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Confidence intervals for variance components and functions of variance components in the random effects model under non-normality

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ABSTRACT

Methods for constructing confidence intervals for the variance components from a random effects model have important applications in a variety of disciplines. A fundamental analysis with random effects models is confidence intervals for the variance components or functions of the variance components. Many methods for constructing confidence intervals are currently being used. These methods work well under normality, equal variance, and equal sample size, but are very sensitive to any violations of these assumptions. This dissertation addresses the problem of constructing confidence intervals for variance components when the random effects or the errors are not normally distributed. The focus is on balanced one way random effects models and four parameters - the between group variance, the ratio of between to within group variance components, the intra-class correlation, and the “stepped-up” reliability - are examined. All of our proposed methods replace the usual estimate of the standard error calculated under the assumption of normality with an estimate calculated under non-normality. For the between group variance, this estimate includes an estimate of the kurtosis of the distribution of the random effect. For the other three parameters, the standard error estimate includes estimates of both the kurtosis of the distribution of the random effect and the kurtosis of the distribution of the errors. If the researcher does not have any information about the distribution of the random effects or the errors, a general kurtosis estimate is used which is based on Pearson’s kurtosis estimator, but with adjustments suggested by Bonett and Shoemaker. If it seems reasonable to assume the random effect or the errors follow a Beta or Gamma distribution, the kurtosis is estimated by first estimating the parameters of these distributions and then using the parameter estimates to estimate the kurtosis. If a previous study has been conducted, kurtosis estimates from the previous study can be pooled with the kurtosis estimates from the current study. Finally, if the researcher can theoretically specify a kurtosis value based on expert knowledge about their field of study, this specified kurtosis value can be used in place of an estimate. Our findings indicate that the proposed methods, especially those that incorporate a researcher's knowledge about the distributions of the random effect and the errors, perform better than the current methods when the normality assumption is violated.

CHAPTER 1. GENERAL INTRODUCTION

Since the variance component model was first formally introduced in the 1930's, it has been used to model experiments in many fields, including Astronomy, Agriculture, Animal Breeding, Biology, Medicine, Engineering, Education, and Psychology, among others. In all of these fields, researchers need both point and interval estimates of the variance components in order to make decisions or test theories.

The classical methods for constructing confidence intervals assume normality of both the random effects and the errors. This dissertation addresses the problem of constructing accurate confidence intervals for variance components when the random effects or the errors are not normally distributed. Under normality, equal variance, and equal sample size, the current confidence interval methods work well, but unlike inferential methods for means that are robust to non-normality, inferential methods for variances are very sensitive to this violation. Mild and difficult to detect levels of non-normality can cause major problems. The focus will be on balanced random effects models. Confidence intervals for four fundamental parameters - the between group variance, the ratio of between to within group variance components, the intra-class correlation, and the "stepped-up" reliability - will be examined.

In this chapter, we provide background information and examples of the one-way random effects ANOVA model. We will also give an overview of the dissertation.

1.1 One-way Random Effects ANOVA Model

The random effects analysis of variance model, or variance component model, can be traced as far back as to the works of the astronomers Airy (1861) and Chauvenet (1863) (as cited in Khuri and Sahai, 1985). Many years later, statisticians re-invented the model beginning with Fisher (1925) who introduced the concept of analysis of variance and Tippet (1931) who clarified the analysis of variance method of variance component estimation (as cited in Khuri and Sahai, 1985).

There are several different quantities that may be of interest in a variance component analysis. One quantity that may be of interest is σ_α^2, the variability between the population group means, which would indicate the degree of similarity among the population means. Another quantity of interest may be the ratio of between to within group variance components, $\theta = \sigma_\alpha^2/\sigma_e^2$. Taking the ratio of these two components gives a standardized measure of the variance of the population group means. This ratio of variance components can be used to obtain another quantity of interest, the intra-class correlation. The formula for the intra-class correlation is

$$\rho_I = \frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \sigma_e^2} = \frac{1}{1 + 1/\theta} \tag{1.1}$$

From this formula, we can see that ρ_I is the ratio of the variance of the random effect to the total variance. That is, it is a measure of the proportion of the variance explained by the random effect and is analagous to R^2 in the regression model.

If the k levels are people randomly selected from a population and each person has n measurements (raters, occasions, alternate forms), then ρ_I is the reliability of a single measurement and it can be used to obtain another quantity of interest, the "stepped-up" reliability, ρ_n (Kristof, 1963). By the Spearman-Brown formula (Furr, 2008),

$$\rho_n = \frac{n\rho_I}{1 + \rho_I(n-1)}. \tag{1.2}$$

and it is the reliability of the average of the n measurements.

The one-way random effects model can be used in several different types of research designs. The most basic of those designs, is the completely randomized design where treatments are randomly assigned to experimental units. In this case the researcher is interested in a factor that has a large number of possible levels and a random sample of treatment levels are selected with the experimental units randomly assigned to the selected treatment levels. In addition to completely randomized designs, the one-way random effects model can also be thought of as a one-way nested classification, that arises from a two-stage cluster sample. In this situation, you have a factor such as "replication" or "sample" that is nested within the levels of the treatment factor. Some examples include the following:

(a) A sample of k formulas of an insecticide is chosen from a large number of formulas whose ratio of active ingredients vary. A random sample of corn fields is randomly assigned to the different formulas and the number of infested plants per acre is recorded. A confidence interval for σ_α^2 is then constructed to examine the variability in the mean number of infested plants per acre between formulas.

(b) A sample of k algebra textbooks is selected from a large collection of algebra textbooks whose teaching approaches vary. A random sample of students is randomly assigned to the different textbooks and end of course exam scores are recorded. A confidence interval for σ_α^2 is then constructed to examine the variability in the mean student exam score between textbooks.

(c) A sample of k individuals is selected from all indivduals taking the driver's exam in Iowa. A random sample of different forms of the driver's test, each form being randomly generated from a large pool of questions, is randomly assigned to the individuals, so that each individual receives n forms of the exam, and the score on each exam is recorded. Confidence intervals for ρ_I and ρ_n are then constructed to examine the reliability of the exam. The confidence interval for ρ_I gives information about the reliability of a single exam score, while the confidence interval for ρ_n gives information about the relability of the average exam score.

(d) A sample of k batches of soap is selected at random from a production line producing a large number of batches. A chemical analysis is done on n randomly selected bars of soap from each batch and the percent of glycerin is recorded for each bar. A confidence interval for σ_α^2 is then constructed to examine the variability in the mean percentage of glycerin per bar between batches.

(e) A sample of k boats fishing the Gulf of Mexico is selected at random from a large fleet of boats. A random sample of n fish from each boat is tested for mercury and the amount of mercury in micrograms per ounce is recorded for each fish. A confidence interval for σ_α^2 is then constructed to examine the variability in the mean amount of mercury in fish between boats.

The remainder of the dissertation is organized as follows: In Chapter 2, the one-way random effects ANOVA model and classical confidence intervals for the four parameters of interest are reviewed. In Chapter 3, the proposed method for σ_α^2 is described and simulation results are presented. Then, the proposed method and simulation results for θ are given, as well as its extension to ρ_I and ρ_n in Chapter 4. Finally, Chapter 5 closes with some general concluding remarks and ideas for future work.

CHAPTER 2. REVIEW OF LITERATURE

In this chapter, we provide the notation required to discuss the problem by first describing the one-way random effects ANOVA model. We will give the ANOVA table, the statistical model, point estimates, and the classical confidence intervals for the four parameters of interest. We will also describe some alternative point estimators and confidence interval methods.

2.1 The Statistical Model

The one-way random effects analysis of variance (ANOVA) model can be written as

$$y_{ij} = \mu + \alpha_i + e_{ij}; \quad i = 1, \ldots, k \qquad j = 1, \ldots, n \tag{2.1}$$

where y_{ij} is the jth observation made at the ith group, μ is an unknown constant representing the overall mean, α_i is the unknown effect due to the ith level, and e_{ij} is a random error. The α_i and e_{ij} are assumed to be independent random variables with mean zero and variance σ^2_α and σ^2_e, respectively. These assumptions imply a compound symmetric covariance structure of the y_{ij} variables within each factor level. This is the notation most commonly used for these types of models and can be found in Scheffé (1959); Burdick and Graybill (1992); Searle, Casella, and McCulloch (1992); and Sahai and Ojeda (2004).

The ANOVA table for the above model is shown in Table 2.1. The ANOVA estimators for σ^2_α, θ, and ρ_I are derived using the standard MSA and MSE estimates and are given by

$$\hat{\sigma}^2_\alpha = n^{-1}(MSA - MSE) \tag{2.2}$$

$$\hat{\theta} = \frac{MSA - MSE}{nMSE} \tag{2.3}$$

$$\hat{\rho}_I = \frac{MSA - MSE}{MSA + (n-1)MSE} \tag{2.4}$$

The estimator for ρ_n can be found using $\hat{\rho_I}$ and the Spearman-Brown formula (Furr 2008) and is given by

$$\hat{\rho_n} = \frac{n\hat{\rho_I}}{1 + \hat{\rho_I}(n-1)}. \tag{2.5}$$

We will use the ANOVA estimators, but several other types of estimators have been proposed. The most popular alternatives are the Maximum Likelihood estimators and the Restricted Maximum Likelihood estimators. Sahai and Ojeda (2004) also describe other estimators including modifications of the Maximum Likelihood estimators, Stein-type estimators, Federer's non-truncated exponential corrector estimators, Naqvi's goodness of fit estimators, Hodges-Lehmann estimators, and minimum variance unbiased estimators. One of the main reasons for choosing one of the alternative estimators is that the ANOVA estimators can give negative values.

Table 2.1 Analysis of Variance for a One-way Random Effects Model

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	Expected Mean Square
Groups	$k-1$	$SSA = n\sum_{i=1}^{k}(\bar{y}_{i.} - \bar{y}_{..})^2$	$MSA = \frac{SSA}{(k-1)}$	$EMSA = \sigma_e^2 + n\sigma_\alpha^2$
Within groups	$k(n-1)$	$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2$	$MSE = \frac{SSE}{k(n-1)}$	$EMSE = \sigma_e^2$
Total	$kn-1$	$SST = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2$		

2.2 Confidence Intervals

2.2.1 Wald Confidence Intervals

Assuming normality and independence of α_i and e_{ij}, the classical method for computing confidence intervals is the Wald method. The Wald confidence intervals for σ_α^2 and θ are

$$\hat{\sigma}_\alpha^2 \pm z_{1-\alpha/2} SE(\hat{\sigma}_\alpha^2) \tag{2.6}$$

and

$$\hat{\theta} \pm z_{1-\alpha/2} SE(\hat{\theta}) \tag{2.7}$$

respectively, where (see Scheffé p. 228, 230)

$$SE(\hat{\sigma}_\alpha^2) = \sqrt{2\left(\frac{(\hat{\sigma}_\alpha^2 + n^{-1}\hat{\sigma}_e^2)^2}{k-1}\right) + 2\left(\frac{(n^{-1}\hat{\sigma}_e^2)^2}{k(n-1)}\right)} \tag{2.8}$$

and

$$SE(\hat{\theta}) = \sqrt{\left(\hat{\theta} + \frac{1}{n}\right)^2 \left(\frac{2}{(k-1)} + \frac{2}{k(n-1)}\right)}. \tag{2.9}$$

The Wald confidence interval for ρ_I can be found by transforming the Wald confidence interval for θ using the relation

$$\rho_I = \frac{1}{1 + 1/\theta} \tag{2.10}$$

and the Wald confidence interval for ρ_n can be found by transforming the Wald confidence interval for ρ_I using the relation in Equation 2.5.

2.2.2 Exact Confidence Intervals

Assuming normality and independence of α_i and e_{ij}, exact confidence intervals for θ, ρ_I, and ρ_n are available. Scheffé (1959 p. 229, 231) gives results for θ and ρ_I, that may be expressed as

$$\left[\frac{\frac{MSA}{MSE} - F_{v_\alpha, v_e, 1-\alpha/2}}{nF_{v_\alpha, v_e, 1-\alpha/2}}, \frac{\frac{MSA}{MSE} - F_{v_\alpha, v_e, \alpha/2}}{nF_{v_\alpha, v_e, \alpha/2}}\right] \tag{2.11}$$

$$\left[\frac{\frac{MSA}{MSE} - F_{v_\alpha, v_e, 1-\alpha/2}}{\frac{MSA}{MSE} + (n-1)F_{v_\alpha, v_e, 1-\alpha/2}}, \frac{\frac{MSA}{MSE} - F_{v_\alpha, v_e, \alpha/2}}{\frac{MSA}{MSE} + (n-1)F_{v_\alpha, v_e, \alpha/2}}\right], \tag{2.12}$$

and Equation 2.12 may be transformed to give the following exact confidence interval for ρ_n

$$\left[\frac{n\left(\frac{MSA}{MSE} - F_{v_\alpha, v_e, 1-\alpha/2}\right)}{n\frac{MSA}{MSE}}, \frac{n\left(\frac{MSA}{MSE} - F_{v_\alpha, v_e, \alpha/2}\right)}{n\frac{MSA}{MSE}}\right] \tag{2.13}$$

where $v_\alpha = k - 1$ and $v_e = k(n-1)$. $F_{v_\alpha, v_e, 1-\alpha/2}$ and $F_{v_\alpha, v_e, \alpha/2}$ are the upper and lower-tail $\alpha/2$-level critical values of the F distribution with v_α and v_e degrees of freedom.

2.2.3 Approximate Normal Theory Confidence Intervals

Moriguti (1954) and Bulmer (1957) independently developed the same confidence limits for σ_α^2. Boardman (1974) showed that these two methods were identical and Bulmer (1957, p.163) showed that the method gives very accurate approximations. Scheffé(1959) gives general formulas for obtaining approximate confidence limits for variance components based on Bulmer's method. These formulas can be used to find approximate confidence limits for σ_α^2. The upper $1-\alpha/2$ confidence limit is given by

$$MSE * g_U(\mathcal{F})/n \tag{2.14}$$

where $\mathcal{F} = MSA/MSE$ and

$$g_U(\mathcal{F}) = F_{\infty,v_\alpha,\alpha/2}\mathcal{F} - 1 + \frac{1}{F_{v_e,v_\alpha,\alpha/2}\mathcal{F}}\left(1 - \frac{F_{\infty,v_\alpha,\alpha/2}}{F_{v_e,v_\alpha,\alpha/2}}\right) \tag{2.15}$$

for $\mathcal{F} \geq 1/F_{v_e,v_\alpha,\alpha/2}$ and $g_U(\mathcal{F}) = 0$ for $\mathcal{F} \leq 1/F_{v_e,v_\alpha,\alpha/2}$. The lower $1-\alpha/2$ confidence limit is given by

$$MSE * g_L(\mathcal{F}) \tag{2.16}$$

where

$$g_L(\mathcal{F}) = \frac{\mathcal{F}}{F_{\infty,v_\alpha,\alpha/2}} - 1 - \frac{F_{v_\alpha,v_e,\alpha/2}}{\mathcal{F}}\left(\frac{F_{v_\alpha,v_e,\alpha/2}}{F_{v_\alpha,\infty,\alpha/2}} - 1\right) \tag{2.17}$$

for $\mathcal{F} \geq 1/F_{v_\alpha,v_e,\alpha/2}$ and $g_L(\mathcal{F}) = 0$ for $\mathcal{F} \leq 1/F_{v_\alpha,v_e,\alpha/2}$.

Researchers in various fields use statistical software packages to analyze their data. Two common software packages used in the field are SAS and SPSS.

According to the SAS/STAT® 9.1 User's Guide, PROC MIXED provides chi-squared based confidence intervals for the variance components using a Satterthwaite approximation (SAS Institute 2004). For the one-way random effects ANOVA model, the approximate upper and lower $1-\alpha$ confidence limits for σ_α^2, given by SAS, are

$$\left[\frac{\nu\hat{\sigma}_\alpha^2}{{\chi^2}_{\nu,\alpha/2}}, \frac{\nu\hat{\sigma}_\alpha^2}{{\chi^2}_{\nu,1-\alpha/2}}\right] \tag{2.18}$$

where $\nu = 2(\frac{\hat{\sigma}_\alpha^2}{SE(\hat{\sigma}_\alpha^2)})^2$, $SE(\hat{\sigma}_\alpha^2)$ is the asymptotic standard error of $\hat{\sigma}_\alpha^2$ assuming normality (Eq 2.8), and $\hat{\sigma}_\alpha^2$ is the REML estimate of σ_α^2 given as

$$max\Big[0, n^{-1}(MSA - MSE)\Big]. \tag{2.19}$$

The User's Guide also says that PROC VARCOMP can be used to obtain confidence intervals for variance components and functions of variance components, specifically θ, if "method" is set equal to "Type I" or "GRR" (SAS Institute 2004). They refer to these confidence intervals as modified large sample (MLS) confidence intervals. In the balanced one-way random effects ANOVA model, the MLS confidence interval for σ_α^2 is the same as Scheffé's approximate normal theory interval, whose lower and upper bounds are given in Equations 2.14 and 2.16, and the MLS confidence interval for θ is the exact confidence interval given in Equation 2.11.

