{ii}(k) + D_{ii} u_i(k) \end{aligned} \tag{7.27}$$

- When $m_{ij} = 1$, define states $x_{ij}(k)$ at vertex v_j

$$z^{-1} H_{ij}(z): \quad \begin{aligned} x_{ij}(k+1) &= A_{ij} x_{ij}(k) + B_{ij} u_j(k) \\ y_{ij}(k) &= C_{ij} x_{ij}(k) \end{aligned} \tag{7.28}$$

- When $m_{ij} \geq 2$, we define states at each vertex on the path π_{ij} as follows

$$z^{-1}H_{ij}(z): \quad \begin{aligned} x_{ij}^{(0)}(k+1) &= A_{ij}x_{ij}^{(0)}(k) + B_{ij}u_j(k) \\ y_{ij}^{(0)}(k) &= C_{ij}x_{ij}^{(0)}(k) \end{aligned} \tag{7.29}$$

  Note that states $x_{ij}^{(0)}(k)$ are defined at vertex v_j and the outputs $y_{ij}^{(0)}(k)$ are passed to vertex $\pi_{ij}(1)$, i.e. the first vertex in the selected path from v_j to v_i. At vertices $\pi_{ij}(p)$, for $p \in \{1,\ldots,m_{ij}-1\}$, we define states $x_{ij}^{(p)}(k)$ corresponding to unit delay systems

$$z^{-1}: \quad \begin{aligned} x_{ij}^{(p)}(k+1) &= y_{ij}^{(p-1)}(k - t_{ij}(p-1)) \\ y_{ij}^{(p)}(k) &= x_{ij}^{(p)}(k). \end{aligned} \tag{7.30}$$

Note that the message received by node $\pi_{ij}(p)$ in the communication path from node v_j to v_i is $y_{ij}^{(p-1)}(k - t_{ij}(p-1))$. This is due to the delay over the communication link from $\pi_{ij}(p-1)$ to $\pi_{ij}(p)$, for all $p \in \{1,\ldots,m_{ij}\}$.

We denote the state vector corresponding to each vertex v_i to be $\tilde{x}_i(k)$, which is formed by appending the states $x_{ii}(k)$, $x_{ji}(k)\ \forall j \in \mathcal{N}_i^+$ and $x_{ab}^{(p)}(k)$ whenever $\pi_{ab}(p) = v_i$ (for $p \in \{0,\ldots,m_{ab}-1\}$), i.e. when vertex v_i is a vertex on the minimum-weight path from some vertex v_b to some other vertex v_a. A network output vector $\tilde{\eta}_{ri}(k)$, for all $r \in \mathcal{N}_i^+$, is formed by appending $y_{ri}(k)$ and $y_{ab}^{(p)}(k)$ whenever $\pi_{ab}(p) = v_i$ and $\pi_{ab}(p+1) = v_r$ (for $p \in \{0,\ldots,m_{ab}-1\}$). Similarly, a network input vector $\tilde{\zeta}_{ij}(k)$, for all $j \in \mathcal{N}_i^-$, is formed by appending $y_{ij}(k-t_{ij})$ and $y_{ab}^{(p)}(k-t_{ij})$ whenever $\pi_{ab}(p) = v_j$ and $\pi_{ab}(p+1) = v_i$ (for $p \in \{0,\ldots,m_{ab}-1\}$). Note that network inputs $\{\tilde{\zeta}_{ij}(k)\}_{i,j}$ and network outputs $\{\tilde{\eta}_{ri}(k)\}_{r,i}$ satisfy the network interconnection equations

$$\tilde{\zeta}_{ij}(k) = \tilde{\eta}_{ij}(k - t_{ij}) \quad \forall j \in \mathcal{N}_i^-. \tag{7.31}$$

At vertex v_i, the output $y_i(k)$ is given by

$$y_i(k) = y_{ii}(k) + \sum_{j:\ m_{ij}=1} y_{ij}(k - t_{ij}) + \sum_{j:\ m_{ij}\geq 2} y_{ij}^{(m_{ij}-1)}(k - t_{ij}(m_{ij}-1)) \tag{7.32}$$

Thus, we can define n sub-systems, $\{\tilde{P}_i\}_i$, each with local states $\tilde{x}_i(k)$, local inputs $u_i(k)$, local outputs $y_i(k)$, network inputs $\tilde{\zeta}_{ij}(k)$ (for all $j \in \mathcal{N}_i^-$) and network outputs $\tilde{\eta}_{ir}(k)$ (for all $r \in \mathcal{N}_i^+$). Following the state-space equations (7.27), (7.28), (7.29), (7.30), (7.32) concerning these states, inputs and outputs at each node, we can see that $\tilde{x}_i(k+1)$ and $y_i(k)$ are linear functions of $\tilde{x}_i(k)$, $u_i(k)$ and $\{\tilde{\zeta}_{ij}(k)\}_{j\in\mathcal{N}_i^-}$; while $\tilde{\eta}_{ri}(k)$ is only a function of $\tilde{x}_i(k)$ (for all $r\in\mathcal{N}_i^+$). Thus, the n sub-systems $\{\tilde{P}_i\}_i$ satisfy the structure given in (3.5) while the network inputs and network outputs satisfy (7.31). Thus the transfer function matrix $P(z)$ is expressed as a networked system $\tilde{P}$ which is a strictly causal interaction of sub-systems $\{\tilde{P}_i\}_i$ over a delay network represented by the given weighted digraph $\mathcal{G}$.

In the second case when $P(z)$ is also a BIBO stable transfer function, we show that the construction procedure used in the previous part of the proof also assures asymptotic stability of $\tilde{P}$.

In order to check asymptotic stability of $\tilde{P}$, we consider the zero-input autonomous system by assuming $u_i(k)=0\ \forall i,k$. First, we shall separate the states defined in (3.18), (3.19), (3.20) and (3.21)into two categories. The first category consists of the states corresponding to the transfer function matrices $P_{ij}(z)$ ($\forall i,j$ such that $m_{ij}\le 1$) that were defined in (3.18) and (3.19). This set of states can be written as $X_1(k)=\mathbf{vert}[x_{ij}(k)]_{i,j:\ m_{ij}\le 1}$. From the state-space equations corresponding to these states, we get

$$X_1(k+1)=\mathbf{diag}[A_{ij}]_{i,j:\ m_{ij}\le 1}X_1(k) \tag{7.33}$$

when $u_i(k)=0$ for all i,k.

