Sampling and interpolation in Hilbert spaces of entire functions

Sa'ud Shehab Al-Sa'di

Iowa State University

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Sampling and interpolation in Hilbert spaces of entire functions

by

Sa’ud Shehab Al-Sa’di

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Eric Weber, Co-major Professor
Justin Peters, Co-major Professor
Stephen Willson
Fritz Keinert
Gary Lieberman

Iowa State University
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DEDICATION

I dedicate this dissertation to my beloved sons, Hashem and Bassel, both of whom, I love so very dearly, and I miss them tremendously. I am proud to be their father, they have taught me the true meaning of life, and gave me love, happiness, and hope.

Also I dedicate this work and give special thanks to my beloved and wonderful mother. Her support, encouragement, and constant love have sustained me throughout my life. Without you this would not have been possible. Finally, I would like to dedicate this thesis to the memory of my father, who was the first encourager and supporter to me in continuing my education.
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ABSTRACT

Sampling theory is the study of spaces of functions which are reconstructible from their values at certain sets of points, which gives rise to a sampling formula for the underlying space. For this, it is necessary to consider spaces of functions whose values at a set of points are well-defined. In this work we consider the sampling and interpolation problems in reproducing kernel Hilbert space of entire functions, called de Branges space. Functions in such space are square integrable on the real line with respect to some weight function, and satisfying some growth conditions. Some sampling and interpolation results in the Paley-Wiener spaces, which are a primary example of de Branges spaces, are reviewed. We develop necessary conditions for sampling and interpolating sequences which generalize some well-known sampling and interpolation results in the Paley-Wiener space. The proofs of the necessary conditions rely very much on the Homogeneous Approximation Property and the Comparison Theorem that we prove in de Branges space. We also give necessary and sufficient conditions for Plancherel-Pólya sequences, and sufficient conditions for interpolation in some de Branges spaces of exponential type.
CHAPTER 1. Introduction

The theory of Hilbert spaces of entire functions was first introduced by L. de Branges in the series of papers [15; 16; 17; 18]. These spaces, which are now called de Branges spaces, generalize the classical Paley-Wiener space which consists of the entire functions of exponential type which are square integrable on the real line. L. de Branges used his construction most notably in his famous proof of the Bieberbach conjecture in 1984. In the theory of differential operators, de Branges used certain properties of these spaces to resolve an important uniqueness problem which is connected with the reconstruction of a canonical system of two differential equations from its monodromy matrix, a problem which defied solution for a long time. Recently, Ortega-Cerdà and K. Seip provided a description of the exponential frames for the Paley-Wiener space, and a related study of sampling and interpolation, by connecting the problem to the de Branges spaces theory of entire functions [40].

A de Branges space consists of entire functions which are square integrable on the real line with respect to some weight function, and which satisfy some growth conditions also with respect to the weight function. A de Branges space is determined by a weight function $E(z)$ which is entire and satisfies the condition

$$|E(\bar{z})| < |E(z)|, \quad \text{Im} \, z > 0.$$  

Such a function is usually called a de Branges function, which coincides with the Hermite-Biehler class of functions. For the Paley-Wiener space, the weight function $E(z)$ has the form $e^{-iaz}$, $a > 0$.

In chapter two, we review basic knowledge of complex and functional analysis. We give basic definitions and theorems that are necessary in this work, including frames and reproducing kernel Hilbert spaces, which will be the framework of our sampling theory. We also give a brief
introduction to the sampling and interpolation theory in reproducing kernel Hilbert spaces.

In chapter three, we introduce the Paley-Wiener spaces of entire functions and their properties, which are the primary example of a de Branges space. The duality between sampling sequences for Paley-Wiener spaces and frames of complex exponential functions for a corresponding $L^2$-space is especially emphasized. We also present the most basic results and a collection of already known facts about sampling and interpolation in the Paley-Wiener spaces, in particular, the necessary and sufficient conditions for sampling and interpolating sequences in terms of Beurling densities. We then give a brief review of the theory of de Branges Hilbert spaces of entire functions, with their relation to entire functions of Hermite-Biehler class. We recall some basic definitions and collect some results and properties of such spaces which are essential for what follows. This includes some equivalent characterizations of de Branges spaces, and a discussion on orthogonal sets in the space.

In chapter four, we answer the questions on sampling and interpolating in de Branges spaces. Given a sequence of real numbers $\{\mu_n\}$, we consider the question of what properties must $\{\mu_n\}$ have in order to be sampling, i.e. that any function in the space can be reconstructed from the values it takes at the points of the sequence $\{\mu_n\}$. We also consider the question of when $\{\mu_n\}$ is interpolating, i.e. when does there exist a function which attains specified values at the points $\{\mu_n\}$. In order to answer these questions, the main tool used is the Beurling density, and the main results will be similar to known results in the Paley-Wiener spaces.

The chapter begins with simple proofs of basic results about Plancherel-Pólya sequences. The proof of the necessary conditions for sampling and interpolating sequences relies on applying our new result on the homogeneous approximation property and the comparison theorem in de Branges spaces which are proved in section 4.2. We then state the theorem describing interpolation in de Branges spaces, more specifically in some de Branges spaces of exponential type. The chapter ends with connecting some of our results to the Feichtinger conjecture about Bessel sequences.
CHAPTER 2. Preliminaries and Overview

In this chapter we introduce the concept of frames in a general Hilbert spaces, then we will focus our attention on a spacial class of frames in reproducing kernel Hilbert spaces, which will play a significant role in sampling theory.

2.1 Frames for Hilbert Spaces

Frame theory arises in applied mathematics, and this appearance is due to the flexibility and redundancy of frames. The concept of frames was introduced by Duffin and Schaeffer [19] in 1952 while working on some problems in nonharmonic Fourier Series, and was greatly popularized by the work of Daubechies and her coauthors [10; 11; 12]. Besides traditional and relevant applications of frames in signal processing, image processing, data compression, sampling theory, recently, frames have been used in numerical analysis in the solution of operator equations, see [9; 44].

Given a Banach space, it is advantageous to find a basis for the space, i.e., a fixed set of vectors \{f_n\} such that any vector \(f\) in the space can be written as \(f = \sum_n c_n f_n\) for some unique choice of scalars \(c_n\). For most of the spaces encountered in ordinary analysis we know that bases exist, but usually we need more than existence. For example, we may want the \(f_n\) to be easily generated in some way or to satisfy some special properties and the \(c_n\) be easy to generate and compute, etc. These conditions can be difficult to satisfy simultaneously.

Let \(H\) be a Hilbert space, and denote by \(\langle f, g \rangle_H\) the inner product, and let \(\|f\|_H = \sqrt{\langle f, f \rangle_H}\) be the norm in \(H\), for \(f, g \in H\). Recall that a set \(\{f_n\}_{n \in I}\) in a Hilbert space \(H\) is said to be an orthonormal set if \(\langle f_n, f_m \rangle = 0\) whenever \(n \neq m\), and \(\langle f_n, f_n \rangle = 1\) for all \(n, m \in I\). An orthonormal basis of a Hilbert space is an orthonormal set such that every element in the space
can be expanded in terms of the basis, in a way that we make precise in the following theorem [20].

**Theorem.** Suppose $\mathcal{H}$ is a Hilbert space and $\{f_n\}_{n \in I}$ is an orthonormal basis in $\mathcal{H}$. Then

$$f = \sum_{n \in I} \langle f, f_n \rangle f_n, \quad \text{for all } f \in \mathcal{H}, \quad (2.1.1)$$

and Parseval’s identity holds

$$\|f\|_{\mathcal{H}}^2 = \sum_{n \in I} |\langle f, f_n \rangle_{\mathcal{H}}|^2, \quad \text{for all } f \in \mathcal{H}. \quad (2.1.2)$$

It is well known that for any separable Hilbert space $\mathcal{H}$, we can construct a (countable) orthonormal basis [27]. Much effort has been expended in finding orthonormal bases for various Hilbert spaces which satisfy additional properties to suit some problem. However, the requirement of orthogonality is very stringent, making it difficult as to find a good orthonormal basis in many situations.

Frames are an alternative to and a generalization of the orthonormal bases, by giving up the requirements of orthogonality and uniqueness of decomposition, we get more freedom in the choice of the $\{f_n\}$ which appear in the expansion (2.1.1), but we still retain good control on the behavior of the coefficients $c_n$’s and the ability to decompose the space [46].

The coefficients $c_n$ are not unique in general, but there is a canonical choice. In all cases, the expansions converge unconditionally (regardless of ordering). A frame for which the coefficients are unique for each $f$ is called a Riesz basis, and is the image of an orthonormal basis under a continuous bijection of $\mathcal{H}$ onto itself. We will briefly recall the definition and basic properties of frames in Hilbert spaces. For more information we refer to [19].

The finite linear span of a sequence of elements $\mathcal{F} = \{f_n\}_{n \in I}$ of $\mathcal{H}$, denoted by $\text{span}(\mathcal{F})$, is the set of all finite linear combinations of the $f_n$’s. We say that $\mathcal{F}$ is complete if the only vector $f$ satisfying $\langle f, f_n \rangle = 0$ for all $n \in I$ is $f \equiv 0$.

**Definition 2.1.1.** ([19]). Let $\mathcal{H}$ be a Hilbert space. A set of vectors $\{f_n; n \in I\} \subseteq \mathcal{H}$ is a frame in $\mathcal{H}$ if there exists constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n \in I} |\langle f, f_n \rangle |^2 \leq B\|f\|^2, \quad (2.1.3)$$
for all $f \in \mathcal{H}$. The constants $A$ and $B$ are called lower and upper frame bounds, respectively. The optimal lower frame bound is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. If $A = B$, then $\{f_n\}_{n \in I}$ is called a tight frame.

Clearly, every orthonormal basis of $\mathcal{H}$ is a tight frame with frame bounds $A = B = 1$. This can be noted from the Parseval’s identity. Moreover, it is easily seen from the lower bound that frames are complete in $\mathcal{H}$.

The frame operator $Sf = \sum_{n \in I} \langle f, f_n \rangle f_n$ is a bounded, positive, and invertible mapping of $\mathcal{H}$ onto itself. The canonical dual frame is $\{\tilde{f}_n\}_{n \in I}$ where $\tilde{f}_n = S^{-1}f_n$. If $\{f_n\}_{n \in I}$ has frame bounds $A, B$, then the canonical dual frame is a frame with frame bounds $\frac{1}{B}, \frac{1}{A}$. Furthermore, for each $f \in \mathcal{H}$ we have the frame expansions

$$f = \sum_{n \in I} \langle f, f_n \rangle \tilde{f}_n = \sum_{n \in I} \langle f, \tilde{f}_n \rangle f_n,$$

with unconditional convergence of these series.

A family $\{f_n\}_{n \in I}$ is a Riesz basis for $\mathcal{H}$ if it is the image of an orthonormal basis for $\mathcal{H}$ under a continuous, invertible mapping of $\mathcal{H}$ onto itself. By [5], $\{f_n\}_{n \in I}$ is a Riesz basis in $\mathcal{H}$ if and only if it is complete and there exist constants $A, B > 0$ such that

$$A \sum_n |a_n|^2 \leq \| \sum_n a_n f_n \|^2 \leq B \sum_n |a_n|^2,$$

for all sequences of scalars $\{a_n\} \in \ell^2$. where $\ell^2$ is the Hilbert space of square summable sequences. More generally, one defines a Riesz sequence to be a set that satisfies (2.1.5) but is not necessarily complete in $\mathcal{H}$, i.e., it will form a Riesz basis for its closed linear span, which might be only a subspace of $\mathcal{H}$.

We now give a necessary and sufficient condition for a frame to be a Riesz basis, we refer to chapter 6 of [5] for proof.

**Theorem 2.1.1.** Let $\{f_n\}_{n \in I}$ be a frame for $\mathcal{H}$. The following are equivalent.

(i) $\{f_n\}_{n \in I}$ is a Riesz basis for $\mathcal{H}$.

(ii) If $\sum_{n \in I} c_n f_n = 0$ for some sequence of scalars $\{c_n\}_{n \in I} \in \ell^2$, then $c_n = 0$ for all $n \in I$.
A sequence \( \{f_n\}_{n \in I} \subset \mathcal{H} \) is called a **Riesz-Fischer sequence** if the problem
\[
\langle f, f_n \rangle_{\mathcal{H}} = a_n, \quad \text{for all } n,
\] (2.1.6)
admits at least one solution \( f \in \mathcal{H} \) whenever \( \{a_n\} \in l^2 \). The following characterization of Riesz-Fischer sequences will play a role in the proofs, see [47] page 154.

**Theorem 2.1.2.** Let \( \{f_n\} \) be a sequence in a Hilbert space \( \mathcal{H} \). Then

(i) If \( \{f_n\} \) is a Riesz-Fischer sequence in \( \mathcal{H} \), then there exists a constant \( c > 0 \) such that
\[
\|f\|^2 \leq \frac{1}{c} \sum_{n} |a_n|^2,
\] (2.1.7)
provided \( \{a_n\} \in l^2 \).

(ii) The sequence \( \{f_n\} \) is a Riesz-Fischer sequence in \( \mathcal{H} \) if and only if there exists \( C > 0 \) such that the inequality
\[
C \sum_{n} |a_n|^2 \leq \| \sum_{n} a_n f_n \|^2,
\] (2.1.8)
holds for every sequence \( \{a_n\} \in l^2 \).

A sequence which satisfies the upper frame bound estimate in (2.1.3), but not necessarily the lower estimate, is called a **Bessel sequence**, and \( B \) is called a **Bessel bound**, i.e.,
\[
\sum_{n \in I} |\langle f, f_n \rangle|^2 \leq B \|f\|^2, \quad \text{for all } f \in \mathcal{H}.
\] (2.1.9)

In applications, a sequence which is a frame is often easily shown to be a Bessel sequence, while the lower frame bound is often more difficult to establish. We will now prove a useful characterization of Bessel sequences, see [5].

**Theorem 2.1.3.** Let \( \{f_n\} \) be a sequence in \( \mathcal{H} \) and \( B > 0 \) be given. Then \( \{f_n\}_{n=1}^\infty \) is a Bessel sequence with Bessel bound \( B \) if and only if
\[
T : l^2 \rightarrow \mathcal{H}, \quad T\{a_n\}_{n=1}^\infty := \sum_{n=1}^\infty a_n f_n,
\]
defines a bounded linear operator and \( \|T\| \leq \sqrt{B} \).
In other words, this theorem says that a sequence \( \{f_n\}_{n=1}^{\infty} \) is a Bessel sequence with Bessel bound \( B \) if and only if for any \( \{a_n\} \in l^2 \), the following inequality holds
\[
\left\| \sum_n a_n f_n \right\|^2 \leq B \sum_n |a_n|^2.
\] (2.1.10)

The proof of this theorem requires the following lemma.

**Lemma 2.1.4.** Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence in a Hilbert space \( \mathcal{H} \), and suppose that
\[
T : l^2 \to \mathcal{H}, \ T\{a_n\}_{n=1}^{\infty} := \sum_{n=1}^{\infty} a_n f_n
\]
defines a bounded linear operator. Then the adjoint operator is given by
\[
T^* : \mathcal{H} \to l^2, \ T^*(f) = \{\langle f, f_n \rangle_{\mathcal{H}}\}_{n=1}^{\infty}.
\] (2.1.11)

Furthermore,
\[
\sum_{n=1}^{\infty} |\langle f, f_n \rangle_{\mathcal{H}}|^2 \leq \|T\|^2 \|f\|^2_{\mathcal{H}}, \text{ for all } f \in \mathcal{H}.
\]

**Proof.** Assume that the operator \( T \) defined above is bounded and linear. Let \( f \in \mathcal{H} \), and \( \{a_n\}_{n=1}^{\infty} \in l^2 \). Then
\[
\langle f, T(\{a_n\}_{n=1}^{\infty}) \rangle_{\mathcal{H}} = \langle f, \sum_{n=1}^{\infty} a_n f_n \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \bar{a}_n \langle f, f_n \rangle_{\mathcal{H}}.
\] (2.1.12)

Since \( T \) is bounded operator, then its adjoint \( T^* \) is also bounded linear operator from \( \mathcal{H} \) to \( l^2 \), and \( \|T^*\| = \|T\| \), see Lemma 4.5.2 of [6]. Since \( T^* f \in l^2 \) for any \( f \in \mathcal{H} \), then we can write \( T^* f \) as
\[
T^* f = \{(T^* f)_n\}_{n=1}^{\infty},
\]
where \( (T^* f)_n \) is the \( n \)-th term of the sequence \( T^* f \). Moreover, since \( T^* \) is bounded we have
\[
\left( \sum_{n=1}^{\infty} |(T^* f)_n|^2 \right)^{1/2} = \|T^* f\|_{l^2} \leq \|T^*\| \|f\|_{\mathcal{H}},
\]
hence, for any \( n \in \mathbb{N} \) we have
\[
|(T^* f)_n| \leq \|T^*\| \|f\|_{\mathcal{H}},
\]
i.e., the map \( f \mapsto (T^* f)_n \) is a bounded map from \( \mathcal{H} \) to \( \mathbb{C} \). Therefore, by Riesz representation theorem, for each \( n \in \mathbb{N} \), there exists \( g_n \in \mathcal{H} \) such that
\[
(T^* f)_n = \langle f, g_n \rangle_{\mathcal{H}}, \ f \in \mathcal{H}
\]
Henec,

\[ T^* f = \{ \langle f, g_n \rangle \}_{n=1}^{\infty} \quad (2.1.13) \]

for some fixed sequence \( \{g_n\}_{n=1}^{\infty} \) in \( \mathcal{H} \).

By the definition of the adjoint operator, we have

\[
\langle f, T \{a_n\}_{n=1}^{\infty} \rangle = \langle T^* f, \{a_n\}_{n=1}^{\infty} \rangle_{l^2}
= \langle \{\langle f, g_n \rangle\}_{n=1}^{\infty}, \{a_n\}_{n=1}^{\infty} \rangle_{l^2}
= \sum_{n=1}^{\infty} \bar{a}_n \langle f, g_n \rangle_{\mathcal{H}},
\]

for all \( \{a_n\}_{n=1}^{\infty} \in l^2 \), and \( f \in \mathcal{H} \). But, by (2.1.12) we also have

\[
\langle f, T \{a_n\}_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} \bar{a}_n \langle f, f_n \rangle_{\mathcal{H}}
\]

therefore,

\[
\sum_{n=1}^{\infty} \bar{a}_n \langle f, g_n \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \bar{a}_n \langle f, f_n \rangle_{\mathcal{H}},
\]

for all \( \{a_n\}_{n=1}^{\infty} \in l^2 \) and all \( f \in \mathcal{H} \). Hence, \( f_n = g_n \) for all \( n \in \mathbb{N} \). Thus, this observation together with (2.1.13) implies that \( T^* \) has the form in (2.1.11).

Since \( T^* \) is bounded, \( \|T^*\| = \|T\| \), and \( T^* \) has the expansion in (2.1.11) we have

\[
\sum_{n=1}^{\infty} |\langle f, f_n \rangle_{\mathcal{H}}|^2 = \| \{\langle f, f_n \rangle_{\mathcal{H}}\}_{n=1}^{\infty} \|_{l^2}^2 = \|T^*f\|^2 \leq \|T^*\|^2 \|f\|^2 = \|T\|^2 \|f\|^2,
\]

completing the proof. \( \square \)

**Proof of Theorem 2.1.3:** Let \( \{f_n\}_{n=1}^{\infty} \) be a Bessel sequence with Bessel bound \( B > 0 \). Let \( \{a_n\}_{n=1}^{\infty} \in l^2 \), i.e., \( \sum_{n=1}^{\infty} |a_n|^2 < \infty \). In order to prove that the operator \( T \) is well defined we need to show that \( \sum_{n=1}^{\infty} a_n f_n \) is convergent. Consider \( n, m \in \mathbb{N} \), with \( n > m \), then

\[
\left\| \sum_{k=1}^{n} a_k f_k - \sum_{k=1}^{m} a_k f_k \right\| = \left\| \sum_{k=m+1}^{n} a_k f_k \right\|
= \sup_{\|f\|=1} \left| \sum_{k=m+1}^{n} a_k \langle f_k, f \rangle \right|
\leq \sup_{\|f\|=1} \sum_{k=m+1}^{n} |a_k \langle f_k, f \rangle|.
\]
\[
\leq \left( \sum_{k=m+1}^{n} |a_k|^2 \right)^{1/2} \sup_{\|f\|=1} \left( \sum_{k=m+1}^{n} |\langle f_k, f \rangle|^2 \right)^{1/2} \\
\leq \sqrt{B} \left( \sum_{k=m+1}^{n} |a_k|^2 \right)^{1/2}
\]

Now, since \( \{a_n\}_{n=1}^{\infty} \in l^2 \), then the sequence of partial sums \( \{\sum_{k=m}^{n} |a_k|^2\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R} \). Therefore, the sequence \( \{\sum_{k=m}^{n} a_k f_k\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathcal{H} \), hence convergent. Thus, \( T(\{a_n\}_{n=1}^{\infty}) \) is well-defined. On the other hand, note that

\[
\|T(\{a_n\}_{n=1}^{\infty})\| = \sup_{\|f\|=1} |\langle T(\{a_n\}_{n=1}^{\infty}), f \rangle| \\
= \sup_{\|f\|=1} \left| \sum_{n=1}^{\infty} a_n f_n, f \right| \\
\leq \sup_{\|f\|=1} \sum_{n=1}^{\infty} |a_n| |\langle f_n, f \rangle| \\
\leq \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \sup_{\|f\|=1} \left( \sum_{n=1}^{\infty} |\langle f_n, f \rangle|^2 \right)^{1/2} \\
\leq \sqrt{B} \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2}
\]

hence, \( T \) is bounded and \( \|T\| \leq \sqrt{B} \). Moreover, by the definition, it is clear that \( T \) is linear.

Conversely, suppose that \( T \) defines a bounded linear operator with \( \|T\| \leq \sqrt{B} \), then by Lemma 2.1.4

\[
\sum_{n=1}^{\infty} |\langle f, f_n \rangle_{\mathcal{H}}|^2 \leq \|T\|^2 \|f\|_{\mathcal{H}}^2 \leq B \|f\|_{\mathcal{H}}^2,
\]

for all \( f \in \mathcal{H} \). That is, \( \{f_n\}_{n=1}^{\infty} \) is a Bessel sequence with Bessel bound \( B \). \( \square \)

Recall that two sequences \( \{f_n\}, \{g_n\} \) are called biorthogonal if \( \langle f_n, g_m \rangle = \delta_{nm} \). It is known that for any Riesz basis there is a unique biorthogonal sequence, which also is a basis, moreover; we have the following lemma, see [5], page 54.

Lemma 2.1.5. Let \( \{f_n\} \) be a Riesz basis of a Hilbert space \( \mathcal{H} \), and \( \{\tilde{f}_n\} \) be its unique biorthogonal sequence. Then, for all \( f, g \in \mathcal{H} \)

\[
\langle f, g \rangle = \sum_{n} \langle f, f_n \rangle \langle \tilde{f}_n, g \rangle
\]

(2.1.14)
Given that \( \{f_n\}_{n \in I} \) is a frame for a Hilbert space \( \mathcal{H} \), the following version of Paley-Wiener theorem for frames gives conditions on a perturbed sequence \( \{g_n\}_{n \in I} \) which implies that it is a frame, see [5], chapter 15.

**Paley-Wiener Theorem for Frames.** Let \( \{f_n\}_{n \in I} \) be a frame for a Hilbert space \( \mathcal{H} \), with bounds \( A, B \), and let \( \{g_n\}_{n \in I} \) be a sequence in \( \mathcal{H} \). If there exists a constant \( R < A \) such that
\[
\sum_{n \in I} |\langle f, f_n - g_n \rangle|^2 \leq R \|f\|^2, \quad \forall f \in \mathcal{H},
\]
then \( \{g_n\} \) is a frame for \( \mathcal{H} \) with bounds
\[
A \left( 1 - \sqrt{\frac{R}{A}} \right)^2, \quad B \left( 1 + \sqrt{\frac{R}{B}} \right)^2.
\]
If \( \{f_n\} \) is a Riesz basis, then \( \{g_n\} \) is a Riesz basis.

**Lemma 2.1.6.** Given a separable Hilbert space \( \mathcal{H} \), and an operator \( T \) on \( \mathcal{H} \). Let \( \{g_n\}_{n=1}^{\infty} \) be a Riesz basis in \( \mathcal{H} \) and \( \{\tilde{g}_n\}_{n=1}^{\infty} \) is its biorthogonal sequence. If \( \mathcal{H} \) is a finite dimensional space, then
\[
\text{trace}(T) = \sum_n \langle T g_n, \tilde{g}_n \rangle. \tag{2.1.15}
\]

**Proof.** Since \( \mathcal{H} \) is a separable Hilbert space it possesses an orthonormal basis, say \( \{f_n\} \), hence, the trace is given by
\[
\text{trace}(T) = \sum_n \langle Tf_n, f_n \rangle.
\]
Given that \( \{g_n\} \) is a Riesz basis in \( \mathcal{H} \), then each element of the orthonormal basis can be expanded as
\[
f_n = \sum_m \langle f_n, \tilde{g}_m \rangle g_m = \sum_k \langle f_n, g_k \rangle \tilde{g}_k.
\]
Moreover, by (2.1.14) we have
\[
\sum_n \langle \tilde{g}_m, f_n \rangle \langle f_n, g_k \rangle = \langle \tilde{g}_m, g_k \rangle = \delta_{mk}.
\]
Therefore,
\[
\sum_n \langle Tf_n, f_n \rangle = \sum_n \langle T \left( \sum_m \langle f_n, \tilde{g}_m \rangle g_m \right), \sum_k \langle f_n, g_k \rangle \tilde{g}_k \rangle
\]
\[ \sum_n \sum_m \sum_k \langle f_n, \tilde{g}_m \rangle T g_m, \langle f_n, g_k \rangle \tilde{g}_m \rangle \]
\[ = \sum_n \sum_m \sum_k \langle f_n, \tilde{g}_m \rangle \langle g_k, f_n \rangle T g_m, \tilde{g}_m \rangle \]
\[ = \sum_m \sum_k \left( \sum_n \langle f_n, \tilde{g}_m \rangle \langle g_k, f_n \rangle \right) T g_m, \tilde{g}_m \rangle \]
\[ = \sum_m \sum_k \delta_{mk} T g_m, \tilde{g}_m \rangle \]
\[ = \sum_m (T g_m, \tilde{g}_m) \]

as desired. \( \square \)

### 2.2 Function Theory

In this section, we give a brief overview of some basic concepts about analytic and entire functions from complex function theory that will be useful throughout this work. Further information can be found in Boas [4], Ya. Levin [33] and [34], amongst others. We denote the set of real numbers by \( \mathbb{R} \), the field of complex numbers by \( \mathbb{C} \), and the upper half-plane \( \mathbb{C}^+ = \{ z \in \mathbb{C}, \ \text{Im}(z) > 0 \} \).

**Definition.** The Hardy space in the upper half-plane \( H^p := H^p(\mathbb{C}^+) \), for \( 0 < p < \infty \), is the set of all analytic functions in the upper half-plane \( \mathbb{C}^+ \) that satisfy

\[
\|f\|_{H^p}^p := \sup_{y>0} \int_{\mathbb{R}} |f(x + iy)|^p dx < \infty.
\]

If \( p = \infty \), then we write \( H^\infty(\mathbb{C}^+) \) for the bounded analytic functions in the upper half-plane. The norm of each \( f \in H^\infty(\mathbb{C}^+) \) is defined by \( \|f\|_{\infty} = \sup_{z \in \mathbb{C}^+} |f(z)| \).

A function is said to be meromorphic on an open set \( U \) of the complex plane if it has no singularities, other than possible poles. An inner function in the upper half-plane is a function \( \Theta(z) \in H^\infty(\mathbb{C}^+) \) such that \( |\Theta(x)| = 1 \) for almost all \( x \in \mathbb{R} \), with respect to the Lebesgue measure. We say that an inner function \( \Theta(z) \) in \( \mathbb{C}^+ \) is a meromorphic inner function if it admits a meromorphic extension to the whole complex plane. It is known (see [24], Lemma 13) that such a function has the form

\[
\Theta(z) = \gamma \exp(-i\tau z)B(z),
\]

(2.2.2)
where \( \tau \leq 0, \gamma \in \mathbb{C} \) with \(|\gamma|=1\), and \( B \) is a Blaschke product with zeros \( \{z_k\} \) of \( \Theta(z) \) in the upper half-plane, tending to infinity, given by

\[
B(z) = \prod_k \left(1 - \frac{z}{z_k}\right) \left(1 - \frac{z}{\overline{z}_k}\right)^{-1}
\]  
(2.2.3)

where the infinite product converges uniformly and absolutely in any compact set of the plane that does not contain points \( \overline{z}_k \) (see lemma 4 in chapter 5 of [34]). If \( \Theta(z) \) has no zeros in the upper half-plane, the product \( B \) is taken equal to 1. Moreover, by Theorem 8 of [14], the zero sequence \( Z = \{z_k\} \) (where \( z_k = x_k + iy_k, y_k > 0 \)) must satisfy the Blaschke condition

\[
\sum_{z_k \in Z} \frac{1}{|z_k|} = \sum_{z_k \in Z} \frac{y_k}{x_k^2 + y_k^2} < +\infty
\]  
(2.2.4)

Given a meromorphic inner function \( \Theta(z) \), there is a well-defined branch of the argument of \( \Theta \) on the real line, that is, a real analytic and strictly increasing function \( \psi \) such that \( \Theta(x) = \exp(i\psi(x)), \ x \in \mathbb{R} \). Indeed, the function \( \psi(x) = \arg \Theta(x) \) is real analytic because \( \log \Theta(z) = \log |\Theta(z)| + i \arg \Theta(z) \) is analytic in a neighborhood of the real line and \( \log |\Theta(z)| = 0 \) on the line, since \( \Theta(z) \) is inner.

To see that \( \psi \) is strictly increasing first notice that the argument of a Blaschke factor \( (z - z_k)/(z - \overline{z}_k) \) is strictly increasing (since it is equal to an arc-tangent). Since for the function \( \exp(-i\tau z), \ \tau < 0 \) we have \( \arg \exp(-i\tau x) = -\tau x \), it is also strictly increasing. Finally, since any meromorphic inner function is a product of Blaschke factors and such an exponential function by (2.2.2), it is argument is a sum of strictly increasing functions and therefore is also strictly increasing.

On the other hand, since \( \Theta(x) = \exp(i\psi(x)), \ x \in \mathbb{R} \), the function \( \psi(x) \) is unique up to an additive constant \( 2\pi k, \ k \in \mathbb{Z} \), thus its derivative is defined uniquely. Moreover, \( \Theta'(x) = i\psi'(x)\Theta(x) \), and

\[
\psi'(x) = \frac{1}{i} \frac{\Theta'(x)}{\Theta(x)} = -\tau + \frac{1}{i} \frac{B'(x)}{B(x)} = -\tau + 2 \sum_k \frac{\Im z_k}{|x - z_k|^2}, \ x \in \mathbb{R}.
\]  
(2.2.5)

**Definition.** Let \( f \) be an analytic function in the upper half-plane \( \mathbb{C}^+ \), then
(i) \( f \) is said to be of bounded type in the upper half-plane \( \mathbb{C}^+ \) if it can be written as a ratio of two analytic bounded functions in \( \mathbb{C}^+ \), i.e., \( f(z) = \frac{p(z)}{q(z)} \).

(ii) The mean type of \( f \) in \( \mathbb{C}^+ \) is defined by

\[
\text{mt}_+(f) := \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y}
\]

The mean type in the lower half-plane can be defined analogously.

The mean type is a generalization of exponential type of functions analytic in the upper half-plane, and not necessary entire. The mean type of a function which is identically zero is taken to be \(-\infty\). Another useful formula for the mean type of functions of bounded type in the upper half-plane, which describe the mean type as an average radial limit in the upper half-plane, is given by

\[
\text{mt}_+(f) = \lim_{r \to \infty} \frac{2}{\pi r} \int_{0}^{\pi} \log |f(re^{i\theta})| \sin \theta \, d\theta
\]

 Clearly, any bounded analytic function in the upper half-plane is of bounded type in \( \mathbb{C}^+ \). The sum and product of two functions which are of bounded type in the upper half-plane are functions of bounded type in the half-plane. In particular, a polynomial is a function of bounded type in the upper half-plane. Moreover, using formula (2.2.6), we can see that any nonzero polynomial \( p(z) = a_n z^n + \cdots + a_1 z + a_0 \) is of zero mean type. Indeed,

\[
\limsup_{y \to \infty} \frac{\log |p(iy)|}{y} = \limsup_{y \to \infty} \frac{\log |a_n (iy)^n + \cdots + a_1 (iy) + a_0|}{y} \\
\leq \limsup_{y \to \infty} \frac{\log(|a_n y^n + \cdots + a_1 y + a_0|)}{y} = 0
\]

The following theorem, which gives a sufficient condition for an analytic function to be of bounded type in the upper half-plane, will be useful in the proofs, see Theorem 11 of [14].

**Theorem 2.2.1.** Let \( f(z) \) be an analytic function in the upper half plane, such that \( |f(z)| \) has a continuous extension to the closed upper half plane. Then \( f(z) \) is of bounded type in the upper half-plane if

\[
\int_{-\infty}^{+\infty} \frac{\log^+ |f(t)|}{1 + t^2} \, dt < \infty,
\]

(2.2.7)
if
\[
\liminf_{r \to \infty} \frac{1}{r^2} \int_0^\pi \log^+ |f(re^{i\theta})| \sin \theta \, d\theta = 0, \tag{2.2.8}
\]
and if
\[
\limsup_{y \to \infty} \frac{\log |f(iy)|}{y} < \infty, \tag{2.2.9}
\]
where \(\log^+ |f(t)| = \max(0, \log |f(t)|)\).

Recall that a function \(f : \mathbb{C} \to \mathbb{C}\) analytic in the whole complex plane \(\mathbb{C}\) is called an entire function. Such function can be represented by a power series
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.2.10}
\]
that converges everywhere, i.e., \(\lim_{n \to \infty} \sqrt[n]{|a_n|} = 0\), where \(a_n = f^{(n)}(0)/n!\).

Since the only bounded entire functions are the constants, it is of interest to consider entire functions with different growth properties. For any entire function \(f(z)\) and any \(r > 0\), let \(M_f(r) := \max_{|z|=r} |f(z)|\). The order of an entire function \(f(z)\), denoted by \(\rho\), is defined by
\[
\rho = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}, \quad 0 \leq \rho \leq \infty
\]
The entire function \(f(z)\) of positive order \(\rho\) is said to be of type \(\sigma\) if
\[
\sigma = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho}, \quad 0 \leq \sigma \leq \infty.
\]

An entire function \(f\) is said to be of normal type if \(0 < \sigma < \infty\), of maximal type if \(\sigma = \infty\), and of minimal type if \(\sigma = 0\).