The MIXED procedure in SPSS® 16.0 uses the delta method to construct approximate Wald-type REML confidence intervals for $log(\sigma_\alpha^2)$ (SPSS 2007). These limits are then inverted to obtain approximate $1-\alpha$ confidence limits for σ_α^2, which are given by

$$exp\left[ln(\hat{\sigma}_\alpha^2) \pm z_{1-\alpha/2}\hat{\sigma}_\alpha^{-2}\sqrt{Var(\hat{\sigma}_\alpha^2)}\right] \tag{2.20}$$

where $Var(\hat{\sigma}_\alpha^2)$ is the asymptotic variance of $\hat{\sigma}_\alpha^2$ assuming normality (Eq 2.8) and $\hat{\sigma}_\alpha^2$ is the REML estimate of σ_α^2.

2.3 Effects of Non-normality

The methods described in Section 2.2 for constructing confidence intervals for the variance components require an assumption of normality. Simulation studies were performed to examine the effect of non-normality on confidence intervals for σ_α^2 and θ.

2.3.1 Monte Carlo Simulation Study Results

Monte Carlo simulation studies were used to evaluate the peformance of some of the methods described in Section 2.2 under non-normality. The simulation studies examined five values of k $(5, 10, 20, 40, 80)$ and five sample sizes per group $(10, 20, 40, 80, 160)$, for a total of 25 different conditions. Monte Carlo trials were generated under each of the 25 conditions. Within each of the 10,000 trials, 95% confidence intervals for σ_α^2 and θ were computed and were classified as capturing or not capturing the true value.

The true value of σ_α^2 was set to 1.0 and the α_i were distributed $\chi^2(5)$ and $t(5)$. Table 2.3 gives the results of the Monte Carlo study when $\alpha_i \sim \chi^2(5)$ and Table 2.4 gives the results when $\alpha_i \sim t(5)$. All of the existing methods give coverage probabilites significantly under the nominal level.

The true value of θ was also set to 1.0 and two cases were considered. The first looked at the case where the α_i were distributed $\chi^2(5)$ and the errors were normally distributed. The second looked at the case where the α_i were distributed $\chi^2(5)$ and the errors were distributed $Beta(1,3)$. Table 2.5 gives the results of the Monte Carlo study for the first case and Table 2.6 gives the results for the second case. In both cases, we see that the Wald method and the Exact method both result in undercoverage. The majority of the coverage probabilities are below 0.90 and for the Wald method, many of the coverage probabilities are even below 0.80.

2.3.2 Detecting Kurtosis

The results from the Monte Carlo simulations show that confidence intervals for σ_α^2 are greatly affected by the kurtosis of the random effect distribution and confidence intervals for θ are greatly affected by both the kurtosis of the random effect distribution and the kurtosis of the distribution of the errors. A power analysis was performed to determine how large k and n would need to be to detect these non-normal distributions with a power of 0.80 using a one-sided test of leptokurtosis (Pearson & Hartley, 1970). We found that the within-study sample size had little effect on the power of the test and that the power was determined primarily by k. The power of the test to detect a $\chi^2(5)$ and t(5) random effect distribution are given in Table 2.2 for sample sizes of 40 per group. The kurtosis of a $\chi^2(5)$ distribution is 5.4 and the kurtosis of a t(5) distribution is 9. It was verified that the leptokurtosis test had the proper power of 0.05 under normal random effect distributions. Note that k needs to be at least 150 to detect the degree of leptokurtosis that would cause serious problems with the currently used methods and studies with $k \geq 150$ are rare.

Table 2.2 Power Results Using an Upper One-sided Kurtosis Test with $\alpha = 0.05$

Distribution	k	Power
chisq(5)	100	0.723
	125	0.774
	150	0.814
	200	0.872
t(5)	100	0.664
	125	0.750
	150	0.804
	200	0.881

Table 2.3 Performance of Confidence Intervals for σ_α^2 for $\alpha_i \sim \chi^2(5)$

		Wald		SPSS		Scheffe		SAS	
		Prob	Avg width	Prob	Avg width	Prob	Avg width	Prob	Avg width
n=10	k=5	0.737	2.70	0.979	40.57	0.912	8.87	0.948	Inf (11)
	k=10	0.774	2.07	0.882	3.40	0.880	3.17	0.932	Inf (1)
	k=20	0.818	1.57	0.866	1.51	0.873	1.71	0.893	1.78
	k=40	0.857	1.18	0.852	1.01	0.860	1.09	0.866	1.10
	k=80	0.884	0.86	0.845	0.70	0.853	0.73	0.853	0.73
n=20	k=5	0.725	2.57	0.900	6.80	0.907	8.33	0.946	Inf (5)
	k=10	0.765	1.98	0.855	2.35	0.883	2.98	0.915	Inf (1)
	k=20	0.814	1.54	0.844	1.45	0.858	1.65	0.869	1.68
	k=40	0.852	1.16	0.837	0.97	0.850	1.03	0.855	1.04
	k=80	0.881	0.86	0.833	0.67	0.838	0.69	0.838	0.69
n=40	k=5	0.720	2.56	0.862	4.85	0.904	8.22	0.946	Inf (1)
	k=10	0.768	1.97	0.845	2.12	0.874	2.96	0.886	3.03
	k=20	0.803	1.52	0.836	1.40	0.855	1.59	0.860	1.61
	k=40	0.844	1.15	0.831	0.94	0.841	1.00	0.843	1.01
	k=80	0.886	0.85	0.827	0.65	0.832	0.68	0.833	0.67
n=80	k=5	0.711	2.48	0.837	3.91	0.896	7.96	0.938	Inf
	k=10	0.758	1.94	0.836	2.17	0.872	2.91	0.876	2.94
	k=20	0.802	1.50	0.829	1.38	0.853	1.58	0.855	1.59
	k=40	0.844	1.14	0.829	0.93	0.838	0.99	0.839	0.99
	k=80	0.885	0.85	0.829	0.64	0.834	0.66	0.834	0.66
n=160	k=5	0.716	2.49	0.841	3.81	0.903	7.99	0.914	3.20
	k=10	0.751	1.91	0.832	2.13	0.868	2.86	0.872	2.88
	k=20	0.798	1.50	0.827	1.37	0.849	1.57	0.850	1.57
	k=40	0.845	1.14	0.825	0.92	0.836	0.98	0.836	0.99
	k=80	0.878	0.85	0.816	0.64	0.823	0.66	0.823	0.66

Table 2.4 Performance of Confidence Intervals for σ_α^2 for $\alpha_i \sim t(5)$

		Wald		SPSS		Scheffe		SAS	
		Prob	Avg width	Prob	Avg width	Prob	Avg width	Prob	Avg width
n=10	k=5	0.751	2.83	0.975	38.51	0.907	8.90	0.845	Inf (16)
	k=10	0.788	2.09	0.878	3.81	0.886	3.16	0.936	Inf (1)
	k=20	0.808	1.43	0.851	1.51	0.864	1.72	0.888	1.79
	k=40	0.793	0.99	0.821	1.02	0.833	1.09	0.843	1.10
	k=80	0.795	0.69	0.812	0.70	0.820	0.73	0.821	0.73
n=20	k=5	0.745	2.70	0.905	7.88	0.911	8.35	0.950	Inf (10)
	k=10	0.779	2.01	0.853	2.24	0.886	2.98	0.915	94 mil
	k=20	0.799	1.36	0.835	1.43	0.861	1.63	0.872	1.66
	k=40	0.795	0.95	0.815	0.97	0.832	1.04	0.835	1.04
	k=80	0.787	0.66	0.801	0.67	0.813	0.70	0.813	0.69
n=40	k=5	0.733	2.61	0.847	4.16	0.905	8.00	0.949	Inf (2)
	k=10	0.775	1.97	0.837	2.16	0.875	2.89	0.890	351 mil
	k=20	0.781	1.33	0.813	1.39	0.842	1.49	0.845	1.60
	k=40	0.792	0.92	0.810	0.94	0.826	1.00	0.828	1.01
	k=80	0.782	0.64	0.793	0.65	0.805	0.67	0.805	0.67
n=80	k=5	0.746	2.63	0.850	3.84	0.909	8.03	0.845	29000
	k=10	0.776	2.00	0.842	2.17	0.875	2.92	0.880	2.95
	k=20	0.789	1.31	0.820	1.37	0.849	1.56	0.851	1.57
	k=40	0.786	0.91	0.806	0.93	0.822	0.99	0.823	0.99
	k=80	0.768	0.64	0.782	0.64	0.791	0.67	0.791	0.67
n=160	k=5	0.737	2.60	0.843	3.97	0.909	7.91	0.922	Inf
	k=10	0.779	1.96	0.836	2.13	0.877	2.86	0.881	2.88
	k=20	0.786	1.31	0.816	1.37	0.843	1.56	0.845	1.57
	k=40	0.777	0.91	0.793	0.92	0.810	0.99	0.810	0.99
	k=80	0.774	0.63	0.782	0.64	0.793	0.66	0.792	0.66

Table 2.5 Performance of Confidence Intervals for θ for $\alpha_i \sim \chi^2(5)$ and $\epsilon_{ij} \sim Normal$

		Wald		Exact	
		Prob	Avg width	Prob	Avg width
n=10	k=5	0.735	2.73	0.910	9.34
	k=10	0.797	2.15	0.892	3.39
	k=20	0.829	1.48	0.878	1.80
	k=40	0.834	1.03	0.860	1.13
	k=80	0.842	0.73	0.851	0.76
n=20	k=5	0.727	2.49	0.912	8.41
	k=10	0.785	1.99	0.886	3.07
	k=20	0.814	1.38	0.861	1.68
	k=40	0.822	0.96	0.851	1.05
	k=80	0.825	0.67	0.834	0.70
n=40	k=5	0.716	2.48	0.901	8.29
	k=10	0.776	1.94	0.876	2.99
	k=20	0.809	1.33	0.855	1.62
	k=40	0.819	0.92	0.844	1.01
	k=80	0.825	0.65	0.832	0.68
n=80	k=5	0.712	2.40	0.903	7.99
	k=10	0.776	1.88	0.875	2.90
	k=20	0.795	1.28	0.851	1.57
	k=40	0.818	0.90	0.847	0.99
	k=80	0.819	0.64	0.833	0.67
n=160	k=5	0.707	2.38	0.902	7.89
	k=10	0.756	1.87	0.867	2.89
	k=20	0.797	1.28	0.851	1.57
	k=40	0.812	0.89	0.838	0.99
	k=80	0.812	0.63	0.822	0.66

Table 2.6 Performance of Confidence Intervals for θ for $\alpha_i \sim \chi^2(5)$ and $\epsilon_{ij} \sim Beta(1,3)$

		Wald		Exact	
		Prob	Avg width	Prob	Avg width
n=10	k=5	0.724	2.72	0.910	9.30
	k=10	0.788	2.13	0.887	3.37
	k=20	0.818	1.48	0.871	1.80
	k=40	0.837	1.03	0.858	1.14
	k=80	0.847	0.72	0.857	0.76
n=20	k=5	0.725	2.56	0.909	8.64
	k=10	0.784	2.02	0.881	3.12
	k=20	0.813	1.38	0.863	1.68
	k=40	0.826	0.96	0.855	1.05
	k=80	0.827	0.67	0.837	0.70
n=40	k=5	0.719	2.47	0.909	8.26
	k=10	0.774	1.91	0.877	2.95
	k=20	0.799	1.31	0.853	1.60
	k=40	0.819	0.93	0.845	1.02
	k=80	0.820	0.65	0.831	0.68
n=80	k=5	0.708	2.41	0.902	8.03
	k=10	0.776	1.90	0.875	2.94
	k=20	0.802	1.30	0.850	1.59
	k=40	0.815	0.90	0.839	0.99
	k=80	0.819	0.64	0.830	0.67
n=160	k=5	0.708	2.42	0.900	8.03
	k=10	0.764	1.85	0.874	2.86
	k=20	0.797	1.28	0.856	1.57
	k=40	0.808	0.89	0.829	0.98
	k=80	0.823	0.63	0.829	0.66

CHAPTER 3. METHODS FOR σ_α^2

3.1 Proposed Method for σ_α^2

The proposed method for σ_α^2 was developed by starting with the chi-squared based method used by SAS. Recall, the approximate upper and lower $1-\alpha$ confidence limits for σ_α^2, given by SAS, are

$$\left[\frac{\nu\hat{\sigma}_\alpha^2}{{\chi^2}_{\nu,\alpha/2}}, \frac{\nu\hat{\sigma}_\alpha^2}{{\chi^2}_{\nu,1-\alpha/2}}\right] \tag{3.1}$$

where $\nu = 2\left(\frac{\hat{\sigma}_\alpha^2}{\sqrt{Var(\hat{\sigma}_\alpha^2)}}\right)^2$. In order to make these limits appropriate for both normal and non-normal distributions, the appoximate variance of $\hat{\sigma}_\alpha^2$ was derived under the assumption of non-normality using results given by Scheffé (1959, p.288, 346) and is given by

$$Var(\hat{\sigma}_\alpha^2) = \left(\frac{2}{k-1}\right)\left(\hat{\sigma}_\alpha^2 + \frac{\hat{\sigma}_e^2}{n}\right)^2 + \left(\frac{2}{k(n-1)}\right)\left(\frac{\hat{\sigma}_e^2}{n}\right)^2 + (\hat{\gamma}_\alpha - 3)\hat{\sigma}_\alpha^4/k \tag{3.2}$$

where $\hat{\gamma}_\alpha$ is Pearson's estimator of kurtosis and is equal to

$$\frac{k\sum(\hat{\mu}_i - \hat{\mu}_T)^4}{\left\{\sum(\hat{\mu}_i - \hat{\mu}_T)^2\right\}^2} \tag{3.3}$$

where $\hat{\mu}_i$ is the mean of group i and $\hat{\mu}_T$ is the overall mean for the groups. Bonett (2006b) notes that unless the sample size is very large, Pearson's estimator tends to have negative bias in leptokurtic distributions and recommends using the following estimator of γ_α, which is asymptotically equivalent to Pearson's estimator:

$$\bar{\gamma}_\alpha = \frac{k\sum(\hat{\mu}_i - m)^4}{\left\{\sum(\hat{\mu}_i - \hat{\mu}_T)^2\right\}^2} \tag{3.4}$$

where m is a trimmed mean with the trim proportion equal to $1/\{2\sqrt{(k-1)}\}$. Note that the trim proportion goes to zero as k goes to infinity. Bonett (2006a) shows that in both symmetric and skewed leptokurtic distributions, $\bar{\gamma}_\alpha$ tends to have less negative bias and a smaller coefficient of variability than Pearson's kurtosis estimator, therefore, Bonett's kurtosis estimator will be used in the proposed method. Shoemaker (2003) shows that for small sample sizes, the term $(\hat{\gamma}_\alpha - 3)\hat{\sigma}^4/k$ is better estimated by $(\hat{\gamma}_\alpha - 2 - (k-3)/k)\hat{\sigma}^4/k$. His small sample adjustment will be used in the proposed method when the kurtosis is estimated. Simulations showed that changing the denominator in the first and last term of Equation 3.2 to $k-2$ also improved the accuracy of the results for small k. This asymptotically equivalent version of Equation 3.2 will be used in the proposed method when the kurtosis is estimated.

3.2 Monte Carlo Study

Monte Carlo simulations were used to estimate coverage probabilities of the Wald, Scheffé, SAS, and the newly proposed confidence intervals for σ^2_α under various conditions. The simulation studies examined five values of k $(5, 10, 20, 40, 80)$ and five sample sizes per group $(10, 20, 40, 80, 160)$, for a total of 25 different conditions. The basic set-up for the simulations was to first simulate a random effect, α_i, for group i, under one of the distributions listed in Table 3.1, and then scale it to have a variance of one. Without loss of generality, the constant μ was set to zero, so α_i does not necessarily have an expectation of zero. Allowing the variance of the distribution of the random effect to vary affected the results of the simulation and will be discussed in Section 3.2.3. After simulating an α_i, n values were simulated from a distribution with mean α_i. This distribution was obtained by choosing one of the distributions in Table 3.1, possibly, but not necessarily the same distribution as the α_i, locating it so that it had a mean of zero and adding α_i. This was repeated for $i = 1, ..., k$ and resulted in k groups of n data points. These data were then used to calculate the various confidence intervals. For each of the 25 conditions, 10,000 Monte Carlo trials were generated, 95% confidence intervals for σ^2_α were calculated, and coverage probabilites were obtained by classifying each interval as capturing or not capturing the true value of σ^2_α.