The second category consists of the states corresponding to all the $P_{ij}(z)$ when $m_{ij}\ge 2$. For example, assume that a shortest path π_{ij} from vertex v_j to vertex v_i has length greater than 1. Then

$$\pi_{ij}=\pi_{ij}(0)\ \pi_{ij}(1)\ \ldots\ \pi_{ij}(m_{ij})$$

where $m_{ij}\ge 2$, $\pi_{ij}(0)=v_j$ and $\pi_{ij}(m_{ij})=v_i$. Corresponding to this path, the states earlier defined in (7.29) and (7.30) are $x_{ij}^{(0)}(k)$, $x_{ij}^{(1)}(k)$, $\ldots$, $x_{ij}^{(m_{ij}-1)}(k)$. Let us define

$$X_{ij}(k)=\mathbf{vert}[x_{ij}^{(p)}(k)]_{p\in\{0,\ldots,m_{ij}-1\}}$$

corresponding to the path π_{ij}. From the state-space equations corresponding to these states, we can see that

$$X_{ij}(k+1)=\begin{bmatrix} A_{ij} & & & & & \\ \mathbf{q}^{t_{ij}(0)}C_{ij} & 0 & & & & \\ & \mathbf{q}^{t_{ij}(1)}I & 0 & & & \\ & & \mathbf{q}^{t_{ij}(2)}I & 0 & & \\ & & & \ddots & \ddots & \\ & & & & \mathbf{q}^{t_{ij}(m_{ij}-2)}I & 0 \end{bmatrix}X_{ij}(k). \tag{7.34}$$

Define $X_2(k) = \mathbf{vert}[X_{ij}(k)]_{\{i,j:2\leq m_{ij}<n\}}$ as the set of states corresponding to $P_{ij}(z)$ when $m_{ij} \geq 2$. Note that $X_1(k)$ and $X_2(k)$ constitute all the states defined corresponding to the n sub-systems $\{\tilde{P}_i\}_i$. From (7.33) and (7.34), we can see that the $A-$matrix corresponding to the dynamics of $\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}$ is block lower triangular with $\{A_{ij}\}_{i,j}$ on the diagonal and the rest of the diagonal terms being zero.

By hypothesis, $P(z)$ is BIBO stable which implies that $\{P_{ij}(z)\}_{i,j}$ are all BIBO stable, which in turn implies that $\{H_{ij}(z)\}_{i,j}$ are all BIBO stable. Note that, we assumed minimal realizations of $P_{ij}(z)$ and $H_{ij}(z)$ in (3.18), (3.19) and (3.20) which implies that the matrices $\{A_{ij}\}_{i,j}$ are all Schur-stable. Thus, we can see that the $A-$matrix of the networked realization $\tilde{P}$ is also Schur-stable based on Lemma 13. $\square$

From Lemma 15 and Theorem 6, we can see that given a weighted digraph $\mathcal{G}$, any networked system that is a strictly causal interaction over $\mathcal{G}$ has a structured transfer function matrix that has sparsity and delay structures corresponding to $\mathcal{G}$ and vice versa. This is true when there are no additional conditions imposed on the systems. If the systems are constrained to be stabilizable and detectable, we notice that Theorem 6 cannot be extended for any general unstable structured transfer function matrix in $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$. Due to this network realizability problem, unstable networked systems (that are stabilizable and detectable) cannot be represented using structured transfer function matrices in $\mathfrak{T}(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$.

7.2 Networked plant model

A networked plant P is modeled as a strictly causal interaction of sub-systems (as in (4.1)) over a given weighted digraph $\mathcal{G}$, with each sub-system including local exogenous input vector $w_i(k)$ and local regulated output vector $z_i(k)$. The state-space description of the sub-systems $\{P_i\}_i$ are given by

$$P_i: \quad \begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^w w_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} B_{ij}^{\zeta}\zeta_{ij}(k) \\ z_i(k) &= C_{ii}^z x_i(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{z\zeta}\zeta_{ij}(k) \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yw} w_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{y\zeta}\zeta_{ij}(k) \\ \eta_{ri}(k) &= C_{ri}^{\eta} x_i(k) \; \forall\, r \in \mathcal{N}_i^+ \end{aligned} \qquad \forall\, i \in \{1,\ldots,n\} \tag{7.35}$$

where $x_i(k)$ denotes the local state vector, $w_i(k)$ local exogenous input vector, $z_i(k)$ local regulated output vector, $u_i(k)$ local control input vector, $y_i(k)$ the local measurement output vector, $\eta_{ri}(k)$ (for all $r \in \mathcal{N}_i^+$) the local network outputs and $\zeta_{ij}(k)$ (for all $j \in \mathcal{N}_i^-$) the local network inputs corresponding to a sub-system P_i. The discrete-time network interaction equations corresponding to the weighted digraph $\mathcal{G}$ are given by

$$\zeta_{ij}(k) = \eta_{ij}(k - t_{ij}) \; \forall \; (v_j, v_i) \in \mathcal{E} \tag{7.36}$$

where t_{ij} denotes the network delay according to (7.1).

Combining (7.35) and (7.36), the network inputs and outputs can be eliminated to give the state-space equations for the sub-systems as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^w w_i(k) + B_i^u u_i(k) + \sum_{j \in \mathcal{N}_i^-} A_{ij} x_j(k - t_{ij}), \\ z_i(k) &= C_{ii}^z x_i(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) + \sum_{j \in \mathcal{N}_i^-} C_{ij}^z x_j(k - t_{ij}) \; \forall \; i \in \{1, \ldots, n\} \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yw} w_i(k) + \sum_{j \in \mathcal{N}_i^-} C_{ij}^y x_j(k - t_{ij}), \end{aligned} \tag{7.37}$$

where $A_{ij} := B_{ij}^{\zeta} C_{ij}^{\eta}$, $C_{ij}^z := D_{ij}^{z\zeta} C_{ij}^{\eta}$ and $C_{ij}^y := D_{ij}^{y\zeta} C_{ij}^{\eta}$. Using the delay shift operator $\mathbf{q}$, the state-space equations in (7.37) can also be concisely written as

$$P : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\mathbf{q}) & B_w & B_u \\ C_z(\mathbf{q}) & D_{zw} & D_{zu} \\ C_y(\mathbf{q}) & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{7.38}$$

where $A(\mathbf{q})$, B_u and C_y are given by (7.20) while

$$[C_z(\mathbf{q})]_{ij} = \begin{cases} C_{ii}^z & \text{if } i = j \\ \mathbf{q}^{t_{ij}} C_{ij}^z & \text{if } (v_j, v_i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{7.39}$$

$$B_w = \mathbf{diag}[B_i^w]_i, \quad D_{zw} = \mathbf{diag}[D_i^{zw}]_i, \quad D_{zu} = \mathbf{diag}[D_i^{zu}]_i, \quad D_{yw} = \mathbf{diag}[D_i^{yw}]_i.$$

Note that $x(k) := \mathbf{vert}[x_i(k)]_i$, $w(k) := \mathbf{vert}[w_i(k)]_i$, $u(k) := \mathbf{vert}[u_i(k)]_i$, $z(k) := \mathbf{vert}[z_i(k)]_i$ and $y(k) := \mathbf{vert}[y_i(k)]_i$ denote the complete state, exogenous input, control input, regulated output and measurement output vectors corresponding to the networked system P and be partitioned according to $\mathcal{P}_x$, $\mathcal{P}_w$,

$\mathcal{P}_u$, $\mathcal{P}_z$ and $\mathcal{P}_y$, respectively. From (7.37), and the partitions of $x(k)$, $w(k)$, $u(k)$, $z(k)$ and $y(k)$, we can see that

$$\begin{aligned} &A(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_x,\mathcal{P}_x), \quad B_w \in S(I,\mathcal{P}_x,\mathcal{P}_w), \quad B_u \in S(I,\mathcal{P}_x,\mathcal{P}_u), \\ &C_z(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_z,\mathcal{P}_x), \quad D_{zw} \in S(I,\mathcal{P}_z,\mathcal{P}_w), \quad D_{zu} \in S(I,\mathcal{P}_z,\mathcal{P}_u), \\ &C_y(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_y,\mathcal{P}_x), \quad D_{yw} \in S(I,\mathcal{P}_y,\mathcal{P}_w). \end{aligned} \tag{7.40}$$

7.3 All internally stabilizing networked controllers

In this section, we extend the parameterization described in Theorem 2 to the case when stabilizing controllers are constrained to be networked systems that are strictly causal interactions over delay networks. In this case, the plant P is also a strictly causal interaction over the given delay network. In order to parameterize internally stabilizing networked controllers, first a model based controller J is chosen to be a networked system based on appropriate $F(\mathbf{q})$ and L. Then Theorem 7 shows that choosing the Youla parameter Q to be a stable networked system will parameterize the stabilizing networked controllers for the given networked plant.