Following Ya. Levine [34], an entire function is said to be of exponential type \(\tau\) if it is of order less than 1 or it is of order 1 and type less than or equal to \(\tau\). An entire function of order 1 and minimal type or of order less than 1 is called an entire function of minimal exponential type. For example, \(f(z) = \sin az, a > 0\) is of order 1 and type \(a\). Indeed,
\[
M_f(r) = \max_{|z|=r} |\sin az| = \max_{|z|=r} |(e^{iaz} - e^{-iaz})/2i| = Ce^{a|\text{Im} z|} = C e^{ar},
\]
for some positive constant \(C\). Hence, the order can be computed as
\[
\rho = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r} = \limsup_{r \to \infty} \frac{\log(\log C + ar)}{\log r} = 1,
\]
and the type is given by
\[
\sigma = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho} = \limsup_{r \to \infty} \frac{\log C + ar}{r} = a,
\]
so \( f(z) = \sin az \) is of exponential type \( a \). Other examples including \( e^{\tau z}, \sin \tau z, \cos \tau z \) and \( \sin \tau z/z \), all are of exponential type \( \tau \). Polynomials are of exponential type zero. The following definition of the exponential type will be used in the proofs.

**Definition.** An entire function \( f(z) \) is said to be of exponential type if the inequality
\[
|f(z)| \leq Ae^{B|z|}
\]
holds for some positive constants \( A \) and \( B \) and all values of \( z \). The infimum of such \( B \) is called the type of \( f \). More specifically, an entire function \( f \) is said to be of exponential type \( \tau \) (\( 0 \leq \tau < \infty \)) if for every \( \epsilon > 0 \), there exists a constant \( A_\epsilon > 0 \) such that
\[
|f(z)| < A_\epsilon e^{(\tau + \epsilon)|z|}, \text{ for all } z \in \mathbb{C}.
\]

By [14] the exponential type of an entire function \( f \) can be computed using the formula
\[
\tau := \limsup_{|z| \to \infty} \frac{\log |f(z)|}{|z|} < \infty. \tag{2.2.11}
\]

Using properties of the logarithmic function one can show that the sum and the product of two entire functions of exponential type are again such functions. Less trivial result is the following, see [34] page 24.

**Lemma 2.2.2.** If \( f_1 \) and \( f_2 \) are entire functions of exponential type, then the quotient \( f_1/f_2 \) is also an entire function of exponential type provided it is entire.

In general, we write \( \tau_f \) for the exponential type of \( f \) when it is necessary to call attention to the particular function that is being considered. By a theorem of M. G. Krein [31], an entire function \( f(z) \) is of exponential type if it is of bounded type in the upper and lower half of the complex plane. In that case, the exponential type of the function is equal to the maximum of its mean types in the upper and lower half planes
\[
\tau_f = \max\{\text{mt}_+(f), \text{mt}_-(f)\}. 
\]
In the study of the growth of an analytic function $f$ of finite order, Phragmén and Lindelöf introduced the notation of the indicator function of $f$, see [33], page 51.

**Definition 2.2.1.** The *indicator function* of a function $f(z)$ of finite order $\rho$ in an angle $\theta_1 \leq \arg z \leq \theta_2$ on the direction in which $z$ tends to infinity, is defined as

$$h_f(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho}, \quad \theta_1 \leq \theta \leq \theta_2$$

For functions of normal type the indicator function is bounded above, and for entire functions of normal type it is bounded below. For entire functions of exponential type $\rho$ is taken to be one, see [4], chapter 5. Moreover, from Definition 2.2.11 of the exponential type $\tau$ of an entire function $f$ and that of its indicator function it readily follows that,

$$h_f(\theta) \leq \tau, \text{ for } \theta_1 \leq \theta \leq \theta_2. \quad (2.2.12)$$

In case of entire functions we take $\theta_1 = -\pi$ and $\theta_2 = \pi$.

The following theorem characterize the indicator function of a quotient of two entire functions of exponential type, see [33] lecture 27.

**Theorem 2.2.3.** Let $f_1(z)$ and $f_2(z)$ be two entire functions of exponential type, with $f_2(z)$ has no zeros in the upper half plane, and also

$$|f_1(x)| \leq |f_2(x)|, \text{ for all } x \in \mathbb{R}.$$  

Then $f(z) = \frac{f_1(z)}{f_2(z)}$ is analytic function of exponential type in the upper half plane. Further,

$$h_f(\theta) = h_{f_1}(\theta) - h_{f_2}(\theta), \text{ for all } \theta \in [0, \pi]$$

Given an analytic function $f(z)$ in $|z| < R$, $R > 0$, let $n(t)$ denote the number of zeros of $f$ in the closed disk $|z| \leq t$, counted according to multiplicity, and

$$N(r) = \int_0^r \frac{n(t)}{t} dt,$$

for all $r > 0$, provided that $n(0) = 0$ (i.e. $f(0) \neq 0$). Jensen’s theorem can be used to relate the distribution of zeros of an entire function to its growth [4].
Jensen’s Theorem. If $f(z)$ is analytic in $\{z : |z| \leq r\}$, $r < R$, and $f(0) \neq 0$, then for $r < R$ we have

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|d\theta - \log |f(0)|$$

(2.2.13)

If $f(0) = 0$, then we can consider the function $\frac{f(z)}{z^m}$, where $m$ is the order of the zero at 0. Then Jensen’s formula (2.2.13) gives

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|d\theta - m \log r - \log |f^{(m)}(0)|$$

(2.2.14)

2.3 Representation of Hermite-Biehler Class Functions

As is known, a polynomial $p(z)$ of degree $n$ has exactly $n$ zeros $\{z_k\}_{k=1}^n \subset \mathbb{C}$ and can be factored in the form

$$p(z) = c \prod_{k=1}^n (z - z_k)$$

(2.3.1)

for some constant $c \in \mathbb{C}$. But unlike polynomials, an entire function may not have zeros at all, unless its order is not an integer, and in such case it must have infinitely many zeros, see [4], page 24. If an entire function has zeros, it can be factored out in terms of it is zeros as in (2.3.1); this can be seen from the following theorem, which is know as the Hadamard factorization theorem for entire functions, see Theorem 13 in chapter 1 of [34], [33].

Hadamard Factorization Theorem. An entire function $f(z)$ of finite order $\rho$ may be represented in the form

$$f(z) = \gamma z^m e^{P_q(z)} \prod_{n=1}^\omega G\left(\frac{z}{z_n}, p\right), \quad \omega \leq \infty$$

(2.3.2)

where $z_1, z_2, \ldots$, are all nonzero roots of the function $f(z)$, $\gamma \in \mathbb{C}$, $p \leq \rho$, $P_q(z)$ is a polynomial in $z$ of degree $q \leq \rho$, $m$ is the multiplicity of the root at the origin, and the function $G$ is defined by

$$G(u, p) = \begin{cases} 
1 - u, & p = 0 \\
(1 - u) \exp \left[ u + \frac{u^2}{2} + \cdots + \frac{u^p}{p!} \right], & p > 0
\end{cases}$$

where $p$ is the smallest integer for which the series $\sum_{n=1}^\infty |z_n|^{-p-1}$ converges.
As an example, we have
\[ \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \]

Given an entire function \( f \) we define the function \( f^* \) as
\[ f^*(z) := \overline{f(z)}, \quad z \in \mathbb{C}. \]

On other words, \( f^*(z) \) is the entire function obtained from \( f(z) \) by replacing all the coefficients in its Taylor series (2.2.10) by their conjugates. An entire function \( f(z) \) is said to be real for real \( z \) if all the coefficients in its Taylor series are real, this implies that \( f^*(z) = f(z) \) for all \( z \in \mathbb{C} \), and consequently, \( f(z) = \overline{f(z)} \) for all \( z \in \mathbb{C} \).

**Lemma 2.3.1.** Let \( f(z) \) be an entire function which is real for real \( z \), then \( f(\overline{z}) = \overline{f(z)} \) for all \( z \in \mathbb{C} \).

**Proof.** Let \( f(z) \) be an entire function which is real for real \( z \). Then, \( f(z) = \sum_{n=1}^{\infty} a_n z^n, \ a_n \in \mathbb{R} \). Note that
\[ \overline{f(z)} = \sum_{n=1}^{\infty} \overline{a_n z^n} = \sum_{n=1}^{\infty} a_n \overline{z^n} = \sum_{n=1}^{\infty} a_n z^n = f(z), \]
i.e., \( f^*(z) = f(z) \) for all \( z \in \mathbb{C} \). Taking complex conjugate for both sides yields \( f(\overline{z}) = \overline{f(z)} \), \( z \in \mathbb{C} \). \( \square \)

An entire function \( E(z) \) is said to be a function of Hermite-Biehler class \( \mathcal{HB} \) if it satisfies
\[ |E(\overline{z})| < |E(z)|, \quad \text{whenever } \text{Im}(z) > 0. \]

We write \( E \in \mathcal{HB} \) if \( E(z) \) has no zeros in the upper half plane \( \text{Im}(z) > 0 \), and \( |E(\overline{z})| \leq |E(z)| \) whenever \( \text{Im}(z) > 0 \). Any polynomial with zeros in the lower half plane is of Hermite-Biehler class. More examples of functions in this space will be given in section 3.2, where they will be called de Branges functions, see for example chapter VII of [34] for an extensive and detailed discussion of such class.

It is known that functions of this class can be characterized in terms of the location of their zeros. More precisely, it is shown that entire functions in the Hermite-Biehler class \( \mathcal{HB} \) have a special form, see [3] and [34] for more details.
Theorem 2.3.2. In order that the entire function \( E(z) \) belong to the class \( \mathcal{HB} \), it is necessary and sufficient that it can be represented as

\[
E(z) = \gamma S(z) e^{-iaz} \prod_n \left( 1 - \frac{z}{z_n} \right) e^{(P_n(z) + P_n^*(z))/2},
\]

(2.3.3)

where \( \gamma \in \mathbb{C} \), with \( |\gamma| = 1 \), \( S(z) \) is an entire function that is real on the real line and has only real zeros, \( P_n(z) = \sum_{k=1}^{n} \frac{z^k}{kz_n^k} \), and \( z_n = x_n - iy_n \), \( y_n > 0 \), for all \( n \), are the zeros of the function \( E(z) \), which are lying in the open lower half plane and satisfying the Blaschke condition

\[
\sum_n |\operatorname{Im} \frac{1}{z_n}| < \infty,
\]

and the number \( a \geq 0 \) is determined by the mean type of \( E^*/E \):

\[
a = -\frac{1}{2} \operatorname{mt} \frac{E^*(z)}{E(z)}.
\]

The following theorem will play a significant role in characterizing entire functions in Hermite-Biehler class that are of exponential type, see [4], page 128-129.

Theorem 2.3.3. Let \( E(z) \) be an entire function of exponential type with zeros in the lower half-plane. If \( h(\theta) \geq h(-\theta) \) for some \( 0 < \theta < \pi \), then \( |E(\bar{z})| \leq |E(z)| \) whenever \( \operatorname{Im}(z) > 0 \).

The next theorem shows that the choice of \( \theta = \pi/2 \) will satisfy the condition of Theorem 2.3.3 for functions of exponential type in the class \( \mathcal{HB} \), see Theorem 7.7.2 of [4].

Theorem 2.3.4. If \( E(z) \) is an entire function of exponential type such that the series \( \sum_n |\operatorname{Im}(1/z_n)| < \infty \), then the indicator function of \( E \) satisfies \( h_E(\theta) - h_E(-\theta) = 2a \sin \theta \), for some \( a \) and all \( \theta \).

Combining these two results we have the following useful theorem, see Theorem 7.8.3 in [4], and Lecture 27 of [33].

Theorem 2.3.5. An entire function \( E(z) \) has the form

\[
E(z) = \gamma z^m e^{bz} e^{-iaz} \prod_n \left( 1 - \frac{z}{z_n} \right) e^{z \operatorname{Re}(\frac{1}{z_n})},
\]

(2.3.4)

with zeros \( \{z_n\}_{n \in \mathbb{Z}} \), and \( a = \frac{1}{2} \left[ h_E\left( \frac{\pi}{2} \right) - h_E\left( -\frac{\pi}{2} \right) \right] \geq 0 \), if and only if \( E(z) \) is of exponential type having no zeros in the upper half-plane and satisfying one of the conditions \( h_E(\theta) \geq h_E(-\theta) \) for some \( \theta \in (0, \pi) \), or \( |E(\bar{z})| \leq |E(z)| \) for \( z \in \mathbb{C}^+ \).
Note that the conditions \( h_E(\theta) - h_E(-\theta) \geq 0 \) for some \( \theta \in (0, \pi) \), and \( |E(\bar{z})| \leq |E(z)| \) for \( z \in \mathbb{C}^+ \) are equivalent by Theorem 2.3.3 and Theorem 2.3.2, by noting that \( a = -\frac{1}{2} \text{mt} \frac{E^*(z)}{E(z)} = \frac{1}{2} \left[ h_E(\frac{\pi}{2}) - h_E(-\frac{\pi}{2}) \right] \geq 0 \).

In what follows we state two theorems which will be used in characterizing analytic functions of bounded type in the upper half-plane, see [43].

**Theorem 2.3.6.** Let \( f(z) \) be a nonzero analytic function of bounded type in the upper half plane \( \mathbb{C}^+ \). Then there exists functions \( B(z) \) Blaschke product, \( G(z) \) outer function, and a real number \( \tau \) such that

\[
f(z) = e^{-i\tau z}B(z)G(z) \frac{S_+(z)}{S_-(z)}
\]

where the functions \( S_{\pm} \) have the form

\[
S_{\pm} = \exp \left( -\frac{1}{\pi i} \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_{\pm}(t) \right),
\]

where \( \mu_{\pm} \) are singular and mutually singular non-negative Borel measures on the real line satisfying

\[
\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_{\pm}(t) < \infty.
\]

The functions \( S_{\pm} \) are inner and \( e^{-i\tau z} \) is inner for \( \tau \) nonpositive. If \( f \) is an inner function the factors \( G(z) \) and \( S_{\pm}(z) \) are constants of modulus one and \( \tau \) is nonpositive.

**Cauchy’s Formula.** Let \( f(z) \) be a function which is analytic and of bounded type and nonpositive mean type in the upper half-plane, and which has a continuous extension to the closed half-plane. If \( \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \), then Cauchy’s formula holds:

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)} dt = \begin{cases} f(z) : \text{Im}(z) > 0 \\ 0 : \text{Im}(z) < 0 \end{cases}
\]

### 2.4 Reproducing Kernel Hilbert Spaces

The notion of a reproducing kernel Hilbert space of functions was first introduced by Aronszajn [1] in 1950, and since then it has become an important technique in mathematical analysis.
These spaces possess additional structure that other Hilbert spaces do not have. This gives, as we will see later, a supplementary point of view that allows one to state and solve new problems.

Let $\mathcal{H}$ be a nonzero Hilbert space of analytic functions on a plane domain $\Omega$ such that, for each point $w \in \Omega$, the linear functional of evaluation at $w$:

$$f \mapsto f(w), \quad f \in \mathcal{H}$$

is bounded on $\mathcal{H}$, i.e., for any $w \in \Omega$ there exists a positive constant $C_w$ such that

$$|f(w)| \leq C_w \|f\|, \quad f \in \mathcal{H}.$$

By continuity of point evaluations and by the Riesz representation theorem, there exists for each $w \in \Omega$ a unique function $K_w \in \mathcal{H}$ such that

$$f(w) = \langle f, K_w \rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}$. The function $K_w$ is called the reproducing kernel at the point $w$, and its norm is the same as that of the corresponding evaluation functional.

In particular, given a Hilbert space of entire functions $\mathcal{H}$, then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS) if there exists a function $K(w, z)$, $w, z \in \mathbb{C}$, with the following properties:

(i) For every $w \in \mathbb{C}$, the function $z \mapsto K(w, z)$ belongs to $\mathcal{H}$, and

(ii) For every $f \in \mathcal{H}$:

$$f(w) = \langle f(t), K(w, t) \rangle_{\mathcal{H}}, \quad \text{for every } w \in \mathbb{C}. \quad (2.4.1)$$

The function $K(w, z)$ is uniquely defined and is called the reproducing kernel of $\mathcal{H}$. Applying property (ii) to the function $K(w, z)$ we get

$$K(w, z) = \langle K(w, t), K(z, t) \rangle_{\mathcal{H}}, \quad \text{for all } w, z \in \mathbb{C},$$

moreover,

$$\|K(w, \cdot)\|^2_{\mathcal{H}} = K(w, w) = \langle K(w, t), K(w, t) \rangle_{\mathcal{H}}, \quad \text{for all } w \in \mathbb{C}. \quad (2.4.2)$$
**Theorem.** Let $H$ be a reproducing kernel Hilbert space, then the reproducing kernel is uniquely determined by the space $H$.

**Proof.** Let $K_1(w,z)$, $K_2(w,z)$ be two reproducing kernels of $H$. Then applying (2.4.1) for $K_1$ and $K_2$, and using the fact that $\|f\|^2 = \langle f, f \rangle$ for all $f \in H$, we get, for all $w \in \mathbb{C}$

\[
\|K_1(,.) - K_2(,.)\|^2 = \langle K_1(w,t) - K_2(w,t), K_1(w,t) - K_2(w,t) \rangle \\
= \langle K_1(w,t) - K_2(w,t), K_1(w,t) \rangle - \langle K_1(w,t) - K_2(w,t), K_2(w,t) \rangle \\
= \langle K_1(w,w) - K_2(w,w) \rangle - \langle K_1(w,w) - K_2(w,w) \rangle \\
= 0
\]

Therefore, $K_1(w,z) = K_2(w,z)$ for all $w,z \in \mathbb{C}$. \qed

Some well-known examples of reproducing kernel Hilbert spaces are given below, see [35].

**Example.**

1. The Paley-Wiener space $PW_\pi$, which consists of entire functions of exponential type at most $\pi$, and are square integrable on the real line, is a RKHS with reproducing kernel $K(w,z) = \frac{\sin \pi (z - \bar{w})}{\pi (z - \bar{w})}$, $w,z \in \mathbb{C}$. More details about Paley-Wiener spaces will be discussed in section 3.1.

2. The Hardy space $H^2(\mathbb{C}^+)$ on the upper half plane $\mathbb{C}^+$ is a RKHS with reproducing kernel

\[
K(w,z) = \frac{1}{2\pi} \frac{i}{z - w}, \quad w,z \in \mathbb{C}^+.
\]

3. The Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which is defined as the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

\[
\|f\|_{H^2(\mathbb{D})}^2 = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty,
\]

is a RKHS with reproducing kernel (called Cauchy kernel)

\[
K(w,z) = \frac{1}{1 - wz}, \quad w,z \in \mathbb{D}.
\]  

(2.4.3)
(4) The Sobolev space $H^s(\mathbb{R})$ of functions $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |\hat{f}(t)|(1 + |t|^2)^s dt < \infty.$$ 

is a RKHS when $s > 1/2$, it is reproducing kernel is given by

$$K(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-it(x-y)}}{(1 + |t|^2)^s} dt, \quad x, y \in \mathbb{R}.$$ 

2.5 Sampling Theory

In many applications, such as mathematics, engineering, and data processing, a function $f$ needs to be reconstructed using its values at a discrete set of points $\{\lambda_n\}_{n \in I}$, for a countable set $I$, namely, from the set $\{f(\lambda_n)\}_{n \in I}$ which are called samples. In this section we give some definitions, basic facts, and an introduction to the sampling theory that will be needed in the study of the sampling theory in de Branges spaces, see [47].

Definition. Let $I$ be a countable index set and $\Lambda = \{\lambda_n\}_{n \in I}$ be a sequence of real numbers. Then:

(1) $\Lambda$ is said to be separated (or $\delta$-uniformly separated) if there exists $\delta > 0$, such that

$$\inf_{n \neq m} |\lambda_n - \lambda_m| \geq \delta > 0.$$ 

The constant $\delta$ is called the separation constant of $\Lambda$.

(2) $\Lambda$ is said to be relatively separated if it is a finite union of uniformly separated sequences.

In this case, $I$ can be partitioned into disjoint sets $I_1, I_2, \ldots, I_N$, such that each sequence $\Lambda_k = \{\lambda_n\}_{n \in I_k}$ is $\delta_k$-uniformly separated, for some $\delta_k > 0$, $k = 1, 2, \ldots, N$.

(3) A separated sequence $\{\lambda_n\}_{n \in I}$ has uniform density $d > 0$ if there is a constant $L$ such that

$$\left|\lambda_n - \frac{n}{d}\right| \leq L, \quad \forall n \in I$$

Throughout our work we will assume that a set $\Lambda$ is separated, unless otherwise stated. Any uniformly separated set can be linearly ordered such that $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$, this fact will be used later in defining the density of such sequences.
In sampling theory, we consider spaces of functions whose value at any point is well-defined, so our framework will have a natural connection to reproducing kernel Hilbert spaces. In particular, the reproducing kernel property (2.4.1) will play a significant role in the theory.

Definition 2.5.1. Let \( \mathcal{H} \) be a reproducing kernel Hilbert space of entire functions with reproducing kernel \( K(w, z) \), then:

1. A sequence \( \{\lambda_n\}_{n \in \mathbb{Z}} \) is said to be a Plancherel-Pólya sequence in \( \mathcal{H} \) if there exist a positive constant \( B \), independent of \( f \), such that
   \[
   \sum_{n \in \mathbb{Z}} \frac{|f(\lambda_n)|^2}{\|K(\lambda_n, \cdot)\|_{\mathcal{H}}^2} \leq B \|f\|_{\mathcal{H}}^2
   \]
   for all \( f \in \mathcal{H} \).

2. A sequence \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) is said to be a sampling sequence for \( \mathcal{H} \) if there exist positive constants \( A \) and \( B \) such that
   \[
   A \|f\|_{\mathcal{H}}^2 \leq \sum_{n \in \mathbb{Z}} \frac{|f(\lambda_n)|^2}{\|K(\lambda_n, \cdot)\|_{\mathcal{H}}^2} \leq B \|f\|_{\mathcal{H}}^2
   \]
   for all \( f \in \mathcal{H} \).

3. A sequence \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \) is said to be an interpolating sequence for \( \mathcal{H} \) if for every sequence of scalars \( \{c_n\} \)
   there exist \( f \in \mathcal{H} \) such that
   \[
   f(\gamma_n) = c_n, \quad \text{for all } n \in \mathbb{Z}, \quad \text{whenever } \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{\|K(\lambda_n, \cdot)\|_{\mathcal{H}}^2} < \infty
   \]

4. A sequence \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \) is said to be a complete interpolating sequence for \( \mathcal{H} \) if for every sequence of scalars \( \{c_n\} \) there exist a unique \( f \in \mathcal{H} \) such that
   \[
   f(\gamma_n) = c_n, \quad \text{whenever } \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{\|K(\lambda_n, \cdot)\|_{\mathcal{H}}^2} < \infty
   \]
   for all \( n \in \mathbb{Z} \).
By the reproducing kernel property (2.4.1) these definitions can be seen from the frame theory viewpoint, and the standard problems of sampling and interpolation in reproducing kernel Hilbert spaces can be rephrased as follows: a sequence \( \Lambda \) is a Plancherel-Pólya sequence in \( \mathcal{H} \) if and only if the corresponding sequence of normalized reproducing kernels is a Bessel sequence. A sequence \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) is a sampling sequence in \( \mathcal{H} \) if and only if the corresponding sequence of normalized reproducing kernels \( \{\frac{K(\lambda_n, .)}{\|K(\lambda_n, .)\|}\}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{H} \), therefore, any function \( f \in \mathcal{H} \) can be reconstructed from its samples on the sequence \( \Lambda \) by the (sampling) formula

\[
f(z) = \sum_{n \in \mathbb{Z}} f(\lambda_n) \tilde{k}_n(z)
\]

where \( \{\tilde{k}_n\}_{n \in \mathbb{Z}} \) is a dual frame of \( \{\frac{K(\lambda_n, .)}{\|K(\lambda_n, .)\|}\}_{n \in \mathbb{Z}} \).

Let \( l^2(K) \) denote the space of sequences of scalars \( \{c_n\}_{n \in \mathbb{Z}} \) that are square summable with respect to the weights \( 1/K(\gamma_n, \gamma_n) \), where \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \), i.e.,

\[
\sum_{n \in \mathbb{Z}} \left|\frac{c_n}{K(\gamma_n, \gamma_n)}\right|^2 < \infty.
\]

If a sequence \( \Gamma \) is an interpolating sequence in a reproducing kernel Hilbert space \( \mathcal{H} \), then by Definition 2.5.1, for any sequence \( \{c_n\} \in l^2(K) \) there exists \( f \in \mathcal{H} \) such that \( f(\gamma_n) = c_n \), for all \( n \in \mathbb{Z} \). This in fact suggests that we can consider the (restriction) map from \( \mathcal{H} \) to \( l^2(K) \) defined by \( T(f) = \{f(\gamma_n)\}_{n \in \mathbb{Z}} \). This implies that the interpolation sequences are related to the condition that the map \( T : \mathcal{H} \to l^2(K) \) is onto, (or equivalently, the map \( f \mapsto \{\langle f, f_n \rangle\} \) maps \( \mathcal{H} \) onto \( l^2 \), by the reproducing kernel property (2.4.1)).

For positive quantities \( \nu \) and \( \omega \), the notation \( \nu \asymp \omega \) will mean that there exist constants \( 0 < C_1, C_2 < \infty \) so that \( C_1 \leq \nu/\omega \leq C_2 \). Recall that a sequence \( \{f_n\} \) in a Hilbert space \( \mathcal{H} \) is called independent if \( f_n \notin \text{span}\{f_k : k \neq n\} \), for all \( n \). The following theorem is an application of Köthe-Toeplitz Theorem in [39] in reproducing kernel Hilbert spaces.

**Theorem 2.5.1.** Let \( \mathcal{H} \) be a RKHS, and \( \{f_n\} \) be the sequence defined in (2.5.1). Let \( K \) be the smallest closed subspace of \( \mathcal{H} \) containing \( \{f_n\} \). Given that \( K(w, z) \) is the reproducing kernel of a Hilbert space \( \mathcal{H} \), define

\[
f_n(z) := \frac{K(\gamma_n, z)}{\|K(\gamma_n, \gamma_n)\|}, \quad \text{for all } n \in \mathbb{Z}
\] (2.5.1)
where $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$. Then the following statements are equivalent:

1. The sequence $\Gamma$ is an interpolating sequence for $\mathcal{H}$.

2. For all $f \in K$, $\|f\|^2 \asymp \sum_n |\langle f, f_n \rangle|^2$, and $\{f_n\}$ is independent.

3. $\|\sum_n a_n f_n\|^2 \asymp \sum_n |a_n|^2$, for all sequences $\{a_n\}$.

Combining property (3) of Theorem 2.5.1 above and the definition of a Riesz sequence, together with inequality (2.1.5), we see that a sequence $\Gamma$ is an interpolating sequence in a RKHS $\mathcal{H}$ if and only if the corresponding normalized reproducing kernels sequence $\{f_n\}$ defined in (2.5.1) is a Riesz sequence in $\mathcal{H}$. Keeping this conclusion in mind, we can see that any Riesz sequence in a RKHS is a Riesz-Fischer sequence. For if $\{a_n\}_{n \in \mathbb{Z}}$ is any sequence in $l^2$, then letting $c_n := a_n \sqrt{K(\gamma_n, \gamma_n)}$, for all $n \in \mathbb{Z}$, we get

$$
\sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} = \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty.
$$

Hence, if $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$ is an interpolating sequence in $\mathcal{H}$, then there exist $f \in \mathcal{H}$ such that $f(\gamma_n) = c_n$, for all $n \in \mathbb{Z}$. By the reproducing kernel property (2.4.1) we have

$$
\langle f(t), K(\gamma_n, t) \rangle = f(\gamma_n) = c_n,
$$

hence,

$$
\langle f, f_n \rangle = \langle f(t), \frac{K(\gamma_n, t)}{\sqrt{K(\gamma_n, \gamma_n)}} \rangle = \frac{c_n}{\sqrt{K(\gamma_n, \gamma_n)}} = a_n.
$$

That is, the sequence $\{f_n\}$ is a Riesz-Fischer sequence in $\mathcal{H}$. It turns out that we can actually do the interpolation with some norm control, i.e., the solution to the interpolation problem $f(\gamma_n) = c_n$ can be chosen so that

$$
\|f\|^2 \leq \frac{1}{c} \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} \tag{2.5.2}
$$

for some constant $c > 0$ (independent of $f$ and $c_n$), this follows directly from Theorem 2.1.2.

On the other hand, since a Riesz basis is just a Riesz sequence which is a frame, then it follows that a sequence $\Lambda$ is a complete interpolating sequence in $\mathcal{H}$ if and only if the corresponding sequence of normalized reproducing kernels $\{f_n\}$ defined in (2.5.1) is Riesz basis in $\mathcal{H}$. Hence, a complete interpolating sequence is an interpolating sequence which is sampling.
Saying that a complete interpolating sequence is both a sampling and interpolating sequence is a way of expressing that it exists as a compromise between two competing density conditions; as we will see later, a sampling sequence should be uniformly “dense”, while an interpolating sequence should be uniformly “sparse”.

From a signal processing point of view, sampling sequences allow us to reconstruct a signal from its values on certain discrete set of points \( \Lambda \). On the other hand, interpolating sequences can be seen as a dual definition of sampling sequences; given a discrete (separated) sequence \( \Lambda \), we want to build a function (signal) carrying the desired information. While a complete interpolating sequence is a minimal sampling sequence, that is, removing any point from the sequence makes it non-sampling, and adding a point makes it non-interpolating.

For the description to be given of sets of sampling and interpolation, one can try to formulate properties of the set \( \left\{ K(\lambda, \cdot)/\|K(\lambda, \cdot)\| \right\}_{\lambda \in \Lambda} \) in terms of the average number of points in the sequence \( \Lambda \) per unit length, which is known as the density of \( \Lambda \). In general, there is not a canonical way to assign a density to an ordered sequence. In fact, different densities are useful for characterizing different properties of sequences. In this work we will use the Beurling’s density concept as generalized by Landau [32]. We consider then separated sets.

The notion of Beurling density is one of the main ingredients in sampling theory, it is a measure of the average number of points of a sequence (set) that lie inside a unit interval. For \( R > 0 \), denote by \( n^+(R) \), \( n^-(R) \) the maximum and minimum of the number of points of a sequence \( \Lambda \) occurring in the interval \([x - R, x + R]\)

\[
n^+(R) = \sup_{x \in \mathbb{R}} \sharp(\Lambda \cap [x - R, x + R]), \tag{2.5.3}
\]

and

\[
n^-(R) = \inf_{x \in \mathbb{R}} \sharp(\Lambda \cap [x - R, x + R]), \tag{2.5.4}
\]

where \( \sharp A \) denotes the cardinality of the set \( A \).

**Definition 2.5.2.** Let \( \Lambda \subset \mathbb{R} \) be a sequence. The upper Beurling density of \( \Lambda \) is defined by

\[
D^+(\Lambda) := \limsup_{R \to \infty} \frac{n^+(R)}{2R} \tag{2.5.5}
\]
and the lower Beurling density of $\Lambda$ is defined by

$$D^{-}(\Lambda) := \liminf_{R \to \infty} \frac{n^{-}(R)}{2R} \quad (2.5.6)$$

If $D^{+}(\Lambda) = D^{-}(\Lambda) = D(\Lambda)$, then the sequence $\Lambda$ is said to have uniform Beurling density $D(\Lambda)$.

If $\Lambda$ is a disjoint union of the sequences $\Lambda_{1}, \Lambda_{2}, \ldots, \Lambda_{N}$, $\Lambda = \bigcup_{k=1}^{N} \Lambda_{k}$, then we always have

$$\sharp(\Lambda \cap [x - R, x + R]) = \sum_{k=1}^{N} \sharp(\Lambda_{k} \cap [x - R, x + R])$$

therefore,

$$D^{+}(\Lambda) = \lim_{R \to \infty} \sup_{x \in \mathbb{R}} \frac{\sharp(\Lambda \cap [x - R, x + R])}{2R}$$

$$= \lim_{R \to \infty} \sup_{x \in \mathbb{R}} (\sum_{k=1}^{N} \frac{\sharp(\Lambda_{k} \cap [x - R, x + R])}{2R})$$

$$\leq \lim_{R \to \infty} \sum_{k=1}^{N} (\sup_{x \in \mathbb{R}} \frac{\sharp(\Lambda_{k} \cap [x - R, x + R])}{2R})$$

$$= \sum_{k=1}^{N} \lim_{R \to \infty} \frac{\sup_{x \in \mathbb{R}} \sharp(\Lambda_{k} \cap [x - R, x + R])}{2R}$$

$$= \sum_{k=1}^{N} \lim_{R \to \infty} \frac{n^{+}_{k}(R)}{2R}$$

$$= \sum_{k=1}^{N} D^{+}(\Lambda_{k})$$

where $n^{+}_{k}(R) := \sup_{x \in \mathbb{R}} \sharp(\Lambda_{k} \cap [x - R, x + R])$, for $1 \leq k \leq N$. A similar computations for the lower Beurling density shows that

$$\sum_{k=1}^{N} D^{-}(\Lambda_{k}) \leq D^{-}(\Lambda) \leq D^{+}(\Lambda) \leq \sum_{k=1}^{N} D^{+}(\Lambda_{k}).$$

The above inequalities may be strict, for example, let $\Lambda = \mathbb{Z}$, $\Lambda_{1} = \mathbb{Z}^{+} = \{ n \in \mathbb{Z}; n \geq 0 \}$, and $\Lambda_{2} = \mathbb{Z}^{-} = \{ n \in \mathbb{Z}; n < 0 \}$. Then $\Lambda = \Lambda_{1} \cup \Lambda_{2}$, and we have

$$D^{+}(\Lambda) = 1 = D^{-}(\Lambda)$$

and

$$D^{+}(\Lambda_{1}) = D^{+}(\Lambda_{2}) = 1, D^{-}(\Lambda_{1}) = D^{-}(\Lambda_{2}) = 0$$
hence, we have

\[ 0 = D^-(\Lambda_1) + D^-(\Lambda_2) < D^-(\Lambda) = D^+(\Lambda) < D^+(\Lambda_1) + D^+(\Lambda_2) = 2 \]

The next result shows that Beurling densities are robust against perturbations [26]. For example, given \( \epsilon > 0 \), let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a sequence such that \( |\lambda_n - an| < \epsilon \) for all \( n \in \mathbb{Z} \), i.e., \( \Lambda \) is perturbation of \( a\mathbb{Z} \), for any nonzero \( a \in \mathbb{R} \). Then the density is unchanged, that is, \( \Lambda \) has uniform Beurling density \( D(\Lambda) = D(a\mathbb{Z}) = 1/a \).

**Lemma 2.5.2.** Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers. Let \( \eta > 0 \), then for each \( \Delta := \{ \lambda_n + \epsilon_n; \epsilon_n \in [-\eta, \eta], n \in \mathbb{Z} \} \), we have

\[ D^+(\Delta) = D^+(\Lambda) \quad \text{and} \quad D^-(\Delta) = D^-(\Lambda) \]

The following well known lemma, see Lemma 7.1.3 of [5]), shows that the assumption that the density of a sequence \( \Lambda \) is finite already places a strong restriction on the separation properties of the sequence \( \Lambda \).

**Lemma 2.5.3.** Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{I}} \) be a sequence of real numbers. Then the following are equivalent:

(a) \( D^+(\Lambda) < \infty \).