Table 3.1 Kurtosis Values for Various Distributions

Distribution	Kurtosis
$Beta(3, 2)$	2.36
$Beta(1, 3)$	3.10
$Beta(9, .5)$	9.56
$\chi^2(5)$	5.4
$Gamma(5, 1)$	4.2
$Gamma(3, 2)$	5.0
$t(5)$	9.0
$Normal$	3.0

It is possible, especially when σ^2_α is small, to obtain negative, zero, or extremely small values of $\hat{\sigma}^2_\alpha$. When this occurrs, ν, in Equation 3.1, can be undefined and SAS will not calculate a confidence interval for σ^2_α. For our proposed method, when $\hat{\sigma}^2_\alpha$ is negative or zero, we set the lower limit equal to zero and and the upper limit is calculated using the standard error of $\hat{\sigma}^2_\alpha$ under normality, as given in Equation 2.8, and by replacing $\hat{\sigma}^2_\alpha$ with the upper limit from the Wald method. That is, the upper limit is $z_{1-\alpha/2}SE(\hat{\sigma}^2_\alpha)$, where $SE(\hat{\sigma}^2_\alpha)$ is the standard error of $\hat{\sigma}^2_\alpha$ under normality. Note that the last term in Equation 3.2 involving kurtosis is zero when $\hat{\sigma}^2_\alpha$ is zero, therefore using the standard error under normality is appropriate. For extremely small values of $\hat{\sigma}^2_\alpha$ that result in values of ν less than one, the proposed method will set $\nu = 1$.

In order to make the simulation results more comparable, a similar adjustment will be made to the Wald method when $\hat{\sigma}^2_\alpha$ is negative. The Wald limits in this situation will be

$$\left[0, z_{1-\alpha/2}SE(\hat{\sigma}^2_\alpha)\right] \tag{3.5}$$

where $SE(\hat{\sigma}^2_\alpha)$ is the standard error of $\hat{\sigma}^2_\alpha$ under normality.

All of the following tables of simulation results give the coverage probabilities as well as an adjusted average width, which is calculated as the average width of the intervals divided by the true value of σ^2_α. This adjustment was made because the average length of the intervals is related to the true value of σ^2_α. If the true value of σ^2_α is larger in one set of simulations than another, we expect the simulations with the larger value of σ^2_α to have wider intervals, everything else being equal. Making this adjustment allows us to compare the average widths from simulations with different values of σ^2_α.

For the SAS confidence intervals, the tables also give the number of trials in which the SAS method was unable to to calculate a confidence interval. This is indicated by a number in parentheses after the adjusted width.

3.2.1 Errors Distributed Normal

The first situation we looked at, was when the errors followed a normal distribution. That is, we simulated α_i from one of the distributions listed in Table 3.1 and then simulated data from a normal distribution with mean α_i. Without loss of generality, the variance of the normal distribution was set to one. We considered four different distributions for the random effect: Normal, $t(5)$, $Beta(9, .5)$, $Beta(3, 2)$, and $\chi^2(5)$. The results are given in Tables 3.2-3.6.

Table 3.2 gives the results for when both the random effect and the errors are distributed Normal. In this case, we expect the Scheffé method to perform the best since the conditions under which the method is appropriate have been met. We are interested in seeing if the proposed method gives reasonable results in this situation. The table shows that, as expected, the Scheffé method performs the best, but the proposed method also performs well when the number of groups is at least ten. The proposed method performs better than both the Wald and SAS methods when both the distributions are Normal. The Wald method consistently gives results below the nominal level, only getting close to the nominal level when there are at least 80 groups. The SAS method does alright when there are at least 20 groups, but gives average widths of Infinity when the k and n are too small.

Table 3.3 gives the results for when the random effect is distributed $t(5)$. Now the proposed method performs the best with the majority of coverage probabilities above 0.90. The other methods consistently obtain coverage probabilities well below the nominal level. The SAS method again gives average widths of infinity when the k and n are too small.

Table 3.4 gives the results for when the random effect is distributed $Beta(9, .5)$. This table also includes a column labeled “Proposed-Beta”, which will be discussed in Section 3.3.3. From this table, we can see that the proposed method performs the best. Although the proposed method tends to have slight undercoverage under these conditions, it still performs better than the Wald, Scheffé, and SAS methods, which consistently give coverage probabilities below 0.80

and in the case of the Wald method, some coverage probabilities even below 0.60.

Table 3.5 gives the results for when the random effect is distributed $Beta(3, 2)$. Except for the cases with the smallest number of groups, the proposed method again peforms the best. In this situation, the Scheffé and SAS methods tend to have overcoverage, while the Wald method has undercoverage until the number of groups is quite large.

Table 3.6 gives the results for when the random effect is distributed $\chi^2(5)$. The proposed method consistently gives coverage probabilities closer to the nominal level than any of the other three methods, with the majority of coverage probabilites above 0.90. The other three methods give coverage probabilities below 0.90. Although the SAS method provides some coverage probabilties that are very close to the nominal level, these probabilities correspond to average widths of infinity.

3.2.2 Errors Distributed Non-Normal

Since the calculation for the variance of $\hat{\sigma}_\alpha^2$ does not include an estimate of the kurtosis of the distribution of the errors, we did not expect the confidence intervals for σ_α^2 to change much when the distribution of the errors was non-normal. To test this assumption we ran a few simulations where both the distribution of the random effect and the distribution of the errors were non-normal. For these simulations we again began by simulating α_i from one of the non-normal distributions in Table 3.1 and then simulated data from a different non-normal distribution from Table 3.1, which was located to have a mean of α_i. The results for a few different combinations of distributions are given in Tables 3.9 and 3.10.

Table 3.9 gives the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed $Beta(1, 3)$. Table 3.10 gives the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed $Gamma(3, 2)$. If we compare these tables to Table 3.5, where the random effect was distributed $\chi^2(5)$ and the errors were distributed Normal, we do not see any differences beyond what would be expected from sampling error.

3.2.3 Varying the value of σ_α^2

As was mentioned at the beginning of this section, changing the variance of the distribution of the random effect affects the results. This is due to the fact that when σ_α^2 is close to zero, we obtain negative estimates of σ_α^2. The closer the true value is to zero, the more negative estimates we obtain. Through simulation, we found that if σ_α^2 is greater than five, we get very few, if any, negative estimates, and therefore our results for values of σ_α^2 greater than five are very similar. To demonstrate how the value of σ_α^2 affects the coverage probabilities, Tables 3.6-3.8 give the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed Normal for $\sigma_\alpha^2 = 0.5, 1.0$, and 5.0.

3.3 Methods Using Prior Kurtosis Information

Any information the researcher can provide about the distribution of the random effect can be helpful in obtaining more accurate estimates of the kurtosis of the random effect and therefore more accurate confidence intervals for the variance of the random effect. In the following sections, we will explain how each of these types of kurtosis estimates can be used with the proposed method and give some simulation results.

3.3.1 Kurtosis Estimates from Previous Studies

One way in which a researcher can obtain information about the distribution of the random effect is from previous studies. If previous study information is available, the kurtosis estimates from these studies can be pooled with the kurtosis estimate from the current study to hopefully provide a more accurate estimate due to a larger sample.

Laylard (1973) notes that a pooled kurtosis value is best estimated by first calculating a pooled estimate of the fourth moment and a pooled estimate of the variance and then plugging these estimates into the formula for kurtosis, rather than pooling the kurtosis values from each study. So, we will calculate the pooled kurtosis as

$$\hat{\gamma}_\alpha^* = \hat{\mu}_{4,pooled}/\widehat{var}^2_{pooled} \tag{3.6}$$

where

$$\hat{\mu}_{4,pooled} = \frac{\sum(\hat{\mu}_i - \hat{\mu}_T)^4 + \sum(\hat{\mu}_{i,prev} - \hat{\mu}_{T,prev})^4}{k+j} \tag{3.7}$$

and

$$\widehat{var}_{pooled} = \frac{\sum(\hat{\mu}_i - \hat{\mu}_T)^2 + \sum(\hat{\mu}_{i,prev} - \hat{\mu}_{T,prev})^2}{k+j} \tag{3.8}$$

where j is the number of groups in the previous study, μ_i is the mean of group i in the current study, μ_T is the overall mean of the current study, $\mu_{i,prev}$ is the mean of group i in the previous study, and $\mu_{T,prev}$ is the overall mean of the previous study. $\hat{\gamma}^*_\alpha$ can then be used in Equation (3.2).

Table 3.11 gives the results for when the random effect is distributed $\chi^2(5)$, the errors are distributed Normal, and there is information from a previous study with k=40 and n=100. This combination of distributions was already considered in Table 3.6 for the Wald, Scheffé, Proposed, and SAS methods, where it was shown that the proposed method performed the best. All of the methods are included in this table for easy comparison, but we are most interested in comparing the proposed method using previous information to the proposed method without using previous information. We see that there does seem to be an advantage when using previous study information, particularly when k is at least 10.

3.3.2 Theoretically Specified Kurtosis Values

A researcher may also be able to theoretically specify a kurtosis value for the distribution of the random effect based on expert knowledge about their field of study. The researcher may know that the type of data they are anlayzing is known to have means that follow a specific distribution. In this situation, the $\hat{\gamma}_\alpha$ in Equation (3.2) will be replaced with the theoretical kurtosis value for the specific distribution.

Table 3.12 gives the results for when the random effect is distributed $\chi^2(5)$, the errors are distributed Normal, and there is a theoretically specified value for the kurtosis of the distribution of the random effect. Again, this combination of distributions was already considered earlier for the Wald, Scheffé, Proposed and SAS methods, where it was shown that the proposed method performed the best. All of the methods are included in this table for easy comparison, but we are most interested in comparing the proposed method using a known kurtosis value to the proposed method using an estimated kurtosis value. The results show that being able to specify a theoretical kurtosis value has an advantage regardless of the values of k and n.

3.3.3 Distribution Based Kurtosis Estimate

Lastly, there are some distributions in which kurtosis can be estimated by first estimating the parameters of the distribution and then using these parameter estimates to calculate a kurtosis value. Some examples of these types of distributions include the beta distribution and the gamma distribution.

If a researcher is looking at data whose values are confined to a finite interval, as would be the case for likert-type data or test score data, it may be reasonable to approximate the distribution of the data with a scaled beta distribution. It is known that the kurtosis of a beta distribution can be calculated as

$$\gamma = 6\left[\frac{(a+b)^2(a+b+1) - ab(a+b+2)}{ab(a+b+2)(a+b+3)}\right] + 3 \tag{3.9}$$

where a and b are the shape parameters of the beta distribution (Evans *et al*, 1993).

The method of moments estimators of the parameters of the beta distribution have a closed form and will be used here. Since we are trying to estimate the kurtosis of the distribution of the random effect, we will need to estimate the parameters of the random effect distribution. If we are assuming the distribution of the random effect can be approximated using a scaled beta distribution on the interval $(0, c)$, then the method of moments estimators of a and b are as follows

$$\hat{a} = \bar{y}_{..}\left[\frac{\bar{y}_{..}(c^3 - c^2\bar{y}_{..}) - c^2\hat{\sigma}^2_\alpha}{c^3\hat{\sigma}^2_\alpha}\right] \tag{3.10}$$

and

$$\hat{b} = (c - \bar{y}_{..})\left[\frac{\bar{y}_{..}(c^3 - c^2\bar{y}_{..}) - c^2\hat{\sigma}^2_\alpha}{c^3\hat{\sigma}^2_\alpha}\right] \tag{3.11}$$

where $\bar{y}_{..}$ is the sample mean of the random effect and $\hat{\sigma}^2_\alpha$ is the ANOVA estimate of the variance of the random effect as given in Equation 2.2. These estimates can be plugged into Equation 3.9, to obtain a kurtosis estimate that can be used in Equation 3.2 to approximate the variance of $\hat{\sigma}^2_\alpha$.

If a researcher is looking at data whose values are on the interval zero to infinity, it may be reasonable to approximate the distribution of the random effect with a gamma distribution. It is known that the kurtosis of a gamma distribution can be calculated as

$$\gamma = 3 + 6/a \tag{3.12}$$

where a is the shape parameter of the gamma distribution (Evans *et al*, 1993). As with the parameters of the beta distribution, the method of moments estimators of the parameters of the gamma distribtuion have a closed form and will be used here. The method of moments estimator of the shape parameter, a, of the gamma distribution is

$$\hat{a} = (\bar{y}_{..}/\hat{\sigma}_\alpha)^2 \tag{3.13}$$

where $\bar{y}_{..}$ is the sample mean of the random effect and $\hat{\sigma}_\alpha$ is the square root of the ANOVA estimate of the variance of the random effect as given in Equation 2.2. This estimate of the shape parameter can be plugged in to Equation 3.12 to obtain a kurtosis estimate that can be used in Equation 3.2 to approximate the variance of $\hat{\sigma}^2_\alpha$.

Table 3.6 gives the results for this method when the random effect is distributed $\chi^2(5)$ and the errors are distributed Normal in the column labeled "Proposed-Gamma". The χ^2 distribution is a special case of the gamma distribution, so estimates of the gamma distribution parameters were used to estimate the kurtosis of the random effect distribution. We already compared the other methods in Section 3.2.1, and found that the proposed method performed the best. Now, comparing the proposed method to the proposed method using gamma parameter estimates, we see that in the majority of cases, the proposed method using gamma parameter estimates performs better than the proposed method, particularly when k is at least 10.

Table 3.4 gives the results for this method when the random effects are distributed $Beta(9, .5)$ and the errors are distributed Normal in the column labeled "Proposed-Beta". Since the random effect is distributed Beta, estimates of the beta distribution parameters were used to estimate the kurtosis of the random effect distribution. We already compared the other methods in Section 3.2.1, and found that the proposed method performed the best. Now, comparing the proposed method to the proposed method using the beta parameter estimates, we see that there is a significant advantage when using the beta parameter estimates, particularly when k is at least 10.

Tables 3.5, 3.9, and 3.10 also include results for the proposed method using parameter based estimates of kurtosis. As in the two tables described above, these three tables also show an advantage when using parameter based estimates of kurtosis.

Table 3.2 Comparison of Methods when Random Effects are Distributed Normal

		Wald		Scheffe		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.822	2.81	0.953	8.78	0.972	184	0.970	Inf (137)
	k=10	0.873	2.07	0.946	3.15	0.969	4.87	0.970	Inf (3)
	k=20	0.904	1.44	0.950	1.71	0.949	1.86	0.959	1.78
	k=40	0.927	0.99	0.950	1.09	0.947	1.10	0.951	1.10
	k=80	0.941	0.69	0.952	0.73	0.947	0.73	0.951	0.73
n=20	k=5	0.816	2.69	0.949	8.32	0.981	147	0.872	Inf (47)
	k=10	0.871	2.02	0.952	3.03	0.960	4.30	0.964	3.19
	k=20	0.907	1.37	0.954	1.64	0.945	1.74	0.957	1.67
	k=40	0.928	0.95	0.952	1.04	0.942	1.04	0.954	1.04
	k=80	0.941	0.66	0.953	0.69	0.947	0.69	0.953	0.69
n=40	k=5	0.822	2.63	0.944	8.11	0.983	135	0.972	Inf (10)
	k=10	0.872	2.01	0.951	2.93	0.950	4.07	0.958	2.99
	k=20	0.906	1.33	0.951	1.60	0.942	1.68	0.953	1.61
	k=40	0.926	0.92	0.951	1.01	0.941	1.01	0.952	1.01
	k=80	0.941	0.65	0.951	0.67	0.944	0.67	0.951	0.67
n=80	k=5	0.821	2.64	0.949	8.01	0.984	138	0.971	Inf (4)
	k=10	0.867	1.98	0.948	2.91	0.943	3.92	0.951	2.94
	k=20	0.910	1.32	0.949	1.58	0.936	1.64	0.950	1.58
	k=40	0.927	0.91	0.947	0.99	0.936	1.00	0.947	1.00
	k=80	0.934	0.64	0.950	0.66	0.941	0.66	0.950	0.66
n=160	k=5	0.815	2.62	0.953	7.91	0.981	128	0.961	Inf (1)
	k=10	0.871	1.98	0.951	2.88	0.942	3.87	0.952	2.89
	k=20	0.904	1.31	0.949	1.56	0.934	1.62	0.950	1.56
	k=40	0.930	0.91	0.947	0.98	0.934	0.98	0.947	0.98
	k=80	0.938	0.63	0.949	0.66	0.939	0.66	0.949	0.66