Theorem 7. *Given a weighted digraph $\mathcal{G}$ and a stabilizable and detectable networked plant P that is a strictly causal interaction over $\mathcal{G}$ with the sub-system dynamics given by (7.35) and the network interaction given by (7.36). Let the state-space representation for P be given by (7.38) with state-space matrices structured and partitioned according to (7.20) and (7.39). Given there exist matrices $F(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_u,\mathcal{P}_x)$ and $L = \mathbf{diag}[L_i]_i \in S(I,\mathcal{P}_x,\mathcal{P}_y)$ such that $(zI - A(z^{-1}) - B_uF(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full-rank for any $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. Then the set of all internally stabilizing FDLTI controllers for P, which are also strictly causal interactions over $\mathcal{G}$, is parametrized by*

$$K = \mathbf{lft}(J,Q), \tag{7.41}$$

where J is a strictly causal interaction over $\mathcal{G}$ with a state-space representation

$$J : \begin{bmatrix} x_J(k+1) \\ \hline u(k) \\ \xi(k) \end{bmatrix} = \left[\begin{array}{c|cc} A(\mathbf{q}) + B_uF(\mathbf{q}) + LC_y(\mathbf{q}) & -L & B_u \\ \hline F(\mathbf{q}) & 0 & I \\ -C_y(\mathbf{q}) & I & 0 \end{array}\right] \begin{bmatrix} x_J(k) \\ \hline y(k) \\ \psi(k) \end{bmatrix} \tag{7.42}$$

and any asymptotically stable networked system Q with it's transfer function matrix in $\mathfrak{T}(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$. Note that the vectors $x_J(k) := \mathbf{vert}[x_i^J(k)]_i$, $\xi(k) := \mathbf{vert}[\xi_i(k)]_i$ and $\psi(k) := \mathbf{vert}[\psi_i(k)]_i$ are partitioned according to $\mathcal{P}_x$, $\mathcal{P}_y$ and $\mathcal{P}_u$, respectively.

Proof. First, we show that J given in (7.42) is in fact a observer-based nominal stabilizing controller for the networked plant P in (7.38). Using the sub-matrices of $A(\mathbf{q})$, B_u, $C_y(\mathbf{q})$, $F(\mathbf{q})$ and L from hypothesis; (7.42) can be written as

$$\begin{aligned}
x_i^J(k+1) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} (A_{ij}+B_i^uF_{ij}+L_iC_{ij}^y)x_j^J(k-t_{ij}) - L_iy_i(k) + B_i^u\psi_i(k),\\
u_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} F_{ij}x_j^J(k-t_{ij})+\psi_i(k),\\
\xi_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} (-C_{ij}^y)x_j^J(k-t_{ij})+y_i(k).
\end{aligned} \tag{7.43}$$

where $t_{ii}=0$ (for all i) and t_{ij} is given by (7.1) (for all $j\in\mathcal{N}_i^-$). Combining (7.38) and (7.43), we can eliminate the variables $\{u_i(k)\}_i$ and $\{y_i(k)\}_i$ to get the dynamics of $T := \mathbf{lft}(P,J)$ as

$$\begin{aligned}
x_i^T(k+1) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} A_{ij}^Tx_j^T(k-t_{ij}) + B_i^T\begin{bmatrix} w_i(k)\psi_i(k)\end{bmatrix}\\
\begin{bmatrix} z_i(k)\xi_i(k)\end{bmatrix} &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} C_{ij}^Tx_j^T(k-t_{ij}) + D_i^T\begin{bmatrix} w_i(k)\psi_i(k)\end{bmatrix}
\end{aligned} \qquad \forall i\in\{1,\dots,n\} \tag{7.44}$$

where $x_i^T(k) = \begin{bmatrix} x_i(k) \\ x_i(k)-x_i^J(k)\end{bmatrix}$ and

$$\begin{aligned}
A_{ij}^T &:= \begin{bmatrix} A_{ij}+B_i^uF_{ij} & -B_i^uF_{ij} \\ 0 & A_{ij}+L_iC_{ij}^y\end{bmatrix}, \quad B_i^T := \begin{bmatrix} B_i^w & B_i^u \\ B_i^w+L_iD_i^{yw} & 0\end{bmatrix},\\
C_{ij}^T &:= \begin{bmatrix} C_{ij}^z+D_i^{zu}F_{ij} & -D_i^{zu}F_{ij} \\ 0 & C_{ij}^y\end{bmatrix}, \quad D_i^T := \begin{bmatrix} D_i^{zw} & D_i^{zu} \\ D_i^{yw} & 0\end{bmatrix}.
\end{aligned} \tag{7.45}$$

Based on hypothesis that $(zI - A(z^{-1}) - B_uF(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full rank for any $z\in\mathbb{C}\backslash\overline{\mathbb{D}}$, we notice that the networked system T in (7.44) is asymptotically stable based on Lemma 13. Thus J is a stabilizing controller of P. From (7.45), we can also see that the transfer function matrix from $\psi_i(k)$ to $\xi_i(k)$ (for any i and j) is a zero matrix.

First, assume that Q is an asymptotically stable networked system with it's transfer function in $\mathfrak{T}(\mathcal{G},\mathcal{P}_u,\mathcal{P}_y)$. Thus, Q has dynamics of the form

$$\begin{aligned} x_i^Q(k+1) &= A_{ii}^Q x_i^Q(k) + B_i^Q \xi_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}^Q x_j^Q(k-t_{ij}) \\ \psi_i(k) &= C_{ii}^Q x_i^Q(k) + D_i^Q \xi_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^Q x_j^Q(k-t_{ij}). \end{aligned} \quad \forall i\in\{1,\dots,n\} \tag{7.46}$$

Since the transfer function from $\psi(k)$ to $\xi(k)$ is zero and Q is asymptotically stable, the closed-loop transfer function $\mathbf{lft}(T,Q)$ is always stable. Using the standard Youla-Kučera parameterization arguments, we can see that the controller given by $K = \mathbf{lft}(J,Q)$ internally stabilizes the given plant P in (7.38) when J is given by (7.42) and Q is an asymptotically stable system. Next, we show that there exists a strictly causal interaction over $\mathcal{G}$ which has the same state-space representation as $\mathbf{lft}(J,Q)$ when Q is a strictly causal interaction over $\mathcal{G}$.