(b) \( \Lambda \) is relatively separated.

(c) For every \( R > 0 \), there exists an integer \( N_R > 0 \) such that

\[ \sup_{n \in \mathbb{Z}} \#(\Lambda \cap [(n-1)R, (n+1)R]) = N_R < \infty \]

(d) For some \( R > 0 \), there exists an integer \( N_R > 0 \) such that

\[ \sup_{n \in \mathbb{Z}} \#(\Lambda \cap [(n-1)R, (n+1)R]) = N_R < \infty \]

We end this section by an example shows that a sequence with finite upper Beurling density doesn’t need to be uniformly separated.

**Example 2.1.** Let \( \Lambda_1 = \mathbb{Z} \), \( \Lambda_2 = \{ n + \frac{1}{n} : n > 1 \} \), and \( \Lambda = \Lambda_1 \cup \Lambda_2 \). Then both of \( \Lambda_1 \) and \( \Lambda_2 \) are uniformly separated, hence \( \Lambda \) is relatively separated, i.e., \( D^+(\Lambda) < \infty \). However, \( \Lambda \) is not uniformly separated.
CHAPTER 3. Hilbert Spaces of Entire Functions

In the beginning of the 60s, de Branges developed a beautiful and formidable theory of Hilbert spaces of entire functions, called de Branges spaces, where the Paley-Wiener spaces are the primary example of such spaces. In this chapter we describe the theory of such Hilbert spaces of entire functions in which the rest of this work is set, and record their most basic properties. Most of this chapter is based on [14] and more details about such spaces can be found there.

We begin by providing an overview of the Paley-Wiener space, and summarize some of the previous work in the sampling theory for the space in order to illustrate how our results in chapter 4 fits in the complete picture.

3.1 Paley-Wiener Spaces

The Paley-Wiener space, denoted by $PW_a$, $a > 0$, consists of entire functions whose Fourier transforms have supports in $[-a, a]$, and whose restrictions to the real axis are square integrable. The Fourier transform of $f$ is the function $\phi \in L^2(-a, a)$ defined by

$$\phi(\xi) := (\mathcal{F}f)(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\xi t} \, dt,$$

where the integral is to be interpreted as a limit in the mean in the $L^2$ sense. With this definition, the Fourier transform is not an isometry, if $\phi \in L^2(-a, a)$ is the Fourier transform of $f \in PW_a$ then

$$f(z) = \int_{-a}^{a} \phi(t)e^{izt} \, dt, \quad \text{and} \quad \|f\|^2_{PW_a} = 2\pi \int_{-a}^{a} |\phi(t)|^2 \, dt = 2\pi \|\phi\|^2_{L^2}.$$

The space $PW_a$ is a vector space under pointwise addition and scalar multiplication, with
inner product defined by
\[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt \]
and norm
\[ \|f\|_{PW_a}^2 = \langle f, f \rangle = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty \]
for all \( f, g \in PW_a \).

We begin with the following fundamental result by Paley and Wiener on bandlimited functions, which gives a nice characterization of the space \( PW_a \). It shows that every function bandlimited by \( a > 0 \) is of exponential type at most \( a \), and every entire function of exponential type at most \( a \) that is square integrable on the real axis belongs to \( PW_a \), see e.g., [41] page 103.

**Paley-Wiener Theorem.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function and \( f|_{\mathbb{R}} \in L^2(\mathbb{R}) \). Let \( a > 0 \), then

(i) there exists \( C > 0 \) such that \( |f(z)| \leq Ce^{a|z|} \) if and only if \( \text{supp}(\hat{f}|_{\mathbb{R}}) \subseteq [-a,a] \),

and

(ii) if one of the two conditions in (i) is fulfilled, then

\[ f(z) = \int_{-a}^{a} (\hat{f}|_{\mathbb{R}})(\xi) e^{iz \xi} \, d\xi. \]  

(3.1.1)

It follows that the space \( PW_a \) can be defined as

\[ PW_a = \{ f|_{\mathbb{R}} \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq [-a,a] \}, \]

where ‘\( \text{supp} \)’ denotes the support of a function, and \( \hat{f} \) stands for the Fourier transform of \( f \).

Given \( f \in PW_a \) then by the Paley-Wiener’s theorem

\[ f(w) = \int_{-a}^{a} \phi(\xi) e^{iw \xi} \, d\xi = \int_{-a}^{a} \phi(-\xi) e^{-iw \xi} \, d\xi = \int_{-a}^{a} \phi(-\xi) e^{i\bar{w} \xi} \, d\xi = \langle \phi(-\xi), e^{i\bar{w} \xi} \rangle_{L^2(-a,a)} \]

for some \( \phi \in L^2(-a,a) \), and all \( w \in \mathbb{C} \). Since
\[
\langle \phi(-\xi), e^{i\bar{w}\xi} \rangle_{L^2(-a,a)} = \langle \mathcal{F}(\phi(-\xi)), \frac{1}{2\pi} \mathcal{F}(e^{i\bar{w}\xi}) \rangle_{PW_a},
\]

\[
\mathcal{F}(\phi(-\xi)) = 2\pi f(t),
\]
and
\[
\mathcal{F}(e^{i\bar{w}\xi}) = \frac{1}{2\pi} \int_{-a}^{a} e^{i\bar{w}\xi} e^{-it\xi} d\xi = \frac{1}{2\pi} \int_{-a}^{a} e^{i\xi(\bar{w}-t)} d\xi = \frac{\sin a(t-\bar{w})}{\pi(t-\bar{w})},
\]
then,
\[
f(w) = \langle \phi(-\xi), e^{i\bar{w}\xi} \rangle_{L^2(-a,a)} = \langle \mathcal{F}(\phi(-\xi)), \frac{1}{2\pi} \mathcal{F}(e^{i\bar{w}\xi}) \rangle_{PW_a} = \langle f(t), \frac{\sin a(t-\bar{w})}{\pi(t-\bar{w})} \rangle_{PW_a}.
\]
Therefore, for all \( f \in PW_a \) we have
\[
f(w) = \langle f(t), K(w,t) \rangle_{PW_a}, \text{ for all } w \in \mathbb{C},
\]
Hence, \( PW_a \) is a reproducing kernel Hilbert space with reproducing kernel \( K(w,z) = \frac{\sin a(z-\bar{w})}{\pi(z-\bar{w})} \).

An interesting property of \( PW_a \) is that if \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) is an uniformly separated sequence, then for any \( f \in PW_a \), the sequence of samples of \( f \) on the points of \( \Lambda \), \( \{f(\lambda_n)\}_{n \in \mathbb{Z}} \), is a square summable sequence. This is in fact equivalent to Definition 2.5.1 of Plancherel-Pólya sequence in the space \( H = PW_a \), where the corresponding reproducing kernel has norm \( \|K(x,.)\|^2 = K(x,x) = a/\pi \) for all \( x \in \mathbb{R} \), by (2.4.2). This classical result is known by Plancherel-Pólya theorem, see [47], page 97. More precisely, it states the following:

**Plancherel-Pólya Theorem.** If \( \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) is a \( \delta \)-uniformly separated sequence, then there exist a positive constant \( B = B(\delta,a) \), independent of \( f \), such that
\[
\sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2 \leq B\|f\|_2^2
\]
for all \( f \in PW_a \).

Given a sequence \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) of real numbers, following Duffin and Schaeffer [19], we say that a system of complex exponentials \( E = \{e^{i\lambda_n t}\}_{n \in \mathbb{Z}} \) is a Fourier frame for \( L^2(-a,a) \) if there exist positive constants \( A \) and \( B \) such that
\[
A\|\phi\|_{L^2}^2 \leq \sum_{n \in \mathbb{Z}} \left| \int_{-a}^{a} \phi(t)e^{-i\lambda_n t}dt \right|^2 \leq B\|\phi\|_{L^2}^2 \quad (3.1.3)
\]
for all $\phi \in L^2(-a,a)$.

On the other hand, by Definition 2.5.1 a sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ is sampling for $PW_a$ if and only if there exists positive constants $A'$ and $B'$ such that

$$A' \|f\|^2_{PW_a} \leq \sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2 \leq B' \|f\|^2_{PW_a}$$ (3.1.4)

for all $f \in PW_a$. We will show that inequalities (3.1.3) and (3.1.4) are equivalent. Given any $f \in PW_a$, then $f^* \in PW_a$, and the Paley-Wiener theorem guarantee the existence of the function $\phi := \hat{f^*}|_{\mathbb{R}} \in L^2(-a,a)$. Hence, apply inequality (3.1.3) to the function $\phi$, and using the fact that $\hat{f^*}(t) = \hat{f}(t)$, we get

$$\sum_{n \in \mathbb{Z}} \left| \int_{-a}^{a} \phi(t) e^{-i\lambda_n t} dt \right|^2 = \sum_{n \in \mathbb{Z}} \left| \int_{-a}^{a} \hat{f^*}(t) e^{-i\lambda_n t} dt \right|^2 = \sum_{n \in \mathbb{Z}} \left| \int_{-a}^{a} \hat{f(t)} e^{i\lambda_n t} dt \right|^2 = \sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2,$$

by (3.1.1)

Hence, the two inequalities (3.1.3) and (3.1.4) are equivalent. This implies that a sequence $\Lambda$ is sampling for $PW_a$ if and only if the system of complex exponentials $\mathcal{E} = \{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ is a Fourier frame for $L^2(-a,a)$.

In 1952, Duffin & Schaeffer [19] gave a sufficient density condition for the system of exponentials $\{e^{i\lambda_n x}\}$ to constitute a frame for $L^2(-a,a)$. They proved the following:

**Theorem 3.1.1.** If $\{\lambda_n\}$ is a sequence of uniform density $d$, then the set of functions $\{e^{i\lambda_n x}\}$ is a frame $L^2(-a,a)$ provided that $0 < a < \pi d$.

The following theorem is fundamental in engineering and digital signal processing because it gives a framework for converting analog signals into sequences of numbers. This theorem was apparently discovered by Shannon and described in a manuscript by 1940, but it was not published until 1949, see Lecture 20 of [33].
Shannon Sampling Theorem. Any $f \in PW_\pi$ can be represented as

$$f(z) = \sum_{n \in \mathbb{Z}} c_n \frac{\sin \pi(z-n)}{\pi(z-n)}$$  \hspace{1cm} (3.1.5)$$

where $c_n = f(n)$, $n \in \mathbb{Z}$, and the convergence is both in $PW_\pi$ and uniform on $\mathbb{R}$. Also we have

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$  

Conversely, given $\{c_n\}_{n \in \mathbb{Z}} \in l^2$, equation (3.1.5) defines the function $f \in PW_\pi$, which solves the interpolation problem $f(n) = c_n$ for all $n \in \mathbb{Z}$.

For the general $a$-bandlimited functions formula (3.1.5) is obtained by a change of variable. For $f \in PW_a$, we have

$$f(z) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{\sin a(z-t_n)}{a(z-t_n)},$$

where $t_n = n\pi/a$.

This theorem makes it possible to recover a bandlimited function with bandlimit at most $a = \pi$ from its values on the integers. It gives the best known simple example of a sampling sequence. In fact, this theorem implies that the set $\Lambda = \mathbb{Z}$ of integers is a complete interpolating sequence for $PW_\pi$. Moreover, the result remains valid for a set $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ whenever

$$\sup_{n \in \mathbb{Z}} |\lambda_n - n| = \delta,$$

(3.1.6) for $\delta > 0$ small enough.

The problem of describing all complete interpolating sequences has been studied since the classical book of Paley and Wiener in 1934, see chapter 7 of [41], where they proved that (3.1.6) holds for $\delta < 1/\pi^2$. In 1964, M. Kadec proved that $\delta < 1/4$ is sufficient. Moreover, this result is sharp in the sense that the constant $1/4$ cannot be replaced by any larger number; a counterexample due to A. Ingham for $\delta = 1/4$ is given in [47], chapter 3.

In 1991, Jaffard [25] provided necessary conditions for sampling sequences in terms of union of uniformly separated sequences and the lower Beurling density, as follows.

**Theorem 3.1.2.** If $\Lambda$ is a set of sampling for $PW_a$ then $\Lambda$ is relatively separated and $D^-(\Lambda) \geq a/\pi$. If $\Lambda$ is relatively separated and $D^-(\Lambda) > a/\pi$, then $\Lambda$ is a sampling sequence in $PW_a$. 

Although Theorem 3.1.2 characterizes sets of sampling for $PW_a$ almost completely in terms of a suitable notion of density, the lower Beurling density $D^{-}(\Lambda)$, this theorem makes no conclusion about whether or not $\Lambda$ is a set of sampling if $\Lambda$ has the critical density $D^{-}(\Lambda) = a/\pi$. However, K. Seip proved that the sequence $\Lambda = \{\lambda_n\} = \{n(1 - |n|^{-1/2})\}_{|n|>1}$ has density 1 and is sampling for $PW_{\pi}$, see Lemma 6.3 of [45]. On the other hand, the example given by Kadec, namely, the sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, where $\lambda_n = n - \frac{1}{4}$ and $\lambda_{-n} = -\lambda_n$ for $n = 1, 2, 3, \ldots$ ($\lambda_0 = 0$), also has density 1, but doesn’t generate a sampling sequence for $PW_{\pi}$, see section 3.3 of [5].

On the other hand, the question of when $\Lambda$ forms a sampling sequence for $PW_a$ was settled by Ortega-Cerdà and Seip in 2002 [40]. They gave a complete characterization of sampling sequences in the Paley-Wiener space $PW_{\pi}$ by connecting the problem with the Hilbert spaces of entire functions theory, denote by $\mathcal{H}(E)$, which will be presented in the next section. They proved the following theorem

**Theorem.** A separated sequence $\Lambda = \{\lambda_n\}$ of real numbers is sampling for $PW_a$ if and only if there exists two entire functions $E, F \in \mathcal{HB}$ such that

(i) $\mathcal{H}(E) = PW_a$,

(ii) $\Lambda$ constitutes the zero sequence of $EF + E^*F^*$.

K. Seip [45] characterized interpolating sequences in $PW_a$. More precisely, he proved the following:

**Theorem 3.1.3.** If $\Gamma = \{\gamma_n\}$ is an interpolating sequence in $PW_a$, then $\Gamma$ is uniformly separated and $D^+(\Gamma) \leq \frac{a}{\pi}$. If $\Gamma$ is uniformly separated and $D^+(\Gamma) < \frac{a}{\pi}$, then $\Gamma$ is an interpolating sequence in $PW_a$.

It follows that if $\Gamma$ is a complete interpolating sequence for $PW_a$, then $D^-(\Gamma) = D^+(\Gamma) = \frac{a}{\pi}$.

The question of when the sequence $\Gamma$ forms a complete interpolating sequence for $PW_a$ was settled by B. S. Pavlov in 1979 [23] using the concept of sine-type functions.

A set $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}}$ is said to be a uniqueness set for $PW_a$ if $F(\gamma_n) = 0$ for all $n \in \mathbb{Z}$ and $F \in PW_a$ imply $F \equiv 0$. Note that the sampling (frame) inequality (2.1.3) implies that a set
of sampling is also a set of uniqueness in $PW_a$, i.e., the corresponding sequence of normalized reproducing kernels is complete in $PW_a$. If a sequence $\Gamma$ is an interpolating sequence which is also a uniqueness set in $PW_a$, then it is a complete interpolating sequence, hence, it must have density $D^+(\Gamma) = \frac{a}{\pi}$. This in fact shows that any separated sequence with density $D^+(\Gamma) < a/\pi$ must be a nonuniqueness set in $PW_a$ by Theorem 3.1.3.

**Theorem 3.1.4.** Let $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be a separated sequence, and $a > 0$. If $D^+(\Gamma) < \frac{a}{\pi}$ then the set $\Gamma$ is not a set of uniqueness in $PW_a$.

### 3.2 Structure of de Branges Space

The foundation of de Branges spaces is the class of Hermite-Biehler functions. Let us recall the definition of this class from chapter two.

**Definition 3.2.1.** An entire function $E(z)$ is said to be of Hermite–Biehler class, denoted by $\mathcal{HB}$, if it satisfies the condition

$$|E(\bar{z})| < |E(z)|,$$

for all $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$.

It should be noted that such a function $E$ is root-free in $\mathbb{C}^+$. When talking about de Branges spaces these functions are called *de Branges functions*. A de Branges function $E(z)$ is said to be *strict* if it has no zeros on the real line, in this case the corresponding space $\mathcal{H}(E)$ is said to be strict de Branges space.

**Example 3.1.** One can show that each of the following functions defines a de Branges function.

1. $E(z) = e^{-iaz}, a > 0$.

2. Any polynomial with no zeros in the upper half plane. Indeed, let $P(z)$ be a polynomial of order $n > 0$ with no zeros in the upper half-plane, then we know that it has exactly $n$ zeros $\{z_k\}_{k=1}^n$ where $z_k = x_k - iy_k$, and $y_k \geq 0$ for all $k = 1, 2, \ldots, n$. Moreover, we have

$$P(z) = c \prod_{k=1}^n p_k(z), \quad p_k(z) = z - z_k$$
for some constant $c \in \mathbb{C}$. Simple computations show that $|p_k(\bar{z})| = |\bar{z} - z_k| < |z - z_k| = |p_k(z)|$ for all $z \in \mathbb{C}^+$ and $k = 1, 2, \ldots, n$. Hence, $|P(\bar{z})| < |P(z)|$ for $z \in \mathbb{C}^+$. That is, $P(z) \in \mathcal{HB}$.

(3) Let $E(z)$ be a de Branges function, and $S(z)$ be an entire function which is real for real $z$, then $E(z)S(z)$ is a de Branges function.

(4) Any finite product of de Branges functions is a de Branges function.

**Remark 3.2.1.** Note that if $z_0 \in \mathbb{C}^-$, then $z - z_0$ is a de Branges function. Also $e^{-iaz}$ is a de Branges function for any $a > 0$. Moreover, by the example above, any finite product of de Branges functions, or of a de Branges function with any entire function $S(z)$ which is real for real $z$, and has only real zeroes, is again a de Branges function. It follows that, for $\gamma \in \mathbb{C}$, the function $E_N(z) = \gamma S(z)e^{-iaz} \prod_{n=1}^{N}(z - z_n)$, is a de Branges function. In fact, Theorem 2.3.2 generalizes these results to the case where the product is an infinite product.

With a function $E \in \mathcal{HB}$, a space of entire functions can be associated.

**Definition 3.2.2.** Given an entire function $E \in \mathcal{HB}$. A de Brange Space associated with $E$, denoted by $\mathcal{H}(E)$, is a Hilbert space which consists of all entire functions $f(z)$ such that

$$||f||^2_E := \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty,$$

(3.2.2)

and $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and nonpositive mean type in the upper half-plane. An inner product is defined by

$$\langle f, g \rangle_E = \int_{\mathbb{R}} \frac{f(t)g(t)}{|E(t)|^2} dt,$$

for all $f, g \in \mathcal{H}(E)$.

The following theorem shows that the space $\mathcal{H}(E)$ always contains nonzero functions.

**Theorem 3.2.1.** Given a de Branges space $\mathcal{H}(E)$, the function

$$K(w, z) = \frac{E(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(w - z)}$$

(3.2.3)
belongs to \( \mathcal{H}(E) \) as a function of \( z \) for every \( w \in \mathbb{C} \), and

\[
f(w) = \langle f(t), K(w, t) \rangle_E, \quad \text{for all } w \in \mathbb{C},
\]

(3.2.4)

for every \( f \in \mathcal{H}(E) \).

**Proof.** First note that \( K(w, z) \) is an entire function of \( z \). We will show that \( K(w, z)/E(z) \) and \( K^*(w, z)/E(z) \) are of bounded type and nonpositive mean type in the upper half-plane. Let \( w \in \mathbb{C} \), then since the ratio \( E^*(z)/E(z) \) is bounded by 1 in the upper half-plane, the function

\[
(\bar{w} - z) \frac{K(w, z)}{E(z)} = \frac{1}{2\pi i} \left( \frac{E(w)}{E(z)} - \frac{E^*(z)}{E(z)} \frac{E(\bar{w})}{E(z)} \right)
\]

is bounded in the upper half-plane. Also, note that \((\bar{w} - z)\) is a polynomial, hence it is of bounded type in the upper half-plane, it follows that the quotient \( K(w, z)/E(z) \) is also of bounded type in the upper half-plane. Moreover, since a nonzero polynomial has a zero mean type and any bounded function has a nonpositive mean type, then the quotient \( K(w, z)/E(z) \) has nonpositive mean type in the upper half-plane. For the same reason, \( K^*(w, z)/E(z) \) is of bounded type and nonpositive mean type in the upper half-plane.

Now we show that \( K(w, z)/E(z) \in L^2(\mathbb{R}) \) for any \( w \in \mathbb{C} \). First note that since \( \overline{E(w)} - \frac{E(t)}{E(t)} E(\bar{w}) \) is a continuous function of \( t \in \mathbb{R} \) and has a zero at \( \bar{w} \) then

\[
\frac{K(w, t)}{E(t)} = \frac{1}{2\pi i (\bar{w} - t)} \left( \frac{E(w)}{E(t)} - \frac{E(t)}{E(t)} \frac{E(\bar{w})}{E(t)} \right)
\]

is a continuous function of \( t \in \mathbb{R} \). Therefore, the integral over bounded subsets of the real line is finite, hence we have

\[
\int_{-\infty}^{\infty} \left| \frac{K(w, t)}{E(t)} \right|^2 dt = \int_{\{t \in \mathbb{R} : |t - \bar{w}| \leq 1\}} \left| \frac{K(w, t)}{E(t)} \right|^2 dt + \int_{\{t \in \mathbb{R} : |t - \bar{w}| > 1\}} \left| \frac{K(w, t)}{E(t)} \right|^2 dt
\]

\[
\leq C + \int_{\{t \in \mathbb{R} : |t - \bar{w}| > 1\}} \left| \frac{K(w, t)}{E(t)} \right|^2 dt
\]

\[
= C + \int_{\{t \in \mathbb{R} : |t - \bar{w}| > 1\}} \left| \frac{E(w) - \frac{E(t)}{E(t)} E(\bar{w})}{2\pi i (\bar{w} - t)} \right|^2 dt
\]

\[
\leq C + \int_{\{t \in \mathbb{R} : |t - \bar{w}| > 1\}} \frac{(|E(w)| + |E(\bar{w})|)^2}{4\pi^2 |\bar{w} - t|^2} dt
\]
\[
C + \frac{(|E(w)| + |E(\bar{w})|)^2}{4\pi^2} \int_{\{t \in \mathbb{R} : |t - \bar{w}| > 1\}} \frac{1}{|\bar{w} - t|^2} dt < \infty.
\]

Therefore, it follows that \(K(w, z)\) belongs to \(\mathcal{H}(E)\) as a function of \(z\) for every \(w \in \mathbb{C}\), by Definition 3.2.2.

To show that \(K(w, z)\) satisfy property (3.2.4), let \(f(z) \in \mathcal{H}(E)\), then we know that the ratio \(f(z)/E(z)\) is analytic in the upper half-plane, and continuous in the closed half-plane (see Proposition 3.2.2 below). Hence, by Cauchy’s Formula in the upper half-plane,

\[
\begin{align*}
\frac{f(z)}{E(z)} &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{1}{E(t) t - z} dt, \\
0 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{1}{E(t) t - \bar{z}} dt,
\end{align*}
\]

for \(z \in \mathbb{C}^+\). The same formula hold also when \(f(z)\) is replaced by \(f^*(z)\), i.e.,

\[
\begin{align*}
\frac{f^*(z)}{E(z)} &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f^*(t) \frac{1}{E(t) t - z} dt, \\
0 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f^*(t) \frac{1}{E(t) t - \bar{z}} dt,
\end{align*}
\]

for \(z \in \mathbb{C}^+\), or equivalently,

\[
\begin{align*}
\frac{f^*(\bar{z})}{E(\bar{z})} &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f^*(t) \frac{1}{E(t) t - \bar{z}} dt, \\
0 &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f^*(t) \frac{1}{E(t) t - z} dt,
\end{align*}
\]

for \(z \in \mathbb{C}^-\). Taking the conjugate for the last equations we have

\[
\begin{align*}
\frac{f(z)}{E(\bar{z})} &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{1}{E(t) t - \bar{z}} dt, \\
0 &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{1}{E(t) t - z} dt,
\end{align*}
\]

for \(z \in \mathbb{C}^-\). It follows that for all nonreal \(z\),

\[
\begin{align*}
f(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) E(z)}{E(t) t - z} dt - \frac{f(t) E^*(z)}{E^*(t) t - z} dt \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) E(z)}{E(t)(t - z)} - \frac{f(t) E^*(z)}{E^*(t)(t - z)} dt \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) E^*(t) E(z) - E(t) E^*(z)}{|E(t)|^2(t - z)} dt \\
&= \langle f(t), K(z, t) \rangle_E
\end{align*}
\]

If \(w\) is real, we choose a sequence \(\{w_n\}\) of nonreal numbers such that \(w = \lim_{n \to \infty} w_n\). Then,

\[
K(w, x)/E(x) = \lim_{n \to \infty} K(w_n, x)/E(x)
\]
Thus, the function $K$ of property about the space $H$ uniformly on every bounded subset of the real axis. If $-\infty < a < b < \infty$, then
\[
\int_a^b |K(w, t) - K(w_n, t)|^2|E(t)|^{-2} dt = \lim_{k \to \infty} \int_a^b |K(w_k, t) - K(w_n, t)|^2|E(t)|^{-2} dt.
\]
On the other hand,
\[
\int_{-\infty}^{\infty} |K(w_k, t) - K(w_n, t)|^2|E(t)|^{-2} dt = \langle K(w_k, t) - K(w_n, t), K(w_k, t) - K(w_n, t) \rangle
\]
\[= K(w_k, w_k) - K(w_k, w_n) - K(w_n, w_k) + K(w_n, w_n).\]
Hence,
\[
\int_a^b |K(w_k, t) - K(w_n, t)|^2|E(t)|^{-2} dt \leq \int_{-\infty}^{\infty} |K(w_k, t) - K(w_n, t)|^2|E(t)|^{-2} dt
\]
\[\leq K(w_k, w_k) - K(w_k, w_n) - K(w_n, w_k) + K(w_n, w_n).\]
Since $K(w, w)$ is a continuous function of $w$ then
\[
\int_a^b |K(w, t) - K(w_n, t)|^2|E(t)|^{-2} dt \leq K(w, w) - K(w, w_n) - K(w_n, w) + K(w_n, w_n).
\]
Since $a$ and $b$ are arbitrary, then $K(w, z) = \lim_{n \to \infty} K(w_n, z)$ in the metric of $\mathcal{H}(E)$. It follows that
\[
f(w) = \lim_{n \to \infty} f(w_n) = \lim_{n \to \infty} \langle f(t), K(w_n, t) \rangle_E = \langle f(t), K(w, t) \rangle_E
\]
for every $f \in \mathcal{H}(E)$.

Lastly, let $w, z \in \mathbb{C}^+$, with $w \neq z$. Since $|E(\bar{z})| < |E(z)|$ for all $z \in \mathbb{C}^+$, then
\[
|K(w, z)| = \frac{|E(w)E(z) - E(\bar{w})E(\bar{z})|}{2\pi i(\bar{w} - z)} \geq \frac{|E(w)||E(z)| - |E(\bar{w})||E(\bar{z})|}{2\pi|\bar{w} - z|} > 0.
\]
Thus, the function $K(w, z)$ defined in (3.2.3) is the reproducing kernel of $\mathcal{H}(E)$, and the space $\mathcal{H}(E)$ contains a nonzero functions. \qed

Sometimes we will write $K_E(w, z)$ (instead of $K(w, z)$) if it is necessary to be more specific about the space $\mathcal{H}(E)$ under consideration. Property 3.2.4 is known as the reproducing kernel property of $\mathcal{H}(E)$. Using the fact that $K_E(w, w) = \langle K_E(w, t), K_E(w, t) \rangle$ for all $w \in \mathbb{C}$, and by applying Cauchy’s inequality on the reproducing property we obtain
\[
|f(w)|^2 \leq ||f||^2_E K_E(w, w)
\]
(3.2.5)
for all \( f(z) \) in \( \mathcal{H}(E) \), and all \( w \in \mathbb{C} \), which means that the point evaluation at any \( w \in \mathbb{C} \) is a bounded linear functional on \( \mathcal{H}(E) \). If \( K_E(w, w) > 0 \) for some \( w \in \mathbb{C} \) then equality holds if and only if \( f(z) = cK_E(w, z) \) for some constant \( c \in \mathbb{C} \).

The canonical example of de Branges spaces is the Paley-Wiener space \( PW_a \), \( a > 0 \). According to Paley-Wiener theorem, the Paley-Wiener space \( PW_a \) can be characterized as the space of entire functions which have exponential type at most \( a \) and are square integrable on the real line \( \mathbb{R} \). In this case we could write \( PW_a = \mathcal{H}(E) \), where \( E(z) = \exp(-iaz) \), where the two spaces are equal as sets, and have equivalent norms. So for any \( f \in PW_a \) we have

\[
||f||^2_{PW_a} = \int_{\mathbb{R}} |f(t)|^2 |E(t)|^2 dt = \int_{\mathbb{R}} |f(t)|^2 dt = ||f||^2_{L^2(\mathbb{R})}
\]

In fact, for \( a = \pi \), Yurri Lyubarskii and K. Seip [36] give a wide range of entire functions \( E \) such that \( PW_\pi = \mathcal{H}(E_\sigma) \) (other than \( \exp(-i\pi z) \)), where the norms \( ||.||_{PW_\pi} \) and \( ||.||_{\mathcal{H}(E_\sigma)} \) are equivalent:

\[
E_\sigma(z) = (z+i) \prod_{k=1}^{\infty} \left( 1 - \frac{z}{k - \sigma - ik - 4\sigma} \right) \left( 1 - \frac{z}{-k + \sigma - ik - 4\sigma} \right)
\]

for \( 0 \leq \sigma < 1/4 \).

Using the definition of the reproducing kernel (3.2.3), a straightforward calculation shows that the corresponding reproducing kernel of \( PW_a \) is

\[
K_a(w, z) = \frac{\sin a(z - \bar{w})}{\pi(z - \bar{w})}
\]

for all \( w, z \in \mathbb{C} \), \( z \neq \bar{w} \). Moreover, by the reproducing kernel property (3.2.4) we have, for \( x \in \mathbb{R} \)

\[
||K_a(x, .)||^2 = \langle K_a(x, t), K_a(x, t) \rangle_E = K_a(x, x) = \lim_{x \to y} K_a(x, y) = \lim_{x \to y} \frac{\sin a(x - y)}{\pi(x - y)} = \frac{a}{\pi}
\]

We provide other examples for de Branges space.

**Example 3.2.**

(1). Any polynomial \( P(z) \) without zeros in \( \mathbb{C}^+ \cup \mathbb{R} \) is a de Branges function, the associated (finite dimensional) de Branges space \( \mathcal{H}(P) \) contains precisely the polynomials whose degree is smaller than that of \( P \). For example, if \( P(z) = (z - z_1)(z - z_2) \) with \( z_1, z_2 \in \mathbb{C}^- \), then the
space $\mathcal{H}(P) = \text{span}\{1, z\}$.

(2). Let $E(z) := \frac{\sin \sqrt{iz}}{\sqrt{iz}}$. The zeros of $E$ are $z_n = -in^2$, $n \in \mathbb{N}$, and are all simple, and $E$ has a factorization

$$E(z) = \prod_{n \in \mathbb{N}} \left(1 + \frac{z}{in^2}\right) \in \mathcal{HB}$$

Moreover, it was proved in [29] that the associated de Branges space is $\mathcal{H}(E) = \mathbb{C}[z]$, where $\mathbb{C}[z]$ is the linear space of all polynomials with coefficients in $\mathbb{C}$.

The finite integral in (3.2.2) implies that the ratio $f(z)/E(z)$ has no singularities on the real axis, hence, the ratio is analytic on the closed upper half plane, for any $f \in \mathcal{H}(E)$. On other words, if $E(z)$ has a zero of order $r > 0$ at a real point $x_o$, then every function in the space $\mathcal{H}(E)$ has a zero of order at least $r$ at $x_o$.

**Proposition 3.2.2.** Let $E(z) \in \mathcal{HB}$ that has a zero of order $r > 0$ at a real point $x_o$. If $f(z)$ is an entire function such that $\int_{-\infty}^{\infty} |f(t)/E(t)|^2 dt < \infty$, then $f(z)$ has a zero of order at least $r$ at $x_o$.

**Proof.** We prove by contradiction. Suppose that $E(z)$ has a zero of order $r > 0$ at some point $x_o \in \mathbb{R}$, then $E(z) = (z - x_o)^r E_o(z)$ for some entire function $E_o(z)$ which does not vanish at $x_o$. Let $f(z)$ be an entire function, then $f(z) = (z - x_o)^m f_o(z)$, for some $0 \leq m < r$, and some entire function $f_o(z)$ which does not vanish at $x_o$. This implies that, give $\epsilon > 0$ there exists $\delta > 0$ such that $|f_o(t)/E_o(t)| > \epsilon$ for all $t \in (x_o - \delta, x_o + \delta)$, so we have

$$\int_{-\infty}^{+\infty} \left|\frac{f(t)}{E(t)}\right|^2 dt = \int_{-\infty}^{+\infty} \left|\frac{f(t)}{(t - x_o)^r E_o(t)}\right|^2 dt$$

$$= \int_{-\infty}^{+\infty} \left\{1 \left(\frac{1}{(t - x_o)^{2r_o}} \left|\frac{f_o(t)}{E_o(t)}\right|\right)^2 dt\right\}, \quad r_o = (r - m)$$

$$\geq \epsilon^2 \int_{x_o-\delta}^{x_o+\delta} \frac{1}{(t - x_o)^{2r_o} dt}$$

$$= \infty$$

(a since $r_o > 0$)

a contradiction, hence $f$ has a zero at $x_o$ of order at least $r$. \hfill \Box

In the context of growth properties of de Branges spaces, the growth of any function $f \in \mathcal{H}(E)$ is governed by the growth of the generating function $E$. The following proposition is a special case of theorem 3.4 in [30].
Proposition 3.2.3. Given an entire function $E \in \mathcal{H}B$. If $E(z)$ is of exponential type $\tau$, then all functions in the space $\mathcal{H}(E)$ are of exponential type less than or equal to $\tau$.

3.3 Characterization of de Branges Spaces

De Branges original definition involved the notions of mean type and bounded type. One useful alternative definition using the reproducing kernel $K(w, z)$ is given in the following theorem, see Theorem 20 of [14].

Theorem 3.3.1. A necessary and sufficient condition that an entire function $f(z)$ belong to $\mathcal{H}(E)$ is that

$$ ||f||^2_E = \int_{\mathbb{R}} |f(t)/E(t)|^2 dt < \infty $$

(3.3.1)

and that

$$ |f(z)|^2 \leq ||f||^2_E K(z, z) $$

(3.3.2)

for all complex $z$.

Proof. Let $f \in \mathcal{H}(E)$, then $||f||^2_E < \infty$ by (3.2.2). Applying the Schwarz inequality to the reproducing kernel property (3.2.4) we get (3.3.2) above.