Table 3.3 Comparison of Methods when Random Effects are Distributed $t(5)$

		Wald		Scheffe		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.716	2.52	0.907	8.86	0.945	248	0.942	Inf (282)
	k=10	0.765	2.04	0.888	3.17	0.961	10.87	0.936	Inf (12)
	k=20	0.799	1.60	0.861	1.72	0.919	3.44	0.887	1.79
	k=40	0.839	1.23	0.840	1.09	0.908	1.88	0.849	1.10
	k=80	0.866	0.93	0.826	0.73	0.913	1.23	0.829	0.73
n=20	k=5	0.694	2.36	0.909	8.18	0.961	214	0.950	Inf (99)
	k=10	0.757	1.97	0.876	3.01	0.937	9.69	0.912	Inf
	k=20	0.801	1.55	0.850	1.64	0.901	3.25	0.862	1.67
	k=40	0.839	1.20	0.831	1.03	0.899	1.84	0.835	1.04
	k=80	0.864	0.93	0.803	0.69	0.907	1.25	0.803	0.69
n=40	k=5	0.695	2.39	0.902	8.21	0.970	218	0.945	Inf (25)
	k=10	0.754	1.96	0.870	2.97	0.911	8.67	0.884	3.04
	k=20	0.801	1.54	0.848	1.60	0.894	3.36	0.851	1.61
	k=40	0.831	1.19	0.825	1.00	0.897	1.76	0.827	1.00
	k=80	0.862	0.91	0.803	0.67	0.901	1.16	0.803	0.67
n=80	k=5	0.689	2.38	0.899	8.14	0.967	218	0.941	Inf (13)
	k=10	0.750	1.92	0.873	2.91	0.904	9.08	0.878	2.94
	k=20	0.791	1.51	0.852	1.57	0.894	3.08	0.856	1.58
	k=40	0.837	1.19	0.822	1.00	0.892	1.77	0.823	1.00
	k=80	0.864	0.91	0.796	0.66	0.906	1.10	0.796	0.66
n=160	k=5	0.689	2.34	0.904	7.98	0.962	204	0.915	957K (3)
	k=10	0.748	1.89	0.874	2.87	0.903	8.69	0.877	2.88
	k=20	0.797	1.50	0.848	1.56	0.887	3.12	0.850	1.56
	k=40	0.831	1.19	0.816	0.99	0.895	1.84	0.817	0.99
	k=80	0.859	0.91	0.792	0.66	0.899	1.16	0.791	0.66

Table 3.4 Comparison of Methods when Random Effects are Distributed $Beta(9, .5)$

		Wald		Scheffe		Proposed-Beta		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.591	2.56	0.815	8.82	0.888	418	0.893	495	0.917	Inf (46)
	k=10	0.653	2.14	0.744	3.14	0.957	70.62	0.945	22.81	0.897	Inf (8)
	k=20	0.729	1.82	0.721	1.73	0.963	7.93	0.876	5.49	0.760	Inf
	k=40	0.796	1.47	0.711	1.09	0.949	2.97	0.885	2.58	0.720	1.10
	k=80	0.840	1.11	0.695	0.73	0.945	1.63	0.892	1.48	0.694	0.73
n=20	k=5	0.564	2.47	0.779	8.37	0.917	368	0.921	534	0.917	Inf (27)
	k=10	0.656	2.13	0.724	3.04	0.962	52.13	0.905	24.19	0.813	Inf
	k=20	0.734	1.82	0.703	1.65	0.950	7.29	0.862	5.86	0.717	2163
	k=40	0.792	1.46	0.683	1.03	0.948	2.85	0.882	2.67	0.686	1.03
	k=80	0.846	1.13	0.687	0.69	0.951	1.60	0.910	1.53	0.687	0.69
n=40	k=5	0.564	2.42	0.760	8.15	0.938	348	0.943	535	0.916	Inf (13)
	k=10	0.642	2.12	0.710	2.97	0.955	44.83	0.854	26.34	0.733	Inf
	k=20	0.721	1.80	0.683	1.59	0.946	6.81	0.854	6.16	0.689	1.61
	k=40	0.789	1.47	0.667	1.00	0.947	2.81	0.885	2.70	0.668	1.00
	k=80	0.843	1.13	0.668	0.67	0.950	1.58	0.909	1.55	0.669	0.67
n=80	k=5	0.557	2.44	0.756	8.13	0.950	341	0.951	568	0.895	Inf (5)
	k=10	0.637	2.07	0.703	2.89	0.946	37.93	0.848	25.80	0.712	Inf
	k=20	0.726	1.78	0.682	1.57	0.944	6.61	0.858	6.13	0.685	1.58
	k=40	0.797	1.48	0.671	0.99	0.948	2.80	0.891	2.80	0.671	1.00
	k=80	0.852	1.15	0.663	0.67	0.949	1.59	0.914	1.58	0.663	0.67
n=160	k=5	0.558	2.44	0.746	8.14	0.953	331	0.928	579	0.780	Inf (2)
	k=10	0.642	2.09	0.698	2.90	0.946	37.22	0.846	27.43	0.703	2.96
	k=20	0.721	1.81	0.665	1.58	0.943	6.57	0.857	6.39	0.667	1.58
	k=40	0.802	1.48	0.670	0.99	0.949	2.78	0.895	2.80	0.670	0.99
	k=80	0.859	1.15	0.665	0.66	0.949	1.59	0.918	1.60	0.665	0.66

Table 3.5 Comparison of Methods when Random Effects are Distributed $Beta(3, 2)$

		Wald		Scheffe		Proposed-Beta		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.810	2.39	0.970	8.66	0.943	11.02	0.983	174	0.985	Inf (7)
	k=10	0.877	1.81	0.973	3.17	0.953	2.71	0.973	4.24	0.987	Inf (1)
	k=20	0.916	1.25	0.974	1.72	0.953	1.44	0.959	1.65	0.980	1.79
	k=40	0.938	0.87	0.978	1.09	0.951	0.91	0.959	0.98	0.980	1.10
	k=80	0.949	0.61	0.979	0.73	0.952	0.61	0.960	0.64	0.979	0.73
n=20	k=5	0.814	2.32	0.969	8.32	0.953	6.72	0.990	151	0.988	Inf (3)
	k=10	0.873	1.69	0.974	3.00	0.953	2.34	0.967	3.66	0.984	1617
	k=20	0.915	1.17	0.978	1.64	0.950	1.32	0.954	1.50	0.981	1.67
	k=40	0.932	0.81	0.977	1.03	0.948	0.85	0.953	0.90	0.978	1.04
	k=80	0.943	0.56	0.979	0.69	0.949	0.57	0.953	0.60	0.979	0.69
n=40	k=5	0.806	2.27	0.969	8.06	0.959	5.56	0.993	143	0.988	Inf (1)
	k=10	0.877	1.64	0.974	2.93	0.949	2.21	0.960	3.43	0.979	3.00
	k=20	0.910	1.12	0.977	1.59	0.947	1.26	0.951	1.43	0.980	1.60
	k=40	0.929	0.77	0.978	1.00	0.948	0.81	0.952	0.86	0.979	1.01
	k=80	0.940	0.54	0.979	0.67	0.946	0.55	0.949	0.57	0.979	0.67
n=80	k=5	0.813	2.25	0.971	7.98	0.958	5.23	0.993	135	0.988	31.84 (1)
	k=10	0.883	1.63	0.974	2.92	0.943	2.16	0.955	3.37	0.976	2.96
	k=20	0.919	1.10	0.979	1.58	0.947	1.24	0.947	1.40	0.979	1.59
	k=40	0.934	0.76	0.980	0.99	0.948	0.80	0.948	0.84	0.981	0.99
	k=80	0.943	0.53	0.981	0.66	0.949	0.54	0.951	0.56	0.980	0.66
n=160	k=5	0.815	2.26	0.970	7.98	0.944	5.11	0.988	132	0.975	Inf
	k=10	0.875	1.60	0.976	2.86	0.947	2.11	0.956	3.31	0.976	2.87
	k=20	0.916	1.09	0.980	1.56	0.947	1.22	0.950	1.37	0.981	1.57
	k=40	0.929	0.75	0.980	0.98	0.949	0.79	0.949	0.83	0.981	0.99
	k=80	0.938	0.52	0.981	0.66	0.947	0.54	0.949	0.55	0.981	0.66

Table 3.6 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$

		Wald		Scheffe		Proposed-Gamma		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.716	2.46	0.912	8.73	0.960	204	0.953	280	0.947	Inf (268)
	k=10	0.776	1.97	0.893	3.14	0.985	11.56	0.968	9.50	0.940	Inf (5)
	k=20	0.816	1.55	0.871	1.73	0.969	3.46	0.926	2.94	0.891	1.80
	k=40	0.849	1.16	0.862	1.08	0.962	1.76	0.918	1.60	0.869	1.09
	k=80	0.885	0.86	0.853	0.73	0.956	1.09	0.924	1.01	0.854	0.73
n=20	k=5	0.710	2.40	0.907	8.39	0.976	168	0.969	271	0.943	Inf (84)
	k=10	0.757	1.92	0.877	3.01	0.978	10.29	0.942	9.16	0.910	11.80 (1)
	k=20	0.803	1.50	0.865	1.64	0.961	3.28	0.904	2.98	0.873	1.67
	k=40	0.852	1.15	0.844	1.04	0.952	1.73	0.904	1.61	0.849	1.04
	k=80	0.876	0.84	0.838	0.69	0.953	1.06	0.915	0.99	0.837	0.69
n=40	k=5	0.702	2.37	0.903	8.20	0.984	166	0.976	261	0.943	Inf (28)
	k=10	0.750	1.88	0.870	2.94	0.965	9.86	0.915	9.37	0.882	3.01
	k=20	0.800	1.47	0.859	1.59	0.960	3.18	0.895	2.93	0.863	1.61
	k=40	0.843	1.13	0.837	1.00	0.955	1.68	0.903	1.59	0.839	1.01
	k=80	0.876	0.84	0.828	0.67	0.953	1.04	0.916	1.00	0.828	0.67
n=80	k=5	0.689	2.33	0.899	8.01	0.985	151	0.978	261	0.942	Inf (7)
	k=10	0.754	1.86	0.877	2.90	0.962	9.77	0.909	9.16	0.882	2.93
	k=20	0.795	1.47	0.845	1.58	0.955	3.16	0.885	2.95	0.848	1.59
	k=40	0.846	1.12	0.841	0.99	0.957	1.67	0.906	1.60	0.841	0.99
	k=80	0.880	0.84	0.831	0.66	0.954	1.04	0.919	1.01	0.931	0.66
n=160	k=5	0.688	2.29	0.899	7.87	0.963	151	0.969	253	0.912	Inf (3)
	k=10	0.746	1.85	0.874	2.88	0.958	9.56	0.901	9.32	0.876	2.90
	k=20	0.800	1.46	0.848	1.58	0.959	3.16	0.887	2.96	0.850	1.58
	k=40	0.844	1.12	0.838	0.98	0.956	1.66	0.904	1.60	0.839	0.98
	k=80	0.876	0.84	0.831	0.66	0.954	1.04	0.918	1.00	0.832	0.66

Table 3.7 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and $\tau^2 = 0.5$

		Wald		Scheffe		Proposed-Gamma		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.729	2.62	0.916	9.52	0.933	308	0.923	332	0.937	Inf (676)
	k=10	0.783	2.08	0.897	3.42	0.977	17.06	0.953	12.98	0.934	Inf (68)
	k=20	0.828	1.64	0.878	1.88	0.981	3.94	0.959	3.20	0.927	Inf (2)
	k=40	0.854	1.22	0.868	1.18	0.964	1.92	0.930	1.66	0.889	1.22
	k=80	0.884	0.88	0.867	0.80	0.955	1.16	0.927	1.04	0.873	0.80
n=20	k=5	0.715	2.48	0.903	8.82	0.958	220	0.946	274	0.939	Inf (292)
	k=10	0.768	2.00	0.885	3.16	0.983	12.06	0.963	9.76	0.933	Inf (9)
	k=20	0.814	1.54	0.871	1.72	0.967	3.42	0.927	2.90	0.893	1.78
	k=40	0.853	1.16	0.860	1.08	0.964	1.76	0.920	1.58	0.866	1.10
	k=80	0.884	0.86	0.847	0.72	0.952	1.10	0.919	1.02	0.850	0.72
n=40	k=5	0.696	2.34	0.907	8.16	0.980	162	0.973	248	0.949	Inf (75)
	k=10	0.745	1.88	0.878	2.96	0.978	10.24	0.938	8.84	0.911	Inf (1)
	k=20	0.813	1.50	0.862	1.64	0.963	3.28	0.906	2.94	0.873	1.66
	k=40	0.848	1.12	0.849	1.02	0.958	1.70	0.907	1.56	0.852	1.04
	k=80	0.879	0.84	0.836	0.68	0.948	1.06	0.917	1.02	0.838	0.70
n=80	k=5	0.693	2.30	0.906	7.92	0.984	150	0.977	248	0.951	Inf (24)
	k=10	0.754	1.88	0.875	2.92	0.967	9.84	0.919	9.42	0.888	2.98
	k=20	0.801	1.48	0.852	1.60	0.962	3.22	0.892	2.94	0.858	1.62
	k=40	0.840	1.12	0.833	1.00	0.951	1.68	0.898	1.58	0.835	1.00
	k=80	0.873	0.84	0.832	0.66	0.953	1.04	0.917	1.00	0.832	0.66
n=160	k=5	0.691	2.30	0.904	7.92	0.987	136	0.978	248	0.942	Inf (7)
	k=10	0.738	1.84	0.868	2.88	0.960	9.44	0.898	9.16	0.874	2.90
	k=20	0.794	1.46	0.850	1.56	0.958	3.12	0.887	2.96	0.852	1.58
	k=40	0.843	1.12	0.836	1.00	0.954	1.68	0.903	1.60	0.837	1.00
	k=80	0.879	0.84	0.834	0.66	0.954	1.04	0.920	1.00	0.834	0.66

Table 3.8 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and $\tau^2 = 5.0$

		Wald		Scheffe		Proposed-Gamma		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.701	2.36	0.901	8.14	0.983	157	0.977	268	0.944	Inf (23)
	k=10	0.752	1.86	0.875	2.92	0.964	9.79	0.914	9.14	0.884	2.97
	k=20	0.798	1.47	0.855	1.60	0.958	3.20	0.889	2.93	0.858	1.60
	k=40	0.848	1.13	0.846	1.01	0.957	1.69	0.906	1.60	0.845	1.01
	k=80	0.885	0.84	0.840	0.68	0.956	1.05	0.923	1.01	0.836	0.67
n=20	k=5	0.702	2.34	0.907	8.04	0.976	142	0.975	266	0.928	Inf (5)
	k=10	0.745	1.85	0.871	2.89	0.960	9.63	0.905	9.14	0.876	2.92
	k=20	0.800	1.45	0.855	1.57	0.963	3.12	0.893	2.95	0.856	1.57
	k=40	0.840	1.12	0.836	0.99	0.956	1.66	0.901	1.58	0.836	0.99
	k=80	0.882	0.83	0.833	0.66	0.953	1.03	0.920	1.00	0.831	0.66
n=40	k=5	0.687	2.33	0.895	7.99	0.963	144	0.964	264	0.904	12,800
	k=10	0.741	1.82	0.874	2.85	0.961	9.15	0.902	8.76	0.876	2.86
	k=20	0.794	1.46	0.850	1.57	0.958	3.13	0.885	2.93	0.850	1.57
	k=40	0.849	1.13	0.842	0.99	0.960	1.67	0.905	1.61	0.842	0.99
	k=80	0.882	0.84	0.833	0.66	0.954	1.03	0.918	1.00	0.832	0.66
n=80	k=5	0.694	2.27	0.906	7.80	0.966	134	0.966	238	0.910	7.85
	k=10	0.750	1.85	0.871	2.88	0.959	9.50	0.897	9.26	0.873	2.89
	k=20	0.797	1.45	0.854	1.56	0.959	3.12	0.890	2.93	0.854	1.56
	k=40	0.842	1.12	0.835	0.98	0.958	1.66	0.901	1.59	0.834	0.98
	k=80	0.882	0.84	0.831	0.66	0.955	1.04	0.925	1.01	0.831	0.66
n=160	k=5	0.692	2.33	0.902	8.01	0.961	149	0.961	258	0.903	8.03
	k=10	0.746	1.85	0.873	2.88	0.962	9.88	0.898	9.56	0.873	2.89
	k=20	0.800	1.45	0.851	1.56	0.957	3.10	0.886	2.93	0.850	1.56
	k=40	0.846	1.12	0.841	0.98	0.958	1.66	0.905	1.60	0.840	0.98
	k=80	0.878	0.83	0.827	0.66	0.954	1.03	0.917	1.00	0.827	0.65

Table 3.9 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors Distributed $Beta(1,3)$