Combining equations in (7.43) and (7.46), we eliminate the variables $\{\xi_i(k)\}_i$ and $\{\psi_i(k)\}_i$ to write the state-space equations corresponding to $K = \mathbf{lft}(J,Q)$ as

$$\begin{aligned} x_i^K(k+1) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} A_{ij}^K x_j^K(k-t_{ij}) + B_i^K y_i(k) \\ u_i(k) &= \sum_{j\in\mathcal{N}_i^-\cup\{i\}} C_{ij}^K x_j^K(k-t_{ij}) + D_i^K y_i(k) \end{aligned} \quad \forall i\in\{1,\dots,n\} \tag{7.47}$$

where $x_i^K(k) = \begin{bmatrix} x_i^J(k) \\ x_i^Q(k) \end{bmatrix}$ and

$$A_{ij}^K := \begin{bmatrix} A_{ij} + B_i^u F_{ij} + L_i C_{ij}^y - B_i^u D_i^Q C_{ij}^y & B_i^u C_{ij}^Q \\ -B_i^Q C_{ij}^y & A_{ij}^Q \end{bmatrix}, \qquad B_i^K := \begin{bmatrix} -L_i + B_i^u D_i^Q \\ B_i^Q \end{bmatrix},$$

$$C_{ij}^K := \begin{bmatrix} F_{ij} - D_i^Q C_{ij}^y & C_{ij}^Q \end{bmatrix}, \qquad D_i^K := D_i^Q.$$

Extending the results of Lemma 5 to networked systems over delay networks, we can see that (7.47) is equivalent to a strictly causal interaction over $\mathcal{G}$ with the same state-space matrices as in (7.47).

On the other hand, given matrices $F(\mathbf{q})$ and L such that $(zI - A(z^{-1}) - B_u F(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full rank for any $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$, standard results on Youla parameterization show that any internally stabilizing controller for the plant P is given by $K = \mathbf{lft}(J,Q)$ where J is given by (7.42) and a stable, causal, FDLTI system Q. Now, assume that K is a strictly causal interaction over $\mathcal{G}$, which implies that K has a state-space realization of the form (7.47). Then, it is easy to see that K internally stabilizes $\hat{J}$ given by

$$\hat{J}: \begin{bmatrix} x_{\hat{J}}(k+1) \\ \hline \psi(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|cc} A(\mathbf{q}) & -L & B_u \\ \hline -F(\mathbf{q}) & 0 & I \\ C_y(\mathbf{q}) & I & 0 \end{array}\right] \begin{bmatrix} x_{\hat{J}}(k) \\ \hline \xi(k) \\ u(k) \end{bmatrix} \tag{7.48}$$

where $x_{\hat{J}}(k)$ is partitioned according to $\mathcal{P}_x$. Following a similar procedure as before, we see that $Q = \mathbf{lft}(\hat{J}, K)$ is a stable strictly causal interaction over $\mathcal{G}$ whenever K is an internally stabilizing networked controller for P. □

Theorem 7 requires matrices $F(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_u,\mathcal{P}_x)$ and $L = \mathbf{diag}[L_i]_i \in S(I,\mathcal{P}_x,\mathcal{P}_y)$ such that $(zI - A(z^{-1}) - B_u F(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full-rank for any $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. These matrices can be obtained using Lemma 7 and Lemma 8. We describe the procedure through a simple example.

7.3.1 Example

Consider a networked system P over a delay network as shown in Fig. 3.1, with dynamics described by (7.3) as

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & \mathbf{q}A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} B_1^u & 0 & 0 \\ 0 & B_2^u & 0 \\ 0 & 0 & B_3^u \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix} \\ \begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} &= \begin{bmatrix} C_{11} & \mathbf{q}C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} D_1^{yu} & 0 & 0 \\ 0 & D_2^{yu} & 0 \\ 0 & 0 & D_3^{yu} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}. \end{aligned} \tag{7.49}$$

In order to find appropriate $F(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_u,\mathcal{P}_x)$ and $L = \mathbf{diag}[L_i]_i \in S(I,\mathcal{P}_x,\mathcal{P}_y)$, we first write the dynamics of P in (7.49) using a network state vector $x_4(k) := x_2(k)$ as

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}_u u(k) \\ y(k) &= \bar{C}_y \bar{x}(k) + D_{yu} u(k) \end{aligned} \tag{7.50}$$

where

$$\bar{x}(k) := \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}, \quad \bar{A} := \begin{bmatrix} A_{11} & 0 & 0 & A_{12} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$
$$\bar{B}_u := \begin{bmatrix} B_1^u & 0 & 0 \\ 0 & B_2^u & 0 \\ 0 & 0 & B_3^u \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{C}_y := \begin{bmatrix} C_{11} & 0 & 0 & C_{12} \\ C_{21} & C_{22} & 0 & 0 \\ 0 & C_{32} & C_{33} & 0 \end{bmatrix}. \tag{7.51}$$

Now, use Lemma 7 to obtain

$$\bar{F} = \begin{bmatrix} F_{11} & 0 & 0 & F_{12} \\ F_{21} & F_{22} & 0 & 0 \\ 0 & F_{32} & F_{33} & 0 \end{bmatrix} \tag{7.52}$$

such that $\bar{A}+\bar{B}_u\bar{F}$ is asymptotically stable. This can be obtained by imposing appropriate sparsity constraints on G and R in (4.13). Following the structure of $\bar{F}$ in (7.52), it is easy to obtain the required $F(\mathbf{q}) \in S(\mathcal{A}(\mathcal{G},\mathbf{q}),\mathcal{P}_u,\mathcal{P}_x)$ from $\bar{F}$ as

$$F(\mathbf{q}) = \begin{bmatrix} F_{11} & \mathbf{q}F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & F_{32} & F_{33} \end{bmatrix}. \tag{7.53}$$

Lemma 13 assures that $(zI - A(z^{-1}) - B_uF(z^{-1}))$ has full rank for all $z \in \mathbb{C}\backslash\bar{\mathbb{D}}$ when $F(\mathbf{q})$ is obtained from $\bar{F}$ such that $\bar{A}+\bar{B}_u\bar{F}$ is asymptotically stable.

Similarly, use Lemma 8 to obtain

$$\bar{L} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.54}$$

such that $\bar{A}+\bar{L}\bar{C}_y$ is asymptotically stable. This again can be obtained by imposing appropriate sparsity constraints on G and R in (4.14). Following the structure of $\bar{L}$ in (7.54), it is easy to obtain the required

$L \in S(I, \mathcal{P}_x, \mathcal{P}_y)$ from $\overline{L}$ as

$$L = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}. \tag{7.55}$$

Lemma 13 assures that $(zI - A(z^{-1}) - LC_y(z^{-1}))$ has full rank for all $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$ when L is obtained from $\overline{L}$ such that $\overline{A} + \overline{L}\overline{C}_y$ is asymptotically stable.