Conversely, let $f$ be an entire function satisfying the two conditions in (3.3.1) and (3.3.2). By Definition 3.2.2 it is sufficient to show that $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and nonpositive mean type in the upper half-plane $\mathbb{C}^+$. It is sufficient to show that the functions $f(z)/E(z)$ and $f^*(z)/E(z)$ satisfying the hypothesis of Theorem 2.2.1, and therefore, are of bounded type in $\mathbb{C}^+$.

To begin with, note that since $E(z)$ has no zeros in the upper half-plane, then $f(z)/E(z)$ is analytic in $\mathbb{C}^+$. Moreover, Proposition 3.2.2 implies that $f(z)/E(z)$ has a continuous extension to the closed half-plane $\mathbb{C}^+ \cup \mathbb{R}$. Since

$$ K(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)} $$

then, for $\text{Im}(z) > 0$ we have

$$ K(z, z) = \frac{|E(z)|^2 - |E(\bar{z})|^2}{4\pi \text{Im}(z)} $$
\[ |f(z)|^2 \leq \|f\|_E^2 K(z, z) \leq \|f\|_E^2 |E(z)|^2. \]

Therefore,
\[ \left| \frac{f(z)}{E(z)} \right|^2 \leq \|f\|_E^2 \frac{1}{4\pi \text{Im}(z)} \]

for \( \text{Im}(z) > 0 \). Now, given that \( \text{Im}(z) = y > 0 \), then for \( 0 < \theta < \pi \)
\[ \left| \frac{f(re^{i\theta})}{E(re^{i\theta})} \right|^2 \leq \|f\|_E^2 \frac{1}{4\pi r \sin \theta} \]

and
\[ \lim_{r \to \infty} \frac{1}{r^2} \int_0^\pi \log^+ \left( \left| \frac{f(re^{i\theta})}{E(re^{i\theta})} \right|^2 \sin \theta \right) \, d\theta \leq \liminf_{r \to \infty} \frac{1}{r^2} \int_0^\pi \log^+ \left( \frac{\|f\|_E^2}{4\pi r \sin \theta} \right) \sin \theta \, d\theta = 0 \]

because
\[ \lim_{r \to \infty} \frac{1}{r^2} \int_0^\pi \log^+ \left( \|f\|_E^2 \sin \theta \right) \sin \theta \, d\theta = 0 \]

and
\[ \lim_{r \to \infty} \frac{1}{r^2} \int_0^\pi \log^+ \left( 4\pi r \sin \theta \right) \sin \theta \, d\theta = 0 \]

On the other hand, since \( f(t)/E(t) \) is square integrable on the real line, then Jensen’s inequality implies that
\[
\int_{-\infty}^{+\infty} \log^+ \left| \frac{f(t)}{E(t)} \right|^2 \frac{dt}{1 + t^2} \leq \int_{-\infty}^{+\infty} \log^+ \left| \frac{f(t)}{E(t)} \right|^2 dt \leq \log^+ \int_{-\infty}^{+\infty} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty
\]

Also, since \( \left| \frac{f(z)}{E(z)} \right|^2 \leq \|f\|_E^2 \frac{1}{4\pi y} \) for \( \text{Im}(z) = y > 0 \), then
\[
\lim_{y \to \infty} \frac{\log \left| \frac{f(iy)}{E(iy)} \right|^2}{y} \leq \lim_{y \to \infty} \frac{\log \left( \|f\|_E^2 / 4\pi y \right)}{y} = \lim_{y \to \infty} \frac{\log \|f\|_E^2}{y} - \lim_{y \to \infty} \frac{\log(4\pi y)}{y} = 0
\]

Hence,
\[
\limsup_{y \to \infty} \frac{\log \left| \frac{f(iy)}{E(iy)} \right|}{y} \leq 0, \quad (3.3.3)
\]

Therefore, the function \( f/E \) satisfies the hypothesis of Theorem 2.2.1, hence it is of bounded type in the upper half-plane.

Moreover, inequality (3.3.3) implies that \( f/E \) is of nonpositive mean type in \( \mathbb{C}^+ \). The same argument applies for the function \( f^*(z)/E(z) \). The theorem is proved.  \( \square \)
Definition (3.2.2) is the original definition for de Branges spaces. The following theorem gives another characterization of de Branges spaces among all Hilbert spaces of entire functions.

**Theorem 3.3.2.** Let $\mathcal{H}$ be a nonzero Hilbert space, whose elements are entire functions, which satisfying the following properties:

1. **(H1)** If $f(z) \in \mathcal{H}$ and $w \in \mathbb{C} \setminus \mathbb{R}$ with $f(w) = 0$, then the function $g(z) = f(z) \frac{z-w}{z-\bar{w}}$ belongs to $\mathcal{H}$, and $\|g\| = \|f\|$. 

2. **(H2)** For every nonreal number $w$, the linear functional defined on the space by $f(z) \mapsto f(w)$ is continuous.

3. **(H3)** If $f(z) \in \mathcal{H}$, then $f^*(z) \in \mathcal{H}$ and $\|f^*\| = \|f\|$. 

Then $\mathcal{H}$ is a de Branges space; there is an entire function $E \in \mathcal{HB}$ such that $\mathcal{H} = \mathcal{H}(E)$, and $\|f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}(E)}$ for all $f \in \mathcal{H}$.

On the other hand, it is easy to show that any de Branges space $\mathcal{H}(E)$ satisfies the above properties. Indeed, let $f(z) \in \mathcal{H}(E)$ and $w \in \mathbb{C} \setminus \mathbb{R}$ with $f(w) = 0$. Set $g(z) := f(z) \frac{z-w}{z-\bar{w}}$. Since $f(w) = 0$ then $g(z)$ is entire function (specifically, it can be extended to an entire function). Note that $\frac{f^*(z)}{E(z)(z-w)}$ and $\frac{f(z)}{E(z)(z-\bar{w})}$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$. First assume that $w \in \mathbb{C}^+$, then $\frac{f(z)}{E(z)(z-w)}$ is analytic, of bounded type, and of nonpositive mean type in $\mathbb{C}^+$ because any polynomial is of bounded type and nonpositive mean type in $\mathbb{C}^+$ and a quotient and product of functions of bounded type is again such function. Hence $\frac{f(z)(z-w)}{E(z)(z-\bar{w})}$ is also of bounded type and nonpositive mean type in $\mathbb{C}^+$. On the other hand, note that the function $\frac{z-w}{z-\bar{w}}$ is analytic in $\mathbb{C}^+$. Further, since $(z-w)$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$, it follows that $\frac{z-w}{z-\bar{w}}$ is also of bounded type and nonpositive mean type in $\mathbb{C}^+$. Hence, $\frac{f(z)(z-w)}{E(z)(z-\bar{w})}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$. That is, $\frac{f(z)(z-w)}{E(z)(z-\bar{w})}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$ for all $w \in \mathbb{C} \setminus \mathbb{R}$ such that $f(w) = 0$.

Let $w \in \mathbb{C}^-$. Note that since $\frac{f^*(z)}{E(z)(z-w)}$ vanish at $\bar{w}$, then $\frac{f^*(z)}{E(z)(z-\bar{w})}$, and hence, $\frac{f^*(z)(z-w)}{E(z)(z-\bar{w})}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$. Further, since $(z-w)$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$, it follows that $\frac{f^*(z)(z-w)}{E(z)(z-\bar{w})}$ is also of bounded type and nonpositive
mean type in $\mathbb{C}^+$. It follows that $\frac{f^*(z)(z-\bar{w})}{E(z)(z-w)}$ is of bounded type and nonpositive mean type in $\mathbb{C}^+$ for all $w \in \mathbb{C} \setminus \mathbb{R}$ with $f(w) = 0$.

On the other hand, note that

$$\int_{\mathbb{R}} \left| g(t) \right|^2 \frac{dt}{E(t)} = \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 \frac{dt}{t-w} = \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 \frac{dt}{t-w}$$

hence, $\|g/E\|_{L^2} = \|f/E\|_{L^2} = \|f\|_E$. Therefore, the function $g \in \mathcal{H}(E)$.

Property (H2) is satisfied by the reproducing kernel property (3.2.4). Let $f \in \mathcal{H}(E)$, then $\frac{f}{E}$ and $\frac{f}{E}$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$. Note that $\frac{f^*}{E} = \frac{f}{E}$, and $|f^*(t)| = |f(t)|$ for all $t \in \mathbb{R}$, hence

$$\int_{\mathbb{R}} \left| \frac{f^*(t)}{E(t)} \right|^2 \frac{dt}{E(t)} = \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 \frac{dt}{E(t)} = \|f\|_E^2$$

Therefore, any de Branges space $\mathcal{H}(E)$ satisfy the above axioms.

**Remark 3.3.1.**

1. It should be noted that the function $E(z)$ in Theorem 3.3.2 such that $\mathcal{H} = \mathcal{H}(E)$ is not uniquely determined. For example, the functions $E_r(z) := rA(z) - i\frac{1}{r}B(z)$, $\forall r \in \mathbb{R}^+$, $E_\theta(z) = e^{i\theta}E(z) := A\theta(z) - iB\theta(z)$, for $0 \leq \theta < 2\pi$, and $E_\beta(z) := [A(z) + \beta B(z)] - iB(z)$, for $\beta \in \mathbb{R}$, all give the same reproducing kernel, and hence, generate the same space $\mathcal{H}(E)$.

2. By noting that

$$f(z) \frac{z-\bar{w}}{z-w} = f(z) + \left( w - \bar{w} \right) \frac{f(z)}{(z-w)}$$

axiom (H1) says that we can divide out zeros of the functions in $\mathcal{H}$, i.e., $\frac{f(z)}{(z-w)}$ belongs to the space $\mathcal{H}$ whenever $f(z)$ belongs to $\mathcal{H}$ and vanishes at a nonreal $w$, while (H2) says that $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{C} \setminus \mathbb{R}$. (H3) is related to the Schwarz reflection principle, as applied to the real axis.

Another implication of axiom (H1) is, one sees that for a nonreal $w_o$, we can always find in $\mathcal{H}$ a function not vanishing at $w_o$. So the evaluation $K(w_o, w_o)$ can not be zero if $w_o$ is non real. Analogue to Proposition 3.2.2, we have the following:
Lemma 3.3.3. Given a strict de Branges space $\mathcal{H}(E)$. If $\alpha \in \mathbb{C}$ with $E(\alpha) \neq 0$, then there exists some $f \in \mathcal{H}(E)$ such that $f(\alpha) \neq 0$.

Proof. Suppose that every function $f \in \mathcal{H}(E)$ vanishes at $\alpha$, then by formula (3.3.4) the function $\frac{f(z)}{z-\alpha}$ belongs to $\mathcal{H}(E)$ for all $f \in \mathcal{H}(E)$ and vanishes at $\alpha$. It follows inductively that $\frac{f(z)}{(z-\alpha)^n} \in \mathcal{H}(E)$ and vanishes at $\alpha$, for all $n \geq 0$. But since $f(z)$ is an entire function, then it must vanish identically, a contradiction since $\mathcal{H}(E)$ contains a nonzero element.

Following [14], any de Branges function $E(z)$ can be written as $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real $z$;

$$A(z) := \frac{1}{2}[E(z) + E^*(z)], \quad B(z) := \frac{i}{2}[E(z) - E^*(z)]$$

Using this notation and the definition of the reproducing kernel in (3.2.3) we can write

$$K_E(w, z) = \frac{B(z)A(w) - A(z)B(w)}{\pi(z - \bar{w})}, \quad \text{for } z \neq \bar{w} \quad (3.3.6)$$

For $z = \bar{w}$, we have

$$K_E(\bar{z}, z) = \frac{1}{\pi}[B'(z)A(z) - A'(z)B(z)]$$

3.4 Orthonormal Basis in $\mathcal{H}(E)$

Given a de Branges function $E(z)$, the function $\Theta(z) := E^*(z)/E(z)$ is a meromorphic inner function. Indeed, since $E(z)$ has no zeros in the upper half-plane, the function $\Theta(z)$ is analytic and bounded by 1 in $\mathbb{C}^+$ by (3.2.1). Moreover, $\Theta(z)$ is unimodular on $\mathbb{R}$. It follows that the function $\Theta(z) \in H^\infty(\mathbb{C}^+)$, and meromorphic in $\mathbb{C}$. The meromorphic extension to the lower half-plane $\mathbb{C}^-$ is by putting $\Theta(z) = 1/\Theta^*(z)$, for $z \in \mathbb{C}^-$.

An important characteristic of the de Branges space $\mathcal{H}(E)$ is the phase function corresponding to the generating function $E$:

Definition. For any entire function $E \in \mathcal{HB}$, there exists a continuous and strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $E(x)e^{i\varphi(x)} \in \mathbb{R}$ for all $x \in \mathbb{R}$, and $E(x)$ can be written as

$$E(x) = |E(x)|e^{-i\varphi(x)}, \quad x \in \mathbb{R} \quad (3.4.1)$$
If a function φ has these properties then it is referred to as a phase function of E (sometimes it is denoted by φ_E if it is necessary to be more specific). It follows that a phase function of E is defined uniquely up to an additive constant, a multiple of 2π. If φ(x) is any such function, then using (3.2.3) and (3.4.1), easy computations gives

$$K(x, x) = \frac{1}{\pi} \varphi'(x) |E(x)|^2$$

(3.4.2)

for all $x \in \mathbb{R}$.

Let $Z = \{z_k\}$ be the zeros set of $E(z)$ in the upper half-plane, repeated according to multiplicity, with $z_k = x_k - iy_k$, $y_k \geq 0$.

Then the sequence $Z$ satisfies the Blaschke condition

$$\sum_{z_k \neq 0} \frac{y_k}{x_k^2 + y_k^2} < +\infty$$

If $E(z) \in \mathcal{HB}$ is of exponential type and has no real zeros, then by Theorem 2.3.5, $E(z)$ has the following canonical representation

$$E(z) = \gamma e^{bz} e^{-iaz} \prod_{z_k \neq 0} \left(1 - \frac{z}{z_k}\right) e^{zR(1/z_k)}$$

(3.4.3)

where $\gamma \in \mathbb{C}$ and $b \in \mathbb{R}$. Also note that the logarithmic derivative of $E$ is given by

$$\frac{E'}{E}(z) = b - ia + \sum_k \left(\frac{1}{z - z_k} + \text{Re} \frac{1}{z_k}\right), \quad z \neq z_k,$$

hence, (3.4.1) implies that

$$\varphi'(x) = -\text{Im} \frac{E'(x)}{E(x)} = a + \sum_k \frac{\text{Im} z_k}{|x - z_k|^2} = a + \sum_k \frac{y_k}{(x - x_k)^2 + y_k^2}$$

(3.4.4)

for all $x \in \mathbb{R}$, where $a = -\frac{1}{2}\text{Im}(E^*/E) \geq 0$ corresponds to the presence of the exponential factor $e^{-iaz}$ in the canonical factorization of $E$ above.

One of the main characteristics of any de Branges space is that it is always has a basis consisting of reproducing kernels corresponding to real points.

**Theorem 3.4.1.** Let $\mathcal{H}(E)$ be a de Branges space and $\varphi(x)$ be a phase function associated with $E$. If $\alpha \in \mathbb{R}$, and $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is a sequence of real numbers, such that $\varphi(\lambda_n) = \alpha + \pi n, \ n \in \mathbb{Z}$, then
1. The functions \( \{ K(\lambda_n, z) \}_{n \in \mathbb{Z}} \) forms an orthogonal set in \( \mathcal{H}(E) \).

2. If \( e^{\imath \alpha} E(z) - e^{-\imath \alpha} E^*(z) \notin \mathcal{H}(E) \), then \( \left\{ \frac{K(\lambda_n, z)}{\|K(\lambda_n, \cdot)\|} \right\}_{n \in \mathbb{Z}} \) is an orthonormal basis for \( \mathcal{H}(E) \).

Moreover, for every \( F(z) \in \mathcal{H}(E) \),

\[
F(z) = \sum_{n \in \mathbb{Z}} F(\lambda_n) \frac{K(\lambda_n, z)}{\|K(\lambda_n, \cdot)\|^2},
\]

and

\[
\|F\|^2 = \sum_{n \in \mathbb{Z}} \left| \frac{F(\lambda_n)}{E(\lambda_n)} \right|^2 \frac{\pi}{\varphi'(\lambda_n)}.
\]

By a Lemma of [15], there is at most one real number \( \alpha \) modulo \( \pi \) such that the function \( e^{\imath \alpha} E(z) - e^{-\imath \alpha} E^*(z) \) belongs to \( \mathcal{H}(E) \). From this theorem, we see that the existence of a sequence \( \{ \lambda_n \}_{n \in \mathbb{Z}} \) such that the set of functions \( \{ K(\lambda_n, \cdot) \}_{n \in \mathbb{Z}} \) forms an orthogonal basis for \( \mathcal{H}(E) \) implies a sampling formula in the space. This means that any \( f \in \mathcal{H}(E) \) can be recovered from its samples \( \{ f(\lambda_n) \}_{n \in \mathbb{Z}} \) by the formula (3.4.5).

The proof of Theorem 3.4.1 requires the following lemma of [14].

**Lemma 3.4.2.** Let \( f(z) \) be a function which is analytic in the complex plane except for isolated singularities at points \( \{ t_n \} \) on the real line. Suppose that \( f^*(z) = f(z) \) and that \( \text{Re}(-\imath f(z)) > 0 \) for \( \text{Im}(z) > 0 \). Then there exist positive numbers \( p_n \) and a nonnegative number \( p \) such that

\[
\frac{f(z) - \overline{f}(w)}{z - \overline{w}} = p + \sum_n p_n \frac{1}{(t_n - z)(t_n - \overline{w})}
\]

for nonreal \( z \) and \( w \). The numbers \( p_n \) are given by

\[
p_n = \lim_{z \to t_n} (t_n - z)f(z)
\]

for every \( n \).

**Proof of Theorem 3.4.1:** Let \( \mathcal{H}(E) \) be a de Branges space and \( K(w, z) \) be the corresponding reproducing kernel. First we will prove the orthogonality. By Theorem 3.2.1 the reproducing kernel \( K(w, z) \) belongs to \( \mathcal{H}(E) \) as a function of \( z \) for all \( w \in \mathbb{C} \), hence any constant multiple of \( K(w, z) \) will belong to \( \mathcal{H}(E) \). In particular, for any real number \( x \), we have \( \frac{K(x, z)}{E(x)} \in \mathcal{H}(E) \). Moreover, since \( E(x) = e^{-\imath \varphi(x)} |E(x)| \), then \( \frac{E(x)}{E(x)} = e^{-\imath \varphi(x)} \) for all \( x \in \mathbb{R} \). Hence,

\[
\frac{K(x, z)}{E(x)} = \frac{1}{E(x)} \frac{E(x)E(z) - E(x)E^*(z)}{2\pi i (x - z)}
\]
Hence, for all \( \alpha \) assume, without loss of generality, that \( \alpha \in \mathbb{C} \). Let \( \lambda \in \mathbb{Z} \) be a sequence of real numbers such that \( \varphi(\lambda_n) = \alpha + n\pi \) for all \( n \in \mathbb{Z} \), and \( \alpha \in [0, \pi) \). Then by the computations above we have

\[
\langle \frac{K(\lambda_n, t)}{E(\lambda_n)} , \frac{K(\lambda_m, t)}{E(\lambda_m)} \rangle_E = \frac{1}{2\pi i} \frac{E(\lambda_n) - e^{i\varphi(\lambda)} E(\lambda_n) - E(\lambda_m)}{E(\lambda_n) - E(\lambda_m)}
\]

\[
= \frac{1}{2\pi i} \frac{E(\lambda_n) - e^{i\varphi(\lambda_n)} E(\lambda_n) - E(\lambda_m)}{E(\lambda_n) - E(\lambda_m)}
\]

\[
= \frac{1 - e^{i\varphi(\lambda_n)} E(\lambda_n) - E(\lambda_m)}{2\pi i (\lambda_n - \lambda_m)}
\]

Hence, \( \langle \frac{K(\lambda_n, t)}{E(\lambda_n)} , \frac{K(\lambda_m, t)}{E(\lambda_m)} \rangle_E = 0 \) whenever \( n \neq m \). That is, the set \( \{ \frac{K(\lambda_n, z)}{E(\lambda_n)} \}_{n \in \mathbb{Z}} \) is an orthogonal set in \( \mathcal{H}(E) \). Consequently, \( \{ K(\lambda_n, z) \}_{n \in \mathbb{Z}} \) is an orthogonal set in \( \mathcal{H}(E) \).

Now we will show that the set \( \{ \frac{K(\lambda_n, z)}{E(\lambda_n)} \}_{n \in \mathbb{Z}} \) is complete in \( \mathcal{H}(E) \). To begin with, we can assume, without loss of generality, that \( \alpha = 0 \), otherwise the general case then follows by considering the function \( e^{i\alpha} E(z) \) instead of \( E(z) \) without changing of the corresponding space.

Let \( E(z) = A(z) - iB(z) \) as in (3.3.5). Let \( f(z) = -A(z)/B(z) \). Since \( A(z) \) and \( B(z) \) are entire functions which are real for real \( z \), and have only real zeros, then \( f^*(z) = f(z) \). Moreover, the singularities of \( f(z) \) are the real points where \( B(z) \) has a zero of higher multiplicity than \( A(z) \).

But since

\[
E(x) = e^{-i\varphi(x)} |E(x)| = (\cos \varphi(x) - i \sin \varphi(x)) |E(x)|
\]

and \( E(x) = A(x) - iB(x) \) then we have

\[
A(x) = |E(x)| \cos \varphi(x),
\]
and

\[ B(x) = |E(x)| \sin \varphi(x). \]

It follows that the singularities of \( f(z) = -A(z)/B(z) \) are the zeros of \( B(z) \), which are the real points \( \{\lambda_n\} \) where \( \varphi(\lambda_n) = n\pi \). On the other hand, since \( K(z, z) > 0 \) for nonreal \( z \), then \( \operatorname{Re}(-if(z)) > 0 \) when \( \operatorname{Im}(z) > 0 \). Indeed, note that by (3.3.6), the reproducing kernel \( K(w, z) \) can be written as

\[ K(w, z) = \frac{\overline{A(w)}B(z) - A(z)\overline{B(w)}}{\pi(z - \bar{w})} \]

therefore,

\[ \frac{f(z) - \overline{f(w)}}{z - \bar{w}} = -\frac{A(z)/B(z) + A(w)/\overline{B(w)}}{z - \bar{w}} = \frac{1}{\overline{B(w)}B(z)} \frac{\overline{A(w)}B(z) - A(z)\overline{B(w)}}{(z - \bar{w})} = \frac{\pi}{\overline{B(w)}B(z)} K(w, z) \]

and

\[ \frac{f(z) - \overline{f(z)}}{z - \bar{z}} = \frac{1}{|B(z)|^2} K(z, z), \]

hence, \( \operatorname{Re}(-if(z)) = \operatorname{Im}f(z) > 0 \) when \( \operatorname{Im}(z) > 0 \). Lemma 3.4.2 implies that there exist positive numbers \( \{p_n\} \) and a nonnegative number \( p \) such that

\[ \frac{f(z) - \overline{f(w)}}{z - \bar{w}} = p + \sum_n p_n \frac{1}{(\lambda_n - z)(\lambda_n - \bar{w})}, \]

for nonreal \( z \) and \( w \). Moreover, the numbers \( p_n \) are given by

\[ p_n = \lim_{z \to \lambda_n} \frac{A(z)(z - \lambda_n)}{B(z)} = \frac{A(\lambda_n)}{B'(\lambda_n)}. \]

So, it follows that we can rewrite the reproducing kernel as

\[ K(w, z) = \frac{p}{\pi} \overline{B(w)}B(z) + \sum_n p_n \frac{B(z)}{(\lambda_n - z)(\lambda_n - \bar{w})} \]

\[ = \frac{p}{\pi} \overline{B(w)}B(z) + \sum_n \frac{A(\lambda_n)}{B'(\lambda_n)} \frac{B(z)}{(z - \lambda_n)(w - \lambda_n) - \bar{w})} \]

(3.4.7)

for all \( z, w \in \mathbb{C} \), and the sum converges in the metric of \( \mathcal{H}(E) \). Indeed, note that since \( \pi \frac{K(\lambda_n, z)}{E(\lambda_n)} = \frac{B(z)}{(z - \lambda_n)} \), then the set of functions \( \{\frac{B(z)}{(z - \lambda_n)}\} \) is an orthogonal sequence in \( \mathcal{H}(E) \).
Moreover,

\[ \|B(t)/(t - \lambda_n)\|_E^2 = \left\langle \frac{K(\lambda_n, t)}{E(\lambda_n)}, \frac{K(\lambda_n, t)}{E(\lambda_n)} \right\rangle_E \]

\[ = \pi^2 \frac{K(\lambda_n, \lambda_n)}{|E(\lambda_n)|^2} \]

\[ = \pi \frac{1}{E(\lambda_n)} \lim_{t \to \lambda_n} \frac{B(t)}{(t - \lambda_n)} \]

\[ = \frac{B'(\lambda_n)}{A(\lambda_n)} \]

where we used the fact \(A(\lambda_n) = (E(\lambda_n) + E(\lambda_n))/2 = E(\lambda_n)\) whenever \(B(\lambda_n) = 0 = E(\lambda_n) - E^*(\lambda_n)\), in the last equality.

Note that since

\[ \sum_n \frac{A(\lambda_n)}{B'(\lambda_n)} \frac{B(w)}{(w - \lambda_n)} \leq K(w, w) \]

then

\[ \sum_n \left\| \frac{A(\lambda_n)}{B'(\lambda_n)} \frac{B(t)}{(t - \lambda_n)} \frac{B(w)}{(w - \lambda_n)} \right\|^2 < \infty, \]

therefore, the orthogonal series in (3.4.7) converges in the metric of \(\mathcal{H}(E)\), and the sum belongs to \(\mathcal{H}(E)\). Since \(K(w, z) \in \mathcal{H}(E)\), it follows that \(\frac{p}{\pi}B(w)B(z)\) belongs to \(\mathcal{H}(E)\).

Let \(F(z) \in \mathcal{H}(E)\) be an element which is orthogonal to \(K(\lambda_n, z)/E(\lambda_n)\) for all \(n\), i.e.,

\[ \langle F(t), \frac{K(\lambda_n, t)}{E(\lambda_n)} \rangle = 0, \text{ for all } n, \text{ or equivalently, } \langle F(t), \frac{B(t)}{(t - \lambda_n)} \rangle = 0, \text{ for all } n. \]

Then

\[ F(w) = \langle F(t), K(\lambda_n, t) \rangle_E \]

\[ = \langle F(t), \frac{p}{\pi}B(w)B(t) + \sum_n \frac{A(\lambda_n)}{B'(\lambda_n)} \frac{B(t)}{(t - \lambda_n)} \frac{B(w)}{(w - \lambda_n)} \rangle_E \]

\[ = \langle F(t), \frac{p}{\pi}B(w)B(t) \rangle_E \]

\[ = \frac{p}{\pi}B(w) \langle F(t), B(t) \rangle_E \]

for all \(w \in \mathbb{C}\). It follows that \(F(z)\) is a constant multiple of \(B(z)\). But since by hypothesis \(B(z) = E(z) - E^*(z) \notin \mathcal{H}(E)\), then \(F(z)\) vanishes identically, hence the set \(\{K(\lambda_n, z)/E(\lambda_n)\}_{n \in \mathbb{Z}}\) is complete.

Since the set \(\{K(\lambda_n, z)/E(\lambda_n)\}_{n \in \mathbb{Z}}\) is a complete orthogonal set in \(\mathcal{H}(E)\), then using the fact that

\[ \|K(\lambda_n, t)\|^2 = K(\lambda_n, \lambda_n) = \frac{1}{\pi} \varphi'(\lambda_n)|E(\lambda_n)|^2 \]
we obtain
\[ F(z) = \sum_{n \in \mathbb{Z}} \langle F(t), \frac{K(\lambda_n, t)}{E(\lambda_n)} \rangle_E \frac{K(\lambda_n, z)}{\|K(\lambda_n, t)/E(\lambda_n)\|} \]

and
\[ \|F\|_2^2 = \sum_{n \in \mathbb{Z}} |\langle F(t), \frac{K(\lambda_n, t)}{E(\lambda_n)} \rangle_E|^2 \]
\[ = \sum_{n \in \mathbb{Z}} \left| \frac{F(\lambda_n)}{E(\lambda_n)} \right|^2 \frac{\pi}{\varphi'(\lambda_n)} \]

for every \( F \in \mathcal{H}(E) \), completing the proof.

### 3.5 Logarithmic Derivative of \( E(z) \)

One of the classical subjects in function theory is the estimation of the norm of the derivative of functions in a given Hilbert space \( \mathcal{H} \) in terms of the norm of the functions in the space, i.e., there exists a constant \( C > 0 \) such that
\[ \|f'\|_\mathcal{H} \leq C \|f\|_\mathcal{H}, \]
for all \( f \in \mathcal{H} \). In case of de Branges space, this problem was first stated by A. D. Baranov [2]. For the Paley-Wiener space \( PW_a \), it is known that
\[ \|f'\|_2 \leq a \|f\|_2, \text{ for all } f \in PW_a. \quad (3.5.1) \]

For general de Branges spaces, he proved the following theorem which gives a sufficient condition for the boundedness of the differentiation operator in \( \mathcal{H}(E) \).

**Theorem 3.5.1.** Let \( E(z) \in \mathcal{H}B \). If \( E'/E \in H^\infty(\mathbb{C}^+) \), then the differentiation operator \( D : f \mapsto f' \) is bounded operator from \( \mathcal{H}(E) \) into \( \mathcal{H}(E) \). Moreover, \( \|D\| \leq C \|E'/E\|_\infty \), where \( C \) is some absolute constant.

As a corollary, he also proved the following result, which will play a significant role in the proofs of chapter 4.
Bernstein Inequality. Let $E(z) \in \mathcal{H}$. If $\frac{E'}{E} \in L^\infty(\mathbb{R})$, then
\[
\|f'\|_2 \leq C \left\| \frac{E'}{E} \right\|_\infty \|f\|_E
\] (3.5.2)
for all $f \in \mathcal{H}(E)$, with $C \leq 4 + \sqrt{6}$.

The following lemma is a direct application of Bernstein inequality.

Lemma 3.5.2. Let $\mathcal{H}(E)$ be a de Branges space. If $E'/E \in L^\infty(\mathbb{R})$, then $(f/E)' \in L^2(\mathbb{R})$ for all $f \in \mathcal{H}(E)$.

Proof. Let $f(z) \in \mathcal{H}(E)$. First note that the ratio $f(t)/E(t)$ is continuous for all $t \in \mathbb{R}$ by Proposition 3.2.2. Using the identity $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ for any $a, b \in \mathbb{R}$ we get
\[
\int_{-\infty}^{\infty} \left| \left( \frac{f(t)}{E(t)} \right)' \right|^2 dt = \int_{-\infty}^{\infty} \left| \frac{f'(t)}{E(t)} - \frac{E'(t) f(t)}{E(t) E(t)} \right|^2 dt \\
\leq 2 \left( \int_{-\infty}^{\infty} \left| \frac{f'(t)}{E(t)} \right|^2 dt + \left\| \frac{E'}{E} \right\|_\infty \int_{-\infty}^{\infty} \left| \frac{f(t)}{E(t)} \right|^2 dt \right) \\
= 2 \left( \left\| \frac{f'}{E} \right\|_2^2 + \left\| \frac{E'}{E} \right\|_\infty \left\| f \right\|_E^2 \right) \\
\leq 2 \left( C^2 \left\| \frac{E'}{E} \right\|_\infty^2 \left\| f \right\|_E^2 + \left\| \frac{E'}{E} \right\|_\infty^2 \left\| f \right\|_E^2 \right) \\
= 2(C^2 + 1) \left\| \frac{E'}{E} \right\|_\infty^2 \left\| f \right\|_E^2
\]
where we used the Bernstein inequality with constant $C \leq 4 + \sqrt{6}$. The right hand side of the last inequality is finite by the assumptions.

Now we present some cases where the logarithmic derivative of a de Branges function $E$ is bounded on the real line, which will be a sufficient condition for the existence of the Bernstein inequality in the space $\mathcal{H}(E)$. This fact will be of great importance in some of our results in the next chapter.

Recall that if $E(z) \in \mathcal{H}$ is of exponential type and has no real zeros, the canonical representation of $E$ given in (2.3.4) implies that the logarithmic derivative of $E$ has the form
\[
\frac{E'(z)}{E(z)} = -ia + b + \sum_n \left( \frac{1}{z-z_n} + \text{Re} \frac{1}{z_n} \right), \quad z \neq z_n
\] (3.5.3)

The following result gives a sufficient and necessary conditions for the boundedness of the function $E'/E$ on the real line in case of de Branges spaces of exponential type, we refer to [3]
for the proof. Some more general results are provided in [2] using the idea of the differential operator.

**Proposition 3.5.3.** Let $E(z)$ be an entire function of the form (2.3.4), then

(i) $\frac{E'}{E} \in L^\infty(\mathbb{R})$ if and only if

$$\sup_{x \in \mathbb{R}} \left( \limsup_{r \to \infty} \left| \sum_{|\bar{z}_n-x|<r} \frac{1}{\bar{z}_n-x} \right| \right) < \infty \quad (3.5.4)$$

(ii) $\frac{E'}{E} \in H^\infty(\mathbb{C}^+)$ if and only if for any $h > 0$

$$\sup_{x \in \mathbb{R}} \left( \limsup_{r \to \infty} \left| \sum_{|\bar{z}_n-x|<r} \frac{1}{\bar{z}_n+ih-x} \right| \right) < \infty \quad (3.5.5)$$

An important implication of the boundedness of the logarithmic derivative of $E$ is the boundedness of the derivative of its phase function.

**Lemma 3.5.4.** Let $E$ be a de Branges function with $\frac{E'}{E} \in L^\infty(\mathbb{R})$, and $\varphi$ be a phase function of $E$, then $\varphi'$ is bounded on $\mathbb{R}$, moreover, $\|\varphi'\|_\infty \leq \|E'/E\|_\infty$.

**Proof.** By the definition of the phase function in (3.4.1), $E(x) = |E(x)|e^{-i\varphi(x)}$, so for all $x \in \mathbb{R}$ the logarithmic derivative of $E$

$$\frac{E'(x)}{E(x)} = -i\varphi'(x) + \frac{|E(x)|'}{|E(x)|},$$

therefore, $\varphi'(x) = -\text{Im}(\frac{E'(x)}{E(x)})$. Hence, 

$$\left| \frac{E'(x)}{E(x)} \right|^2 = |\varphi'(x)|^2 + \left| \frac{|E(x)|'}{|E(x)|} \right|^2,$$

for all $x \in \mathbb{R}$, and $\|\varphi'\|_\infty \leq \|E'/E\|_\infty$. $\square$

However, the boundedness of the derivative of the phase function of $E$ could be attained without the need of Lemma 3.5.4. To begin with, note that if $z_n = x_n - iy_n$ is a zero of a given $E \in \mathcal{HB}$ then, by (3.4.4), we have

$$\varphi'(x_n) = a + \sum_k \frac{y_k}{(x_n-x_k)^2} - \frac{y_k}{y_n} \geq \frac{1}{y_n},$$
and this becomes large if \( y_n \) is small. Therefore, the condition that \( \inf_k y_k > 0 \) cannot be dropped if we want \( \varphi'(x) \) to be bounded. On the other hand, note that \( \varphi'(x) \geq a \) for all \( x \in \mathbb{R} \), hence, if \( a \neq 0 \) then \( \varphi'(x) \) is bounded away from zero. The following theorem gives necessary and sufficient conditions for boundedness of \( \varphi'(x) \), see [22].