		Wald		Scheffe		Proposed-Gamma		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.713	2.44	0.909	8.68	0.958	203	0.951	271	0.943	Inf (274)
	k=10	0.770	2.00	0.883	3.18	0.982	12.02	0.963	10.02	0.932	Inf (11)
	k=20	0.812	1.54	0.864	1.72	0.967	3.43	0.921	2.98	0.887	1.79
	k=40	0.855	1.17	0.855	1.09	0.959	1.78	0.914	1.61	0.863	1.10
	k=80	0.881	0.85	0.851	0.73	0.957	1.09	0.920	1.01	0.853	0.72
n=20	k=5	0.699	2.39	0.902	8.34	0.974	177	0.968	279	0.946	Inf (106)
	k=10	0.761	1.91	0.877	3.00	0.981	10.53	0.938	9.25	0.912	Inf (1)
	k=20	0.812	1.49	0.861	1.64	0.962	3.26	0.905	2.93	0.871	1.67
	k=40	0.843	1.14	0.847	1.03	0.953	1.71	0.907	1.59	0.850	1.03
	k=80	0.884	0.85	0.838	0.69	0.951	1.07	0.916	1.01	0.838	0.69
n=40	k=5	0.694	2.35	0.905	8.11	0.981	155	0.975	267	0.945	Inf (34)
	k=10	0.757	1.88	0.878	2.95	0.969	10.03	0.920	9.19	0.891	3.02
	k=20	0.807	1.48	0.864	1.61	0.965	3.21	0.899	2.91	0.868	1.62
	k=40	0.848	1.13	0.840	1.01	0.955	1.70	0.906	1.60	0.842	1.01
	k=80	0.874	0.85	0.825	0.67	0.954	1.05	0.917	1.01	0.826	0.67
n=80	k=5	0.686	2.34	0.900	8.03	0.986	152	0.977	265	0.938	Inf (2)
	k=10	0.748	1.84	0.877	2.87	0.963	10.25	0.907	9.23	0.883	2.91
	k=20	0.792	1.46	0.850	1.57	0.958	3.13	0.888	2.88	0.853	1.58
	k=40	0.850	1.14	0.837	1.00	0.957	1.68	0.906	1.62	0.838	1.00
	k=80	0.881	0.83	0.830	0.66	0.954	1.03	0.918	1.00	0.830	0.66
n=160	k=5	0.696	2.34	0.904	8.05	0.967	148	0.970	261	0.917	Inf
	k=10	0.750	1.83	0.876	2.86	0.965	9.30	0.902	9.07	0.878	2.88
	k=20	0.788	1.44	0.855	1.55	0.958	3.09	0.885	2.89	0.856	1.56
	k=40	0.846	1.12	0.841	0.99	0.957	1.66	0.906	1.60	0.841	0.99
	k=80	0.879	0.83	0.828	0.66	0.951	1.03	0.918	1.00	0.828	0.66

Table 3.10 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors Distributed $Gamma(3,2)$

		Wald		Scheffe		Proposed-Gamma		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.711	2.49	0.907	8.82	0.960	226	0.951	296	0.942	Inf (258)
	k=10	0.770	1.96	0.886	3.13	0.984	11.34	0.963	9.14	0.934	Inf (9)
	k=20	0.820	1.56	0.870	1.74	0.968	3.49	0.923	3.02	0.888	1.81
	k=40	0.848	1.16	0.855	1.09	0.960	1.78	0.917	1.61	0.862	1.10
	k=80	0.879	0.86	0.846	0.73	0.954	1.10	0.916	1.02	0.847	0.73
n=20	k=5	0.702	2.42	0.904	8.45	0.978	176	0.969	270	0.944	Inf (78)
	k=10	0.751	1.92	0.874	3.01	0.979	10.69	0.941	9.48	0.911	33.8
	k=20	0.810	1.51	0.863	1.65	0.959	3.30	0.904	2.97	0.872	1.68
	k=40	0.843	1.14	0.843	1.03	0.957	1.72	0.906	1.59	0.848	1.04
	k=80	0.881	0.85	0.836	0.69	0.954	1.06	0.917	1.01	0.837	0.69
n=40	k=5	0.694	2.35	0.903	8.13	0.984	154	0.977	258	0.945	Inf (19)
	k=10	0.752	1.89	0.876	2.95	0.964	10.03	0.914	9.61	0.889	3.02
	k=20	0.810	1.48	0.860	1.60	0.962	3.20	0.897	2.94	0.864	1.62
	k=40	0.844	1.14	0.844	1.01	0.956	1.69	0.906	1.61	0.845	1.01
	k=80	0.878	0.84	0.832	0.67	0.954	1.05	0.918	1.00	0.832	0.67
n=80	k=5	0.697	2.36	0.901	8.13	0.984	152	0.978	260	0.938	Inf (6)
	k=10	0.751	1.86	0.874	2.90	0.964	9.50	0.906	9.30	0.881	2.93
	k=20	0.794	1.45	0.851	1.57	0.961	3.13	0.890	2.93	0.854	1.57
	k=40	0.840	1.12	0.838	0.99	0.954	1.67	0.901	1.59	0.838	0.99
	k=80	0.877	0.83	0.832	0.66	0.953	1.04	0.917	0.99	0.832	0.66
n=160	k=5	0.688	2.32	0.903	7.97	0.966	139	0.968	254	0.912	9.51 (2)
	k=10	0.743	1.85	0.868	2.88	0.959	9.55	0.897	9.31	0.870	2.90
	k=20	0.791	1.45	0.850	1.56	0.957	3.12	0.885	2.92	0.851	1.56
	k=40	0.843	1.13	0.837	0.99	0.957	1.67	0.905	1.61	0.837	0.99
	k=80	0.880	0.85	0.827	0.66	0.951	1.04	0.919	1.02	0.827	0.66

Table 3.11 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors Distributed Normal with Previous Study information

		Wald		Scheffe		Proposed-Previous		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.771	3.10	0.909	8.67	0.960	356	0.951	277	0.946	Inf (15)
	k=10	0.824	2.32	0.887	3.17	0.980	30.46	0.963	9.60	0.933	Inf
	k=20	0.858	1.76	0.869	1.74	0.965	3.96	0.924	2.99	0.887	1.81
	k=40	0.877	1.24	0.861	1.08	0.950	1.75	0.919	1.59	0.867	1.09
	k=80	0.899	0.88	0.856	0.73	0.935	1.05	0.917	1.00	0.856	0.72
n=20	k=5	0.766	3.04	0.903	8.29	0.976	333	0.968	261	0.947	Inf (13)
	k=10	0.828	2.27	0.886	3.02	0.979	27.17	0.944	9.47	0.917	32mil
	k=20	0.853	1.70	0.861	1.64	0.954	3.97	0.901	2.94	0.872	1.67
	k=40	0.874	1.22	0.942	1.03	0.942	1.70	0.903	1.57	0.846	1.03
	k=80	0.899	0.88	0.843	0.69	0.940	1.04	0.921	1.01	0.844	0.69
n=40	k=5	0.773	3.01	0.903	8.12	0.984	324	0.977	262	0.948	Inf (1)
	k=10	0.818	2.26	0.879	2.96	0.972	29.40	0.918	9.39	0.889	876 (1)
	k=20	0.857	1.67	0.860	1.59	0.955	3.87	0.898	2.93	0.863	1.61
	k=40	0.877	1.21	0.842	1.01	0.942	1.72	0.907	1.59	0.843	1.01
	k=80	0.897	0.87	0.835	0.67	0.939	1.04	0.921	1.01	0.835	0.67
n=80	k=5	0.779	3.03	0.897	8.12	0.987	319	0.977	267	0.937	Inf
	k=10	0.829	2.25	0.875	2.92	0.969	27.58	0.910	9.39	0.880	2.96
	k=20	0.853	1.68	0.853	1.58	0.950	3.88	0.888	2.97	0.855	1.59
	k=40	0.881	1.20	0.844	0.99	0.946	1.70	0.903	1.58	0.844	0.99
	k=80	0.891	0.86	0.824	0.66	0.930	1.04	0.910	1.01	0.824	0.66
n=160	k=5	0.771	2.93	0.907	7.85	0.985	304	0.972	252	0.918	7.97
	k=10	0.815	2.20	0.875	2.85	0.968	27.56	0.901	9.17	0.878	2.87
	k=20	0.850	1.67	0.851	1.57	0.954	3.70	0.887	2.97	0.852	1.57
	k=40	0.877	1.20	0.833	0.99	0.941	1.69	0.903	1.60	0.833	0.99
	k=80	0.897	0.87	0.820	0.66	0.932	1.04	0.916	1.01	0.820	0.66

Table 3.12 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors Distributed Normal with Known Kurtosis

		Wald		Scheffe		Proposed-Known		Proposed		SAS	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.770	3.05	0.910	8.68	0.946	61.34	0.952	279	0.947	Inf (302)
	k=10	0.839	2.40	0.884	3.16	0.971	8.52	0.967	10.13	0.934	Inf (10)
	k=20	0.886	1.94	0.871	1.71	0.970	3.17	0.921	2.91	0.891	1.78
	k=40	0.913	1.37	0.860	1.09	0.961	1.73	0.916	1.60	0.866	1.10
	k=80	0.932	0.96	0.851	0.73	0.955	1.08	0.921	1.01	0.850	0.73
n=20	k=5	0.771	3.02	0.898	8.37	0.963	46.37	0.970	266	0.943	Inf (95)
	k=10	0.841	2.38	0.878	3.02	0.972	7.59	0.943	9.60	0.911	65.28
	k=20	0.886	1.90	0.862	1.63	0.965	3.02	0.902	2.93	0.872	1.66
	k=40	0.917	1.34	0.846	1.03	0.958	1.67	0.910	1.59	0.849	1.04
	k=80	0.936	0.95	0.840	0.69	0.955	1.05	0.921	1.00	0.841	0.69
n=40	k=5	0.770	2.90	0.908	7.93	0.976	39.68	0.979	254	0.952	Inf (25)
	k=10	0.845	2.35	0.874	2.94	0.971	7.25	0.914	9.29	0.885	3.12
	k=20	0.889	1.87	0.857	1.59	0.963	2.95	0.893	2.88	0.861	1.60
	k=40	0.918	1.32	0.841	1.00	0.959	1.64	0.906	1.59	0.843	1.00
	k=80	0.932	0.94	0.827	0.68	0.954	1.04	0.917	1.02	0.828	0.68
n=80	k=5	0.771	3.00	0.895	8.13	0.971	39.04	0.975	262	0.938	Inf (8)
	k=10	0.844	2.36	0.876	2.93	0.970	7.16	0.905	9.73	0.883	2.96
	k=20	0.889	1.87	0.857	1.57	0.962	2.93	0.887	2.93	0.869	1.58
	k=40	0.916	1.32	0.837	0.99	0.957	1.63	0.900	1.57	0.839	0.99
	k=80	0.930	0.92	0.830	0.66	0.956	1.02	0.914	1.00	0.830	0.66
n=160	k=5	0.775	2.96	0.903	7.97	0.975	37.67	0.970	261	0.913	264K (1)
	k=10	0.840	2.34	0.869	2.88	0.968	7.03	0.901	9.29	0.871	2.90
	k=20	0.886	1.87	0.848	1.56	0.960	2.91	0.881	2.91	0.849	1.57
	k=40	0.918	1.31	0.833	0.99	0.955	1.63	0.900	1.59	0.834	0.99
	k=80	0.935	0.92	0.827	0.66	0.956	1.02	0.917	1.00	0.827	0.66

CHAPTER 4. METHODS FOR θ

4.1 Proposed Method for θ

The proposed method for θ was developed by starting with an approximate variance stablizing transformation $ln(\hat{\theta})$ for $\hat{\theta}$ and the classical Wald method. The Wald method was used to obtain confidence limits for $ln(\theta)$ and then these limits were inverted to obtain confidence limits for θ, which are

$$exp\left[ln(\hat{\theta}) \pm z_{1-\alpha/2}\sqrt{Var(\hat{\theta})}/\hat{\theta}\right] \tag{4.1}$$

The multivariate delta method and results given by Scheffé (1959, p.288, 346) can be used on $\hat{\theta} = \frac{SSA}{SSA+SSE}$ to find the approximate variance of $\hat{\theta}$ under non-normality. The approximate variance of $\hat{\theta}$ under non-normality is

$$Var(\hat{\theta}) = \left(\theta + \frac{1}{n}\right)^2 \left(\frac{2}{k-1} + \frac{2}{k(n-1)}\right) + \theta^2 \left(\frac{\gamma_\alpha}{k} + \frac{\gamma_e}{nk}\right) \tag{4.2}$$

where γ_α is the kurtosis of the random effect and γ_e is the kurtosis of the data within each factor level. Bonett's kurtosis estimate, given in Equation 3.4, and Shoemaker's small sample adjustment will be used to estimate γ_α and γ_e.

4.2 Burch Method

In a recent article, Burch (2011), proposed a REML-based confidence interval for θ. He notes that assuming regularity conditions, which relate to k approaching infinity while n remains fixed (Burch and Harris, 2001), Jiang (1996) established asymptotic normality of REML estimators in non-normal applications. Jiang (1996, 2005) also derived the asymptotic covariance matrix for REML estimators, combining these results gives

$$\hat{\theta} \sim N\left(\theta, \frac{(1+n\theta)^2}{n^2}\left(\frac{Var(SSE)}{k^2(n-1)^2\sigma_2^4} + \frac{Var(SSA)}{(k-1)^2(\sigma_2^2 + n\sigma_1^2)^2}\right)\right) \tag{4.3}$$

assuming that the covariance of SSE and SSA is negligible. Furthermore, since SSE and SSA are quadratic forms, one can show that

$$Var(SSE) \approx k(n-1)\sigma_2^4 \left(\frac{n-1}{n}\kappa + 2 \right) \tag{4.4}$$

and

$$Var(SSA) \approx (k-1)(\sigma_2^2 + n\sigma_1^2)^2 \left(\frac{k-1}{kn}\kappa + 2 \right) \tag{4.5}$$

where

$$\kappa = E\left[\left(\frac{1}{\sqrt{\sigma_2^2}}(Y_{ij} - \bar{Y_{i.}}) + \frac{1}{\sqrt{\sigma_2^2 + n\sigma_1^2}}(\bar{Y_{i.}} - \mu) \right)^4 \right] - 3. \tag{4.6}$$

Expression (4.3) simplifies to

$$\hat{\theta} \sim N\left(\theta, \frac{(1+\theta)^2}{n^2} 2 \left(\frac{\kappa}{kn} + \frac{kn-1}{k(n-1)(k-1)} \right) \right) \tag{4.7}$$

In order to address the practical problem of $\hat{\theta}$ being postively skewed, Burch applies the natural logarithmic transformation suggested by many authors including Bonett (2006c), Cleveland (1984), and Laylard (1973), to bring in the right tail of the distribution. Applying this transformation to the REML estimator of θ using Slutsky's theorem, gives the following

$$log(1+n\hat{\theta}) \sim N\left(log(1+n\theta), 2\left(\frac{\kappa}{kn} + \frac{kn-1}{k(n-1)(k-1)} \right) \right) \tag{4.8}$$

Burch notes that $log(1+n\hat{\theta})$ is essentially Fisher's z-transformation as described by Ramasundarahettige et al. (2009), which is known to correct for left skewness.

As a first attempt at estimating κ, Burch suggests using

$$\hat{\hat{\kappa}} = \hat{\kappa} + 3\left(1 - \frac{1}{k^2n^2}\left(\frac{k^2(n-1)^3}{k(n-1)+2} + 2k(n-1)(k-1) + \frac{(k-1)^3}{k+1} \right) \right) \tag{4.9}$$

where

$$\hat{\kappa} = \frac{1}{kn} \sum_{i=1}^{a} \sum_{i=1}^{b} \left(\frac{1}{\sqrt{MSE}}(Y_{ij} - \bar{Y_{i.}}) + \frac{1}{\sqrt{MSA}}(\bar{Y_{i.}} - \bar{Y_{..}}) \right)^4 - 3 \tag{4.10}$$

is the plug-in estimator of κ using Equation 4.6.

$\hat{\hat{\kappa}}$ is a bias-corrected estimator of κ if the random effect and errors are normally distributed. Burch notes that even with the bias correction term, $\hat{\hat{\kappa}}$ overestimates κ for platykurtic distributions and severely underestimates κ for leptokurtic distributions. He suggests replacing κ in

Equation 4.8 with a function of $\hat{\hat{\kappa}}$, $g(\hat{\hat{\kappa}})$, so that

$$\widehat{Var}(log(1+n\hat{\theta})) = 2\left(\frac{g(\hat{\hat{\kappa}})}{kn} + \frac{kn-1}{k(n-1)(k-1)}\right). \tag{4.11}$$

Combining all this, Burch gives what he calls a REML-based confidence interval for θ as

$$\left(\frac{1}{n}\left[(1+n\hat{\theta})e^{-z_{1-\alpha/2}\sqrt{\widehat{Var}(log(1+n\hat{\theta}))}} - 1\right], \frac{1}{n}\left[(1+n\hat{\theta})e^{z_{1-\alpha/2}\sqrt{\widehat{Var}(log(1+n\hat{\theta}))}} - 1\right]\right) \tag{4.12}$$

where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution.

Through simulation, Burch empirically determines that

$$g(\hat{\hat{\kappa}}) = 2.0\hat{\hat{\kappa}} + 0.5\hat{\hat{\kappa}}^2 \tag{4.13}$$

helps correct for the overestimation of κ in platykurtic distributions and the severe underestimation of κ in leptokurtic distributions.

Burch's method will be compared to the proposed method in the following simulations.