7.4 Optimal solution for networked controller design problem

Let $\mathcal{G}$ denote the weighted digraph representing a general delay network interaction. Given a networked plant P with sub-system dynamics following (7.35) that are interacting over a network specified by (7.36). Following the discussion in Section 4.3, the norm-minimizing network control problems where the controller is constrained to be a strictly causal interaction over the given $\mathcal{G}$ can be written as

$$\begin{aligned} \min \quad & \|T_{zw}\|_\alpha \\ \text{subject to} \quad & K \quad \text{is a strictly causal interaction over } \mathcal{G}, \\ & T_{zw} \quad \text{is asymptotically stable} \end{aligned} \tag{7.56}$$

where $T_{zw} = \mathbf{lft}(P,K)$ denotes the closed-loop mapping from $w(k)$ to $z(k)$, and $\alpha = 2$ or ∞. Based on Theorem 7, the set of internally stabilizing networked controllers that are strictly causal interactions over $\mathcal{G}$ are parameterized as $K = \mathbf{lft}(J,Q)$ where J is given by (7.42) and Q is a stable networked system over $\mathcal{G}$ with $\mathbf{tf}(Q) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. If there exist matrices $F(\mathbf{q})$ and L such that $(zI - A(z^{-1}) - B_u F(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full rank for any $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$, then the set of all closed-loop transfer function matrices from $w(k)$ to $z(k)$ for an internally stabilizing networked controller (which is a strictly causal interaction over $\mathcal{G}$) can be given by

$$\mathfrak{C}_{zw} := \{T_{11}(z) + T_{12}(z)Q(z)T_{21}(z) : \; Q(z) = \mathbf{tf}(Q), \; Q \in \mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)\} \tag{7.57}$$

where

$$\begin{aligned} \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} := \begin{bmatrix} D_{zw} & D_{zu} \\ D_{yw} & 0 \end{bmatrix} + \sum_{k=0}^{\infty} \begin{bmatrix} C_z(z^{-1}) + D_{zu}F(z^{-1}) & -D_{zu}F(z^{-1}) \\ 0 & C_y(z^{-1}) \end{bmatrix} \cdot \\ \begin{bmatrix} A(z^{-1}) + B_uF(z^{-1}) & -B_uF(z^{-1}) \\ 0 & A(z^{-1}) + LC_y(z^{-1}) \end{bmatrix}^k \begin{bmatrix} B_w & B_u \\ B_w + LD_{yw} & 0 \end{bmatrix} z^{-k-1} \end{aligned} \tag{7.58}$$

From (7.58), it is easy to note that $T_{22}(z) = 0$. The norm-minimization networked control problem in (7.56) can be written as

$$\begin{array}{ll} \min & \|T_{zw}\|_{\alpha} \\ \text{subject to} & T_{zw} \in \mathfrak{C}_{zw} \end{array} \qquad \text{for } \alpha = 2 \text{ or } \infty \tag{7.59}$$

where $\mathfrak{C}_{zw}$ is given by (7.57) which can equivalently by written as

$$\begin{array}{ll} \min & \|T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)\|_{\alpha} \\ \text{subject to} & Q(z) \in \mathfrak{T}^{s}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y) \end{array} \qquad \text{for } \alpha = 2 \text{ or } \infty. \tag{7.60}$$

This problem is exactly the same as (4.20) and the vectorization idea used in Section 4.3 can be used to write the $\mathcal{H}_2$ networked control problem as an unconstrained $\mathcal{H}_2$ problem

$$\begin{array}{ll} \min & \left\|\mathbf{vec}(T_{11}(z)) + (T_{21}(z)' \otimes T_{12}(z))S(z)H(z)\right\|_2 \\ \text{subject to} & H(z) \in \mathcal{RH}_{\infty}^{a \times 1}, \end{array} \tag{7.61}$$

where $S(z)$ and $H(z)$ are given by (4.22). The unconstrained convex optimization problem in (7.61) can be solved using standard techniques. Let $H^{\star}(z)$ denote the solution of the optimization problem (7.61). Then the corresponding optimal $Q^{\star}(z)$ is given by $Q^{\star}(z) = \mathbf{vec}^{-1}(S(z)H^{\star}(z)) \in \mathfrak{T}^{s}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Following the proof of Theorem 7, the corresponding internally stabilizing controller $K^{\star}$ is obtained based on J given by (7.42) and $Q^{\star}(z)$. From Theorem 7 and the problem formulation in (7.56), we can see that $K^{\star}$ thus designed is the optimal internally stabilizing networked controller that is a strictly causal interaction over $\mathcal{G}$ for the given networked plant P.

CHAPTER 8. Numerical examples

8.1 Example for Theorem 1

Let a unit-weight digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be given (as shown in Fig. 3.2(b)), where $\mathcal{V} = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{(v_1, v_2), (v_2, v_1), (v_2, v_3)\}$. Let $\mathcal{P}_u = (1,1,1)$ and $\mathcal{P}_y = (1,1,1)$. Let the transfer function matrix of a stable structured system over $\mathcal{G}$ be given by

$$P(z) = \begin{bmatrix} \frac{z+1}{z-0.5} & \frac{0.5}{z-0.8} & 0 \\ \frac{-0.1}{z-0.5} & \frac{z+0.1}{z-0.1} & 0 \\ \frac{1}{(z-0.1)(z-0.8)} & \frac{0.3}{z-0.8} & \frac{z-0.2}{z-0.5} \end{bmatrix}. \tag{8.1}$$

Note that (8.1) satisfies the delay and sparsity constraints (3.11) corresponding to the digraph $\mathcal{G}$. Thus $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. Following the notation from Theorem 1, we write the minimal state-space realizations

$$P_{11}(z) = \frac{z+1}{z-0.5} \rightarrow \left[\begin{array}{c} x_{11}(k+1) \\ \hline y_{11}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.5 & 1 \\ \hline 1.5 & 1 \end{array}\right] \left[\begin{array}{c} x_{11}(k) \\ \hline u_1(k) \end{array}\right],$$

$$P_{12}(z) = \frac{0.5}{z-0.8} \rightarrow \left[\begin{array}{c} x_{12}(k+1) \\ \hline y_{12}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.8 & 0.5 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} x_{12}(k) \\ \hline u_2(k) \end{array}\right],$$

$$P_{21}(z) = \frac{-0.1}{z-0.5} \rightarrow \left[\begin{array}{c} x_{21}(k+1) \\ \hline y_{21}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.5 & 0.25 \\ \hline -0.4 & 0 \end{array}\right] \left[\begin{array}{c} x_{21}(k) \\ \hline u_1(k) \end{array}\right],$$

$$P_{22}(z) = \frac{z+0.1}{z-0.1} \rightarrow \left[\begin{array}{c} x_{22}(k+1) \\ \hline y_{22}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.1 & 0.5 \\ \hline 0.4 & 1 \end{array}\right] \left[\begin{array}{c} x_{22}(k) \\ \hline u_2(k) \end{array}\right],$$

$$z^{-1}H_{31}(z) = zP_{31}(z) = \frac{z}{(z-0.1)(z-0.8)} \rightarrow \left[\begin{array}{c} x_{31}^{(0)}(k+1) \\ \hline y_{31}^{(0)}(k) \end{array}\right] = \left[\begin{array}{cc|c} 0.9 & -0.32 & 1 \\ 0.25 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_{31}^{(0)}(k) \\ \hline u_1(k) \end{array}\right],$$

$$z^{-1} \rightarrow \left[\begin{array}{c} x_{31}^{(1)}(k+1) \\ \hline y_{31}^{(1)}(k) \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} x_{31}^{(1)}(k) \\ \hline y_{31}^{(0)}(k) \end{array}\right],$$

$$P_{32}(z) = \frac{0.3}{z-0.8} \rightarrow \left[\begin{array}{c} x_{32}(k+1) \\ \hline y_{32}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.8 & 0.5 \\ \hline 0.6 & 0 \end{array}\right] \left[\begin{array}{c} x_{32}(k) \\ \hline u_2(k) \end{array}\right],$$

$$P_{33}(z) = \frac{z-0.2}{z-0.5} \rightarrow \left[\begin{array}{c} x_{33}(k+1) \\ \hline y_{33}(k) \end{array}\right] = \left[\begin{array}{c|c} 0.5 & 0.5 \\ \hline 0.6 & 1 \end{array}\right] \left[\begin{array}{c} x_{33}(k) \\ \hline u_3(k) \end{array}\right].$$