**Theorem 3.5.5.** Let \( E \) be a de Branges function, with zeros \( \{z_n\}_{n \in \mathbb{Z}}, z_n = x_n - iy_n, 0 < a_1 \leq y_n \leq a_2 < \infty \). Let \( \varphi \) be a corresponding phase function of \( E \). Then

(a). \( \varphi'(x) \) is uniformly bounded from above on \( \mathbb{R} \) if and only if there exists a real number \( d > 0 \) and an integer \( N > 0 \) such that any rectangle

\[ R_{x,d} := [x, x + d] \times [a_1, a_2], \quad \text{with} \ x \in \mathbb{R} \]

contains at most \( N \) points of \( \bar{z}_n \).

(b). \( \varphi'(x) \) is uniformly bounded away from zero on \( \mathbb{R} \) if there exists a real number \( d > 0 \) such that any rectangle \( R_{x,d} \) contains at least one \( \bar{z}_n \).

(c). \( \varphi'(x) \) is uniformly bounded from above and uniformly bounded away from zero on \( \mathbb{R} \) if and only if the set \( \{\bar{z}_n\} \) satisfy both conditions of (a) and (b) simultaneously.

**Proof.** (a). Let \( d > 0 \) and \( N > 0 \) such that the rectangle \( R_{x,d} \) contains at most \( N \) points of \( \bar{z}_n \). Note that the set \( \{[kd, (k+1)d]\}_{k \in \mathbb{Z}} \) covers \( \mathbb{R} \). Set

\[ R_k := R_{kd,d} = [kd, (k+1)d] \times [a_1, a_2], \quad \text{and} \ R_k^* := R_{k-1} \cup R_k \cup R_{k+1}, \quad \text{for} \ k \in \mathbb{Z} \]

By (3.4.4), \( \varphi' \) has the form

\[ \varphi'(x) = a + \sum_{z_n \in R_k^*} \frac{y_n}{(x - x_n)^2 + y_n^2} \]

Now, given any \( x \in \mathbb{R} \), there exists \( k \in \mathbb{Z} \) such that \( x \in [kd, (k+1)d] \). Since \( a_1 \leq y_n \) we have

\[ \sum_{z_n \in R_k^*} \frac{y_n}{(x - x_n)^2 + y_n^2} \leq \sum_{z_n \in R_k^*} \frac{y_n}{y_n^2} = \sum_{z_n \in R_k^*} \frac{1}{y_n} \leq \frac{3N}{a_1} \]

Moreover, since \( y_n \leq a_2 \), then

\[ \sum_{z_n \notin R_k} \frac{y_n}{(x - x_n)^2 + y_n^2} \leq \sum_{z_n \notin R_k} \frac{a_2}{(x - x_n)^2} \leq \sum_{m=1}^{\infty} \frac{2a_2N}{d^2 m^2} = \frac{2a_2N \pi^2}{d^2 6} \]
Therefore, for all $x \in \mathbb{R}$

$$\varphi'(x) = a + \sum_{\bar{z}_n \in R_k^*} \frac{y_n}{(x - x_n)^2 + y_n^2} + \sum_{\bar{z}_n \notin R_k^*} \frac{y_n}{(x - x_n)^2 + y_n^2}$$

$$\leq a + \frac{3N}{a_1} + \frac{a_2 N \pi^2}{3d^2} < \infty$$

Conversely, assume that $\varphi'(x)$ is bounded above on $\mathbb{R}$ by some constant $M > 0$. Let $d = 1$, and let $N_k$ denote the number of points $\bar{z}_n$ in $R_k$. Since for any $x \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $x \in [k, k+1]$, we have

$$\varphi'(x) \geq \sum_{\bar{z}_n \in R_k} \frac{y_n}{(x - x_n)^2 + y_n^2} \geq \sum_{\bar{z}_n \in R_k} \frac{a_1}{1 + a_2^2} \geq N_k \frac{a_1}{1 + a_2^2}$$

Hence,

$$N_k \leq \frac{1 + a_2^2}{a_1} \varphi'(x) \leq \frac{1 + a_2^2}{a_1} M$$

for all $k \in \mathbb{Z}$. That is, the number of points in any rectangle $R_{x,d}$ is at most $N = \frac{1 + a_2^2}{a_1} M$, for any $x \in \mathbb{R}$.

(b). Let $d > 0$, then by assumption, number of points $\bar{z}_n$ in any rectangle $R_k$ is at least one. Given any $x \in \mathbb{R}$, there exists $k_0 \in \mathbb{Z}$ such that $x \in [k_0d, (k_0+1)d]$. Since $a_1 \leq y_n \leq a_2$ for all $n$, we have

$$\varphi'(x) \geq \sum_{n} \frac{y_n}{(x - x_n)^2 + y_n^2} \geq \frac{y_{k_0}}{(x - x_{k_0})^2 + y_{k_0}^2} \geq \frac{a_1}{d^2 + a_2^2}$$

hence, $\varphi'(x) \geq \frac{a_1}{d^2 + a_2^2}$, for all $x \in \mathbb{R}$.

(c). Suppose that $\varphi'(x)$ is uniformly bounded from above and uniformly bounded away from zero on $\mathbb{R}$. Then the set $\{\bar{z}_n\}$ have the property in (a), so we only have to check the property stated in (b). Let $d, N > 0$ be the constants from part (a). Suppose that property (b) is not satisfied by the set $\{\bar{z}_n\}$, then for any $\delta > 0$ there is a rectangle $R = R_{x,\delta}$ that has no points from $\{\bar{z}_n\}$. Take a big integer $L$ and put $\delta = (2L + 1)d$, where $d$ is the number from part (a). Let $c$ be the center of $[x, x + \delta]$. Then, for all $n$ we have

$$|c - x_n| \geq \frac{\delta}{2} + md = \frac{1}{2}(2L + 1)d + md \geq (L + m)d,$$
for some integer \( m \geq 0 \). Also, since each rectangle of length \( d \) contains at most \( N \) points of the set \( \{ \bar{z}_n \} \), we have

\[
\varphi'(c) = \sum_{\bar{z}_n \notin R} \frac{y_n}{(c-x_n)^2 + y_n^2} \\
\leq \sum_{\bar{z}_n} \frac{a_2}{(c-x_n)^2} \\
\leq \sum_{m=0}^{\infty} \frac{a_2N}{(\delta/2 + md)^2} \\
\leq \sum_{m=0}^{\infty} \frac{a_2N}{(L+m)^2d^2} \\
= \frac{a_2N}{d^2} \sum_{m \geq L} \frac{1}{m^2} = O\left( \frac{1}{L} \right)
\]

But since \( L \) is arbitrary large then this contradicts the assumption that \( \varphi' \) bounded away from zero on \( \mathbb{R} \).

Conversely, if the set \( \{ \bar{z}_n \} \) satisfy both conditions of (a) and (b) simultaneously, then \( \varphi'(x) \) is bounded above by part (a), and bounded away from zero by part (b).

\[
\square
\]

The condition stated in part (b) above is not necessary, for example, consider a function \( E(z) \in HB \) with \( n^2 \) repeated zeros at points \( \pm n^2 + i, n \geq 1 \). First note that \( \varphi'(x) \) is an even function. For \( 0 \leq x < 1 \) we have

\[
\varphi'(x) \geq \sum_{n} \frac{y_n}{(x-x_n)^2 + y_n^2} \geq \frac{y_1}{(x-x_1)^2 + y_1^2} = \frac{1}{(x-1)^2 + 1} > \frac{1}{2}
\]

On the other hand, if \( x \geq 1 \), then for \( x \in [k^2 - k - 1, k^2 + k + 1] \), \( k \geq 2 \), we have

\[
\varphi'(x) \geq \sum_{\bar{z}_n \in R_k} \frac{y_n}{(x-x_n)^2 + y_n^2} \\
\geq \frac{y_k}{(x-x_k)^2 + y_k^2} \\
= \frac{k^2}{(x-k^2)^2 + 1} \\
= \frac{1}{2} \frac{2k^2}{k^2 + 2k + 2} \\
\geq \frac{1}{2} \frac{k^2 + 2k}{k^2 + 2k + 2}
\]
\[ \varphi'(x) \geq \frac{2}{\delta}. \]

Therefore, \( \varphi'(x) \) is bounded away from zero on \( \mathbb{R} \). However, there is no \( d > 0 \) for which the condition stated in part (b) is satisfied.

Functions \( E \in \mathcal{HB} \) that satisfy \( E'/E \in L^\infty(\mathbb{R}) \), as well as \( E'/E \in L^\infty(\mathbb{R}) \) and \( 0 < \delta \leq \varphi'(x) \) play a central role in chapter 4. We present here a large class of such functions.

**Example 3.3.** Let \( E \) have the form

\[ E(z) = \gamma e^{bz} e^{-iaw} \prod_{n \in \mathbb{Z}} \left( 1 - \frac{z}{z_n} \right) e^{z \text{Re}(\frac{1}{n})}, \tag{3.5.6} \]

and let the zeros \( z_n \) satisfy the following conditions:

(a). \( z_n = \beta n + w_n \), for all \( n \in \mathbb{Z} \), where \( \beta > 0 \), and the sequence \( \{w_n\}_{n \in \mathbb{Z}} \) is bounded,

(b). \( \text{Im}(w_n) \leq \eta < 0 \).

Then \( \frac{E'}{E} \in L^\infty(\mathbb{R}) \).

**Proof.** Without loss of generality we may assume that \( \beta = 1 \) (by applying the change of variable \( z \to \beta z \)).

First we will assume that \( w_n = -i, \ n \in \mathbb{Z} \). Then the corresponding de Branges function, call it \( E_o \), will have zeros at \( z_n = n + w_n = n - i, \ n \in \mathbb{Z} \). Moreover, we can assume, without loss of generality, that \( E_o \) has the form

\[ E_o(z) = \prod_{n \in \mathbb{Z}} \left( 1 - \frac{z}{z_n} \right) e^{z(n/(n^2+1))} \]

then, by (3.5.3) we have

\[ \frac{E_o'(x)}{E_o(x)} = \sum_{n \in \mathbb{Z}} \left( \frac{1}{x - n + i} + \frac{n}{n^2 + 1} \right) \]

Now, recall that the function \( \cot \pi z \) has the following expansion (see [13])

\[ \cot \pi z = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{n \in \mathbb{Z}, n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right) \]

therefore,

\[ \cot \pi(x + i) = \frac{1}{\pi(x + i)} + \frac{1}{\pi} \sum_{n \in \mathbb{Z}, n \neq 0} \left( \frac{1}{x - n + i} + \frac{1}{n} \right) \]
Using the fact that \( \cot \pi(x + i) \) is uniformly bounded for all \( x \in \mathbb{R} \) (note that \( \frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n} = O(\frac{1}{n^2}) \)), it follows that \( \frac{E'(x)}{E(x)} \in L^\infty(\mathbb{R}) \).

Now, let \( E \) be the de Branges function with zeros \( z_n = n + w_n \), where \( w_n = u_n + iv_n \), \( n \in \mathbb{Z} \).

Since \( \{w_n\} \) is bounded, then \( \{u_n\} \) and \( \{v_n\} \) are bounded real sequences. So, we have

\[
\frac{E'(x)}{E(x)} - \frac{E'_o(x)}{E_o(x)} = \sum_{n \in \mathbb{Z}} \left( \frac{1}{x - n + \bar{w}_n} - \frac{1}{x - n + i} \right) + \sum_{n \in \mathbb{Z}} \left( \frac{n + u_n}{(n + u_n)^2 + v_n^2} - \frac{n}{n^2 + 1} \right)
\]

The modulus of the first sum equals

\[
\left| \sum_{n \in \mathbb{Z}} \frac{i + \bar{w}_n}{(x - n - \bar{w}_n)(x - n + i)} \right| \leq \sup_{n \in \mathbb{Z}} (|w_n| + 1) \sum_{n \in \mathbb{Z}} \frac{1}{|x - n + i| \cdot |x - n - \bar{w}_n|} \\
\leq \sup_{n \in \mathbb{Z}} (|w_n| + 1) \left( \sum_{n \in \mathbb{Z}} \frac{1}{(x - n)^2 + 1} \right)^{1/2} \cdot \left( \sum_{n \in \mathbb{Z}} \frac{1}{(x - n - u_n)^2 + v_n^2} \right)^{1/2}
\]

where we used the Cauchy inequality in the last inequality. Since \( \inf_n v_n^2 > 0 \), then both of the sums in the last inequality are bounded uniformly for each \( x \in \mathbb{R} \). Moreover, the sum

\[
\sum_{n \in \mathbb{Z}} \left( \frac{n + u_n}{(n + u_n)^2 + v_n^2} - \frac{n}{n^2 + 1} \right)
\]

does not depend on \( x \), and it converges. Therefore, \( E'/E \in L^\infty(\mathbb{R}) \).

The proof of the next corollary is a direct application of part (b) of Theorem 3.5.5.

**Corollary 3.5.6.** In Example 3.3, if \( w_n = u_n + iv_n \) where \( u_n \in [\alpha_1, \alpha_2] \) and \( v_n \in [a_1, a_2] \), \( a_1 > 0 \) for all \( n \in \mathbb{Z} \), then \( E'/E \in L^\infty(\mathbb{R}) \). and \( \varphi'(x) \) is bounded away from zero.
CHAPTER 4. Sampling and Interpolation in de Branges space

The main results of this chapter is to give some Beurling density conditions for sampling and interpolation in de Branges space, which have analogues in the setting of band-limited functions discussed in section 3.1. A sequence which is sampling in \( \mathcal{H}(E) \) is often easily shown to be a Plancherel-Pólya sequence, while the lower sampling bound, in Definition 2.5.1, is often more difficult to establish. We first prove a useful characterization of Plancherel-Pólya sequence in the space \( \mathcal{H}(E) \).

4.1 Plancherel-Pólya Sequences in de Branges Spaces

In this section we solve the problem of characterizing those separated sequences of real numbers \( \{\mu_n\}_{n\in\mathbb{Z}} \) for which the corresponding system of normalized reproducing kernels forms a Bessel sequence in the space \( \mathcal{H}(E) \).

Recall that a sequence \( \{\mu_n\}_{n\in\mathbb{Z}} \) of real numbers is a Plancherel-Pólya sequence in \( \mathcal{H}(E) \) if there exists a positive constant \( B \), independent of \( f \), such that

\[
\sum_{n\in\mathbb{Z}} \frac{|f(\lambda_n)|^2}{\|K(\lambda_n, \cdot)\|_{\mathcal{H}}^2} \leq B \|f\|_{\mathcal{H}}^2
\]

for all \( f \in \mathcal{H}(E) \), where \( K(w, z) \) is the reproducing kernel of \( \mathcal{H}(E) \).

The following two lemmas will be useful in the proofs.

**Lemma 4.1.1.** Let \( \mathcal{H}(E) \) be a de Brange space, and \( \varphi(x) \) be the corresponding phase function of \( E(z) \). Let \( \alpha \in [0, \pi) \), and \( \{\lambda_n\}_{n\in\mathbb{Z}} \) be a sequence of real numbers such that \( \varphi(\lambda_n) = \alpha + n\pi \), \( n \in \mathbb{Z} \). If \( 0 < \delta \leq \varphi'(x) \leq M \), for all \( x \in \mathbb{R} \), then

\[
\frac{\pi}{M} \leq \lambda_{n+1} - \lambda_n \leq \frac{\pi}{\delta}
\]

(4.1.1)

for all \( n \in \mathbb{Z} \).
Proof. Let \( n \in \mathbb{Z} \), then by the Mean Value Theorem there exist a point \( \nu_n \) between \( \lambda_n \) and \( \lambda_{n+1} \) such that
\[
\frac{\varphi(\lambda_{n+1}) - \varphi(\lambda_n)}{\lambda_{n+1} - \lambda_n} = \varphi'(\nu_n),
\]
hence,
\[
\lambda_{n+1} - \lambda_n = \frac{\varphi(\lambda_{n+1}) - \varphi(\lambda_n)}{\varphi'(\nu_n)}.
\]
Since \( \varphi(\lambda_n) = \alpha + n\pi \), then \( \varphi(\lambda_{n+1}) - \varphi(\lambda_n) = \pi \). Also, since \( \delta \leq \varphi'(x) \leq M \) for all \( x \in \mathbb{R} \), then
\[
\frac{\pi}{M} \leq \frac{\varphi(\lambda_{n+1}) - \varphi(\lambda_n)}{\varphi'(\nu_n)} \leq \frac{\pi}{\delta},
\]
Thus, \( \frac{\pi}{M} \leq \lambda_{n+1} - \lambda_n \leq \frac{\pi}{\delta} \), as desired. \( \square \)

Lemma 4.1.2. Let \( \mathcal{H}(E) \) be a de Brange space, and \( \varphi(x) \) be the corresponding phase function of \( E(z) \), and \( K(w, z) \) be the corresponding reproducing kernel. Let \( \alpha \in [0, \pi) \), and \( \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers such that \( \varphi(\lambda_n) = \alpha + n\pi \), \( n \in \mathbb{Z} \). Let \( f_n(z) = \frac{K(\lambda_n, z)}{E(\lambda_n)} \), \( n \in \mathbb{Z} \), \( z \in \mathbb{C} \). If \( 0 < \delta \leq \varphi'(x) \leq M \), for all \( x \in \mathbb{R} \), then
\[
\frac{\delta}{\pi} \| f \|^2_E \leq \sum_n |\langle f, f_n \rangle|^2 \leq \frac{M}{\pi} \| f \|^2_E,
\]
for all \( f \in \mathcal{H}(E) \), i.e., the sequence \( \{ f_n \}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{H}(E) \).

Proof. Let \( f \in \mathcal{H}(E) \). Since \( \varphi(\lambda_n) = \alpha + n\pi \), \( n \in \mathbb{Z} \), then by Theorem 3.4.1 the corresponding normalized reproducing kernels \( \{ \frac{K(\lambda_n, \cdot)}{\| K(\lambda_n, \cdot) \|} \} \) is an orthonormal basis in \( \mathcal{H}(E) \), thus we have
\[
\sum_n \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)} = \sum_n \left| \langle f, \frac{K(\lambda_n, \cdot)}{\| K(\lambda_n, \cdot) \|} \rangle \right|^2 = \| f \|^2_E.
\]
Using the fact that \( K(x, x) = \frac{1}{\pi} \varphi'(x) |E(x)|^2 \) for all \( x \in \mathbb{R} \), and that \( \varphi'(x) \leq M \), we obtain
\[
\sum_n |\langle f, f_n \rangle|^2 = \sum_n \left| \langle f(t), \frac{K(\lambda_n, t)}{E(\lambda_n)} \rangle \right|^2
\]
\[
= \sum_n \frac{|f(\lambda_n)|^2}{\| E(\lambda_n) \|^2}
\]
\[
= \sum_n \frac{\pi |f(\lambda_n)|^2}{\varphi'(\lambda_n) \| E(\lambda_n) \|^2} \frac{\varphi'(\lambda_n)}{\pi}
\]
\[
= \sum_n \frac{|f(\lambda_n)|^2 \varphi'(\lambda_n)}{\pi K(\lambda_n, \lambda_n)}
\]
\[
\sum_n \left| \langle f_n, f \rangle \right|^2 = \sum_n \frac{|f(\lambda_n)|^2 \varphi'(\lambda_n)}{K(\lambda_n, \lambda_n)} \geq \frac{\delta}{\pi} \sum_n \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)} = \frac{\delta}{\pi} \|f\|_E^2.
\]

Similarly, since \(0 < \delta \leq \varphi'(x)\), we also get

\[
\sum_n \left| \langle f_n, f \rangle \right|^2 = \sum_n \frac{|f(\lambda_n)|^2 \varphi'(\lambda_n)}{K(\lambda_n, \lambda_n)} \geq \frac{\delta}{\pi} \sum_n \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)} = \frac{\delta}{\pi} \|f\|_E^2.
\]

Since \(f\) is arbitrary, then the sequence \(\{f_n\}_{n \in \mathbb{Z}}\) is a frame for \(\mathcal{H}(E)\), completing the proof. \(\square\)

We now state and proof some charactrizations of Plancherel-Pólya sequences in the space \(\mathcal{H}(E)\).

**Theorem 4.1.3.** Let \(\mathcal{H}(E)\) be a de Branges space where \(E\) has no real zeros, \(\frac{E'}{E} \in L^\infty(\mathbb{R})\), and \(\varphi'(x)\) is bounded away from zero. Let \(\{\lambda_n\}_{n \in \mathbb{Z}}\), \(\{\mu_n\}_{n \in \mathbb{Z}}\) be two sequences of real numbers, such that \(\varphi(\lambda_n) = \alpha + n\pi\) for all \(n \in \mathbb{Z}\). If \(\lambda_n \leq \mu_n \leq \lambda_{n+1}\), for all \(n \in \mathbb{Z}\), then \(\{\mu_n\}_{n \in \mathbb{Z}}\) is a Plancherel-Pólya sequence in \(\mathcal{H}(E)\).

**Proof.** Since \(\varphi'\) is bounded away from zero, then there exist \(\delta > 0\) such that \(\varphi'(x) \geq \delta\), for all \(x \in \mathbb{R}\). Also, since \(\frac{E'}{E} \in L^\infty(\mathbb{R})\) then, by Lemma 3.5.4, \(\varphi'(x) \leq M\) for all \(x \in \mathbb{R}\), for some \(M > 0\). Let \(\lambda_n \leq \mu_n \leq \lambda_{n+1}\), and \(\varphi(\lambda_n) = \alpha + n\pi\) for all \(n \in \mathbb{Z}\). Then by Lemma (4.1.1) we have \(\lambda_{n+1} - \lambda_n \leq \frac{\pi}{\delta}\). Consequently, \(\max_n |\mu_n - \lambda_n| \leq \rho \leq \frac{\pi}{\delta}\).

Set \(f_n(z) = \frac{K(\lambda_n, z)}{E(\lambda_n)}\), and \(g_n(z) = \frac{K(\mu_n, z)}{E(\mu_n)}\), \(n \in \mathbb{Z}\). We need to show that there exist a constant \(B_\mu > 0\), such that

\[
\sum_n \left| \langle f, \frac{K(\mu_n, \cdot)}{K(\mu_n, \cdot)} \rangle \right|^2 \leq B_\mu \|f\|^2
\]

for all \(f \in \mathcal{H}(E)\).

To begin with, note that given \(f \in \mathcal{H}(E)\), the function \(f(t)/E(t)\) is continuous and differentiable for all \(t \in \mathbb{R}\). Hence, using Hölder inequality we get

\[
|\langle f, g_n - f_n \rangle|^2 = \left| \langle f, \frac{K(\mu_n, \cdot)}{E(\mu_n)} - \frac{K(\lambda_n, \cdot)}{E(\lambda_n)} \rangle \right|^2
\]
\[ = \left| \left\langle f, \frac{K(\mu_n, \cdot)}{E(\mu_n)} \right\rangle - \left\langle f, \frac{K(\lambda_n, \cdot)}{E(\lambda_n)} \right\rangle \right|^2 \]
\[ = \left| \frac{f(\mu_n)}{E(\mu_n)} - \frac{f(\lambda_n)}{E(\lambda_n)} \right|^2 \]
\[ = \left| \int_{\mu_n}^{\lambda_n} \left( \frac{f(t)}{E(t)} \right)' dt \right|^2 \]
\[ \leq \int_{\mu_n}^{\lambda_n} \left( \frac{f(t)}{E(t)} \right)'^2 dt \cdot \int_{\mu_n}^{\lambda_n} 1 dt \]
\[ \leq \max_n |\mu_n - \lambda_n| \int_{\mu_n}^{\lambda_n} \left( \frac{f(t)}{E(t)} \right)'^2 dt \]
\[ \leq \rho \int_{\mu_n}^{\lambda_n+1} \left( \frac{f(t)}{E(t)} \right)'^2 dt \]

Hence, by Lemma 3.5.2 we get

\[ \sum_n |\langle f, g_n - f_n \rangle|^2 \leq \rho \sum_n \int_{\mu_n}^{\lambda_n+1} \left( \frac{f(t)}{E(t)} \right)'^2 dt \]
\[ \leq \rho \int_{-\infty}^{\infty} \left( \frac{f(t)}{E(t)} \right)'^2 dt \]
\[ \leq 2\rho \left( C^2 \left\| \frac{E'}{E} \right\|_{\infty}^2 \left\| f \right\|_{E}^2 + \left\| \frac{E'}{E} \right\|_{\infty}^2 \left\| f \right\|_{E}^2 \right) \]

Therefore,

\[ \sum_n |\langle f, g_n - f_n \rangle|^2 \leq R \left\| f \right\|_{E}^2 \tag{4.1.3} \]

where \( \rho \leq \pi/\delta \), and \( R = 2\rho(C^2 + 1)\left\| \frac{E'}{E} \right\|_{\infty}^2 \). Since \( \varphi(\lambda_n) = \alpha + n\pi \) for all \( n \in \mathbb{Z} \), then by Lemma 4.1.2, the sequence \( \{f_n\}_{n \in \mathbb{Z}} \) is a frame with frame bounds \( \delta/\pi \) and \( M/\pi \). Therefore, Minkowski inequality implies

\[ \left( \sum_n |\langle f, g_n \rangle|^2 \right)^{\frac{1}{2}} \leq \left( \sum_n |\langle f, g_n - f_n \rangle|^2 \right)^{\frac{1}{2}} + \left( \sum_n |\langle f, f_n \rangle|^2 \right)^{\frac{1}{2}} \]
\[ \leq \sqrt{R} \left\| f \right\| + \sqrt{\frac{M}{\pi}} \left\| f \right\| \]
\[ = \left( \sqrt{R} + \sqrt{\frac{M}{\pi}} \right) \left\| f \right\| \]

for every \( f \in \mathcal{H}(E) \). It follows that

\[ \sum_n \left| \left\langle f, \frac{K(\mu_n, \cdot)}{\|K(\mu_n, \cdot)\|} \right\rangle \right|^2 = \sum_n \frac{|f(\mu_n)|^2}{K(\mu_n, \mu_n)} \]
\[ \sum_n \pi |f(\mu_n)|^2 \varphi'(\mu_n)|E(\mu_n)|^2 \]
\[ \leq \frac{\pi}{\delta} \sum_n |f(\mu_n)|^2 |E(\mu_n)|^2 \]
\[ = \frac{\pi}{\delta} \sum_n |\langle f, g_n \rangle|^2 \]
\[ \leq \frac{\pi}{\delta} (\sqrt{R} + \sqrt{\frac{M}{\pi}})^2 \|f\|_E^2 \]  
(4.1.4)

for all \( f \in \mathcal{H}(E) \). That is, the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) is a Plancherel-Pólya sequence in \( \mathcal{H}(E) \), with bound (at most) \( B_\mu = \frac{\pi}{\delta} (\sqrt{R} + \sqrt{\frac{M}{\pi}})^2 \), completing the proof.

\[ \square \]

**Proposition 4.1.4** (Perturbation of Plancherel-Pólya Sequence in \( \mathcal{H}(E) \)).

Let \( \mathcal{H}(E) \) be a de Branges space where \( E \) has no real zeros, \( \frac{E'}{E} \in L^\infty(\mathbb{R}) \), and \( 0 < \delta \leq \varphi'(x) \) for all \( x \in \mathbb{R} \). Let \( \mathcal{N} = \{\nu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be a \( \delta_0 \)-uniformly separated sequence. Let

\[ \mathcal{M} := \{\nu_n + \epsilon_n : \epsilon_n \in [-\eta, \eta], n \in \mathbb{Z}\}, \]

where \( 0 < \eta < \delta_0/2 \). If \( \mathcal{N} \) is a Plancherel-Pólya sequence in \( \mathcal{H}(E) \) with bound \( B_\nu \), then \( \mathcal{M} \) is also a Plancherel-Pólya sequence in \( \mathcal{H}(E) \) with bound \( B_\mu = B_\mu(B_\nu, \eta) \).

**Proof.** Let \( \mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \), then \( |\nu_n - \mu_n| = |\epsilon_n| \leq \eta \), for all \( n \in \mathbb{Z} \). Let \( \mathcal{M}_1 = \{\mu_n \in \mathcal{M} : \epsilon_n \geq 0\} \) and \( \mathcal{M}_2 = \{\mu_n \in \mathcal{M} : \epsilon_n < 0\} \), then \( \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \). Since the union of Plancherel-Pólya sequences is again such sequence, it is enough to show that \( \mathcal{M}_1 \) is Plancherel-Pólya sequences in \( \mathcal{H}(E) \) (the same proof will apply for \( \mathcal{M}_2 \)). Therefore, without loss of generality we may assume that \( \epsilon_n \geq 0 \) for all \( n \).

First note that since \( \frac{E'}{E} \in L^\infty(\mathbb{R}) \), then \( \varphi'(x) \leq M \) for all \( x \in \mathbb{R} \) by Lemma 3.5.4. Set \( f_n(z) = \frac{K(\nu_n, z)}{E(\nu_n)} \), and \( g_n(z) = \frac{K(\mu_n, z)}{E(\mu_n)} \), \( n \in \mathbb{Z} \). Let \( f \in \mathcal{H}(E) \). Following the same computations in the proof of Theorem 4.1.3 we get

\[ \sum_n |\langle f, g_n - f_n \rangle|^2 \leq 2 \max |\nu_n - \mu_n| \left( C^2 \left\| \frac{E'}{E} \right\|_\infty^2 \|f\|_E^2 + \left\| \frac{E'}{E} \right\|_\infty^2 \|f\|_E^2 \right) \]
\[ \leq R \|f\|_E^2 \]
where \( R = 2\eta(C^2 + 1)\| E \|_\infty^2 \).

Since \( N \) is a Plancherel-Pólya sequence in \( \mathcal{H}(E) \) with bound \( B_\nu \), then
\[
\sum_n |f(\nu_n)|^2 = \sum_n |\langle f, K(\nu_n, \nu_n) \rangle|^2 \leq B_\nu \| f \|^2.
\]

Hence, by inequality (4.1.2) we have
\[
\sum_n |\langle f, f_n \rangle|^2 \leq \frac{M}{\pi} \sum_n |f(\nu_n)|^2 \leq \frac{M}{\pi} B_\nu \| f \|^2.
\]

Therefore,
\[
\left( \sum_n |\langle f, g_n \rangle|^2 \right)^{\frac{1}{2}} \leq \left( \sum_n |\langle f, g_n - f_n \rangle|^2 \right)^{\frac{1}{2}} + \left( \sum_n |\langle f, f_n \rangle|^2 \right)^{\frac{1}{2}} \leq \sqrt{R} \| f \| + \sqrt{\frac{M}{\pi} B_\nu} \| f \|
\]
\[
= \left( \sqrt{R} + \sqrt{\frac{M}{\pi} B_\nu} \right) \| f \|
\]

Let \( B = \sqrt{R} + \sqrt{\frac{M}{\pi} B_\nu} \). Inequality (4.1.4) implies that
\[
\sum_n \left| \langle f, K(\mu_n, \mu_n) \rangle \right|^2 \leq \frac{\pi}{\delta} \sum_n |\langle f, g_n \rangle|^2 \leq \frac{\pi}{\delta} B^2 \| f \|^2.
\]

Since \( f \in \mathcal{H}(E) \) is arbitrary, this implies that the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) is a Plancherel-Pólya sequence in \( \mathcal{H}(E) \), with bound (at most) \( B_\mu = \frac{\pi}{\delta} B^2 \), completing the proof.

The next proposition, which relates the phase function \( \varphi \) of \( E(z) \) and the Beurling densities of a given sequence, will be useful in the proofs:

**Proposition 4.1.5.** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a phase function of \( E(z) \) satisfying \( 0 < \delta \leq \varphi'(x) \leq M \), for all \( x \in \mathbb{R} \). Let \( \mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R} \). Then
\[
\frac{1}{M} D^-(\mathcal{M}) \leq D^-(\varphi(\mathcal{M})) \leq \frac{1}{\delta} D^-(\mathcal{M}) \tag{4.1.5}
\]
and
\[
\frac{1}{M} D^+(\mathcal{M}) \leq D^+(\varphi(\mathcal{M})) \leq \frac{1}{\delta} D^+(\mathcal{M}) \tag{4.1.6}
\]
Proof. Let \( r > 0 \), first we will show that

\[
[\varphi(x - r), \varphi(x + r)] \subseteq [\varphi(x) - Mr, \varphi(x) + Mr]
\]  \hspace{1cm} (4.1.7)

for all \( x \in \mathbb{R} \), or equivalently, \( \varphi(x) - Mr \leq \varphi(x - r) \leq \varphi(x + r) \leq \varphi(x) + Mr \) for all \( x \in \mathbb{R} \). To begin with, let \( x \in \mathbb{R} \), then since \( \varphi \) is continuously differentiable on \( \mathbb{R} \) we have

\[
\varphi(x + r) = \varphi(x) + \int_{x}^{x+r} \varphi'(x) \, dx
\]

\[
\leq \varphi(x) + \int_{x}^{x+r} M \, dx
\]

\[
= \varphi(x) + Mr
\]

hence, \( \varphi(x + r) \leq \varphi(x) + Mr \). On the other hand, we have

\[
\varphi(x) = \varphi(x - r + r)
\]

\[
= \varphi((x - r) + r)
\]

\[
\leq \varphi(x - r) + Mr, \quad \text{(by applying the result above for } x - r),
\]

and we get \( \varphi(x) - Mr \leq \varphi(x - r) \). Since \( \varphi \) is a nondecreasing function and \( r > 0 \), then \( \varphi(x - r) < \varphi(x + r) \). Therefore, \( \varphi(x) - Mr \leq \varphi(x - r) \leq \varphi(x + r) \leq \varphi(x) + Mr \) for all \( x \in \mathbb{R} \).