4.3 Monte Carlo Study

Monte Carlo simulation studies were used to estimate coverage probabilities of the Exact, Burch, and the newly proposed confidence intervals for θ under various conditions. The simulation studies examined five values of k $(5, 10, 20, 40, 80)$ and five sample sizes per group $(10, 20, 40, 80, 160)$, for a total of 25 different conditions. The basic set-up for the simulations was to first simulate a random effect, α_i, for group i, under one of the distributions listed in Table 3.1. Without loss of generality, the constant μ was set to zero, so α_i does not necessarily have an expectation of zero. After simulating an α_i, n values were simulated from a distribution with mean α_i. This distribution was obtained by choosing one of the distributions in Table 3.1, possibly, but not necessarily, the same distribution as the α_i, locating it so that it had a mean of zero and then adding α_i. This was repeated for $i = 1, ..., k$ and resulted in k groups of n data points. These data were then used to calculate the various confidence intervals. For each of the 25 conditions, 10,000 Monte Carlo trials were generated, 95% confidence intervals for θ were calculated, and coverage probabilites were obtained by classifying each interval as capturing or not capturing the true value of θ. Allowing the variance to vary affects the results, but since we need to consider the variance of the random effect, as well as the variance of the distribution of the errors, considering a range of true θ values makes the most sense. Burch (2011) showed that the results seem to vary depending on where ρ_I is between 0 and 1, with the best results ocurring when ρ_I is close to 0.50. Therefore, for each combination of distributions, we will consider three values for ρ_I $(0.25, 0.50, 0.75)$.

All of the following tables of simulation results give the coverage probabilities as well as an adjusted average width, which is calculated as the average width of the intervals divided by the true value of θ. This adjustment was made because the average length of the intervals is related to the true value of θ. If the true value of θ is larger in one set of simulations than another, we expect the simulations with the larger value of θ to have wider intervals, everything else being equal. Making this adjustment allows us to compare the average widths from simulations with different values of θ.

4.3.1 Errors Distributed Normal

The first situation we looked at, was when the errors followed a normal distribution. That is, we simulated α_i from one of the distributions listed in Table 3.1 and then simulated data from a normal distribution with mean α_i. Without loss of generality, the variance of the normal distribution was set to one. We considered two different distributions for the random effect: Normal and $\chi^2(5)$. The results are given in Tables 4.1-4.4.

Table 4.1 gives the results for when both the random effect and the errors are distributed Normal and $\rho_I = 0.50$. In this case, we expect the Exact method to perform the best since the conditions under which the method is appropriate have been met. We are interested in seeing if the proposed method gives reasonable results in this situation. The table shows that, as expected, the Exact method performs the best, but the proposed method also performs well. The Burch method performs about the same as the proposed method when both the distributions are Normal.

Tables 4.2-4.4 give the results for when the random effect is distributed $\chi^2(5)$, with each table representing a different value of ρ_I. Regardless of the value of ρ_I, the proposed method performs the best the majority of the time, particularly when k is at least 10. The Burch method gives coverage probabilities that are consistently below 0.90.

4.3.2 Errors Distributed Non-Normal

Since the calculation for the variance of θ includes both an estimate of the kurtosis of the distribution of the random effect and an estimate of the kurtosis of the distribution of the errors, we expect the confidence intervals for θ to change when the distribution of the errors is non-normal. To test this assumption, we ran simulations where both the distribution of the random effect and the distribution of the errors were non-normal. For these simulations we again began by simulating α_i from one of the non-normal distributions in Table 3.1 and then simulated data from a different non-normal distribution from Table 3.1, which was located to have a mean of α_i. The results for a few different combinations of distributions are given in Tables 4.5-4.10. Again, for each combination of distributions, three values of ρ_I were considered.

Tables 4.5-4.7 give the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed $Beta(1, 3)$, with each table representing a different value of ρ_I. These results are similar to what we saw when the errors were normally distributed. Regardless of the value of ρ_I, the proposed method performs the best the majority of the time, particularly when k is at least 10. The Burch method gives coverage probabilities that are consistently below 0.90.

Tables 4.8-4.10 give the results for when the random effect is distributed $Gamma(3, 2)$ and the errors are distributed $Gamma(5, 1)$, with each table representing a different value of ρ_I. The proposed method performs the best when k is at least 20, with the Exact method having a slight advantage with k is small. The Burch method again gives coverage probabilities that are consistently below 0.90.

4.4 Methods Using Prior Kurtosis Information

Any information the researcher can provide about the distribution of the random effect or the distribution of the errors can be helpful in obtaining more accurate estimates of the kurtosis of the random effect and the kurtosis of the errors and therefore more accurate confidence intervals for the ratio of variances. In the following sections, we will explain how each of these types of kurtosis estimates can be used with the proposed method and give some simulation results.

4.4.1 Kurtosis Estimates from Previous Studies

One way in which a researcher can obtain information about the distribution of the random effect is from previous studies. If previous study information is available, the kurtosis estimates from these studies can be pooled with the kurtosis estimate from the current study to hopefully provide a more accurate estimate due to a larger sample.

As was mentioned in Section 3.3.1, Laylard (1973) notes that a pooled kurtosis value is best estimated by first calculating a pooled estimate of the fourth moment and a pooled estimate of the variance and then plugging these estimates into the formula for kurtosis, rather than pooling the kurtosis values from each study. A pooled kurtosis will be calculated for both the random effect and the errors using Equations 3.6-3.8. These pooled kurtosis values will then be used in Equation 4.2.

Table 4.11 gives the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed Normal with $\rho_I = 0.50$ and there is information from a previous study with k=40 and n=100. This combination of distributions was already considered in Table 4.3 for the Exact, Proposed, and Burch methods, where it was shown that the proposed method performed the best the majority of the time. All of the methods are included in this table for easy comparison, but we are most interested in comparing the proposed method using previous information to the proposed method without using previous information. We see that there is an advantage when using previous study information, with coverage probabilities fairly close to the nominal level for all values of k and n.

Table 4.12 gives the results for when the random effect is distributed $\chi^2(5)$ and the errors are distributed $Beta(1,3)$ with $\rho_I = 0.50$ and there is information from a previous study with k=40 and n=100. This combination of distributions was already considered in Table 4.6 for the Exact, Proposed, and Burch methods, where it was shown that the proposed method performed the best the majority of the time. All of the methods are included in this table for easy comparison, but we are most interested in comparing the proposed method using previous information to the proposed method without using previous information. We see that there is an advantage when using previous study information, with coverage probabilities fairly close to the nominal level for all values of k and n.

4.4.2 Theoretically Specified Kurtosis Values

A researcher may also be able to theoretically specify a kurtosis value for the random effect and/or the errors based on expert knowledge about their field of study. The researcher may know that the type of data they are anlayzing is known to have means that follow a specific distribution and/or errors that follow a specific distribution. In this situation, the $\hat{\gamma}_\alpha$ and $\hat{\gamma}_e$ in Equation 4.2 will be replaced with the theoretical kurtosis values for the specific distributions.

Table 4.13 gives the results when the random effect is distributed $\chi^2(5)$, the errors are distributed Normal, and there is a theoretically specified value for the kurtosis of the distribution of the random effect and the distribution of the errors. The table shows that there is a great advantage when using theoretically specified values for the kurtoses of the distributions. The covereage probablities when using the known values are very close to the nominal value for all values of k and n.

Table 4.14 gives the results when the random effect is distributed $\chi^2(5)$, the errors are distributed $Beta(1,3)$, and there is a theoretically specified value for the kurtosis of the distribution of the random effect and the distribution of the errors. The table shows that there is a great advantage when using theoretically specified values for the kurtoses of the distributions. The covereage probablities when using the known values are very close to the nominal value for all values of k and n.

4.4.3 Distribution Based Kurtosis Estimate

Lastly, there are some distributions in which kurtosis can be estimated by first estimating the parameters of the distribution and then using these parameter estimates to calculate a kurtosis value. Some examples of these types of distributions include the beta distribution and the gamma distribution. The details on how to calculate this kurtosis estimate for the distribution of the random error were given in Section 3.3.3. When constructing confidence intervals for θ, an estimate of the kurtosis of the distribution of the errors is also needed and the equations for calculating this estimate are given below.

If we are assuming the distribution of errors can be approximated using a scaled beta distribution on the interval $(0, c)$, then the method of moments estimators of a and b for each group are

$$\hat{a} = \bar{y}_{i.} \left[\frac{\bar{y}_{i.}(c^3 - c^2\bar{y}_{i.}) - c^2\hat{\sigma_i}^2}{c^3\hat{\sigma_i}^2} \right] \tag{4.14}$$

and

$$\hat{b} = (c - \bar{y}_{i.}) \left[\frac{\bar{y}_{i.}(c^3 - c^2\bar{y}_{i.}) - c^2\hat{\sigma_i}^2}{c^3\hat{\sigma_i}^2} \right] \tag{4.15}$$

where $\bar{y}_{i.}$ is the sample mean of the ith group and $\hat{\sigma_i}^2$ is the sample variance of the ith group. We can obtain a kurtosis estimate for each group by plugging these estimates into Equation 3.9. We then average the group kurtoses to obtain a kurtosis estimate that can be used in Equation 4.2 to approximate the variance of $\hat{\theta}$.

If we are assuming the distribution of errors can be approximated using a gamma distribution, then the method of moments estimator of a for each group is

$$\hat{a} = (\bar{y}_{i.}/\hat{\sigma}_i)^2 \tag{4.16}$$

where $\bar{y}_{i.}$ is the sample mean of the ith group and $\hat{\sigma}_i$ is the square root of the sample variance of the ith group. We can obtain a kurtosis estimate for each group by plugging this estimate of the shape parameter into Equation 3.12. We then average the group kurtoses to obtain a kurtosis estimate that can be used in Equation 4.2 to approximate the variance of $\hat{\theta}$.

Table 4.3 gives the results for this method when the random effect is distributed $\chi^2(5)$ and the errors are distributed Normal with $\rho_I = 0.50$, in the column labeled "Proposed-Gamma". The χ^2 distribution is a special case of the gamma distribution, so estimates of the gamma distribution parameters were used to estimate the kurtosis of the random effect distribution. There are no parameter based estimates for the Normal distribution, so the kurtosis of the distribution of the errors was estimated as described in the proposed method. We already compared the other methods in Section 4.3.1, and found that the proposed method performed the best the majority of the time. Now, comparing the proposed method to the proposed method using gamma parameter estimates, we see that the proposed method using gamma parameter estimates performs better than the proposed method, giving coverage probabilities that are very close to the nominal level for all values of k and n.

Table 4.6 gives the results for this method when the random effects are distributed $\chi^2(5)$ and the errors are distributed $Beta(1,3)$ with $\rho_I = 0.50$, in the column labeled "Proposed-Gamma/Beta". The random effect is distributed χ^2, so estimates of the gamma distribution parameters were used to estimate the kurtosis of the random effect distribution. The errors are distributed $Beta$, so the beta distribution paramters were used to estimate the kurtosis of the distribution of the errors. We already compared the other methods in Section 4.3.2, and found that the proposed method performed the best the majority of the time. Now, comparing the proposed method to the proposed method using the parameter estimates, we see that there is a significant advantage when using the parameter estimates. Except for the case when both k and n are small, the parameter estimate method gives coverage probabilities that are very close to the nominal level.

Table 4.9 gives the results for this method when the random effects are distributed $Gamma(3,2)$ and the errors are distributed $Gamma(5,1)$ with $\rho_I = 0.50$, in the column labeled "Proposed-Gamma". The random effect and the errors are distributed $Gamma$, so estimates of the gamma distribution parameters were used to estimate the kurtosis of both the random effect distribution and the distribution of the errors. We already compared the other methods in Section 4.3.2, and found that the proposed method performed the best the majority of the time. Now, comparing the proposed method to the proposed method using the parameter estimates, we see that there is a significant advantage when using the parameter estimates. The parameter estimate method gives coverage probabilities very close to the nominal level for all values of k and n.

Tables 4,2, 4,4, 4.5, 4.7, 4.8, and 4.10 also include results for the proposed method using parameter based estimates of kurtosis. As in the tables described above, these tables also show an advantage when using paraemter based estimates of kurtosis, although there does seem to be more variation in the coverage probabilities when ρ_I is further from 0.50.

4.5 Methods for ρ_I and ρ_n

Confidence intervals for ρ_I and ρ_n can be found by transforming the confidence intervals for θ using the relations given in Equations 1.1 and 1.2. Since these are monotonic functions of θ, the coverage probablities given in Tables 4.1-4.14 also apply to these two parameters. These parameters are of most interest in reliability studies. The classic paper of Shrout and Fleiss (1979) outlines guidelines for choosing among six different forms of the intraclass correlation. In their paper ρ_I is denoted by $ICC(1,1)$ and ρ_n is denoted by $ICC(1,n)$, where n is the number of measurements per person.

Two methods have recently been proposed for constructing confidence intervals for coefficient alpha, a special case of ρ_n. The first is what has been called the normal-theory (NT) interval estimator proposed by van Zyl, Neudecker, and Nel (2000). This estimator does not require the assumption of a compound symmetric covariance matrix, just the assumption that the items comprising the test are normally distributed. The second is an asymptotically distribution-free (ADF) confidence interval proposed by Yuan, Guarnaccia, and Hayslip (2003). This confidence interval estimates the variance of coefficient alpha using fourth moments and therefore does not require that the sampling distribution be known. It should be noted that both of these methods are based on large sample theory and therefore large samples will be needed in order for either of them to provide accurate confidence intervals. Maydeu-Olivares, Coffman, and Hartman (2007) perform a simulation study in which they conclude ADF confidence intervals are accurate for sample sizes (k) over 100. Our results show that our proposed methods work well when the sample size is at least 20, so in reliability studies, when k is usually at least 30 (where k is the sample size), the methods proposed in this chapter should work quite well.

Table 4.1 Comparison of Methods when Random Effects are Distributed Normal and Errors are Distributed Normal, $\rho = 0.5$

		Proposed		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.910	7.40	0.952	9.19	0.909	4.62
	k=10	0.950	2.97	0.952	3.35	0.934	2.56
	k=20	0.942	1.70	0.949	1.81	0.943	1.61
	k=40	0.940	1.10	0.948	1.14	0.944	1.08
	k=80	0.941	0.75	0.946	0.76	0.943	0.74
n=20	k=5	0.924	6.63	0.947	8.55	0.901	4.16
	k=10	0.931	2.65	0.947	3.11	0.930	2.33
	k=20	0.933	1.54	0.950	1.68	0.941	1.48
	k=40	0.935	1.01	0.947	1.05	0.943	0.99
	k=80	0.941	0.69	0.948	0.71	0.946	0.68
n=40	k=5	0.934	6.30	0.951	8.32	0.908	4.00
	k=10	0.927	2.50	0.950	2.97	0.932	2.22
	k=20	0.930	1.47	0.950	1.62	0.939	1.42
	k=40	0.935	0.96	0.949	1.01	0.945	0.95
	k=80	0.941	0.66	0.950	0.68	0.947	0.66
n=80	k=5	0.933	6.09	0.951	8.05	0.910	3.84
	k=10	0.930	2.45	0.955	2.92	0.932	2.17
	k=20	0.927	1.45	0.949	1.60	0.940	1.40
	k=40	0.935	0.94	0.949	0.99	0.946	0.93
	k=80	0.943	0.65	0.952	0.67	0.951	0.65
n=160	k=5	0.928	6.07	0.950	7.96	0.903	3.79
	k=10	0.921	2.44	0.950	2.90	0.930	2.15
	k=20	0.928	1.42	0.951	1.57	0.946	1.38
	k=40	0.935	0.94	0.951	0.99	0.947	0.93
	k=80	0.940	0.64	0.950	0.66	0.948	0.64

Table 4.2 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed Normal, $\rho = 0.25$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.840	17.25	0.857	6.89	0.921	10.94	0.905	5.55
	k=10	0.939	4.47	0.960	6.19	0.907	3.89	0.882	2.98
	k=20	0.968	2.43	0.992	3.05	0.899	2.15	0.888	1.91
	k=40	0.929	1.56	0.962	1.82	0.887	1.35	0.882	1.27
	k=80	0.924	1.04	0.955	1.17	0.881	0.89	0.878	0.87
n=20	k=5	0.909	8.76	0.923	13.67	0.915	9.43	0.862	4.60
	k=10	0.962	3.56	0.988	4.97	0.894	3.36	0.865	2.52
	k=20	0.912	2.13	0.959	2.64	0.876	1.84	0.857	1.62
	k=40	0.908	1.41	0.953	1.62	0.863	1.15	0.857	1.09
	k=80	0.915	0.97	0.947	1.07	0.855	0.77	0.847	0.75
n=40	k=5	0.941	7.79	0.955	11.83	0.909	8.63	0.849	4.15
	k=10	0.901	3.40	0.949	4.59	0.885	3.12	0.853	2.33
	k=20	0.890	2.01	0.951	2.47	0.867	1.69	0.848	1.48
	k=40	0.903	1.35	0.952	1.53	0.850	1.06	0.837	1.00
	k=80	0.910	0.94	0.947	1.02	0.838	0.71	0.832	0.69
n=80	k=5	0.923	7.48	0.936	10.86	0.906	8.25	0.844	3.94
	k=10	0.879	3.31	0.945	4.44	0.875	2.99	0.840	2.23
	k=20	0.879	2.01	0.949	2.43	0.855	1.64	0.838	1.43
	k=40	0.899	1.35	0.956	1.50	0.845	1.02	0.835	0.96
	k=80	0.915	0.93	0.953	1.00	0.841	0.68	0.834	0.66
n=160	k=5	0.903	7.23	0.922	10.39	0.901	7.98	0.833	3.80
	k=10	0.877	3.30	0.942	4.40	0.972	2.96	0.836	2.20
	k=20	0.880	1.96	0.947	2.35	0.858	1.58	0.841	1.39
	k=40	0.896	1.35	0.950	1.49	0.833	1.01	0.826	0.94
	k=80	0.915	0.93	0.952	0.98	0.832	0.67	0.826	0.65