In the graph $\mathcal{G}$, the shortest path (with length 2) from vertex v_1 to vertex v_3 is given by $v_1 \rightarrow v_2 \rightarrow v_3$ and the corresponding states are defined by $x_{31}^{(0)}(k)$ and $x_{31}^{(1)}(k)$. Thus the path $\pi_{31} = v_1\, v_2\, v_3$ and $l_{31} = 2$. Following the proof of Theorem 1, we define state vectors corresponding to each node to be

$$\tilde{x}_1(k) = \begin{bmatrix} x_{11}(k) \\ x_{21}(k) \\ x_{31}^{(0)}(k) \end{bmatrix}, \quad \tilde{x}_2(k) = \begin{bmatrix} x_{12}(k) \\ x_{22}(k) \\ x_{32}(k) \\ x_{31}^{(1)}(k) \end{bmatrix}, \quad \tilde{x}_3(k) = x_{33}(k).$$

The outgoing messages from each node are given by

$$\tilde{\eta}_{21}(k) = \begin{bmatrix} y_{21}(k) \\ y_{31}^{(0)}(k) \end{bmatrix}, \quad \tilde{\eta}_{12}(k) = \begin{bmatrix} y_{12}(k) \end{bmatrix}, \quad \tilde{\eta}_{32}(k) = \begin{bmatrix} y_{32}(k) \\ y_{31}^{(1)}(k) \end{bmatrix},$$

and the outputs at each node are given by

$$\begin{aligned} y_1(k) &= y_{11}(k) + y_{12}(k), \\ y_2(k) &= y_{21}(k) + y_{22}(k), \\ y_3(k) &= y_{31}^{(1)}(k) + y_{32}(k) + y_{33}(k). \end{aligned}$$

Since the network represented by $\mathcal{G}$ is noiseless and has zero-delay, the incoming message vectors at each vertex are given by

$$\tilde{\zeta}_{12}(k) = \tilde{\eta}_{12}(k), \quad \tilde{\zeta}_{21}(k) = \tilde{\eta}_{21}(k), \quad \tilde{\zeta}_{32}(k) = \tilde{\eta}_{32}(k). \tag{8.2}$$

Using the state-space matrices of $P_{ij}(z)$, the dynamics at each vertex v_i are defined as a sub-system $\tilde{P}_i$ given by

$$\begin{aligned}
\tilde{P}_1 : \left[\begin{array}{c} \tilde{x}_1(k+1) \\ \hline y_1(k) \\ \hline \tilde{\eta}_{21}(k) \end{array}\right] &= \left[\begin{array}{cccc|c|c} 0.5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0.9 & -0.32 & 1 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \\ \hline 1.5 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & -0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \tilde{x}_1(k) \\ \hline u_1(k) \\ \hline \tilde{\zeta}_{12}(k) \end{array}\right], \\
\tilde{P}_2 : \left[\begin{array}{c} \tilde{x}_2(k+1) \\ \hline y_2(k) \\ \hline \tilde{\eta}_{12}(k) \\ \hline \tilde{\eta}_{32}(k) \end{array}\right] &= \left[\begin{array}{cccc|c|cc} 0.8 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0.4 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \tilde{x}_2(k) \\ \hline u_2(k) \\ \hline \tilde{\zeta}_{21}(k) \end{array}\right], \\
\tilde{P}_3 : \left[\begin{array}{c} \tilde{x}_3(k+1) \\ \hline y_3(k) \end{array}\right] &= \left[\begin{array}{c|c|cc} 0.5 & 0.5 & 0 & 0 \\ \hline 0.6 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{c} \tilde{x}_3(k) \\ \hline u_3(k) \\ \hline \tilde{\zeta}_{32}(k) \end{array}\right].
\end{aligned} \tag{8.3}$$

The sub-systems $\{\tilde{P}_i\}_i$ in (8.3) interacting over the network interconnection (8.2) describes the networked system $\tilde{P}$ corresponding to $P(z)$. Combining the equations in (8.3) and (8.2), we get the state-space representation for $\tilde{P}$ as (A, B_u, C_y, D_{yu}) where

$$A = \left[\begin{array}{cccc|cccc|c} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & -0.32 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{array}\right], \quad B_u = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0.25 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0.5 \end{array}\right]$$

$$C_y = \left[\begin{array}{cccc|cccc|c} 1.5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -0.4 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 1 & 0.6 \end{array}\right], \quad D_{yu} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

when the state, input and output vectors are given by $\tilde{x}(k) = \mathbf{vert}[\tilde{x}_i(k)]_i$, $u(k) = \mathbf{vert}[u_i(k)]_i$ and $y(k) = \mathbf{vert}[y_i(k)]_i$, respectively. Note that A is Schur-stable and $(A, B_u, C_y, D_{yu}) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ where $\mathcal{P}_x = (4, 4, 1)$, $\mathcal{P}_u = (1, 1, 1)$ and $\mathcal{P}_y = (1, 1, 1)$. By calculating the transfer function matrix corresponding to $\tilde{P}$, we can see that $\mathbf{tf}(\tilde{P}) = P(z)$.

8.2 Example for designing networked controllers over zero-delay networks

Using this example we explain the concepts and algorithms discussed in Chapter 3 and Chapter 4 to solve a $\mathcal{H}_2$ networked control problem. We consider a strictly causal interaction of 3 sub-systems over a zero-delay directed communication network represented by a unit-weight digraph $\mathcal{G}$ given in Fig. 3.2. Let the 3 sub-systems $\{P_i\}_{i \in \{1,2,3\}}$ of the form (4.1) be expressed in their state-space representation as given below

$$
\begin{aligned}
P_1: \left[\begin{array}{c} x_1(k+1) \\ \hline z_1(k) \\ \hline y_1(k) \\ \hline \eta_{21}(k) \end{array}\right] &= \left[\begin{array}{cc|cc|c|cc} 0.1 & 0.8 & 0.1 & 0 & 0.1 & 0.4 & 0 \\ 0.3 & -0.5 & 0.1 & 0 & 0.1 & 0 & 0.2 \\ \hline 0.2 & 0.1 & 0 & 1 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.2 & 0.1 & 0 & 1 & 0 & 0.3 & 0.2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1(k) \\ \hline w_1(k) \\ \hline u_1(k) \\ \hline \zeta_{12}(k) \end{array}\right], \\
P_2: \left[\begin{array}{c} x_2(k+1) \\ \hline z_2(k) \\ \hline y_2(k) \\ \hline \eta_{12}(k) \\ \eta_{32}(k) \end{array}\right] &= \left[\begin{array}{cc|cc|c|cc} -0.6 & 1.3 & 0 & 0 & 0 & 1.4 & 0 \\ 0.5 & 0.2 & 0.2 & 0 & 0.2 & 0 & -0.3 \\ \hline 0.1 & 0.1 & 0 & 1 & 0 & 0.1 & -0.3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.1 & 0.1 & 0 & 1 & 0 & 0.1 & -0.3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_2(k) \\ \hline w_2(k) \\ \hline u_2(k) \\ \hline \zeta_{21}(k) \end{array}\right], \\
P_3: \left[\begin{array}{c} x_3(k+1) \\ \hline z_3(k) \\ \hline y_3(k) \end{array}\right] &= \left[\begin{array}{cc|cc|c|cc} 1.2 & 0 & 0.4 & 0 & 0.4 & 0.1 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0 & 0 & -0.8 \\ \hline 0.1 & 0.4 & 0 & 1 & 0 & -0.1 & 0.3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.1 & 0.4 & 0 & 1 & 0 & -0.1 & 0.3 \end{array}\right] \left[\begin{array}{c} x_3(k) \\ \hline w_3(k) \\ \hline u_3(k) \\ \hline \zeta_{32}(k) \end{array}\right],
\end{aligned}
\tag{8.4}
$$

and the zero-delay network interconnection in (4.2) is given by

$$\zeta_{12}(k) = \eta_{12}(k), \quad \zeta_{21}(k) = \eta_{21}(k), \quad \zeta_{32}(k) = \eta_{32}(k).$$