Using the fact that \( \varphi \) is bijective and relation (4.1.7), we get

\[
\#(\mathcal{M} \cap [x - r, x + r]) = \#(\varphi(\mathcal{M}) \cap [\varphi(x - r), \varphi(x + r)])
\]

\[
\leq \#(\varphi(\mathcal{M}) \cap [\varphi(x) - Mr, \varphi(x) + Mr])
\]

for all \( x \in \mathbb{R} \). Hence,

\[
\inf_{x \in \mathbb{R}} \frac{\#(\mathcal{M} \cap [x - r, x + r])}{Mr} \leq \inf_{x \in \mathbb{R}} \frac{\#(\varphi(\mathcal{M}) \cap [\varphi(x) - Mr, \varphi(x) + Mr])}{Mr}
\]

\[
= \inf_{y \in \mathbb{R}} \frac{\#(\varphi(\mathcal{M}) \cap [y - Mr, y + Mr])}{Mr}
\]

for all \( r > 0 \). Taking liminf as \( r \to \infty \) yields

\[
\frac{1}{M} D^- (\mathcal{M}) = \liminf_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\#(\mathcal{M} \cap [x - r, x + r])}{Mr} \leq \liminf_{r \to \infty} \inf_{y \in \mathbb{R}} \frac{\#(\varphi(\mathcal{M}) \cap [y - Mr, y + Mr])}{Mr}
\]

\[
= D^- (\varphi(\mathcal{M})).
\]
Again, let $r > 0$, we will show that

$$[\varphi(x) - \delta r, \varphi(x) + \delta r] \subseteq [\varphi(x - r), \varphi(x + r)]$$

(4.1.8)

for all $x \in \mathbb{R}$, or equivalently, $\varphi(x - r) \leq \varphi(x) - \delta r \leq \varphi(x) + \delta r \leq \varphi(x + r)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$, then

$$\varphi(x + r) = \varphi(x) + \int_x^{x+r} \varphi'(x)dx$$

$$\geq \varphi(x) + \int_x^{x+r} \delta dx$$

$$= \varphi(x) + \delta r$$

hence, $\varphi(x + r) \geq \varphi(x) + \delta r$. On the other hand, we have

$$\varphi(x) = \varphi(x - r + r)$$

$$= \varphi((x - r) + r)$$

$$\geq \varphi(x - r) + \delta r,$$

and we get $\varphi(x) - \delta r \geq \varphi(x - r)$. Therefore, $\varphi(x - r) \leq \varphi(x) - \delta r \leq \varphi(x) + \delta r \leq \varphi(x + r)$ for all $x \in \mathbb{R}$.

Again, using the fact that $\varphi$ is bijective, and relation (4.1.8), we get

$$\sharp(M \cap [x - r, x + r]) = \sharp((M) \cap [\varphi(x - r), \varphi(x + r)])$$

$$\geq \sharp((M) \cap [\varphi(x) - \delta r, \varphi(x) + \delta r])$$

for all $x \in \mathbb{R}$. Hence,

$$\inf_{x \in \mathbb{R}} \frac{\sharp(M \cap [x - r, x + r])}{\delta r} \geq \inf_{x \in \mathbb{R}} \frac{\sharp((M) \cap [\varphi(x) - \delta r, \varphi(x) + \delta r])}{\delta r}$$

$$\equiv \inf_{y \in \mathbb{R}} \frac{\sharp((M) \cap [y - \delta r, y + \delta r])}{\delta r}$$

for all $r > 0$. Taking liminf as $r \to \infty$ yields

$$\frac{1}{\delta} D^-(M) = \lim_{r \to \infty} \inf_{x \in \mathbb{R}} \frac{\sharp(M \cap [x - r, x + r])}{\delta r} \geq \lim_{r \to \infty} \inf_{y \in \mathbb{R}} \frac{\sharp((M) \cap [y - \delta r, y + \delta r])}{\delta r}$$

$$= D^-(\varphi(M))$$
Similar computations show that
\[
\frac{1}{M} D^+(M) \leq D^+(\varphi(M)) \leq \frac{1}{\delta} D^+(M)
\]

The next theorem gives sufficient conditions for a separated sequence to be a Plancherel-Pólya sequence in $\mathcal{H}(E)$ in terms of the upper Beurling density of the sequence.

**Theorem 4.1.6.** Given a de Brange space $\mathcal{H}(E)$ with $E$ has no real zeros, $\frac{E'}{E} \in L^\infty(\mathbb{R})$, and the derivative of the corresponding phase function of $E$ is bounded away from zero. Let $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. If $D^+(\mathcal{M}) < \infty$, then $\mathcal{M}$ is a Plancherel-Pólya sequence in $\mathcal{H}(E)$.

**Proof.** Assume that $\frac{E'}{E} \in L^\infty(\mathbb{R})$, then by Lemma 3.5.4 there is a constant $M > 0$ such that $\varphi'(x) \leq M < \infty$, for all $x \in \mathbb{R}$. Also, since $\varphi'$ is bounded away from zero on $\mathbb{R}$, then there exist $\delta > 0$ such that $\delta \leq \varphi'(x)$, for all $x \in \mathbb{R}$. By Theorem 3.4.1 we can find a sequence $\{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$, such that $\varphi(\lambda_n) = \alpha + n\pi$ for all $n \in \mathbb{Z}$, for some $\alpha \in [0, \pi)$ and the corresponding normalized reproducing kernels is an orthonormal basis in $\mathcal{H}(E)$. Set $k_{\lambda_n}(z) = \frac{K(\lambda_n, z)}{||K(\lambda_n, \cdot)||}$, and $k_{\mu_n}(z) = \frac{K(\mu_n, z)}{||K(\mu_n, \cdot)||}$, $n \in \mathbb{Z}$. We need to show that there is some constant $B_\mu > 0$, such that

\[
\sum_{n \in \mathbb{Z}} |\langle f, k_{\mu_n} \rangle|^2 \leq B_\mu ||f||^2, \quad \text{for every } f \in \mathcal{H}(E).
\]

Since $D^+(\mathcal{M}) < \infty$, then $D^+(\varphi(\mathcal{M})) < \infty$ by Proposition 4.1.5. Lemma 2.5.3 implies that the number of points of the sequence $\{\varphi(\mathcal{M})\}$ in any interval of a given finite length is bounded, that is, given $R > 0$ there exist an integer $N_R > 0$ such that

\[
\sup_{y \in \mathbb{R}} \sharp(\{\varphi(\mathcal{M})\} \cap [y, y + R]) \leq \sup_{y \in \mathbb{R}} \sharp(\{\varphi(\mathcal{M})\} \cap [y - R, y + R]) \leq N_R < \infty.
\]

In particular, for $R = \pi$, then there exist $N_\pi \in \mathbb{N}$, such that

\[
\sharp(\{\varphi(\mathcal{M})\} \cap [\alpha + k\pi, \alpha + (k + 1)\pi]) \leq N_\pi, \quad \text{for all } k \in \mathbb{Z}
\]

or equivalently,

\[
\sharp(\{\varphi(\mathcal{M})\} \cap [\varphi(\lambda_k), \varphi(\lambda_{k+1})]) \leq N_\pi, \quad \text{for all } k \in \mathbb{Z}
\]
Since the function $\varphi$ is bijective we can trace the points $\varphi(\mu_n)$ back to get
\[
\sharp(\mathcal{M} \cap [\lambda_k, \lambda_{k+1}]) \leq N_{\pi}, \quad \text{for all } k \in \mathbb{Z}.
\]

This means that we can partition the sequence $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ into a finite number of (disjoint) subsequences $\mathcal{M}_j$ in a way such that for each $1 \leq j \leq N_{\pi}$ there is at most one point of the sequence $\mathcal{M}_j$ in $[\lambda_k, \lambda_{k+1}]$, for all $k \in \mathbb{Z}$:
\[
\mathcal{M} = \bigcup_{j=1}^{N_{\pi}} \mathcal{M}_j, \quad \mathcal{M}_j := \{\mu_i^{(j)}\}_{i \in \mathbb{Z}}, \quad j = 1, 2, \ldots, N_{\pi}
\]

For $j \in \{1, 2, \ldots, N_{\pi}\}$, define the index set $I_j := \{k_i \in \mathbb{Z} : \sharp(\mathcal{M}_j \cap [\lambda_k, \lambda_{k+1}]) = 1\}$. Then the sequences $\mathcal{M}_j$ and $\Lambda_j := \{\lambda_{k_i}\}_{k_i \in I_j}$ are interlaced. Since $\varphi(\lambda_{k_i}) = \alpha + k_i \pi$ for all $k_i \in I_j$, then
\[
\sum_{k_i \in I_j} |\langle f, k_{\lambda_{k_i}} \rangle|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, k_{\lambda_n} \rangle|^2 = \|f\|^2.
\]

Therefore, applying Theorem 4.1.3 for the sequences $\mathcal{M}_j$ and $\Lambda_j$, $j = 1, 2, \ldots, N_{\pi}$, implies that the sequence $\mathcal{M}_j$ is a Plancherel-Pólya sequence for $\mathcal{H}(E)$, with bound at most $B_{\mu} := \frac{\pi}{\delta} \left(\sqrt{\frac{2\pi}{\delta}} \sqrt{C^2 + 1} \|E'\|_{\infty} + \sqrt{\frac{M}{\pi}}\right)^2$, for every $j = 1, 2, \ldots, N_{\pi}$, where $C$ is the Bernstein inequality constant. Hence, the sequence $\{\mu_n\}_{n \in \mathbb{Z}}$ is a Plancherel-Pólya sequence for $\mathcal{H}(E)$, with bound at most $N_{\pi}B_{\mu}$. \hfill \Box

The following lemmas show that in many cases we can restrict our considerations to spaces $\mathcal{H}(E)$ with $E(x) \neq 0$ for all $x \in \mathbb{R}$. For suppose that $E(x_o) = 0$ for some $x_o \in \mathbb{R}$, then Proposition 3.2.2 shows that every function $f \in \mathcal{H}(E)$ must have a zero at $x_o$ with multiplicity greater than or equal to that of $E$. This implies that we can divide all functions in the space by $(z - x_o)$ and get a new de Branges space $\mathcal{H}(E_1)$ where $E(z) = (z - x_o)E_1(z)$, and $E_1 \in \mathcal{HB}$. The reproducing kernel for the new space, as we will show in the next lemma, is given by $K_{E_1}(w, z) = \frac{K_{E}(w, z)}{(z - x_o)(w - x_o)}$. Continuing in this way we can assume that that $E_1(z)$ has no real zeros. Moreover, we will show that the two spaces $\mathcal{H}(E)$ and $\mathcal{H}(E_1)$ share the sampling and interpolation properties.
Lemma 4.1.7. Given two entire functions \( S, E_1 \) with \( E_1 \in \mathcal{HB} \), and \( S(z) \) real for real \( z \), then \( SE_1 \in \mathcal{HB} \), and

\[
\mathcal{H}(SE_1) = S\mathcal{H}(E_1) = \{ S(z)f(z) : f(z) \in \mathcal{H}(E_1) \} \tag{4.1.9}
\]

with reproducing kernel

\[
K_{SE_1}(w, z) = \bar{S}(w)S(z)K_{E_1}(w, z), \text{ for all } w, z \in \mathbb{C}. \tag{4.1.10}
\]

Proof. First we note that \( S(z)E_1(z) \) is an entire function. Since \( S(z) \) is real for real \( z \), then Lemma 2.3.1 implies that \( S^*(z) = S(z) \), or equivalently, \( S(\bar{z}) = \bar{S}(z) \) for all \( z \in \mathbb{C} \). Hence, for all \( z \in \mathbb{C}^+ \)

\[
|\!(SE_1)(\bar{z})| = |S(z)||E_1(\bar{z})| = |S(z)||E_1(\bar{z})| < |S(z)||E_1(z)| = |(SE_1)(z)|,
\]

Thus, \( |(SE_1)(\bar{z})| < |(SE_1)(z)| \) for all \( z \in \mathbb{C}^+ \). Therefore, \( SE_1 \in \mathcal{HB} \), and hence, the space \( \mathcal{H}(SE_1) \) is de Branges space.

Let \( \mathcal{H}(E_1), \mathcal{H}(SE_1) \) denote the de Branges spaces associated with \( E_1 \) and \( SE_1 \), respectively, and \( K_{E_1}(w, z), K_{SE_1}(w, z) \) the corresponding reproducing kernels of \( \mathcal{H}(E_1) \) and \( \mathcal{H}(SE_1) \), respectively. Hence, using the definition of the reproducing kernel in (3.2.3), we have for all \( z, w \in \mathbb{C} \)

\[
K_{SE_1}(w, z) = \frac{\bar{S}(w)E_1(\bar{w})S(z)E_1(z) - \bar{S}(w)E_1(\bar{w})S^*(z)E^*_1(z)}{2\pi i(\bar{w} - z)} = \frac{S(\bar{w})\bar{E}_1(\bar{w})S(z)E_1(z) - S(\bar{w})E_1(\bar{w})S(z)E^*_1(z)}{2\pi i(\bar{w} - z)} = \frac{S(\bar{w})S(z)E_1(\bar{w})E_1(z) - E_1(\bar{w})E^*_1(z)}{2\pi i(\bar{w} - z)},
\]

therefore,

\[
K_{SE_1}(w, z) = \bar{S}(w)S(z)K_{E_1}(w, z), \ \forall w, z \in \mathbb{C}. \tag{4.1.11}
\]

Given any \( g \in \mathcal{H}(SE_1) \), the function \( f(z) := g(z)/S(z) \) belongs to the space \( \mathcal{H}(E_1) \). Indeed, note that

\[
\int_{\mathbb{R}} \left| \frac{f(t)}{E_1(t)} \right|^2 dt = \int_{\mathbb{R}} \frac{|g(t)|^2}{|S(t)|^2|E_1(t)|^2} dt = \int_{\mathbb{R}} \left| \frac{g(t)}{SE_1(t)} \right|^2 dt = \|g\|^2_{SE_1} < \infty, \tag{4.1.12}
\]
hence, \(\|f\|_{E_1} = \|g\|_{SE_1}\). Moreover, using the fact that \(|g(z)|^2 \leq \|g\|_{SE_1}^2 K_{SE_1}(z, z)\) for all \(z \in \mathbb{C}\), and formula (4.1.11), then

\[
|f(z)|^2 = \left| \frac{g(z)}{S(z)} \right|^2 \leq \frac{1}{|S(z)|^2} \|g\|_{SE_1}^2 K_{SE_1}(z, z) = \|Sf\|_{SE_1}^2 K_{E_1}(z, z).
\]

It follows that \(f \in \mathcal{H}(E_1)\) by Theorem 3.3.1. Let \(w \in \mathbb{C}\), using the reproducing kernel property (3.2.4) we get

\[
g(w) = \langle g(t), K_{SE_1}(w, t) \rangle_{SE_1} = \int \frac{g(t)K_{SE_1}(w, t)}{|S(t)E_1(t)|^2} dt = \int \frac{g(t)S(w)S(t)K_{E_1}(w, t)}{|S(t)E_1(t)|^2} dt = S(w) \int \frac{(g(t)/S(t))K_{E_1}(w, t)}{|E_1(t)|^2} dt = S(w) \langle \frac{g(t)}{S(t)}, K_{E_1}(w, t) \rangle_{E_1} = S(w) \langle \frac{S}{g}, f \rangle_E = S(w) f(w).
\]

Therefore, any \(g \in \mathcal{H}(SE_1)\) can be written as \(g(z) = S(z)f(z)\), for some \(f \in \mathcal{H}(E_1)\). Conversely, for any \(f \in \mathcal{H}(E_1)\), the function \(Sf\) belongs to the space \(\mathcal{H}(SE_1)\). Indeed,

\[
\|Sf\|_{SE_1}^2 = \int \frac{|S(t)f(t)|^2}{|S(t)E_1(t)|^2} dt = \int \frac{|f(t)|^2}{|E_1(t)|^2} dt = \|f\|_{E_1}^2 < \infty, \quad (4.1.13)
\]

also by (4.1.11) we get

\[
|S(z)f(z)|^2 \leq |S(z)|^2 \|f\|_{E_1}^2 K_{E_1}(z, z) = \|Sf\|_{SE_1}^2 K_{SE_1}(z, z),
\]
for all \( z \in \mathbb{C} \). Thus, the space \( \mathcal{H}(SE_1) \) is well defined and given by (4.1.9), with reproducing kernel defined in (4.1.10).

Conversely, any \( E \in \mathcal{HB} \) can be factored out into a product of two entire functions \( E_1(z) \in \mathcal{HB} \), has no real zeros, and \( S(z) \), real for real \( z \) and has only real zeros.

**Lemma 4.1.8.** Let \( \mathcal{H}(E) \) be a de Branges space. Then \( E(z) = S(z)E_1(z) \) where \( E_1(z) \in \mathcal{HB} \), has no real zeros, and \( S(z) \) is an entire function, which is real for real \( z \), has only real zeros, and the zeros of \( E_1 \) are exactly the zeros of \( E \) in the lower half-plane. Moreover, the map \( f(z) \mapsto S(z)f(z) \) is an isometric transformation of \( \mathcal{H}(E_1) \) onto \( \mathcal{H}(E) \)

**Proof.** Since \( E(z) \in \mathcal{HB} \) then by Theorem 2.3.2 \( E \) has the following representation

\[
E(z) = \gamma S(z)e^{-iaz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{(P_n(z)+P_n^*(z))/2}, \tag{4.1.14}
\]

where \( \gamma \in \mathbb{C} \), with \( |\gamma| = 1 \), \( S(z) \) is an entire function that is real on the real line and has only real zeros, and \( \{z_n\} \) is the zeros set of \( E \) in the open lower half-plane. Denote the product of factors in (4.1.14) except \( S(z) \), by \( E_1(z) \). Thus, \( E(z) = S(z)E_1(z) \). Clearly, \( E_1(z) \) is entire function, and has no real zeros. Moreover, \( E_1 \in \mathcal{HB} \). Indeed, let \( z \in \mathbb{C}^+ \), then we know that \( |E(\bar{z})| < |E(z)| \). Since \( S(\bar{z}) = \overline{S(z)} \) by Lemma 2.3.1, then for all \( z \in \mathbb{C}^+ \)

\[
|S(z)||E_1(\bar{z})| = |S(\bar{z})E_1(\bar{z})| = |E(\bar{z})| < |E(z)| = |S(z)E_1(z)| = |S(z)||E_1(z)|.
\]

Therefore, \( |E_1(\bar{z})| < |E(z)| \) for all \( z \in \mathbb{C}^+ \), that is, \( E_1 \in \mathcal{HB} \), and hence, the space \( \mathcal{H}(E_1) \) is defined.

To prove the second part, let the map \( T : \mathcal{H}(E_1) \to \mathcal{H}(E) \) be such that \( T(f)(z) = S(z)f(z) \). Then, \( T \) is well-defined. For if \( f(z) \in \mathcal{H}(E_1) \), then \( S(z)f(z) \) is entire function, and

\[
\int_{\mathbb{R}} \left| \frac{S(t)f(t)}{E(t)} \right|^2 dt = \int_{\mathbb{R}} \left| \frac{S(t)}{E(t)} \right|^2 \left| f(t) \right|^2 dt = \int_{\mathbb{R}} \left| \frac{f(t)}{E_1(t)} \right|^2 dt = \|f\|^2_{E_1} < \infty.
\]

Hence, \( \|Sf\|_E = \|f\|_{E_1} \). Note also that \( T \) is linear. For all \( f, g \in \mathcal{H}(E_1) \)

\[
\langle Tf, Tg \rangle_E = \langle Sf, Sg \rangle_E = \int_{\mathbb{R}} \frac{S(t)f(t)S(t)g(t)}{|S(t)|^2|E_1(t)|^2} dt = \int_{\mathbb{R}} \left| \frac{f(t)g(t)}{E_1(t)} \right|^2 dt = \langle f, g \rangle_{E_1}.
\]

Same computations as in the proof of Lemma 4.1.7 show that

\[
K_E(w, z) = \bar{S}(w)S(z)K_{E_1}(w, z), \ \forall w, z \in \mathbb{C},
\]
where $K_E(w, z)$, $K_{E_1}(w, z)$ are the corresponding reproducing kernels of $\mathcal{H}(E)$ and $\mathcal{H}(E_1)$, respectively. Using the fact that $|f(z)|^2 \leq \|f\|_{E_1}^2 K_{E_1}(z, z)$ for all $z \in \mathbb{C}$, and formula (4.1.11), then

$$|(Sf)(z)|^2 = |S(z)|^2 |f(z)|^2 \leq |S(z)|^2 \|f\|_{E_1}^2 K_{E_1}(z, z) = \|Sf\|_{E_1}^2 |S(z)|^2 K_{E_1}(z, z) = \|Sf\|_{E_1}^2 K_E(z, z).$$

Hence, $Sf \in \mathcal{H}(E)$ for all $f \in \mathcal{H}(E_1)$ by Theorem 3.3.1. Again, similar computations as in the proof of Lemma (4.1.7) show that any function $g \in \mathcal{H}(E)$ can be written as $g(z) = S(z)f(z)$ for (unique) $f \in \mathcal{H}(E_1)$, completing the proof.

\[\square\]

**Lemma 4.1.9.** Suppose that $E_1(z) \in \mathcal{HB}$, and $S(z)$ is an entire function which is real for real $z$. Let $\{\mu_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Then $\{\mu_n\}_{n \in \mathbb{Z}}$ is a Plancherel-Pólya sequence in $\mathcal{H}(E_1)$ if and only if $\{\mu_n\}_{n \in \mathbb{Z}}$ is a Plancherel-Pólya sequence in $\mathcal{H}(SE_1)$. Moreover, the two bounds are equal.

**Proof.** First assume that $\{\mu_n\}_{n \in \mathbb{Z}}$ is a Plancherel-Pólya sequence in $\mathcal{H}(E_1)$. Then there exists a constant $B_1 > 0$ such that

$$\sum_n \left| \langle f, K_{E_1}(\mu_n, \cdot) \rangle_{E_1} \right|^2 \leq B_1 \|f\|^2_{E_1}$$

for all $f \in \mathcal{H}(E_1)$.

Given $g \in \mathcal{H}(SE_1)$, then by Lemma 4.1.7 there is some $f \in \mathcal{H}(E_1)$ such that $g = Sf$. Thus, using (4.1.11) we obtain

$$\sum_n \frac{|g(\mu_n)|^2}{\|K_{SE_1}(\mu_n, \cdot)\|^2} = \sum_n \frac{|\langle g(t), K_{SE_1}(\mu_n, t) \rangle_{SE_1} |^2}{\|K_{SE_1}(\mu_n, \cdot)\|^2} = \sum_n \frac{|\langle S(t)f(t), S(\mu_n)S(t)K_{E_1}(\mu_n, t) \rangle_{SE_1} |^2}{|S(\mu_n)|^2 K_{E_1}(\mu_n, \mu_n)} = \sum_n \frac{|\langle S(t)f(t), S(t)K_{E_1}(\mu_n, t) \rangle_{SE_1} |^2}{K_{E_1}(\mu_n, \mu_n)} = \sum_n \frac{|\langle f(t), K_{E_1}(\mu_n, t) \rangle_{E_1} |^2}{\|K_{E_1}(\mu_n, \cdot)\|^2_{E_1}}.$$
\[
\leq B_1 \|f\|^2_{E_1} \\
= B_1 \|g\|^2_{E_1}\]

where we used (4.1.12) in the last equality. Therefore, the sequence \(\{\mu_n\}_{n \in \mathbb{Z}}\) is a Plancherel-Pólya sequence for \(\mathcal{H}(SE_1)\) with bound \(B_1\).

Conversely, suppose that \(\{\mu_n\}_{n \in \mathbb{Z}}\) is a Plancherel-Pólya sequence for \(\mathcal{H}(SE_1)\). Then there exists \(B > 0\) such that
\[
\sum_n \frac{|\langle g(t), K_{SE_1}(\mu_n, t) \rangle_{SE_1}|^2}{\|K_{SE_1}(\mu_n, \cdot)\|^2_{SE_1}} \leq B \|g\|^2_{SE_1},
\]
for all \(g \in \mathcal{H}(SE_1)\). Let \(f \in \mathcal{H}(E_1)\), the function \(Sf \in \mathcal{H}(SE_1)\) by Lemma 4.1.7. Hence, by (4.1.11), (4.1.13) and (4.1.16) we get
\[
\sum_n \frac{|\langle (Sf)(t), K_{SE_1}(\mu_n, t) \rangle_{SE_1}|^2}{\|K_{SE_1}(\mu_n, \cdot)\|^2_{SE_1}} \leq B \|Sf\|^2_{SE_1} = B \|f\|^2_{E_1},
\]
and
\[
\sum_n \frac{|f(\mu_n)|^2}{\|K_{E_1}(\mu_n, \cdot)\|^2_{E_1}} = \sum_n \frac{|\langle f(t), K_{E_1}(\mu_n, t) \rangle_{E_1}|^2}{\|K_{E_1}(\mu_n, \cdot)\|^2_{E_1}} \\
= \sum_n \frac{|\langle f(t), \tilde{S}(t)\tilde{S}(t)K_{E_1}(\mu_n, t) \rangle_{SE_1}|^2}{\|K_{E_1}(\mu_n, \cdot)\|^2_{E_1}} \\
= \sum_n \frac{|\langle \tilde{S}(t)f(t), \tilde{S}(\mu_n)\tilde{S}(t)K_{E_1}(\mu_n, t) \rangle_{SE_1}|^2}{\|S(\mu_n)\|^2K_{E_1}(\mu_n, \mu_n)} \\
= \sum_n \frac{|\langle (Sf)(t), K_{SE_1}(\mu_n, t) \rangle_{SE_1}|^2}{\|K_{SE_1}(\mu_n, \cdot)\|^2_{SE_1}} \leq B \|Sf\|^2_{SE_1} = B \|f\|^2_{E_1}.
\]

Therefore, the sequence \(\{\mu_n\}_{n \in \mathbb{Z}}\) is a Plancherel-Pólya sequence for \(\mathcal{H}(E_1)\) with bound \(B\), completing the proof.

\[\square\]

**Remark** 4.1.1. Note that if \(E(z) = S(z)E_1(z)\), where \(E_1(z) \in \mathcal{HB}\) has no real zeros, and \(S(z)\) is entire function which is real for real \(z\), having only real zeros, then
\[
\frac{E'}{E} = \frac{S'}{S} + \frac{E_1'}{E_1}
\]

Hence, if \(S(z)\) has real zeros then \(E'/E \notin L^\infty(\mathbb{R})\). However, the above two lemmas implies that Theorem 4.1.3 and Theorem 4.1.6 still valid if we only require that \(E_1'/E_1 \in L^\infty(\mathbb{R})\) instead.
Therefore, in Theorem 4.1.3 and Theorem 4.1.6 we can drop the assumption that \( E \) has to be nonzero on the real line from the hypothesis.

Moreover, if \( \varphi_1(x) \) is the corresponding phase function of \( E_1(z) \), and \( \varphi(x) \) is the corresponding phase function of \( E(z) = S(z)E_1(z) \), then \( \varphi_1(x) \) and \( \varphi(x) \) differs in a constant. Indeed, since we know that

\[
K_{SE}(x, x) = S^2(x)K_{E_1}(x, x), \quad \text{and} \quad K_{E_1}(x, x) = \frac{1}{\pi} \varphi_1'(x)|E_1(x)|^2,
\]

and \( K_{SE}(x, x) = \frac{1}{\pi} \varphi'(x)|S(x)E_1(x)|^2 \), for all \( x \in \mathbb{R} \), then \( \varphi_1'(x) = \varphi'(x) \) for all \( x \in \mathbb{R} \). Therefore, using the definition of the upper Beurling density it follows that \( D^+ (\varphi(\{ \mu_n \})) = D^+ (\varphi_1(\{ \mu_n \})) \).

Similar computations as in the proof of Lemma 4.1.9 for the lower bound shows that the same is true in case of sampling (by replacing the “\( \leq \)” signs by “\( \geq \)” in the above proof).

**Lemma 4.1.10.** Given an entire functions \( E \in \mathcal{HB} \), with \( E(z) = S(z)E_1(z) \) where \( S(z) \) is real for real \( z \) and has only real zeros, and \( E_1(z) \) has no real zeros. Let \( \{ \mu_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers. Then \( \{ \mu_n \}_{n \in \mathbb{Z}} \) is a sampling sequence in \( \mathcal{H}(E) \) if and only if \( \{ \mu_n \}_{n \in \mathbb{Z}} \) is sampling sequence in \( \mathcal{H}(E_1) \).

**Lemma 4.1.11.** Given an entire functions \( E \in \mathcal{HB} \), with \( E(z) = S(z)E_1(z) \) where \( S(z) \) is real for real \( z \) and has only real zeros, and \( E_1(z) \) has no real zeros. Let \( \Gamma = \{ \gamma_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers. Then \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E) \) if and only if it is an interpolating sequence in \( \mathcal{H}(E_1) \).

**Proof.** Assume first that \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E_1) \). Let \( \{ c_n \} \) be a sequence such that

\[
\sum_n \frac{|c_n|^2}{K_E(\gamma_n, \gamma_n)} < \infty. \tag{4.1.18}
\]

We will find a function \( F(z) \in \mathcal{H}(E) \) which solves the interpolation problem \( F(\gamma_n) = c_n \) for all \( n \in \mathbb{Z} \). Note that if \( K_E(\gamma_n, \gamma_n) = 0 \) for some \( n \), then \( c_n \) must be zero in order that the sum in (4.1.18) be finite. Therefore, if \( S(\gamma_n) = 0 \) for some \( n \in \mathbb{Z} \), then by (4.1.10) we have \( K_E(\gamma_n, \gamma_n) = 0 \), and hence, \( c_n = 0 \).
Define the sequence \( a_n := \frac{c_n}{S(\gamma_n)} \) for all \( n \in \mathbb{Z} \). By the discussion above, if \( S(\gamma_n) = 0 \) for some \( n \) then we let \( a_n = 0 \). Using (4.1.10) we get
\[
\sum_n \frac{|a_n|^2}{K_{E_1}(\gamma_n, \gamma_n)} = \sum_n \frac{|c_n|^2}{S(\gamma_n)^2 K_{E_1}(\gamma_n, \gamma_n)} = \sum_n \frac{|c_n|^2}{K_E(\gamma_n, \gamma_n)} < \infty.
\]
Since \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E_1) \), there exist at least one function \( f_o \in \mathcal{H}(E_1) \) such that \( f_o(\gamma_n) = a_n \), for all \( n \). Let \( F(z) := S(z)f_o(z) \), then \( F \in \mathcal{H}(E) \) by Lemma 4.1.7, moreover, we have
\[
F(\gamma_n) = S(\gamma_n)f_o(\gamma_n) = S(\gamma_n)a_n = c_n
\]
for all \( n \). That is, \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E) \).

Conversely, assume that \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E_1) \). Let \( \{c_n\} \) be any sequence such that
\[
\sum_n \frac{|c_n|^2}{K_{E_1}(\gamma_n, \gamma_n)} < \infty.
\]
We will find a function \( f_o(z) \in \mathcal{H}(E_1) \) such that \( f_o(\gamma_n) = a_n \) for all \( n \in \mathbb{Z} \). Define a sequence \( a_n := c_nS(\gamma_n) \) for all \( n \in \mathbb{Z} \). Then using (4.1.10)
\[
\sum_n \frac{|a_n|^2}{K_E(\gamma_n, \gamma_n)} = \sum_n \frac{|c_n|^2|S(\gamma_n)|^2}{K_{E_1}(\gamma_n, \gamma_n)} = \sum_n \frac{|c_n|^2}{S(\gamma_n)^2 K_{E_1}(\gamma_n, \gamma_n)} = \sum_n \frac{|c_n|^2}{K_{E_1}(\gamma_n, \gamma_n)} < \infty
\]
Therefore, since \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E) \), there exist at least one function \( F \in \mathcal{H}(E) \) such that \( F(\gamma_n) = a_n \), for all \( n \). Since \( F \in \mathcal{H}(E) \) then by Lemma 4.1.7, there exist an entire function \( f_o \in \mathcal{H}(E_1) \), such that \( F(z) = S(z)f_o(z) \) for all \( z \in \mathbb{C} \), hence we have
\[
f_o(\gamma_n) = \frac{F(\gamma_n)}{S(\gamma_n)} = \frac{a_n}{S(\gamma_n)} = c_n
\]
for all \( n \). That is, \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E_1) \).
In this section we prove that the Homogeneous Approximation Property (HAP) holds for the reproducing kernel in $\mathcal{H}(E)$, we then use this result to prove the Comparison Theorem which will hold for frames formed by the reproducing kernels of the space. The Homogeneous Approximation Property and the Comparison Theorem were introduced by Ramanathan and Steger [42] in the context of Gabor frames. A version of the HAP for frames of translates of band-limited functions was proved by Gröchenig and Razafinjato in [21].

Let $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be a sequence such that the corresponding normalized reproducing kernels $k_{\gamma_n} = \frac{K(\gamma_n, \cdot)}{\|K(\gamma_n, \cdot)\|}$ form a Riesz basis in $\mathcal{H}(E)$, and $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ be such that the corresponding normalized reproducing kernels $k_{\mu_n} = \frac{K(\mu_n, \cdot)}{\|K(\mu_n, \cdot)\|}$ form a frame in $\mathcal{H}(E)$. Let $r > 0$, and $y \in \mathbb{R}$, and define the index sets

$$J_r(y) = \{n \in \mathbb{Z} : |\gamma_n - y| \leq r\}, \quad I_r(y) = \{n \in \mathbb{Z} : |\mu_n - y| \leq r\}$$

Define the subspaces

$$V_r(y) := \text{span}\{k_{\gamma_n} : n \in J_r(y)\}$$

and

$$W_r(y) := \text{span}\{\tilde{k}_{\mu_n} : n \in I_r(y)\}$$

where $\{\tilde{k}_{\mu_n}\}$ is the canonical dual frame of $\{k_{\mu_n}\}$.

Denote the corresponding orthogonal projections by

$$P_{y,r} : \mathcal{H}(E) \rightarrow V_r(y), \quad Q_{y,r} : \mathcal{H}(E) \rightarrow W_r(y)$$

For all $f \in \mathcal{H}(E)$:

$$\|f - Q_{y,r}f\| = \inf_{c_n} \|f - \sum_{n \in J_r(y)} c_n \tilde{k}_{\mu_n}\|$$

The following simple lemma will be needed in the proof of the Homogeneous Approximation Property in $\mathcal{H}(E)$. 

4.2 Homogeneous Approximation Property in $\mathcal{H}(E)$
Lemma 4.2.1. Let \( \mathcal{H}(E) \) be a de Branges space with reproducing kernel \( K(w, z) \). If \( \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) is a sequence such that \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{H}(E) \), then for each \( x \in \mathbb{R} \) the sequence \( \{k_{\mu_n}(x)\} \in l^2 \).

Proof. Since \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{H}(E) \), there exists \( A, B > 0 \) such that

\[
A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f(t), k_{\mu_n}(t) \rangle_{E}|^2 \leq B\|f\|^2,
\]

(4.2.1)

for all \( f(z) \in \mathcal{H}(E) \). In particular, let \( x \in \mathbb{R} \), and set \( f(z) = K(x, z) \), then \( f \in \mathcal{H}(E) \). Moreover,

\[
f(x) = K(x, x) = \|K(x, .)\|_E^2 < \infty.
\]

Hence, applying (4.2.1) for the function \( f \), and using the reproducing kernel property (3.2.4), we get

\[
A\|K(x, .)\|_E^2 \leq \sum_{n \in \mathbb{Z}} |k_{\mu_n}(x)|^2 \leq B\|K(x, .)\|_E^2.
\]

That is, the sequence \( \{k_{\mu_n}(x)\}_{n \in \mathbb{Z}} \in l^2 \).

\[
\text{Theorem 4.2.2. (Homogeneous Approximation Property in } \mathcal{H}(E) \text{).}
\]

Let \( \mathcal{H}(E) \) be a de Branges space such that the phase function of \( E(z) \) satisfying \( 0 < \delta \leq \varphi'(x) \) for all \( x \in \mathbb{R} \). Let \( \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be a separated sequence such that \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) is a frame in \( \mathcal{H}(E) \). Then given \( \epsilon > 0 \) there exists \( R = R(\epsilon) > 0 \) such that for all \( y \in \mathbb{R} \) and all \( r > 0 \)

\[
\sup_{|x-y| \leq r} \|k_x(.) - Q_{y,r}k_x(.)\| < \epsilon,
\]

(4.2.2)

where \( k_x(z) = \frac{K(x, z)}{\|K(x, .)\|} \), \( x \in \mathbb{R} \).