Table 4.3 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed Normal, $\rho = 0.5$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.939	8.76	0.951	13.15	0.914	9.40	0.860	4.72
	k=10	0.924	3.59	0.963	4.88	0.899	3.32	0.865	2.53
	k=20	0.909	2.14	0.958	2.62	0.878	1.81	0.862	1.61
	k=40	0.912	1.41	0.955	1.59	0.865	1.13	0.885	1.07
	k=80	0.921	0.98	0.957	1.06	0.858	0.76	0.855	0.74
n=20	k=5	0.942	7.92	0.969	11.58	0.910	8.63	0.852	4.20
	k=10	0.890	3.39	0.947	4.55	0.880	3.09	0.847	2.33
	k=20	0.889	2.04	0.951	2.46	0.868	1.68	0.846	1.48
	k=40	0.899	1.38	0.951	1.54	0.846	1.06	0.838	1.00
	k=80	0.913	0.94	0.952	1.01	0.841	0.70	0.836	0.68
n=40	k=5	0.912	7.52	0.928	11.08	0.901	8.30	0.838	3.99
	k=10	0.883	3.34	0.947	4.41	0.879	2.98	0.846	2.23
	k=20	0.878	1.98	0.946	2.38	0.856	1.61	0.838	1.41
	k=40	0.900	1.35	0.952	1.50	0.848	1.02	0.837	0.96
	k=80	0.914	0.93	0.952	0.99	0.834	0.68	0.832	0.66
n=80	k=5	0.903	7.31	0.924	10.42	0.903	8.03	0.834	3.83
	k=10	0.870	3.19	0.942	4.24	0.878	2.87	0.832	2.14
	k=20	0.872	1.95	0.948	2.35	0.857	1.58	0.831	1.38
	k=40	0.896	1.32	0.951	1.47	0.838	0.99	0.828	0.93
	k=80	0.914	0.93	0.950	0.99	0.826	0.67	0.822	0.65
n=160	k=5	0.902	7.32	0.925	10.45	0.899	8.05	0.843	3.83
	k=10	0.868	3.20	0.941	4.27	0.874	2.89	0.836	2.14
	k=20	0.873	1.98	0.949	2.35	0.847	1.57	0.829	1.38
	k=40	0.894	1.34	0.952	1.47	0.835	0.99	0.822	0.93
	k=80	0.917	0.93	0.951	0.98	0.830	0.66	0.826	0.64

Table 4.4 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed Normal, $\rho = 0.75$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.926	8.22	0.942	11.98	0.909	8.85	0.852	4.45
	k=10	0.889	3.47	0.947	4.61	0.886	3.14	0.853	2.40
	k=20	0.889	2.09	0.951	2.50	0.867	1.70	0.849	1.51
	k=40	0.910	1.41	0.955	1.55	0.854	1.07	0.846	1.02
	k=80	0.926	0.97	0.955	1.03	0.848	0.71	0.847	0.70
n=20	k=5	0.911	7.74	0.931	11.00	0.908	8.38	0.849	4.08
	k=10	0.878	3.33	0.941	4.42	0.876	2.99	0.845	2.25
	k=20	0.882	2.00	0.951	2.39	0.858	1.62	0.840	1.42
	k=40	0.901	1.35	0.950	1.50	0.843	1.02	0.835	0.96
	k=80	0.916	0.94	0.950	0.99	0.839	0.68	0.834	0.66
n=40	k=5	0.897	7.51	0.919	10.75	0.895	8.14	0.830	3.91
	k=10	0.875	3.20	0.941	4.26	0.880	2.90	0.839	2.16
	k=20	0.876	1.98	0.946	2.37	0.852	1.59	0.834	1.39
	k=40	0.902	1.35	0.955	1.49	0.848	1.00	0.838	0.94
	k=80	0.914	0.93	0.950	0.98	0.833	0.67	0.828	0.65
n=80	k=5	0.903	7.10	0.926	10.17	0.906	7.87	0.846	3.75
	k=10	0.869	3.24	0.943	4.30	0.873	2.91	0.836	2.16
	k=20	0.878	1.96	0.951	2.34	0.856	1.57	0.838	1.37
	k=40	0.893	1.34	0.949	1.48	0.839	0.99	0.826	0.93
	k=80	0.919	0.93	0.954	0.98	0.833	0.66	0.827	0.64
n=160	k=5	0.900	7.38	0.925	10.62	0.903	8.05	0.837	3.83
	k=10	0.869	3.26	0.943	4.32	0.872	2.90	0.839	2.15
	k=20	0.874	1.95	0.947	2.33	0.850	1.56	0.833	1.37
	k=40	0.899	1.34	0.953	1.47	0.842	0.98	0.830	0.92
	k=80	0.915	0.93	0.951	0.98	0.830	0.66	0.824	0.64

Table 4.5 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed $Beta(1,3)$, $\rho = 0.25$

		Proposed		Proposed-Gamma/Beta		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.847	17.21	0.866	79.12	0.923	11.23	0.914	5.81
	k=10	0.939	4.53	0.964	10.45	0.910	3.96	0.886	3.07
	k=20	0.969	2.45	0.996	3.79	0.897	2.15	0.884	1.92
	k=40	0.936	1.57	0.973	1.97	0.889	1.34	0.884	1.28
	k=80	0.930	1.05	0.960	1.21	0.884	0.90	0.882	0.88
n=20	k=5	0.906	8.89	0.920	29.17	0.915	9.44	0.860	4.65
	k=10	0.966	3.61	0.990	6.77	0.898	3.35	0.867	2.53
	k=20	0.911	2.13	0.973	3.01	0.880	1.84	0.863	1.63
	k=40	0.913	1.41	0.963	1.71	0.863	1.15	0.856	1.09
	k=80	0.921	0.97	0.960	1.09	0.859	0.77	0.856	0.76
n=40	k=5	0.940	7.93	0.954	17.85	0.905	8.71	0.846	4.20
	k=10	0.901	3.41	0.971	5.56	0.884	3.11	0.852	2.33
	k=20	0.891	2.03	0.964	2.68	0.866	1.70	0.846	1.49
	k=40	0.901	1.36	0.957	1.59	0.854	1.07	0.845	1.01
	k=80	0.920	0.94	0.958	1.03	0.852	0.71	0.847	0.69
n=80	k=5	0.923	7.41	0.963	13.50	0.908	8.19	0.846	3.91
	k=10	0.883	3.22	0.957	4.76	0.883	2.95	0.846	2.20
	k=20	0.882	1.99	0.956	2.48	0.859	1.61	0.841	1.42
	k=40	0.896	1.34	0.953	1.51	0.840	1.02	0.832	0.96
	k=80	0.915	0.93	0.955	1.00	0.831	0.68	0.826	0.66
n=160	k=5	0.908	7.41	0.940	12.03	0.902	8.08	0.835	3.85
	k=10	0.871	3.25	0.949	4.54	0.875	2.92	0.834	2.17
	k=20	0.879	1.96	0.953	2.42	0.857	1.59	0.838	1.39
	k=40	0.898	1.32	0.956	1.49	0.843	1.00	0.833	0.94
	k=80	0.911	0.93	0.951	0.98	0.831	0.67	0.823	0.65

Table 4.6 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed $Beta(1,3)$, $\rho = 0.5$

		Proposed		Proposed-Gamma/Beta		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.963	8.87	0.951	56.30	0.913	9.38	0.862	4.82
	k=10	0.925	3.67	0.996	9.12	0.891	3.34	0.864	2.59
	k=20	0.907	2.17	0.985	3.41	0.871	1.81	0.858	1.62
	k=40	0.912	1.43	0.968	1.79	0.858	1.14	0.850	1.08
	k=80	0.924	0.98	0.958	1.10	0.856	0.76	0.855	0.74
n=20	k=5	0.938	7.81	0.969	24.86	0.907	8.54	0.847	4.19
	k=10	0.894	3.42	0.979	6.36	0.879	3.10	0.854	2.35
	k=20	0.886	2.03	0.968	2.81	0.859	1.67	0.845	1.48
	k=40	0.905	1.36	0.961	1.61	0.858	1.05	0.851	0.99
	k=80	0.920	0.96	0.955	1.04	0.841	0.71	0.838	0.69
n=40	k=5	0.911	7.45	0.967	16.51	0.897	8.16	0.840	3.93
	k=10	0.882	3.32	0.967	5.33	0.878	2.99	0.845	2.24
	k=20	0.880	1.95	0.960	2.55	0.857	1.60	0.841	1.41
	k=40	0.898	1.35	0.955	1.54	0.838	1.01	0.826	0.95
	k=80	0.916	0.93	0.952	1.00	0.839	0.68	0.836	0.66
n=80	k=5	0.905	7.33	0.949	13.02	0.905	7.97	0.833	3.81
	k=10	0.872	3.20	0.954	4.70	0.876	2.90	0.836	2.16
	k=20	0.879	1.97	0.956	2.46	0.852	1.59	0.835	1.39
	k=40	0.895	1.34	0.954	1.50	0.839	1.00	0.831	0.94
	k=80	0.920	0.93	0.953	0.99	0.830	0.67	0.826	0.65
n=160	k=5	0.900	7.18	0.936	11.63	0.900	7.96	0.839	3.79
	k=10	0.867	3.17	0.945	4.43	0.871	2.86	0.832	2.13
	k=20	0.872	1.95	0.950	2.40	0.846	1.57	0.830	1.38
	k=40	0.896	1.32	0.956	1.47	0.838	0.98	0.827	0.92
	k=80	0.916	0.93	0.949	0.98	0.832	0.66	0.826	0.64

Table 4.7 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed $Beta(1,3)$, $\rho = 0.75$

		Proposed		Proposed-Gamma/Beta		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.931	8.28	0.980	52.19	0.910	8.79	0.855	4.51
	k=10	0.897	3.51	0.993	8.70	0.884	3.14	0.855	2.43
	k=20	0.891	2.11	0.985	3.29	0.861	1.70	0.848	1.53
	k=40	0.907	1.40	0.969	1.73	0.845	1.07	0.837	1.02
	k=80	0.922	0.97	0.956	1.06	0.843	0.71	0.841	0.70
n=20	k=5	0.907	7.58	0.984	23.85	0.906	8.26	0.842	4.09
	k=10	0.881	3.36	0.979	6.20	0.877	3.00	0.845	2.27
	k=20	0.880	2.01	0.968	2.76	0.857	1.62	0.841	1.43
	k=40	0.902	1.35	0.961	1.58	0.851	1.01	0.842	0.96
	k=80	0.920	0.95	0.954	1.02	0.833	0.68	0.832	0.67
n=40	k=5	0.902	7.35	0.965	16.19	0.897	8.04	0.837	3.88
	k=10	0.877	3.29	0.967	5.25	0.875	2.94	0.843	2.20
	k=20	0.877	1.94	0.961	2.52	0.855	1.58	0.837	1.39
	k=40	0.896	1.35	0.955	1.52	0.833	1.00	0.826	0.94
	k=80	0.913	0.93	0.949	0.99	0.829	0.67	0.824	0.65
n=80	k=5	0.901	7.29	0.952	13.07	0.903	8.02	0.842	3.83
	k=10	0.867	3.20	0.953	4.67	0.873	2.87	0.833	2.14
	k=20	0.872	1.94	0.950	2.42	0.851	1.56	0.832	1.37
	k=40	0.894	1.35	0.951	1.50	0.829	0.99	0.820	0.93
	k=80	0.914	0.93	0.952	0.98	0.826	0.66	0.821	0.64
n=160	k=5	0.905	7.26	0.941	11.46	0.906	7.94	0.847	3.78
	k=10	0.866	3.16	0.947	4.42	0.872	2.85	0.834	2.12
	k=20	0.874	1.94	0.951	2.37	0.852	1.56	0.832	1.36
	k=40	0.898	1.33	0.953	1.47	0.833	0.98	0.827	0.92
	k=80	0.914	0.93	0.950	0.98	0.829	0.66	0.822	0.64

Table 4.8 Comparison of Methods when Random Effects are Distributed $Gamma(3, 2)$ and Errors are Distributed $Gamma(5, 1)$, $\rho = 0.25$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.972	20.60	0.866	28.40	0.921	11.25	0.981	6.58
	k=10	0.936	4.32	0.965	5.91	0.914	3.94	0.910	3.37
	k=20	0.965	2.39	0.991	2.91	0.899	2.14	0.913	2.11
	k=40	0.936	1.54	0.963	1.76	0.895	1.35	0.916	1.41
	k=80	0.933	1.05	0.955	1.15	0.891	0.90	0.917	0.97
n=20	k=5	0.901	7.63	0.925	12.17	0.918	9.28	0.883	4.94
	k=10	0.955	3.38	0.989	4.71	0.903	3.38	0.892	2.73
	k=20	0.916	2.03	0.960	2.49	0.891	1.83	0.896	1.72
	k=40	0.916	1.37	0.954	1.56	0.876	1.16	0.891	1.16
	k=80	0.923	0.94	0.954	1.03	0.870	0.77	0.887	0.80
n=40	k=5	0.932	6.73	0.958	10.74	0.911	8.74	0.866	4.38
	k=10	0.886	3.07	0.952	4.29	0.894	3.12	0.871	2.42
	k=20	0.884	1.90	0.950	2.34	0.875	1.69	0.870	1.54
	k=40	0.913	1.30	0.955	1.46	0.868	1.06	0.877	1.04
	k=80	0.921	0.91	0.954	0.98	0.862	0.71	0.871	0.72
n=80	k=5	0.896	6.28	0.939	9.76	0.913	8.14	0.850	3.98
	k=10	0.871	2.92	0.947	4.05	0.894	2.97	0.867	2.26
	k=20	0.882	1.88	0.952	2.29	0.874	1.64	0.865	1.46
	k=40	0.898	1.27	0.950	1.42	0.860	1.02	0.857	0.97
	k=80	0.920	0.89	0.954	0.95	0.853	0.68	0.856	0.67
n=160	k=5	0.882	6.17	0.931	9.61	0.914	8.06	0.853	3.88
	k=10	0.861	2.91	0.944	4.01	0.888	2.92	0.861	2.20
	k=20	0.872	1.84	0.952	2.23	0.868	1.59	0.853	1.41
	k=40	0.899	1.27	0.954	1.40	0.859	1.00	0.851	0.95
	k=80	0.917	0.89	0.949	0.94	0.846	0.67	0.846	0.65

Table 4.9 Comparison of Methods when Random Effects are Distributed $Gamma(3, 2)$ and Errors are Distributed $Gamma(5, 1)$, $\rho = 0.5$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.929	8.01	0.953	12.47	0.913	9.65	0.881	5.43
	k=10	0.914	3.47	0.962	4.69	0.896	3.37	0.892	2.85
	k=20	0.906	2.06	0.954	2.48	0.884	1.81	0.895	1.77
	k=40	0.916	1.39	0.952	1.55	0.867	1.14	0.893	1.18
	k=80	0.927	0.96	0.953	1.02	0.863	0.76	0.895	0.81
n=20	k=5	0.923	6.86	0.971	10.60	0.912	8.67	0.875	4.54
	k=10	0.882	3.15	0.949	4.28	0.890	3.12	0.876	2.51
	k=20	0.883	1.96	0.952	2.36	0.975	1.69	0.878	1.58
	k=40	0.906	1.31	0.956	1.46	0.864	1.06	0.873	1.06
	k=80	0.916	0.91	0.950	0.97	0.854	0.70	0.872	0.73
n=40	k=5	0.888	6.33	0.933	9.60	0.914	8.10	0.859	4.06
	k=10	0.868	2.99	0.948	4.09	0.892	2.98	0.870	2.31
	k=20	0.876	1.87	0.949	2.27	0.876	1.62	0.870	1.47
	k=40	0.904	1.29	0.952	1.43	0.862	1.02	0.865	0.99
	k=80	0.919	0.89	0.951	0.95	0.856	0.68	0.862	0.68
n=80	k=5	0.877	6.22	0.934	9.57	0.912	8.07	0.859	3.95
	k=10	0.857	2.91	0.942	4.01	0.885	2.92	0.858	2.22
	k=20	0.873	1.84	0.952	2.24	0.870	1.59	0.859	1.42
	k=40	0.900	1.27	0.953	1.40	0.858	1.00	0.853	0.95
	k=80	0.916	0.88	0.952	0.94	0.848	0.67	0.849	0.67
n=160	k=5	0.866	6.18	0.919	9.46	0.904	7.95	0.844	3.83
	k=10	0.857	2.90	0.944	3.98	0.886	2.90	0.858	2.18
	k=20	0.867	1.81	0.948	2.19	0.868	1.60	0.852	1.38
	k=40	0.896	1.25	0.950	1.39	0.856	0.98	0.849	0.93
	k=80	0.919	0.89	0.951	0.94	0.848	0.66	0.846	0.65