By interconnecting the three sub-systems over the network, we get the networked system P with following state-space matrices

$$A = \left[\begin{array}{cc|cc|cc} 0.1 & 0.8 & 0.4 & 0 & 0 & 0 \\ 0.3 & -0.5 & 0 & 0.2 & 0 & 0 \\ \hline 1.4 & 0 & -0.6 & 1.3 & 0 & 0 \\ 0 & -0.3 & 0.5 & 0.2 & 0 & 0 \\ \hline 0 & 0 & 0.1 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & -0.8 & 0.3 & 0.4 \end{array}\right],$$

$$B_u = \left[\begin{array}{c|c|c} 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ \hline 0 & 0 & 0.4 \\ 0 & 0 & 0 \end{array}\right], \quad C_y = \left[\begin{array}{cc|cc|cc} 0.2 & 0.1 & 0.3 & 0.2 & 0 & 0 \\ \hline 0.1 & -0.3 & 0.4 & 0.1 & 0 & 0 \\ \hline 0 & 0 & -0.1 & 0.3 & 0.1 & 0.4 \end{array}\right].$$

The other state-space matrices can also be obtained from the sub-system dynamics and the network interconnection. Note that P is an unstable system since A has eigenvalues outside the unit disc. For comparison purpose, an optimal internally stabilizing centralized controller K_{central} is computed using standard techniques and the corresponding optimal cost is given by $\|\mathbf{lft}(P, K_{\text{central}})\|_2 = 25.6203$. Following Lemma 7 and Lemma 8, we obtain following matrices F and L so that $A + B_u F$ and $A + LC_y$ are Schur stable.

$$F = \left[\begin{array}{cc|cc|cc} -4.4408 & 2.1392 & -0.0012 & -3.5507 & 0 & 0 \\ \hline 2.9020 & -1.9631 & -4.7372 & 1.5855 & 0 & 0 \\ \hline 0 & 0 & -0.4561 & 0.4437 & -3.0970 & -0.1546 \end{array}\right],$$

$$L = \left[\begin{array}{c|c|c} -1.9583 & 0 & 0 \\ -0.3447 & 0 & 0 \\ \hline 0 & 11.1308 & 0 \\ 0 & -3.8995 & 0 \\ \hline 0 & 0 & -1.3975 \\ 0 & 0 & -0.1351 \end{array}\right].$$

Note that F is structured according to $\mathcal{A}(\mathcal{G})$ while L is block-diagonal. We can construct the following observer-based networked controller

$$K_{\text{nom}} = \left[\begin{array}{c|c} A + B_u F + LC_y & -L \\ \hline F & 0 \end{array}\right] \tag{8.5}$$

using the matrices F and L. Note that K_{nom} is a stabilizing controller that is a strictly causal interaction over $\mathcal{G}$. Also, note that K_{nom} is a full-order controller. In this example, this nominal networked controller is unstable and gives a performance cost of $\|\mathbf{lft}(P, K_{\text{nom}})\|_2 = 157.7915$.

In order to find an optimal networked controller, we first use Theorem 2 to parameterize the set of all internally stabilizing networked controllers for the given networked plant based on the matrices F and L. Then following the formulation given in Section 4.3, we obtain the optimal internally stabilizing networked controller K_{opt} that is a strictly causal interaction over the given network. The performance cost $\left\|\mathbf{lft}(P, K_{\text{opt}})\right\|_2$ for this optimal controller is 54.2338. The optimal controller is not presented in the thesis due to its large order but we shall present some information about the controller to better appreciate the optimal solution.

First, the order of the optimal networked controller is 62 where the sub-systems K_1, K_2 and K_3 have order 22, 24 and 16, respectively. Note that in the case of centralized problem, the optimal controller can be full-order, i.e. it has order 6. The networked controller has larger order to compensate for the lack of full communication. The optimal cost provided by our optimal networked controller can also be used as a bound in designing sub-optimal reduced-order networked controllers.

Second, the optimal networked controller is non-minimal but is stabilizable and detectable such that the closed-loop system is internally stable. Last but not least, we note that the optimal networked controller is unstable with two unstable poles at 1.1629. So, if we had used a transfer function based approach (for example, [8]) to design an optimal stabilizing controller with a structured transfer function matrix, it is not known how to realize the unstable transfer function matrix as a stabilizing networked controller over the given network. In essence, we provide an optimal stabilizing networked controller and also provide a methodology to implement it over the given network even when the stabilizing controller is unstable.

For the same plant, using the results from Chapter 5, we also found a full-order internally stabilizing networked controller K_{full} that is a strictly causal interaction over the given $\mathcal{G}$. The full-order controller

$K_{\text{full}} = (A_K, B_K, C_K, D_K)$ is given by the following state-space matrices

$$A_K = \left[\begin{array}{cc|cc|cc} 0.3723 & -0.0925 & 7.847 & -2.169 & 0 & 0 \\ 1.707 & -0.972 & 3.992 & -0.4528 & 0 & 0 \\ \hline 1.676 & -1.164 & -19.14 & 6.447 & 0 & 0 \\ 5.281 & -3.675 & 59.69 & 20.12 & 0 & 0 \\ \hline 0 & 0 & -2.787 & 1.056 & -5.641 & 0 \\ 0 & 0 & 3.217 & -1.216 & 7.391 & 7.107 \end{array}\right],$$

$$B_K = \left[\begin{array}{c|c|c} -0.1409 & 0 & 0 \\ 0.1209 & 0 & 0 \\ \hline 0 & 0.3864 & 0 \\ 0 & 1.225 & 0 \\ \hline 0 & 0 & -0.0513 \\ 0 & 0 & 0.0582 \end{array}\right], \quad D_K = \left[\begin{array}{c|c|c} -5.48 & 0 & 0 \\ \hline 0 & -26.12 & 0 \\ \hline 0 & 0 & 1.092 \end{array}\right],$$

$$C_K = \left[\begin{array}{cc|cc|cc} -2.813 & 1.224 & -76.91 & 19.84 & 0 & 0 \\ \hline -77.33 & 55.54 & 911.4 & -300.1 & 0 & 0 \\ \hline 0 & 0 & 68.49 & -21.67 & 160.6 & 137.3 \end{array}\right]$$

and the performance cost $\|\mathbf{lft}(P, K_{\text{full}})\|_2$ for this full-order controller is 95.9587.