Proof. Let \( \{\tilde{k}_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) be the canonical dual frame of \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) in \( \mathcal{H}(E) \). Let \( y \in \mathbb{R} \), \( r > 0 \), and \( x \in \mathbb{R} \) such that \( |x-y| \leq r \). First we will show that (4.2.2) holds when the function \( k_x(z) \) is replace by the function \( \frac{K(x, z)}{E(x)} \).

Since the function \( \frac{K(x, z)}{E(x)} \in \mathcal{H}(E) \) (as a function of \( z \)) for all \( x \in \mathbb{R} \), then it can be expanded as

\[
\frac{K(x, z)}{E(x)} = \sum_{n \in \mathbb{Z}} \frac{K(x, t)}{E(x)} \frac{K(\mu_n, t)}{\|K(\mu_n, .)\|} \tilde{k}_{\mu_n}(z) = \sum_{n \in \mathbb{Z}} \frac{K(\mu_n, x)}{E(x)\|K(\mu_n, .)\|} \tilde{k}_{\mu_n}(z)
\]
Let $R > 0$ (to be determined). Since $Q_{y,r+R}$ is an orthogonal projection, then we have

$$
\left\| \frac{K(x, \cdot)}{E(x)} - Q_{y,r+R} \frac{K(x, \cdot)}{E(x)} \right\| \leq \left\| \sum_{n \in \mathbb{Z}} \frac{K(\mu_n, x)}{E(x)} \|K(\mu_n, \cdot)\| \tilde{k}_{\mu_n}(z) - \sum_{n \in I_{r+R}(y)} \frac{K(\mu_n, x)}{E(x)} \|k_{\mu_n}(z) \right\|
$$

$$
= \left\| \sum_{n \not\in I_{r+R}(y)} \frac{K(\mu_n, x)}{E(x)} \|k_{\mu_n}(z) \right\|
$$

But since $\{ \frac{K(\mu_n, z)}{\|K(\mu_n, \cdot)\|} \}$ is a frame then by inequality (2.10) there exists a constant $C > 0$ such that

$$
\left\| \frac{K(x, \cdot)}{E(x)} - Q_{y,r+R} \frac{K(x, \cdot)}{E(x)} \right\|^2 \leq \frac{C}{|E(x)|^2} \sum_{n \not\in I_{r+R}(y)} \frac{K(\mu_n, x)}{\|K(\mu_n, \cdot)\|^2}
$$

(4.2.3)

where the sum in the right hand side of the last inequality is finite by Lemma 4.2.1.

Let $L_x(R) := \{ n \in \mathbb{Z} : |x - \mu_n| \leq R \}$. By the assumption $|x - y| \leq r$, if $n \not\in I_{r+R}(y)$, then $|\mu_n - y| > r + R$, so we have $|x - \mu_n| > R$, i.e., $n \not\in L_x(R)$. This implies that $L_x(R) \subseteq I_{r+R}(y)$ whenever $|x - y| \leq r$. Hence inequality (4.2.3) becomes

$$
\left\| \frac{K(x, \cdot)}{E(x)} - Q_{y,r+R} \frac{K(x, \cdot)}{E(x)} \right\|^2 \leq \frac{C}{|E(x)|^2} \sum_{n \not\in L_x(R)} \frac{K(\mu_n, x)}{\|K(\mu_n, \cdot)\|^2}
$$

(4.2.4)

Since $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$, and using the fact that $K(x, x) = \frac{1}{\pi} \varphi'(x)|E(x)|^2$ for all $x \in \mathbb{R}$, we have

$$
\frac{K(\mu_n, x)}{\|K(\mu_n, \cdot)\|^2} = \frac{|K(\mu_n, x)|^2}{K(\mu_n, \mu_n)} = \frac{\pi |K(\mu_n, x)|^2}{\varphi'(\mu_n)|E(\mu_n)|^2} \leq \frac{\pi}{\delta} \left| \frac{K(\mu_n, x)}{E(\mu_n)} \right|^2
$$

(4.2.5)

On the other hand, by the definition of the phase function $\varphi$ of $E(z)$ in (3.4.1), we have $E(x)/E(x) = e^{-2i\varphi(x)}$, which implies that

$$
\left| \frac{K(\mu_n, x)}{E(\mu_n)} \right|^2 = \frac{\left| E(x)E(\mu_n) - E(x)E(\mu_n) \right|^2}{2i\pi(x - \mu_n)\overline{E(\mu_n)}}
$$

$$
= \left| \frac{\overline{E(x)e^{-2i\varphi(\mu_n)} - E(x)}^2}{2i\pi(x - \mu_n)} \right|
$$

$$
= \left| \frac{E(x)^2}{4\pi^2} \left| e^{-2i\varphi(\mu_n)} - e^{-2i\varphi(x)} \right|^2 \right|
$$

$$
= \left| \frac{E(x)^2}{4\pi^2} \left( \left| e^{-2i\varphi(\mu_n)} \right| + \left| e^{-2i\varphi(x)} \right| \right)^2 \right|
$$

$$
\leq \left| \frac{E(x)^2}{\pi^2} \frac{1}{(x - \mu_n)^2} \right|
$$
Using the last inequality together with inequalities (4.2.4) and (4.2.5) we get

\[
\left\| \frac{K(x,\cdot)}{E(x)} - Q_{y,r+R} \frac{K(x,\cdot)}{E(x)} \right\|^2 \leq \frac{C}{\|E(x)\|^2} \left( \sum_{n \notin L_x(R)} \left| \frac{K(\mu_n, x)}{E(\mu_n)} \right|^2 \right)
\]

\[
\leq \frac{C}{\pi \delta} \sum_{n \notin L_x(R)} \frac{1}{(x - \mu_n)^2}
\]

Note that since the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) is separated, then for each \( x \in \mathbb{R} \) and for each finite \( R > 0 \), the index set \( L_x(R) \) is finite. Hence, we may assume that \( L_x(R) = \{n_1, n_2, \ldots, n_L\} \) with \( n_1 < n_2 < \cdots < n_L \). Again, since the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) is separated then we can find \( \rho > 0 \) such that \( |\mu_{n+1} - \mu_n| \geq \rho > 0 \), for all \( n \in \mathbb{Z} \). Let \( n > n_L \). Then \( |x - \mu_n| > R \) and \( \mu_n > \mu_{n_L} \). Note that since \( |x - \mu_{n_L}| \leq R \) then we should have \( x < \mu_n \). Let \( j \) be a nonnegative integer such that \( \mu_{n-j} \) be the first element to the right of \( x \), of the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \), such that \( |x - \mu_{n-j}| > R \). Since the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) is (strictly) increasing, we also have

\[
|x - \mu_{n-j+1}| > R + \rho, \quad |x - \mu_{n-j+2}| > R + 2\rho, \quad \ldots, \quad |x - \mu_{n-j+(j-1)}| > R + (j-1)\rho,
\]

\[
|x - \mu_n| > R + j\rho, \quad |x - \mu_{n+1}| > R + (j+1)\rho, \quad \ldots.
\]

In general, \( |x - \mu_{n+k}| > R + (j + k)\rho \), for all \( k \geq -j \), whenever \( n > n_L \).

On the other hand, if \( n < n_1 \), then \( |x - \mu_n| > R \) and \( \mu_n < \mu_{n_1} \). Note that since \( |x - \mu_{n_1}| \leq R \) then we should have \( \mu_n < x \). Let \( m \) be a nonnegative integer such that \( \mu_{n+m} \) be the first element to the left of \( x \), of the sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \), such that \( |x - \mu_{n+m}| > R \). We also have

\[
|x - \mu_{n+m-1}| > R + \rho, \quad |x - \mu_{n+m-2}| > R + 2\rho, \quad \ldots, \quad |x - \mu_{n+m-(m-1)}| > R + (m-1)\rho,
\]

\[
|x - \mu_n| > R + m\rho, \quad |x - \mu_{n-1}| > R + (m+1)\rho, \quad \ldots.
\]

In general, \( |x - \mu_{n+k}| > R + (m-k)\rho \), for all \( k \leq m \), whenever \( n < n_1 \).

Hence,

\[
\sum_{n \notin L_x(R)} \frac{1}{(x - \mu_n)^2} = \sum_{n < n_1} \frac{1}{(x - \mu_n)^2} + \sum_{n > n_L} \frac{1}{(x - \mu_n)^2}
\]

\[
= \sum_{k = -\infty}^{m} \frac{1}{(x - \mu_{n+k})^2} + \sum_{k = -j}^{\infty} \frac{1}{(x - \mu_{n+k})^2}
\]

\[
< \sum_{k = -\infty}^{m} \frac{1}{(R + (m-k)\rho)^2} + \sum_{k = -j}^{\infty} \frac{1}{(R + (j+k)\rho)^2}
\]
\[= \sum_{k=0}^{\infty} \frac{1}{(R + k\rho)^2} + \sum_{k=0}^{\infty} \frac{1}{(R + k\rho)^2} = 2 \sum_{k=0}^{\infty} \frac{1}{(R + k\rho)^2}.\]

Inequality (4.2.6) implies that
\[
\left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\| < \frac{2C}{\pi \delta} \sum_{k=0}^{\infty} \frac{1}{(R + k\rho)^2}
\]
for all \(x \in \mathbb{R}\) such that \(|x - y| \leq r\), therefore,
\[
\sup_{|x-y| \leq r} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\|^2 < \frac{2C}{\pi \delta} \sum_{k=0}^{\infty} \frac{1}{(R + k\rho)^2}.
\]

So, since the latter sum is finite then we can choose \(R = R(\epsilon) > 0\) sufficiently large so that the latter sum is less than \(\epsilon \pi \delta / 2C\). That is, the homogeneous approximation property holds for the function \(K(x,z)/E(x)\), for all \(x \in \mathbb{R}\).

Now, note that
\[
\left\| \frac{K(x,.)}{K(x,.)} - Q_{y,r+R} \frac{K(x,.)}{K(x,.)} \right\| = \frac{|E(x)|}{\sqrt{K(x,x)}} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\|
\]
\[
= \sqrt{\frac{\pi}{\varphi'(x)}} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\|
\]
\[
\leq \sqrt{\frac{\pi}{\delta}} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\|
\]

Given \(\epsilon > 0\), choose \(R = R(\epsilon) > 0\), as above, such that
\[
\sup_{|x-y| \leq r} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\| < \sqrt{\frac{\delta}{\pi}} \epsilon
\]
we get
\[
\sup_{|x-y| \leq r} \left\| \frac{K(x,.)}{K(x,.)} - Q_{y,r+R} \frac{K(x,.)}{K(x,.)} \right\| \leq \sqrt{\frac{\pi}{\delta}} \sup_{|x-y| \leq r} \left\| \frac{K(x,.)}{E(x)} - Q_{y,r+R} \frac{K(x,.)}{E(x)} \right\|
\]
\[
< \sqrt{\frac{\pi}{\delta}} \epsilon \sqrt{\frac{\delta}{\pi}} = \epsilon
\]
for all \(r > 0\), and all \(y \in \mathbb{R}\), which proves the theorem. \(\Box\)
The Homogeneous Approximation Property has several implications on the geometry of the sequence $M$, in particular, it shows a relationship between the Beurling densities of a frame and the Beurling densities of any orthonormal basis or Riesz basis in $H(E)$. This further yields necessary conditions for the existence of sampling and interpolation sequences as we will see later.

The following theorem is consistent with the fact that frames provide redundant non-orthogonal expansions in Hilbert space, accordingly, they should be “denser” than orthonormal bases (Riesz basis).

**Theorem 4.2.3. (Comparison Theorem).**

Let $H(E)$ be a de Branges space, and the corresponding phase function of $E$ satisfies $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$. Suppose that $M = \{\mu_n\}$, $\Gamma = \{\gamma_n\} \subseteq \mathbb{R}$ are two separated sequences, such that $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is a frame in $H(E)$, and $\{k_{\gamma_n}(z)\}_{n \in \mathbb{Z}}$ is a Riesz basis for a closed subspace of $H(E)$. Then for every $\epsilon > 0$, there exists $R = R(\epsilon) > 0$, such that for all $r > 0$ and $y \in \mathbb{R}$, we have

$$(1 - \epsilon) \#(\Gamma \cap [y - r, y + r]) \leq \#(M \cap [y - r - R, y + r + R]).$$

Therefore,

$$D^-(\Gamma) \leq D^-(M), \quad \text{and} \quad D^+(\Gamma) \leq D^+(M).$$

**Proof.** Let $k_{\gamma_n}(z) = \frac{K(\gamma_n, z)}{\|K(\gamma_n, .)\|}$, $n \in \mathbb{Z}$, and $\tilde{k}_{\gamma_n}$ denote the biorthogonal basis of $k_{\gamma_n}$, that is, $\langle k_{\gamma_n}, \tilde{k}_{\gamma_m} \rangle = \delta_{n,m}$. Since any Riesz basis is uniformly bounded and the biorthogonal sequence of a Riesz basis is also a Riesz basis, then there exists a constant $C_o > 0$ such that $\|\tilde{k}_{\gamma_n}\| \leq C_o$, for all $n \in \mathbb{Z}$.

Given $\epsilon > 0$, choose $R = R(\epsilon) > 0$ such that the homogeneous approximation property holds for the function $k_x(z)$, $x \in \mathbb{R}$, with $\epsilon/C_o$, i.e., for all $r > 0$ and $y \in \mathbb{R}$

$$\sup_{|x - y| \leq r} \|k_x(.) - Q_{y,r+R}k_x(.)\| < \epsilon/C_o.$$

Given $r > 0$ and $y \in \mathbb{R}$ we define the operators $T_{y,r} : V_r(y) \to V_r(y)$ by

$$T_{y,r} = P_{y,r} Q_{y,r+R}. \quad (4.2.7)$$
By definition, the sequence \( \{k_{\gamma_n}\}_{n \in J_r(y)} \) is a basis for \( V_r(y) \) (since \( \{k_{\gamma_n}\} \) is Riesz sequence and hence linearly independent). Moreover, the dual basis of \( \{k_{\gamma_n}\}_{n \in J_r(y)} \) in \( V_r(y) \) is the projection of \( \{\tilde{k}_{\gamma_n}\}_{n \in J_r(y)} \) under \( P_{y,r} \), that is \( \{P_{y,r}\tilde{k}_{\gamma_n}\}_{n \in J_r(y)} \). Indeed, for \( n, m \in J_r(y) \) we have

\[
\langle k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_m} \rangle = \langle P^*_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_m} \rangle = \langle P_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_m} \rangle = \langle k_{\gamma_n}, \tilde{k}_{\gamma_m} \rangle = \delta_{nm}.
\]

Note that since \( \text{dom}(T_{y,r}) = V_r(y) \) then

\[
T_{y,r} = P_{y,r}Q_{y,r+R} = P_{y,r}Q_{y,r+R}P_{y,r}, \tag{4.2.8}
\]

hence, \( T_{y,r} \) is self-adjoint. Moreover, since \( V_r(y) \) is finite dimensional then the trace of \( T_{y,r} \) is finite. By the biorthogonality of the sequences \( \{k_{\gamma_n}\}_{n \in J_r(y)} \) and \( \{P_{y,r}\tilde{k}_{\gamma_n}\}_{n \in J_r(y)} \) and Lemma 2.1.6, the trace of \( T_{y,r} \) can be written as

\[
\text{tr}(T_{y,r}) = \sum_{n \in J_r(y)} \langle T_{y,r}k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle = \sum_{n \in J_r(y)} \langle P_{y,r}T_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle = \sum_{n \in J_r(y)} \langle T_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle
\]

Since \( P_{y,r} \) is an orthogonal projection, it is self adjoint, i.e., \( P^*_{y,r} = P_{y,r} \), so we have

\[
\langle T_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle = \langle P_{y,r}Q_{y,r+R}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle = \langle Q_{y,r+R}k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle = \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n} + k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle = \langle k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle + \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle
\]

\[
= \langle P_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle + \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle = \langle k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle + \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle = 1 + \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle
\]

where we used the fact that \( P_{y,r}k_{\gamma_n} = k_{\gamma_n} \), for all \( n \in J_r(y) \). So we have

\[
\langle T_{y,r}k_{\gamma_n}, \tilde{k}_{\gamma_n} \rangle - 1 = \langle Q_{y,r+R}k_{\gamma_n} - k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n} \rangle \tag{4.2.9}
\]
Applying the Cauchy–Schwarz inequality to the right hand side of the previous equation, and using the fact that \( \|P_{y,r}\| = 1 \) to get \( \|P_{y,r}\| \|\tilde{k}_{\gamma_n}\| = \|\hat{k}_{\gamma_n}\| \leq C_o \), we obtain

\[
\langle Q_{y,r} + R k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n}\rangle \quad \leq \quad \langle Q_{y,r} + R k_{\gamma_n}, P_{y,r}\hat{k}_{\gamma_n}\rangle
\]

\[
\leq \sup_{|x-y| \leq r} \|k_x(.) - Q_{y,r} + R k_x(.)\| \|P_{y,r}\| \|\hat{k}_{\gamma_n}\|
\]

\[
< \left( \frac{\epsilon}{C_o} \right) C_o
\]

whenever \( |\gamma_n - y| \leq r \). Therefore, by (4.2.9) we get

\[
|\langle T_{y,r} k_{\gamma_n}, \hat{k}_{\gamma_n}\rangle - 1| = |\langle Q_{y,r} + R k_{\gamma_n}, P_{y,r}\tilde{k}_{\gamma_n}\rangle| \leq \epsilon \quad \text{(4.2.10)}
\]

Now, note that

\[
\left| \sum_{n \in J_r(y)} 1 - \sum_{n \in J_r(y)} \langle T_{y,r} k_{\gamma_n}, \tilde{k}_{\gamma_n}\rangle \right| = \left| \sum_{n \in J_r(y)} (1 - \langle T_{y,r} k_{\gamma_n}, \tilde{k}_{\gamma_n}\rangle) \right|
\]

\[
\leq \sum_{n \in J_r(y)} \left| 1 - \langle T_{y,r} k_{\gamma_n}, \tilde{k}_{\gamma_n}\rangle \right|
\]

Hence, by the definition of the trace of \( T_{y,r} \) and (4.2.10) we have

\[
\sum_{n \in J_r(y)} 1 - \text{tr}(T_{y,r}) \leq \left| \sum_{n \in J_r(y)} 1 - \sum_{n \in J_r(y)} \langle T_{y,r} k_{\gamma_n}, \tilde{k}_{\gamma_n}\rangle \right| \leq \sum_{n \in J_r(y)} \epsilon
\]

Therefore, we can estimate a lower bound to the trace of \( T_{y,r} \) by

\[
\text{tr}(T_{y,r}) \geq \sum_{n \in J_r(y)} (1 - \epsilon) = (1 - \epsilon) \#(\Gamma \cap [y - r, y + r])
\]

On the other hand, since the operator norm of \( T_{y,r} \) satisfies \( \|T_{y,r}\| = \|P_{y,r} Q_{y,r} + R\| \leq \|P_{y,r}\| \|Q_{y,r} + R\| = 1 \), all the eigenvalues of \( T_{y,r} \) have modulus less than or equal to 1, this in turn provides us with an upper bound for the trace of \( T_{y,r} \). Indeed,

\[
\text{tr}(T_{y,r}) = \sum_{\text{non-zero eigenvalues of } T_{y,r}} \leq \text{rank}(T_{y,r})
\]

Also, since \( \text{rank}(T_{y,r}) = \dim(\text{range}(T_{y,r})) = \dim(\text{range}(P_{y,r} Q_{y,r} + R)) \leq \dim(W_{y+r}) \), then

\[
\text{tr}(T_{y,r}) \leq \dim(W_{y+r}) \leq \# \{ \mu_n : |\mu_n - y| \leq r + R \}
\]

\[
= \#(M \cap [y - r - R, y + r + R])
\]
Therefore, combining these two estimates of the trace of $T_{y,r}$ we get

$$(1 - \epsilon)\sharp(\Gamma \cap [y - r, y + r]) \leq \sharp(\mathcal{M} \cap [y - r - R, y + r + R])$$

for all $r > 0$ and all $y \in \mathbb{R}$. Moreover,

$$(1 - \epsilon)\frac{\sharp(\Gamma \cap [y - r, y + r])}{2r} \leq \frac{(2r + 2R)}{2r} \frac{\sharp(\mathcal{M} \cap [y - r - R, y + r + R])}{(2r + 2R)}$$

taking the infimum over all $y \in \mathbb{R}$ for both sides

$$(1 - \epsilon)\inf_{y \in \mathbb{R}} \frac{\sharp(\Gamma \cap [y - r, y + r])}{2r} \leq \frac{(2r + 2R)}{2r} \inf_{y \in \mathbb{R}} \frac{\sharp(\mathcal{M} \cap [y - r - R, y + r + R])}{(2r + 2R)}$$

and by taking liminf as $r \to \infty$ yields the estimates

$$(1 - \epsilon)D^- (\Gamma) \leq D^- (\mathcal{M})$$

Since $\epsilon$ is arbitrary, we conclude that

$$D^- (\Gamma) \leq D^- (\mathcal{M})$$

A similar calculations shows that

$$D^+ (\Gamma) \leq D^+ (\mathcal{M})$$

It should be noted that if the phase function $\varphi$ satisfying $0 < \delta \leq \varphi' \leq M$, then by Proposition 4.1.5, the result of the Comparison Theorem can be stated, equivalently, in terms of the density of values of the phase function at the sequences $\Gamma$ and $\mathcal{M}$ in the theorem. More precisely, we have the following:

**Corollary 4.2.4.** Let $\mathcal{H}(E)$ be a de Branges space, and the corresponding phase function of $E$ satisfies $0 < \delta \leq \varphi' (x) \leq M$ for all $x \in \mathbb{R}$. Suppose that $\mathcal{M} = \{\mu_n\}, \Gamma = \{\gamma_n\} \subseteq \mathbb{R}$ are two separated sequences, such that $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is a frame in $\mathcal{H}(E)$, and $\{k_{\lambda_n}(z)\}_{n \in \mathbb{Z}}$ is a Riesz basis for a closed subspace of $\mathcal{H}(E)$. Then

$$D^- (\varphi(\Gamma)) \leq \frac{M}{\delta} D^- (\varphi(\mathcal{M})),$$

and

$$D^+ (\varphi(\Gamma)) \leq \frac{M}{\delta} D^+ (\varphi(\mathcal{M})).$$
4.3 Necessary Density Conditions For Sampling and Interpolation in $\mathcal{H}(E)$

Now that we have proved the Homogeneous Approximation Property and the Comparison Theorem, we have all the powerful tools needed to prove one of our main results. Recall that a sequence $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ is said to be a sampling sequence for a de Branges space $\mathcal{H}(E)$ if there exist positive constants $A$ and $B$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{n \in \mathbb{Z}} |f(\mu_n)|^2 \leq B\|f\|_{\mathcal{H}}^2$$

for all $f \in \mathcal{H}(E)$, where $K(w, z)$ is the reproducing kernel of $\mathcal{H}(E)$. Moreover, this is equivalent to the corresponding sequence of normalized reproducing kernels $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ forming a frame for $\mathcal{H}(E)$. Hence, any function $f \in \mathcal{H}(E)$ can be reconstructed from its samples on the sequence $\mathcal{M}$ by the sampling formula

$$f(z) = \sum_{n \in \mathbb{Z}} f(\mu_n) \tilde{k}_{\mu_n}(z)$$

where $\{\tilde{k}_{\mu_n}\}_{n \in \mathbb{Z}}$ is any dual frame of $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$.

The following theorem gives a necessary density condition for a sequence to be sampling in $\mathcal{H}(E)$.

**Theorem 4.3.1.** Let $E \in \mathcal{HB}$, with phase function satisfying $0 < \delta \leq \varphi'(x)$, for all $x \in \mathbb{R}$. If $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ is a uniformly separated sampling sequence in $\mathcal{H}(E)$, then $D^-(\mathcal{M}) \geq \frac{\delta}{\pi}$.

**Proof.** Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be such that $\varphi(\lambda_n) = \alpha + n\pi$, for all $n \in \mathbb{Z}$, for some $\alpha \in [0, \pi)$. Thus, the corresponding normalized reproducing kernels $\{k_{\lambda_n}(z)\}_{n \in \mathbb{Z}}$ forms an orthonormal set for $\mathcal{H}(E)$. Moreover, using the fact that $D^-(a\mathbb{Z}) = \frac{1}{a}$, for any $a \neq 0$, we have

$$D^-(\{\varphi(\lambda_n)\}) = D^-(\{\alpha + n\pi\}) = D^-(\{n\pi\}) = D^-(\pi\mathbb{Z}) = \frac{1}{\pi}.$$

On the other hand, if $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$, then by proposition (4.1.5) we have

$$D^-(\{\varphi(\lambda_n)\}) \leq \frac{1}{\delta} D^-(\Lambda),$$

hence, $D^-(\Lambda) \geq \frac{\delta}{\pi}$. If $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}}$ is a sampling sequence in $\mathcal{H}(E)$, then by the Comparison Theorem we have

$$D^-(\Lambda) \leq D^-(\mathcal{M}),$$
therefore, $D^-(M) \geq \frac{\delta}{\pi}$, as desired.

\begin{theorem}
Let $E \in \mathcal{HB}$, with phase function satisfying $0 < \delta \leq \varphi'(x) \leq M < \infty$, for all $x \in \mathbb{R}$. If $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$ is a uniformly separated interpolating sequence in $\mathcal{H}(E)$, then $D^+(\Gamma) \leq \frac{M}{\pi}$.
\end{theorem}

\begin{proof}
Let $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$ be an interpolating sequence in $\mathcal{H}(E)$, then the corresponding normalized reproducing kernels $\{k_{\gamma_n}(z)\}_{n \in \mathbb{Z}}$ is a Riesz basis for some subspace of $\mathcal{H}(E)$. Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be such that $\varphi(\lambda_n) = \alpha + n\pi$, for all $n \in \mathbb{Z}$, for some $\alpha \in [0, \pi)$. Then the corresponding normalized reproducing kernels of $\lambda_n$'s forms an orthonormal set for $\mathcal{H}(E)$, hence a frame. Moreover,

$$D^+(\{\varphi(\lambda_n)\}) = D^+(\{\alpha + n\pi\}) = D^+(\{n\pi\}) = D^+(n\pi) = \frac{1}{\pi},$$

On the other hand, since $\varphi'(x) \leq M$ for all $x \in \mathbb{R}$, then by proposition (4.1.5) we have

$$\frac{1}{M} D^+(\Lambda) \leq \frac{1}{M} D^+(\{\varphi(\lambda_n)\}),$$

hence, $D^+(\Lambda) \leq \frac{M}{\pi}$. The Comparison Theorem implies that

$$D^+(\Gamma) \leq D^+(\Lambda) \leq \frac{M}{\pi}.$$

\end{proof}

\begin{theorem}
Let $E \in \mathcal{HB}$, with phase function satisfying $0 < \delta \leq \varphi'(x) \leq M$, for all $x \in \mathbb{R}$. If $\Gamma = \{\gamma_n\}_{n \in \mathbb{Z}}$ is a uniformly separated complete interpolating sequence in $\mathcal{H}(E)$, then

$$\frac{\delta}{\pi} \leq D^-(\Gamma) \leq D^+(\Gamma) \leq \frac{M}{\pi}.$$

\end{theorem}

\begin{proof}
If $\Gamma$ is a complete interpolating sequence in $\mathcal{H}(E)$, then its both sampling and interpolating sequence, hence, by Theorem 4.3.2 and Theorem 4.3.3 it follows that $\frac{\delta}{\pi} \leq D^-(\Gamma)$ and $D^+(\Gamma) \leq \frac{M}{\pi}$, that is, $\frac{\delta}{\pi} \leq D^-(\Gamma) \leq D^+(\Gamma) \leq \frac{M}{\pi}$.

\end{proof}

\begin{remark}
In case of the Paley-Wiener space $PW_a = \mathcal{H}(E)$ where $E(z) = e^{-iaz}$, $a > 0$, the corresponding phase function is $\varphi(x) = ax$, so $\delta = M = a$ in this case. Theorem 4.3.1 says that
\end{remark}
if a separated sequence $M = \{\mu_n\}_{n \in \mathbb{Z}}$ is a sampling sequence in $PW_a$ then the lower Beurling density $D^-(M) \geq \frac{\delta}{\pi} = \frac{2}{\pi}$, which is consistent with the result obtained by Jaffard in Theorem 3.1.2. Theorem 4.3.2 says that if a separated sequence $M = \{\mu_n\}_{n \in \mathbb{Z}}$ is an interpolating sequence in $PW_a$ then the upper Beurling density $D^+(M) \leq \frac{M}{\pi} = \frac{\alpha}{\pi}$, which is also consistent with the result obtained by K. Seip in Theorem 3.1.3 for $a = \pi$. Also, Theorem 4.3.3 says that if a separated sequence $M = \{\mu_n\}_{n \in \mathbb{Z}}$ is a complete interpolating sequence in $PW_a$ then the Beurling densities $D^+(M) = D^+(M) = \frac{\alpha}{\pi}$, this is again consistent with the result obtained in Theorem 3.1.3 for $a = \pi$.

If $\{\lambda_n\}_{n \in \mathbb{Z}}$ is a sequence of real numbers satisfying $\varphi(\lambda_n) = \alpha + n\pi$, $n \in \mathbb{Z}$, then Lemma 4.1.2 shows that the sequence $\{f_n\}_{n \in \mathbb{Z}}$, where $f_n(z) = \frac{K(\lambda_n, z)}{E(\lambda_n)}$, is a frame in $H(E)$ with frame bounds $A = \frac{\delta}{\pi}$ and $B = \frac{M}{\pi}$. Moreover, we have proved in the first part of Theorem 3.4.1 that the sequence $\{f_n\}_{n \in \mathbb{Z}}$ is an orthogonal set in $H(E)$. Therefore, Theorem 2.1.1 now implies that the sequence $\{f_n\}_{n \in \mathbb{Z}}$ is a Riesz basis, by noting that $\{f_n\}_{n \in \mathbb{Z}}$ satisfies condition (ii) of that theorem.

This observation together with the Paley-Wiener Theorem for frames, and inequality (4.1.3), gives some conditions on a perturbed sequence $\{\mu_n\}$ to be complete interpolating sequence.

**Corollary 4.3.4.** Let $H(E)$ be a de Branges space, with $E'/E \in L^\infty(\mathbb{R})$ and $\varphi(x)$ be the corresponding phase function of $E$ with $0 < \delta \leq \varphi'(x)$ for all $x \in \mathbb{R}$. Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers such that $\varphi(\lambda_n) = \alpha + n\pi$, $n \in \mathbb{Z}$. If $\{\mu_n\}$ is a sequence such that

$$\max_n |\mu_n - \lambda_n| \leq \rho < \frac{\delta}{2\pi(C^2 + 1)\|E'/E\|_\infty^2} \quad (4.3.1)$$

where $C$ is the Bernstein inequality constant, then $\{\mu_n\}_{n \in \mathbb{Z}}$ is a complete interpolating sequence in $H(E)$.

Any uniformly separated sequence is a relatively separated sequence. However, Example 2.1 shows that relatively separated sequence is not always uniformly separated. In the next two lemmas we prove that in case of sampling and interpolating sequences we do not need to assume that the sequence is separated.
Lemma 4.3.5. Let $\mathcal{H}(E)$ be a de Branges space with $E' \in H^\infty(C^+)$ and $0 < \delta \leq \varphi'(x)$. Let $\Lambda = \{\gamma_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ be an interpolating sequence in $\mathcal{H}(E)$, then $\Lambda$ is uniformly separated.

Proof. Let $k \in \mathbb{Z}$, and define a sequence $\{c_n\}_{n \in \mathbb{Z}}$ by $c_k = E(\gamma_k) \sqrt{\varphi'(\gamma_k)}$, and $c_n = 0$ for all $n \neq k$. Then by (3.4.2) we get

$$\sum_{n} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} = \pi < \infty.$$ 

Since $\Lambda$ is an interpolating sequence in $\mathcal{H}(E)$ then, by (2.5.1) and (2.5.2), there exist $f \in \mathcal{H}(E)$ such that $f(\gamma_n) = c_n$ for all $n \in \mathbb{Z}$, and

$$\|f\|_E^2 \leq \sum_{n} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} = \frac{\pi}{c}, \quad (4.3.2)$$

for some constant $c > 0$. In particular, $f(\gamma_k) = E(\gamma_k) \sqrt{\varphi'(\gamma_k)}$, and $f(\gamma_n) = 0$ for all $n \neq k$.

Then, by the Mean Value Theorem, we have for $n \neq k$

$$\sqrt{\delta} \leq \sqrt{\varphi'(\gamma_k)} = \left| \frac{f(\gamma_k)}{E(\gamma_k)} \right| = \left| \frac{f(\gamma_k)}{E(\gamma_k)} - \frac{f(\gamma_n)}{E(\gamma_n)} \right| = |\gamma_k - \gamma_n| \left| \left( \frac{f'}{E} \right)(t) \right|, \quad (4.3.3)$$

for some point $t$ between $\gamma_n$ and $\gamma_k$.

Since $E' \in H^\infty(C^+)$, then the differentiation operator is bounded on $\mathcal{H}(E)$, i.e. $f' \in \mathcal{H}(E)$ for all $f \in \mathcal{H}(E)$. Moreover, Bernstein inequality (3.5.2) gives

$$\|f'\|_E \leq C\|E'/E\|_\infty\|f\|_E,$$

for some constant $C > 0$. Therefore, for the above $f$ and $t$, by (3.2.5) applying to $f'$, and the fact that $\varphi'(t) \leq M$ (by Lemma 3.5.4) we get

$$|f'(t)| \leq \|f'\|_E \sqrt{K(t, t)}$$

$$\leq C\|E'/E\|_\infty\|f\|_E \sqrt{\varphi'(t)} |E(t)|$$

$$\leq C\|E'/E\|_\infty\|f\|_E \frac{M}{\pi} |E(t)|,$$

hence,

$$\left| \frac{f'}{E}(t) \right| \leq C \frac{M}{\pi} \|E'/E\|_\infty\|f\|_E$$
On the other hand, by (3.2.5) applying to \( f \)

\[
|f(t)| \leq \|f\|_E \sqrt{K(t,t)} = \frac{1}{\sqrt{\pi}} \|f\|_E \sqrt{\varphi'(t)} |E(t)| \leq \sqrt{\frac{M}{\pi}} \|f\|_E |E(t)|,
\]

hence,

\[
\left| \frac{f}{E}(t) \right| \leq \sqrt{\frac{M}{\pi}} \|f\|_E.
\]

Therefore,

\[
\left| \left( \frac{f}{E} \right)'(t) \right| = \left| \frac{f'}{E}(t) - \frac{E'}{E}(t) \frac{f}{E}(t) \right| \leq \left| \frac{f'}{E}(t) \right| + \left| \frac{E'}{E}(t) \right| \left| \frac{f}{E}(t) \right| \leq C \sqrt{\frac{M}{\pi}} \|E'/E\|_\infty \|f\|_E + \|E'/E\|_\infty \sqrt{\frac{M}{\pi}} \|f\|_E \leq (C + 1) \sqrt{\frac{M}{\pi}} \|E'/E\|_\infty \|f\|_E.
\]

So, by (4.3.3) and the last inequality we get

\[
\sqrt{\delta} \leq |\gamma_k - \gamma_n| (C + 1) \sqrt{\frac{M}{\pi}} \|E'/E\|_\infty \|f\|_E.
\]

Now using the fact that \( \|f\|^2 \leq \pi/c \) from (4.3.2), we get

\[
\sqrt{\delta} \leq |\gamma_k - \gamma_n| \sqrt{\frac{M}{c}} (C + 1) \sqrt{\frac{M}{\pi}} \|E'/E\|_\infty.
\]

Therefore,

\[
|\gamma_k - \gamma_n| \geq \delta_o
\]

where \( \delta_o = \sqrt{\delta} \left( \sqrt{\frac{M/c}{(C + 1)\|E'/E\|_\infty}} \right)^{-1} \). That is, the sequence \( \Gamma \) is uniformly separated. \( \square \)

**Lemma 4.3.6.** Let \( \mathcal{H}(E) \) be a de Branges space with \( \frac{E'}{E} \in H^\infty(\mathbb{C}^+) \) and \( 0 < \delta \leq \varphi'(x) \). Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be a sampling in \( \mathcal{H}(E) \). Then there is a uniformly separated set \( \Lambda' \subset \Lambda \) such that \( \Lambda' \) is sampling in \( \mathcal{H}(E) \).