Table 4.10 Comparison of Methods when Random Effects are Distributed $Gamma(3, 2)$ and Errors are Distributed $Gamma(5, 1)$, $\rho = 0.75$

		Proposed		Proposed-Gamma		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.910	7.57	0.949	11.55	0.914	8.91	0.877	5.00
	k=10	0.889	3.31	0.953	4.40	0.894	3.16	0.888	2.65
	k=20	0.891	1.99	0.951	2.37	0.875	1.70	0.881	1.65
	k=40	0.908	1.37	0.947	1.49	0.862	1.07	0.882	1.11
	k=80	0.922	0.94	0.951	0.99	0.854	0.71	0.883	0.76
n=20	k=5	0.880	6.63	0.936	10.18	0.914	8.11	0.862	4.07
	k=10	0.861	2.98	0.944	4.06	0.886	2.93	0.865	2.28
	k=20	0.872	1.87	0.947	2.25	0.867	1.59	0.861	1.45
	k=40	0.899	1.27	0.951	1.41	0.862	1.00	0.864	0.97
	k=80	0.916	0.91	0.953	0.96	0.847	0.68	0.869	0.70
n=40	k=5	0.881	6.71	0.935	10.27	0.908	8.25	0.862	4.14
	k=10	0.861	2.97	0.944	4.06	0.886	2.92	0.865	2.27
	k=20	0.869	1.83	0.946	2.22	0.866	1.58	0.860	1.43
	k=40	0.903	1.28	0.952	1.41	0.858	1.00	0.862	0.98
	k=80	0.918	0.90	0.951	0.94	0.848	0.67	0.854	0.67
n=80	k=5	0.880	6.51	0.931	9.84	0.908	8.00	0.855	3.90
	k=10	0.859	2.97	0.946	4.05	0.886	2.91	0.858	2.21
	k=20	0.872	1.84	0.948	2.23	0.867	1.58	0.857	1.41
	k=40	0.897	1.26	0.953	1.40	0.857	0.99	0.852	0.95
	k=80	0.917	0.88	0.951	0.93	0.843	0.66	0.848	0.65
n=160	k=5	0.876	6.49	0.929	10.03	0.908	8.04	0.851	3.87
	k=10	0.849	2.89	0.942	3.98	0.883	2.88	0.854	2.16
	k=20	0.861	1.81	0.947	2.19	0.862	1.55	0.846	1.37
	k=40	0.894	1.27	0.951	1.39	0.849	0.98	0.845	0.93
	k=80	0.910	0.88	0.949	0.93	0.840	0.66	0.841	0.64

Table 4.11 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed Normal with Previous Study Information, $\rho = 0.5$

		Proposed		Proposed-Previous		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.931	8.50	0.942	12.32	0.910	9.18	0.858	4.62
	k=10	0.923	3.64	0.972	4.63	0.895	3.34	0.863	2.55
	k=20	0.908	2.12	0.947	2.41	0.878	1.81	0.866	1.60
	k=40	0.911	1.42	0.942	1.52	0.863	1.14	0.853	1.07
	k=80	0.921	0.97	0.936	1.00	0.855	0.76	0.849	0.74
n=20	k=5	0.938	7.77	0.967	11.36	0.906	8.52	0.847	4.15
	k=10	0.889	3.46	0.951	4.41	0.877	3.11	0.845	2.34
	k=20	0.888	2.03	0.943	2.32	0.863	1.68	0.838	1.48
	k=40	0.900	1.37	0.938	1.46	0.851	1.06	0.843	0.99
	k=80	0.917	0.95	0.935	0.98	0.841	0.71	0.835	0.68
n=40	k=5	0.912	7.56	0.954	11.01	0.903	8.28	0.836	3.98
	k=10	0.879	3.34	0.951	4.25	0.876	3.00	0.841	2.24
	k=20	0.879	2.00	0.939	2.29	0.851	1.62	0.834	1.42
	k=40	0.895	1.35	0.935	1.44	0.842	1.02	0.833	0.96
	k=80	0.914	0.94	0.936	0.97	0.832	0.68	0.825	0.66
n=80	k=5	0.905	7.24	0.949	10.14	0.903	7.95	0.838	3.80
	k=10	0.874	3.24	0.948	4.15	0.880	2.92	0.841	2.17
	k=20	0.874	1.97	0.939	2.24	0.850	1.58	0.830	1.39
	k=40	0.900	1.33	0.936	1.42	0.841	1.00	0.831	0.93
	k=80	0.913	0.93	0.936	0.96	0.831	0.67	0.822	0.65
n=160	k=5	0.901	7.35	0.946	10.69	0.900	8.09	0.836	3.85
	k=10	0.876	3.20	0.949	4.07	0.877	2.86	0.838	2.13
	k=20	0.870	1.94	0.938	2.21	0.851	1.56	0.831	1.36
	k=40	0.898	1.34	0.934	1.42	0.840	0.99	0.830	0.93
	k=80	0.911	0.92	0.933	0.95	0.828	0.66	0.820	0.64

Table 4.12 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed $Beta(1,3)$ with Previous Study Information, $\rho = 0.5$

		Proposed		Proposed-Previous		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.932	8.82	0.945	12.20	0.912	9.35	0.860	4.81
	k=10	0.924	3.67	0.966	4.61	0.889	3.33	0.863	2.58
	k=20	0.909	2.16	0.948	2.45	0.874	1.81	0.862	1.62
	k=40	0.917	1.42	0.944	1.51	0.864	1.13	0.858	1.08
	k=80	0.924	0.98	0.938	1.01	0.854	0.76	0.855	0.75
n=20	k=5	0.936	7.75	0.966	11.16	0.907	8.42	0.843	4.14
	k=10	0.895	3.39	0.956	4.34	0.889	3.09	0.856	2.34
	k=20	0.887	2.00	0.944	2.30	0.868	1.66	0.846	1.47
	k=40	0.901	1.38	0.935	1.47	0.844	1.05	0.836	1.00
	k=80	0.917	0.94	0.934	0.97	0.842	0.70	0.839	0.69
n=40	k=5	0.913	7.66	0.957	11.24	0.904	8.41	0.845	4.05
	k=10	0.879	3.22	0.949	4.12	0.879	2.93	0.873	2.19
	k=20	0.874	1.98	0.941	2.26	0.855	1.61	0.834	1.41
	k=40	0.901	1.33	0.937	1.33	0.845	1.01	0.834	0.95
	k=80	0.918	0.93	0.937	0.96	0.837	0.68	0.833	0.66
n=80	k=5	0.902	7.52	0.944	10.73	0.895	8.13	0.836	3.88
	k=10	0.881	3.33	0.947	4.24	0.875	2.96	0.845	2.21
	k=20	0.876	1.96	0.937	2.25	0.856	1.58	0.831	1.38
	k=40	0.894	1.34	0.936	1.42	0.840	0.99	0.828	0.93
	k=80	0.919	0.93	0.956	0.96	0.832	0.67	0.828	0.65
n=160	k=5	0.905	7.47	0.948	10.92	0.903	8.17	0.842	3.89
	k=10	0.870	3.18	0.948	4.01	0.876	2.87	0.839	2.14
	k=20	0.871	1.94	0.939	2.21	0.852	1.57	0.832	1.37
	k=40	0.894	1.34	0.931	1.43	0.831	0.99	0.822	0.93
	k=80	0.913	0.92	0.932	0.95	0.826	0.66	0.817	0.64

Table 4.13 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed *Normal* with Known Kurtosis, $\rho = 0.5$

		Proposed		Proposed-Gamma/Beta		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.948	9.48	0.945	11.05	0.906	9.36	0.857	4.70
	k=10	0.923	3.60	0.973	4.41	0.897	3.31	0.862	2.53
	k=20	0.903	2.14	0.957	2.50	0.873	1.81	0.860	1.61
	k=40	0.915	1.43	0.959	1.57	0.867	1.14	0.860	1.08
	k=80	0.920	0.98	0.953	1.05	0.857	0.76	0.853	0.74
n=20	k=5	0.949	7.79	0.977	9.03	0.906	8.49	0.846	4.13
	k=10	0.898	3.39	0.964	4.16	0.890	3.10	0.860	2.33
	k=20	0.894	2.01	0.962	2.36	0.872	1.68	0.850	1.47
	k=40	0.899	1.40	0.952	1.52	0.843	1.07	0.834	1.00
	k=80	0.918	0.95	0.951	1.00	0.946	0.70	0.842	0.68
n=40	k=5	0.912	7.54	0.959	8.83	0.905	8.36	0.843	4.01
	k=10	0.878	3.27	0.958	4.02	0.876	2.98	0.843	2.22
	k=20	0.880	1.99	0.953	2.32	0.854	1.62	0.835	1.42
	k=40	0.897	1.33	0.956	1.46	0.841	1.01	0.832	0.95
	k=80	0.917	0.93	0.954	0.98	0.837	0.68	0.826	0.66
n=80	k=5	0.902	7.35	0.951	8.48	0.899	8.04	0.835	3.84
	k=10	0.871	3.29	0.954	3.99	0.870	2.94	0.839	2.19
	k=20	0.875	1.95	0.956	2.27	0.857	1.58	0.836	1.38
	k=40	0.901	1.35	0.955	1.46	0.841	1.00	0.833	0.94
	k=80	0.913	0.94	0.952	0.98	0.829	0.67	0.823	0.65
n=160	k=5	0.902	7.31	0.952	8.45	0.902	8.01	0.840	3.81
	k=10	0.868	3.28	0.857	3.98	0.867	2.93	0.836	2.17
	k=20	0.878	1.94	0.958	2.26	0.855	1.57	0.835	1.37
	k=40	0.897	1.32	0.957	1.45	0.842	0.99	0.830	0.93
	k=80	0.912	0.93	0.952	0.97	0.830	0.66	0.827	0.64

Table 4.14 Comparison of Methods when Random Effects are Distributed $\chi^2(5)$ and Errors are Distributed $Beta(1,3)$ with Known Kurtosis, $\rho = 0.5$

		Proposed		Proposed-Gamma/Beta		Exact		Burch	
		Prob	AvgW	Prob	AvgW	Prob	AvgW	Prob	AvgW
n=10	k=5	0.937	8.80	0.942	10.03	0.910	9.20	0.854	4.74
	k=10	0.927	3.70	0.977	4.50	0.895	3.37	0.871	2.61
	k=20	0.908	2.17	0.957	2.49	0.874	1.81	0.861	1.62
	k=40	0.917	1.44	0.957	1.57	0.865	1.13	0.856	1.08
	k=80	0.921	0.98	0.952	1.05	0.857	0.76	0.855	0.74
n=20	k=5	0.942	8.07	0.964	9.23	0.906	8.68	0.852	4.26
	k=10	0.895	3.36	0.963	4.12	0.883	3.07	0.852	2.32
	k=20	0.892	2.06	0.958	2.38	0.864	1.69	0.851	1.49
	k=40	0.903	1.39	0.953	1.51	0.849	1.06	0.841	1.00
	k=80	0.914	0.94	0.953	1.00	0.842	0.70	0.844	0.68
n=40	k=5	0.912	7.52	0.954	8.74	0.901	8.27	0.845	3.99
	k=10	0.885	3.27	0.962	4.00	0.886	2.96	0.851	2.22
	k=20	0.879	1.99	0.958	2.31	0.861	1.62	0.843	1.42
	k=40	0.895	1.33	0.956	1.46	0.843	1.01	0.830	0.95
	k=80	0.920	0.94	0.952	0.99	0.838	0.68	0.834	0.66
n=80	k=5	0.909	7.40	0.952	8.57	0.902	8.12	0.942	3.88
	k=10	0.872	3.29	0.958	3.98	0.873	2.93	0.836	2.19
	k=20	0.874	1.94	0.958	2.27	0.853	1.58	0.835	1.38
	k=40	0.897	1.34	0.956	1.46	0.839	1.00	0.831	0.94
	k=80	0.918	0.94	0.953	0.98	0.833	0.67	0.828	0.65
n=160	k=5	0.899	7.13	0.950	8.23	0.906	7.80	0.835	3.72
	k=10	0.869	3.22	0.957	3.93	0.876	2.89	0.837	2.15
	k=20	0.875	1.94	0.963	2.26	0.853	1.57	0.834	1.37
	k=40	0.895	1.33	0.952	1.45	0.837	0.99	0.827	0.93
	k=80	0.918	0.93	0.956	0.97	0.831	0.66	0.824	0.64

CHAPTER 5. CONCLUSION AND FUTURE WORK

Over the past several decades, the random effects model has been widely used to model experiments in many different fields. In all of these fields, it is important for researchers to have both point and interval estimates of the variance components in order to make decisions and test theories. The results given in Section 2.3 illustrated how sensitive the classical confidence interval methods are to the violation of non-normality. The Wald, Exact, and Scheffé methods, as well as the method proposed by Burch (2001), consistently give coverage probabilities well below the nominal level. In this chapter we summarize the methods presented in this work and discuss how they can produce more accurate confidence intervals when the assumption of non-normality is violated. Limitations of the proposed methods and ideas for future investigation and related work are also noted.

In Chapter 3, we proposed a method for constructing a confidence interval for σ_α^2 based on the chi-squared distribution. We derived the approximate variance of $\hat{\sigma}_\alpha^2$ under the assumption of non-normality. This included an estimate of the kurtosis of the distribution of the random effect, for which Bonett's (2006b) recommended estimator was used. A small sample adjustment suggested by Shoemaker (2003) was also used. Through simulation studies, we were able to show that the proposed method performed better than the Wald, Scheffé, and SAS methods, the majority of the time.

In Chapter 4, we proposed a method for constructing a confidence interval for θ using an approximate variance stabilizing transformation, $ln(\hat{\theta})$, and the classical Wald method. We derived the approximate variance of θ under the assumption of non-normality. This included estimates of both the kurtosis of the distribution of the random effect and the kurtosis of the distribution of the errors. Bonett's (2006b) recommended estimator of kurtosis was used for both the random effect and the errors. Shoemaker's (2003) small sample adjustment was also used. Through simulation studies, we were able to show that the proposed method performed better than the Exact method and the mehod proposed by Burch (2011), the majority of the time.

For both σ_α^2 and θ, we also looked at how incorporating information the researcher may have about the distributions of the random effect or the errors affects the accuracy of the confidence intervals. We first considered the case where the researcher had previous study information that could be used to obtain an estimate of kurtosis that could then be pooled with the current study's kurtosis estimate. This pooled estimate should be more accurate and lead to more accurate estimates of the variance and therefore more accurate confidence intervals.

We then considered the case where the researcher may be able to theoretically specify kurtosis values based on expert knowledge in their field. If they knew that the random effect or errors should theoretically follow a known distribution, the kurtosis of that specific distribution could be used in the variance formula, rather than an estimate.

Lastly, we considered the case where it may be reasonable to approximate the distribution of the random effect or the distribution of the errors with a distribution whose parameters can be used to estimate kurtosis. We looked at the beta distribution, in which estimates of the shape and scale parameter can be used to calculate an estimate of kurtosis and the gamma distribution, in which an estimate of the shape parameter can be used to calculate an estimate of the kurtosis.

All of these estimates of kurtosis using prior information were used with the proposed methods for σ_α^2 and θ. Simulation studies showed that these methods that incorporated additional knowledge about the data had a great advantage over all the other methods.

There are several different paths that could be pursued in future research to extend the results of this thesis. The first possibilty is to look at two-factor random effects models. The methods for some of the variance components will be straight forward extensions of the one-way model, while others will take more work. The two-factor model would also allow us to look at quantities such as the differnce in means, which is often a quantity of interest.

Another possibility is to look at random effects meta-analysis models. These models have become increasingly popular in recent years because methods for combining effect-size estimates from multiple studies have important applications in a variety of disciplines. Through out this research, we have assumed a balanced model. In a meta-analysis, it is very unlikely that all of the studies will have the same sample size. If the sample sizes, are not approximately equal, it will be necessary to adjust the method to account for the unbalanced design.

Finally, a deeper investigation could be made into the cases with a small number of groups. Although we used Shoemaker's (2003) small sample adjustment, none of the methods we looked at consistently give good results when there are a small number of groups, especially when the sample size per group is also small. Maybe this is simply due to the fact that there is not enough information to get an accurate estimate, but more research into alternative small sample adjustments could be done.

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