8.3 Example for designing networked controllers over general delay networks

Using this example we explain the concepts and algorithms discussed in Chapter 7 to solve a $\mathcal{H}_2$ networked control problem in the general delay network case. We consider a strictly causal interaction of 3 sub-systems over a directed delay network represented by a weighted digraph $\mathcal{G}$ given in Fig. 3.1. Let the 3 sub-systems $\{P_i\}_{i\in\{1,2,3\}}$ be expressed in their state-space representation given by (8.4) interacting over a delay network interconnection given by

$$\zeta_{12}(k) = \eta_{12}(k-1), \quad \zeta_{21}(k) = \eta_{21}(k), \quad \zeta_{32}(k) = \eta_{32}(k).$$

By interconnecting the three sub-systems over the network, we get the dynamics of the networked system P with following state-space matrices

$$A(\mathbf{q}) = \left[\begin{array}{cc|cc|cc} 0.1 & 0.8 & 0.4\mathbf{q} & 0 & 0 & 0 \\ 0.3 & -0.5 & 0 & 0.2\mathbf{q} & 0 & 0 \\ \hline 1.4 & 0 & -0.6 & 1.3 & 0 & 0 \\ 0 & -0.3 & 0.5 & 0.2 & 0 & 0 \\ \hline 0 & 0 & 0.1 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & -0.8 & 0.3 & 0.4 \end{array}\right],$$

$$B_u = \left[\begin{array}{c|c|c} 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ \hline 0 & 0 & 0.4 \\ 0 & 0 & 0 \end{array}\right], \quad C_y(\mathbf{q}) = \left[\begin{array}{cc|cc|cc} 0.2 & 0.1 & 0.3\mathbf{q} & 0.2\mathbf{q} & 0 & 0 \\ \hline 0.1 & -0.3 & 0.4 & 0.1 & 0 & 0 \\ \hline 0 & 0 & -0.1 & 0.3 & 0.1 & 0.4 \end{array}\right].$$

The other state-space matrices can also be obtained from the sub-system dynamics and the network interconnection. Note that P is an unstable system since $(zI - A(z^{-1}))$ looses rank when $z = 1.2$. For comparison purpose, an optimal internally stabilizing centralized controller K_{central} is computed using standard techniques and the corresponding optimal cost is given by $\|\mathbf{lft}(P, K_{\text{central}})\|_2 = 3.5035$. Following the procedure described in Section 7.3.1, we obtain the following matrices $F(\mathbf{q})$ and L so that $(zI - A(z^{-1}) - B_u F(z^{-1}))$ and $(zI - A(z^{-1}) - LC_y(z^{-1}))$ have full-rank for all $z \in \mathbb{C} \backslash \overline{\mathbb{D}}$.

$$F = \left[\begin{array}{cc|cc|cc} -1.6363 & 1.6519 & -1.245\mathbf{q} & -1.458\mathbf{q} & 0 & 0 \\ \hline 2.9983 & 0.0202 & -3.9174 & 1.8998 & 0 & 0 \\ \hline 0 & 0 & -0.4093 & 0.4682 & -3.0687 & -0.1586 \end{array}\right],$$

$$L = \left[\begin{array}{c|c|c} -1.5276 & 0 & 0 \\ 0.1740 & 0 & 0 \\ \hline 0 & 11.2678 & 0 \\ 0 & -5.8945 & 0 \\ \hline 0 & 0 & -1.4985 \\ 0 & 0 & -0.1939 \end{array}\right].$$

Note that $F(\mathbf{q})$ is structured according to $\mathcal{A}(\mathcal{G},\mathbf{q})$ while L is block-diagonal. We can construct the following observer-based networked controller

$$K_{\text{nom}} = \left[\begin{array}{c|c} A(\mathbf{q}) + B_u F(\mathbf{q}) + LC_y(\mathbf{q}) & -L \\ \hline F(\mathbf{q}) & 0 \end{array}\right] \tag{8.6}$$

using the matrices $F(\mathbf{q})$ and L. Note that K_{nom} is a stabilizing controller that is a strictly causal interaction over $\mathcal{G}$. Also note that K_{nom} is a full-order controller. In this example, this nominal networked controller is unstable and gives a performance cost of $\|\mathbf{lft}(P,K_{\text{nom}})\|_2 = 130.4313$.

In order to find an optimal networked controller, we first use Theorem 7 to parameterize the set of all internally stabilizing networked controllers for the given networked plant based on the matrices $F(\mathbf{q})$ and L. Then following the formulation given in Section 7.4, we obtain the optimal internally stabilizing networked controller K_{opt} that is a strictly causal interaction over the given delay network. The performance cost $\left\|\mathbf{lft}(P,K_{\text{opt}})\right\|_2$ for this optimal controller is 3.5266. In this example, the K_{opt} was found to be asymptotically stable but with a large order.

CHAPTER 9. Conclusions

In this thesis, we studied the class of networked systems that are made of finite-dimensional, linear, time-invariant, causal, discrete-time sub-systems interacting over a noiseless, pure-delay, discrete-time network, all sharing the same clock. We first studied the case when the discrete-time network has no delays. We showed that the networked systems built on a zero-delay network can be represented using systems with structured state-space or transfer function matrices, in general. But in the case when the networked systems are constrained to be stabilizable and detectable, we point out that structured transfer function matrices cannot be used to represent the networked systems due to the problem of network realizability. "Given an unstable structured transfer function matrix, it is not known how to realize it as a stabilizable and detectable networked system over a given network."

Next, we studied the networked control problems where the controller is required to be a networked system that internally stabilizes a given plant. In this scenario, we observed that transfer function based approaches are not suitable to solve the networked control problems since the stabilizing controllers obtained as solutions to such approaches can in general be unstable. And due to the network realizability problem, such solutions may not be realizable over the given network while assuring stabilizability and detectability. Instead, we used the relationship between networked systems and structured systems to parameterize all internally stabilizing networked controllers using the state-space form of Youla-Kučera parameterization. Thus, synthesizing optimal networked controllers is shown to be a constrained convex optimization problem. In the case of $\mathcal{H}_2$ networked control, the constrained convex optimization problem is reduced to an unconstrained convex optimization problem which can easily be solved using standard techniques.

Since the optimal networked controllers can possibly have a large order, we also provide methodologies to design full-order internally stabilizing networked controllers by extending the results of [24]. We also solved the networked estimation problem by posing it as an equivalent networked control problem and use the results obtained for networked control. Next, we studied the networked systems when the network interaction can have any arbitrary delay structure. Using the shift delay operator used in [9], we extended the framework developed for systems over zero-delay networks to systems over any general delay networks. Finally, we provided numerical examples to describe the main results of the thesis.

We thus studied the problem of designing networked controllers for networked plants when both plant and controller are constrained to be on the same network. Since the transfer function approaches can not address the network realizability problem, we proposed a state-space approach for parameterizing all internally stabilizing networked controllers that allows one to synthesize optimal networked controllers that stabilize the given plant and can be expressed as sub-systems interacting over the given network.

9.1 Directions for future work

As future research work, it would be interesting to study the network realization problem in more detail. One can also look at model reduction techniques that assure stabilizability and detectability while reducing the order of a networked system. Presently, the networked controller design procedure proposed in this thesis is centralized, i.e. the controller can be designed only with the complete knowledge about the networked plant model. One can study distributed desing and synthesis techniques that allow more scalability to the networked controller design problem. Since the framework used for networked controller design is based on classical Youla parameterization, many of the results in control theory that are based on Youla parameterization may be extended to networked control.

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