**Proof.** If \( \Lambda \) is uniformly separated then we are done, so assume it is not. Since \( \Lambda \) is sampling in \( \mathcal{H}(E) \) then it is a Plancherel-Polya sequence, hence it is relatively separated, by Theorem

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4.4.5. Thus, it can be written as a finite disjoint union of uniformly separated sequences
\[ \Lambda_k = \{ \lambda_n^{(k)} \}_{n \in \mathbb{Z}} \text{ each with separation constant } \delta_k > 0, \text{ i.e.} \]
\[ \Lambda = \bigcup_{k=1}^{N} \Lambda_k, \quad \inf_{\lambda_i^{(k)}, \lambda_j^{(k)} \in \Lambda_k} |\lambda_i^{(k)} - \lambda_j^{(k)}| \geq \delta_k > 0, \quad k = 1, 2, \ldots, N \quad (4.3.4) \]

Let \( \delta_o = \min\{\delta_1, \delta_2, \ldots, \delta_N\} \), and \( 0 < \epsilon < \delta_o/4 \). We will construct an \( \epsilon \)-uniformly separated subset \( \Lambda' \) of \( \Lambda \) such that for every \( \lambda \in \Lambda \) there exists \( \lambda' \in \Lambda' \) with
\[ |\lambda - \lambda'| \leq \epsilon \quad (4.3.5) \]

First, let \( \Lambda'_1 := \Lambda_1 \), and define the sets \( \Lambda'_k := \Lambda'_{k-1} \cup \Lambda_k^e \) where
\[ \Lambda_k^e =\{ \lambda_n^{(k)} \in \Lambda_k : |\lambda_n^{(k)} - \lambda'| > \epsilon, \forall \lambda' \in \Lambda'_{k-1} \} \subseteq \Lambda_k \]
for \( k = 2, 3, \ldots, N \).

Now, we claim that \( \Lambda'_k \) is \( \epsilon \)-uniformly separated for all \( k = 1, 2, \ldots, N \). Indeed, \( \Lambda'_1 := \Lambda_1 \)
which is \( \delta_o \)-uniformly separated, hence it is \( \epsilon \)-uniformly separated. Let \( \lambda', \mu' \in \Lambda'_2 = \Lambda'_1 \cup \Lambda'_2 \),
then we have three cases:

case (1): If \( \lambda', \mu' \in \Lambda'_1 \), then \( |\lambda' - \mu'| > \epsilon \) since \( \Lambda'_1 \) is \( \epsilon \)-uniformly separated.

case (2): If \( \lambda', \mu' \in \Lambda'_2 \subseteq \Lambda_2 \), then \( |\lambda' - \mu'| \geq \delta_o > \epsilon \) since \( \Lambda_2 \) is \( \delta_o \)-uniformly separated.

case (3): If \( \lambda' \in \Lambda'_1 \) and \( \mu' \in \Lambda'_2 \), then by the definition of \( \Lambda'_2 \) we have \( |\mu' - \lambda'| > \epsilon \) because \( |\mu' - \lambda''| > \epsilon \) for all \( \lambda'' \in \Lambda'_1 \).

It follows that \( \Lambda'_2 \) is \( \epsilon \)-uniformly separated. Continuing in this way we have that \( \Lambda'_k \) is \( \epsilon \)-uniformly separated for \( k = 1, 2, 3, \ldots, N \). Let \( \Lambda' := \Lambda'_N \), then \( \Lambda' \subset \Lambda \) and is \( \epsilon \)-uniformly separated.

Now we will prove (4.3.5). Let \( \lambda \in \Lambda \), then there exist a unique \( l \in \{1, 2, \ldots, N\} \) and \( m \in \mathbb{Z} \)
such that \( \lambda = \lambda_m^{(l)} \in \Lambda_l \). If \( \lambda \in \Lambda' \) then we are done. So, suppose that \( \lambda \notin \Lambda' \). Then \( \lambda \notin \Lambda'_k \) for all \( k = 1, 2, 3, \ldots, N \). In particular, \( \lambda = \lambda_m^{(l)} \notin \Lambda'_l = \Lambda'_{l-1} \cup \Lambda'_l \). Hence
\[ \lambda = \lambda_m^{(l)} \notin \Lambda'_l = \{ \lambda_n^{(l)} \in \Lambda_l : |\lambda_n^{(l)} - \lambda'| > \epsilon, \forall \lambda' \in \Lambda'_{l-1} \} \]
it follows that \( |\lambda - \lambda'| \leq \epsilon \) for some \( \lambda' \in \Lambda'_{l-1} \), this proves (4.3.5). Example 4.1 and Example 4.2 below illustrate this idea.
On other words, given \( \lambda_n^{(k)} \in \Lambda \), there exist \( \lambda' := \lambda_n^{(k)'} \in \Lambda \) such that

\[ |\lambda_n^{(k)} - \lambda_n^{(k)'}| \leq \epsilon \]

for all \( k = 1, 2, \ldots, N \) and \( n \in \mathbb{Z} \).

Let \( f \in \mathcal{H}(E) \) be arbitrary. Since the ratio \( f(x)/E(x) \) is continuous on \( \mathbb{R} \) then, for \( 0 < |\lambda_n^{(k)} - \lambda_n^{(k)'}| < \epsilon \), the Mean value theorem gives

\[
\left| \frac{f(\lambda_n^{(k)})}{E(\lambda_n^{(k)})} - \frac{f(\lambda_n^{(k)'})}{E(\lambda_n^{(k)'})} \right| = \left| \left( \frac{f}{E} \right)'(\mu_n^{(k)}) \right| |\lambda_n^{(k)} - \lambda_n^{(k)'}| \leq \epsilon \left| \left( \frac{f}{E} \right)'(\mu_n^{(k)}) \right|
\]

for some point \( \mu_n^{(k)} \) between \( \lambda_n^{(k)} \) and \( \lambda_n^{(k)'} \). Note that since \( \epsilon < \delta_o/4 \) then for a fixed \( k \) there could be at most two points of the sequence \( \{\mu_n^{(k)}\}_{n \in \mathbb{Z}} \) between any two points of \( \Lambda_k \). It follows that, for each \( k = 1, 2, \ldots, N \), the sequence \( \{\mu_n^{(k)}\}_{n \in \mathbb{Z}} \) is uniformly separated with separation constant at least \( \delta_o/2 \), i.e., it is \( \delta_o/2 \)-uniformly separated.

Since \( \Lambda \) is sampling in \( \mathcal{H}(E) \) then there exists \( A_\Lambda, B_\Lambda > 0 \) such that

\[
A_\Lambda \|f\|_E^2 \leq \sum_{n \in \mathbb{Z}} \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)} \leq B_\Lambda \|f\|_E^2
\]

for all \( f \in \mathcal{H}(E) \). Hence, the subsequence \( \Lambda' \) satisfies the upper sampling inequality, i.e.,

\[
\sum_{\lambda' \in \Lambda'} \frac{|f(\lambda')|^2}{K(\lambda', \lambda')} \leq B_\Lambda \|f\|_E^2
\]

for all \( f \in \mathcal{H}(E) \). We now show that there exist a constant \( A' > 0 \) such that

\[
A' \|f\|_E^2 \leq \sum_{\lambda' \in \Lambda'} \frac{|f(\lambda')|^2}{K(\lambda', \lambda')}
\]

for all \( f \in \mathcal{H}(E) \).

Using the inequality \( |a + b|^2 \leq 2(|a|^2 + |b|^2) \) for any any \( a, b \in \mathbb{R} \), the assumption (4.3.4), and the fact that \( 0 < \delta \leq \varphi'(x) \) for all \( x \in \mathbb{R} \) we get

\[
A_\Lambda \|f\|_E^2 \leq \sum_{n \in \mathbb{Z}} \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)} = \sum_{n \in \mathbb{Z}} \left| \frac{f(\lambda_n)}{E(\lambda_n)} \right|^2 \frac{\pi}{\varphi'(\lambda_n)} \leq \frac{\pi}{\delta} \sum_{n \in \mathbb{Z}} \left| \frac{f(\lambda_n)}{E(\lambda_n)} \right|^2
\]
\[
\pi \sum_{k=1}^{N} \frac{\lambda_{n}^{(k)}}{E(\lambda_{n}^{(k)})} \leq \frac{2\pi}{\delta} \left[ \sum_{k=1}^{N} \sum_{\lambda_{n}^{(k)}} \left| \frac{f(\lambda_{n}^{(k)})}{E(\lambda_{n}^{(k)})} \right|^{2} \right] \]

Note that since \(\epsilon < \delta_{o}/4\) then for each fixed \(k\) the points \(\lambda_{n}^{(k)}\)’s are distinct. To prove this, let \(\lambda_{n}^{(k)}, \lambda_{m}^{(k)} \in \Lambda_{k}\) for \(n \neq m\), and let \(\lambda_{n}^{(k)} = \lambda_{m}^{(k)} = \lambda \in \Lambda\) such that \(|\lambda_{n}^{(k)} - \lambda| \leq \epsilon\) and \(|\lambda_{m}^{(k)} - \lambda| \leq \epsilon\), then since \(\Lambda_{k}\) is \(\delta_{o}\)-uniformly separated we get

\[\delta_{o} \leq |\lambda_{n}^{(k)} - \lambda_{m}^{(k)}| \leq |\lambda_{n}^{(k)} - \lambda| + |\lambda_{m}^{(k)} - \lambda| \leq 2\epsilon\]

hence, \(\delta_{o} \leq 2\epsilon\), a contradiction. However, a repetition of the \(\lambda_{n}^{(k)}\)’s is possible for different \(k\)’s, i.e., it is possible to have \(|\lambda_{n}^{(k)} - \lambda| \leq \epsilon\) and \(|\lambda_{m}^{(l)} - \lambda| \leq \epsilon\) for \(k \neq l\) and \(\lambda \in \Lambda\), see Example 4.2.

Now, in case of repetitions of some of the points \(\lambda_{n}^{(k)}\)’s, note that

\[
\sum_{\lambda' \in \Lambda'} \left( \frac{f(\lambda')}{E(\lambda')} \right)^{2} \leq \sum_{k=1}^{N} \sum_{\lambda_{n}^{(k)}} \left( \frac{f(\lambda_{n}^{(k)}')}{E(\lambda_{n}^{(k)}')} \right)^{2} \leq N \sum_{\lambda' \in \Lambda'} \left( \frac{f(\lambda')}{E(\lambda')} \right)^{2}
\]

and the sum in the left has no repetitions of the \(\lambda_{n}^{(k)}\)’s. So, substituting the last inequality in (4.3.6) we have

\[
A_{\Lambda} \|f\|_{E}^{2} \leq \frac{2\pi}{\delta} \left[ \epsilon^{2} \sum_{k=1}^{N} \sum_{n} \left( \frac{f(\lambda_{n}^{(k)})}{E(\lambda_{n}^{(k)})} \right)^{2} \right] + N \sum_{\lambda' \in \Lambda'} \left( \frac{f(\lambda')}{E(\lambda')} \right)^{2}
\]

(4.3.7)

Since \(\Lambda\) is a sampling sequence in \(H(E)\), and \(\Lambda_{k} = \{\lambda_{n}^{(k)}\}_{n \in \mathbb{Z}} \subset \Lambda\), then \(\Lambda_{k}\) is a Plancherel-Pólya sequence in \(H(E)\), with bound \(B_{\Lambda_{k}} \leq B_{\Lambda}\), for all \(k = 1, 2, \ldots, N\). On the construction of the sequence \(\{\mu_{n}^{(k)}\}\) above, we have \(|\lambda_{n}^{(k)} - \mu_{n}^{(k)}| \leq \epsilon\), for all \(k = 1, 2, \ldots, N\), and all \(n\). Hence, it follows by Proposition 4.1.4 that the sequence \(\{\mu_{n}^{(k)}\}_{n \in \mathbb{Z}}\) is a Plancherel-Pólya sequence in \(H(E)\), with bound \(B_{\mu}^{(k)}\), independent of the choice of \(\epsilon\), more precisely, \(B_{\mu}^{(k)} \leq \frac{\pi}{2} \left( \sqrt{\frac{2\pi}{\delta}} \sqrt{C^{2} + 1} \|E'\|_{\infty} + \sqrt{\frac{M}{\pi}} B_{\Lambda_{k}} \right)^{2}\), for all \(k = 1, 2, \ldots, N\). Let \(B_{\mu} := \sum_{k=1}^{N} B_{\mu}^{(k)}\).

Since \(\frac{E'}{E} \in H^{\infty}(\mathbb{C}^{+})\) then the differentiation operator is bounded on \(H(E)\), i.e., \(f' \in H(E)\) whenever \(f \in H(E)\), by Theorem 3.5.1. Note that for all \(x \in \mathbb{R}\)

\[
\left( \frac{f'}{E} \right)(x) = \left| \frac{f'(x)}{E(x)} - \frac{E'(x)}{E(x)} \frac{f(x)}{E(x)} \right|
\]
\[ \sum_{k=1}^{N} \sum_{n} \left( \frac{f'}{E}(\mu_n^{(k)}) \right)^2 \leq 2 \sum_{k=1}^{N} \sum_{n} \left( \frac{f'}{E}(\mu_n^{(k)}) \right)^2 + 2 \left\| \frac{E'}{E} \right\|_{\infty}^2 \sum_{k=1}^{N} \sum_{n} \left( \frac{f}{E}(\mu_n^{(k)}) \right)^2 \]

\[ \leq \frac{2M}{\pi} \sum_{k=1}^{N} \sum_{n} \left( \frac{f'}{E}(\mu_n^{(k)}) \right)^2 + \frac{2M}{\pi} \left\| \frac{E'}{E} \right\|_{\infty}^2 \sum_{k=1}^{N} \sum_{n} \left( \frac{f}{E}(\mu_n^{(k)}) \right)^2 \]

\[ \leq \frac{2M}{\pi} B_{\mu} \| f' \|_{E}^2 + \frac{2M}{\pi} B_{\mu} \| \frac{E'}{E} \|_{\infty}^2 \| f \|_{E}^2 \]

\[ = \frac{2M}{\pi} B_{\mu}(C^2 + 1) \left\| \frac{E'}{E} \right\|_{\infty}^2 \| f \|_{E}^2 \]

Using the last inequality and (4.3.7) we get

\[ A_\lambda \| f \|_{E}^2 \leq \frac{2\pi}{\delta} \left[ \epsilon^2 \sum_{k=1}^{N} \sum_{n} \left( \frac{f}{E}(\mu_n^{(k)}) \right)^2 + N \sum_{\lambda' \in \Lambda'} \left| f(\lambda') \right|^2 \right] \]

\[ \leq \frac{2\pi}{\delta} \left[ \epsilon^2 \frac{2M}{\pi} B_{\mu}(C^2 + 1) \right] \left\| \frac{E'}{E} \right\|_{\infty}^2 \| f \|_{E}^2 \]

\[ = \frac{4M}{\delta} \epsilon^2 B_{\mu}(C^2 + 1) \left\| \frac{E'}{E} \right\|_{\infty}^2 \| f \|_{E}^2 \]

Thus,

\[ \frac{\delta}{2MN} (A_\lambda - \frac{4M}{\delta} \epsilon^2 B_{\mu}(C^2 + 1) \| E'/E \|_{\infty}^2) \| f \|_{E}^2 \leq \sum_{\lambda' \in \Lambda'} \left| f(\lambda') \right|^2 \frac{K(\lambda', \lambda')}{\lambda'} \]

It follows that \( \Lambda' \) is a sampling sequence if \( \epsilon < (A_\lambda - \frac{4M}{\delta} \epsilon^2 B_{\mu}(C^2 + 1) \| E'/E \|_{\infty}^2)^{1/2} \), with lower frame bound \( A' := \frac{\delta}{2MN} (A_\lambda - \frac{4M}{\delta} \epsilon^2 B_{\mu}(C^2 + 1) \| E'/E \|_{\infty}^2) \) and upper frame bound \( B_{\Lambda} \).

\[ \square \]

**Remark 4.3.2.** Lemma 4.3.6 implies that if the sequence \( \mathcal{M} \) in Theorem 4.3.1 is a sampling sequence in \( \mathcal{H}(E) \), which is not necessary uniformly separated, then the result that
$D^-(\mathcal{M}) \geq \frac{\delta}{\pi}$ still holds, if the underlying de Branges function $E$ satisfying $E'/E \in H^\infty(\mathbb{C}^+)$ with $\text{mt}(E^*/E) \neq 0$. Because, in this case, if $\mathcal{M}'$ is a uniformly separated subset of $\mathcal{M}$, which is sampling then by Theorem 4.3.1 we have $D^-(\mathcal{M}) \geq D^-(\mathcal{M}') \geq \frac{\delta}{\pi}$, as desired.

The following examples illustrate the method of constructing the subsequence $\Lambda'$ in Lemma 4.3.6.

**Example 4.1.** Let $\Lambda$ be a sequence defined by

$$\Lambda = \left\{ n, n + \frac{n+1}{n+2} \right\}_{n=0}^{\infty} = \left\{ 0, \frac{1}{2}, 1, \frac{2}{3}, 2, \frac{3}{4}, 3, \frac{4}{5}, \ldots \right\}.$$ 

Then it is clear that $\Lambda$ is not uniformly separated, as the distance between $n + \frac{n+1}{n+2}$ and the next element in the list going to 0 as $n \to \infty$. Let $\Lambda_1 = \left\{ \frac{1}{2}, \frac{2}{3}, 2, \frac{3}{4}, \ldots \right\}$ and $\Lambda_2 = \{ 0, 1, 2, 3, \ldots \}$, then $\Lambda_1$ is $\delta_1$-uniformly separated with $\delta_1 = 1$, and $\Lambda_2$ is $\delta_2$-uniformly separated with $\delta_2 = 1$.

So, $\Lambda = \Lambda_1 \cup \Lambda_2$, and $\Lambda$ is relatively separated. Let $\delta_o = 1$, and $0 < \epsilon < 1/4$, then a subset $\Lambda'$ of $\Lambda$ satisfying property (4.3.5) would be

$$\Lambda' = \left\{ 0, \frac{1}{2}, 1, \frac{2}{3}, 2, \frac{3}{4}, 3, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \ldots \right\} = \Lambda'_1 \cup \Lambda'_2 = \Lambda'_2$$

where $\Lambda'_1 = \Lambda_1$, and $\Lambda'_2 = \{ 0, 1, 2, 3 \}$. Therefore, $\Lambda'$ is $\epsilon$-uniformly separated sequence, for any $0 < \epsilon < \frac{1}{4}$. Thus, if $\Lambda$ is sampling in some de Branges space $\mathcal{H}(E)$ satisfying the hypothesis of Lemma 4.3.6, then $\Lambda'$ will be a uniformly separated sampling sequence in $\mathcal{H}(E)$.

**Example 4.2.** Let $\Lambda$ be a sequence defined by

$$\Lambda = \left\{ n, n + 2 + \frac{1}{n+3}, n + \frac{n+1}{n+2} \right\}_{n=0}^{\infty}$$

Then $\Lambda$ is not uniformly separated, for the same reason in the above example. Let $\Lambda_1 = \{ n + \frac{n+1}{n+2} \}_{n=0}^{\infty}$, $\Lambda_2 = \{ n \}_{n=0}^{\infty}$, and $\Lambda_3 = \{ n + 2 + \frac{1}{n+3} \}_{n=0}^{\infty}$, then $\Lambda_k$ is $\delta_k$-uniformly separated with $\delta_k = 1$, for $k = 1, 2, 3$.

So, for $\delta_o = 1$, and $0 < \epsilon < 1/4$, then a subset $\Lambda'$ of $\Lambda$ satisfying property (4.3.5) would be

$$\Lambda' = \left\{ 0, \frac{1}{2}, 1, \frac{2}{3}, 2, \frac{3}{4}, 3, \frac{4}{5}, 4, \frac{5}{6}, 5, \frac{6}{7}, 6, \frac{7}{8}, 7, \frac{8}{9}, \ldots \right\}$$

where

$$\Lambda'_1 = \Lambda_1 = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \ldots \right\}.$$
\[ \Lambda_2' = \Lambda_1' \cup \Lambda_2', \Lambda_2' = \{0, 1, 2, 3\}, \]
\[ \Lambda_3' = \Lambda_2' \cup \Lambda_3', \Lambda_3' = \{2\frac{1}{3}, 3\frac{1}{3}\}. \]

Therefore, \( \Lambda' \) is \( \epsilon \)-uniformly separated sequence for any \( 0 < \epsilon < \frac{1}{4} \). Note that \( \lambda' = 7\frac{8}{5} \in \Lambda' \) is within \( \epsilon \) of \( \lambda^{(2)}_8 = 8 \) and \( \lambda^{(3)}_6 = 8\frac{1}{5} \).

### 4.4 De Branges Spaces of Exponential Type

Since the elements of a de Branges space are entire functions, it is a natural task to study de Branges spaces from the viewpoint of growth properties of their elements. In this section we consider such de Branges spaces whose elements possess a certain growth behaviour, specifically, all are of exponential type. To begin with, we will set up some notations and prove a couple of results concerning such spaces.

Recall that an entire function \( f \) is of exponential type if there exist constants \( A, B > 0 \) such that \( |f(z)| \leq Ae^{B|z|} \) for all \( z \in \mathbb{C} \), and that the exponential type is defined as the infimum of all \( B \)'s above. The main reason for our interest in functions of exponential type is that if the structure function of a de Branges space is of exponential type then any function in the space is of exponential type, as Proposition 3.2.3 shows.

**Proposition 4.4.1.** Let \( \mathcal{H}(E) \) be a de Branges space where \( E(z) \) is of exponential type \( \tau_E \). Let \( \mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \) be a sequence of real numbers. If \( D^- (\mathcal{M}) > \tau_E \), then the sequence \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) is complete.

**Proof.** Given a nonzero function \( f \in \mathcal{H}(E) \), with \( f \) orthogonal to every element in the sequence \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \), then
\[ f(\mu_n) = \langle f(t), K(\mu_n, t) \rangle_E = 0, \] for all \( n \), hence, \( f(\mu_n) = 0 \) for all \( n \).

Since \( f \in \mathcal{H}(E) \), then by Proposition 3.2.3 \( f \) is of exponential type \( \tau_f \leq \tau_E \). So, given \( \epsilon > 0 \) there exist \( A_{\epsilon} > 0 \) such that
\[ |f(z)| \leq A_{\epsilon} e^{(\tau_f + \epsilon)|z|}, \]
for all $z \in \mathbb{C}$. First, if $\tau_f \leq \tau_E$ then we can find $\epsilon' > 0$ such that $\tau_f + \epsilon' = \tau_E$. Hence, there exist $A_{\epsilon'} > 0$ such that

$$|f(z)| \leq A_{\epsilon'} e^{(\tau_f + \epsilon')|z|} = A_{\epsilon'} e^{\tau_E|z|}, \text{ for all } z \in \mathbb{C}.$$  

Next we will use Jensen’s Theorem, and for this we may assume, without loss of generality, that $|f(0)| \neq 0$, otherwise, we just use the general Jensen’s formula (2.2.14) instead (by considering the function $\frac{f(z)}{z^m}$, where $m$ is the order of the zero at 0). In fact we can assume that $|f(0)| = 1$ (or we consider the function $f(z)/|f(0)|$). Therefore, by Jensen’s Theorem, we have

$$N(r) = \int_{0}^{r} \frac{n(t)}{t} dt \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta - \log |f(0)|$$

$$\leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta$$

$$\leq \log A_{\epsilon'} + \tau_E r.$$  

where $n(t)$ denotes the number of zeros of $f$ in the closed disk $|z| \leq t$, counted according to multiplicity. For any $t > 0$, recall that, by (2.5.4), $n^{-}(t) := \inf_{x \in \mathbb{R}} \sharp(\{\mu_n\} \cap [x - t, x + t])$, hence,

$$n^{-}(t) = \inf_{x \in \mathbb{R}} \sharp(\{\mu_n\} \cap [x - t, x + t])$$

$$\leq \sharp(\{\mu_n\} \cap [-t, t]),$$  

(by taking $x = 0$)

$$\leq n(t)$$

So, for all $r > 0$ we have

$$\int_{0}^{r} \frac{n^{-}(t)}{t} dt \leq \int_{0}^{r} \frac{n(t)}{t} dt \leq \log A_{\epsilon'} + \tau_E r$$

Since $D^{-}\{\mu_n\} > \tau_E$, then there exists $r_o > 0$ such that $\frac{n^{-}(r)}{r} > \tau_E$, for all $r \geq r_o$. Therefore,

$$\int_{0}^{r_o} \frac{n^{-}(t)}{t} dt + \int_{r_o}^{r} \frac{n^{-}(t)}{t} dt = \int_{0}^{r} \frac{n^{-}(t)}{t} dt \leq \log A_{\epsilon'} + \tau_E r$$

hence,

$$c + \tau_E(r - r_o) < \log A_{\epsilon'} + \tau_E r$$
where \( c = \int_0^r n(t)/t \, dt < \infty \). Dividing by \( r \) and taking the limit as \( r \to \infty \) we get \( \tau_E < \tau_E \), a contradiction.

If \( \tau_f = \tau_E \), then since \( D^{-}(\{ \mu_n \}) > \tau_E \), we can find \( \epsilon_o > 0 \) small enough, such that \( D^{-}(\{ \mu_n \}) - \tau_E > \epsilon_o \). Hence, for any \( \epsilon \) with \( 0 < \epsilon < \epsilon_o \) there exists \( r_\epsilon > 0 \) such that

\[
\frac{n(r)}{r} > \tau_E + \epsilon
\]

for all \( r \geq r_\epsilon \). Fix \( \epsilon' \), \( \epsilon'' \) with \( 0 < \epsilon' < \epsilon'' < \epsilon_o \), then there exists \( r_{\epsilon''} > 0 \) such that \( \frac{n(r)}{r} > \tau_E + \epsilon'' \)

for all \( r \geq r_{\epsilon''} \), and there exists \( A_{\epsilon'} > 0 \) such that

\[
|f(z)| \leq A_{\epsilon'} e^{(\tau_E + \epsilon')|z|}
\]

for all \( z \in \mathbb{C} \). It follows that

\[
\int_0^{r_{\epsilon''}} \frac{n(t)}{t} \, dt + \int_{r_{\epsilon''}}^r \frac{n(t)}{t} \, dt = \int_0^r \frac{n(t)}{t} \, dt \leq \log A_{\epsilon'} + (\tau_E + \epsilon')r
\]

hence,

\[
C + (\tau_E + \epsilon'')(r - r_{\epsilon''}) < \log A_{\epsilon'} + (\tau_E + \epsilon')r
\]

Dividing by \( r \) and taking the limit as \( r \to \infty \) we get \( \tau_E + \epsilon'' < \tau_E + \epsilon' \), which implies that \( \epsilon'' < \epsilon' \), a contradiction. So in both cases it follows that \( f \equiv 0 \). Thus, the sequence \( \{ k_{\mu_n}(z) \}_{n \in \mathbb{Z}} \) is complete. \( \square \)

Recall that if an entire function \( E \in \mathcal{HB} \) is of exponential type and has no real zeros then by (2.3.4), \( E(z) \) has the following representation

\[
E(z) = \gamma e^{bz} e^{-iaz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z Re(\frac{1}{z_n})} , \quad a = -\frac{1}{2} \text{mt} \frac{E^*}{E} \tag{4.4.1}
\]

where \( \gamma \in \mathbb{C} \), \( b \in \mathbb{R} \), \( a \geq 0 \), and \( \{ z_n \}_{n \in \mathbb{Z}} \) is the zeros set of \( E \) in the lower half-plane satisfying

\[
\sum_n |\text{Im} \frac{1}{z_n}| < \infty \tag{4.4.2}
\]

**Definition 4.4.1.** Denote the product of factors in (4.4.1) except \( e^{-iaz} \), by \( E_o(z) \), i.e.,

\[
E_o(z) = \gamma e^{bz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z Re(\frac{1}{z_n})} . \tag{4.4.3}
\]
Then \( E_o(z) = \frac{E(z)}{e^{-iaz}} \) is an entire function (since \( e^{-iaz} \) has no zeros), moreover, the zeros of \( E_o(z) \) satisfy (4.4.2).

The next lemma shows that the function \( E_o(z) \) is a function of exponential type of Hermite-Beihler class.

**Lemma 4.4.2.** Let \( E(z) \in \mathcal{HB} \) be of exponential type that has the representation in (4.4.1), and \( E_o(z) \) as defined in Definition 4.4.1. Then \( E_o(z) \) is entire function of exponential type, has no zeros in the upper half-plane, and \(|E_o(\bar{z})| \leq |E_o(z)| \) for all \( z \in \mathbb{C}^+ \).

**Proof.** Let \( E(z) \) be an entire function of exponential type \( \tau_E \). Note that \( e^{-iaz} \) is entire function of exponential type \( a \) which has no zeros, and \( E_o(z) = \frac{E(z)}{e^{-iaz}} \) for all \( z \in \mathbb{C} \), thus, \( E_o(z) \) is entire function which is a quotient of two entire functions of exponential type, and hence, it is of exponential type \( \tau_o \) by Lemma 2.2.2, with \( \tau_E \leq a + \tau_o \). Moreover, \( E_o(z) \) has no zeros in the upper half-plane, since the zeros of \( E(z) \) and \( E_o(z) \) are the same.

To show that \(|E_o(\bar{z})| \leq |E_o(z)| \) for all \( z \in \mathbb{C}^+ \), it is sufficient by Theorem 2.3.3 to find some \( \theta \in (0, \pi) \) such that \( h_o(\theta) \geq h_o(-\theta) \), where \( h_o(\theta) \) is the indicator function of \( E_o(z) \).

Let \( h_E(\theta) \) and \( h_a(\theta) \) be the indicator functions of \( E(z) \), \( e^{-iaz} \), respectively. Since \(|E(\bar{z})| < |E(z)| \) for all \( z \in \mathbb{C}^+ \), then it follows by Theorem 2.3.5 that \( a = \frac{1}{2} [h_E(\frac{\pi}{2}) - h_E(-\frac{\pi}{2})] \geq 0 \). On the other hand, by Definition 2.2.1 of the indicator function we have

\[
h_o(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r} = \limsup_{r \to \infty} \frac{\log |e^{-ia(re^{i\theta})}|}{r} = \limsup_{r \to \infty} \frac{\log |e^{-ia(r \cos \theta + ir \sin \theta)}|}{r} = \limsup_{r \to \infty} \frac{\log |e^{ar \sin \theta}|}{r} = \limsup_{r \to \infty} \frac{ar \sin \theta}{r} = a \sin \theta
\]

for \( \theta \in [-\pi, \pi] \). Hence, \( h_o(\frac{\pi}{2}) = a \), and \( h_o(-\frac{\pi}{2}) = -a \). Since \( E_o(z) = \frac{E(z)}{e^{-iaz}} \), then

\(|E(x)| = |e^{-iax}| |E_o(x)| = |E_o(x)|, \) for all \( x \in \mathbb{R} \).

So, applying Theorem 2.2.3 (with \( f_1(z) = E(z) \) and \( f_2(z) = E_o(z) \)) we obtain

\( h_o(\theta) = h_E(\theta) - h_a(\theta) \), for all \( \theta \in [0, \pi] \).
Therefore,
\[ h_o\left(\frac{\pi}{2}\right) - h_o\left(-\frac{\pi}{2}\right) = \left( h_E\left(\frac{\pi}{2}\right) - h_E\left(-\frac{\pi}{2}\right) \right) - \left( h_a\left(\frac{\pi}{2}\right) - h_a\left(-\frac{\pi}{2}\right) \right) \]
\[ = 2a - 2a = 0 \]

Thus, by Theorem 2.3.3, \(|E_o(\bar{z})| \leq |E_o(z)|\) for all \(z \in \mathbb{C}^+\), completing the proof. \(\square\)

**Lemma 4.4.3.** Let \(E(z) = e^{-iaz}E_o(z)\), where \(E_o(z)\) as defined in Definition 4.4.1 and \(a = -\frac{1}{2} \text{mt} \frac{E^*_E}{E} \neq 0\). Then for any \(F(z) \in \mathcal{H}(e^{-iaz})\), the function \(f(z) = F(z)E_o(z)\) belongs to the space \(\mathcal{H}(E)\).

**Proof.** Let \(F \in \mathcal{H}(e^{-iaz})\), and \(f(z) := F(z)E_o(z)\), then \(f(z)\) is entire function. To show that \(f \in \mathcal{H}(E)\) we need to verify the conditions of Definition 3.2.2. Note that since \(F \in \mathcal{H}(e^{-iaz})\), then \(F(z)e^{-iaz}\) and \(F^*(z)e^{-iaz}\) are of bounded type and nonpositive mean type in the upper half-plane by Definition 3.2.2. Moreover, \(\frac{F(t)}{e^{-iat}} \in L^2(\mathbb{R})\). Since \(|E_o(\bar{z})| \leq |E_o(z)|\) for \(z \in \mathbb{C}^+\) by Lemma 4.4.2, and \(E_o(z)\) has no zeros in the upper half-plane, then the ratio \(\frac{E^*_o(z)}{E_o(z)}\) is analytic in \(\mathbb{C}^+\). Thus,
\[ \left| \frac{E^*_o(z)}{E_o(z)} \right| \leq 1, \quad \text{for all } z \in \mathbb{C}^+, \quad (4.4.4) \]
i.e., \(\frac{E^*_o(z)}{E_o(z)}\) is bounded in \(\mathbb{C}^+\), hence of bounded type in \(\mathbb{C}^+\). Now, we claim that the ratio \(\frac{E^*_o(z)}{E_o(z)}\) has a nonpositive mean type in \(\mathbb{C}^+\). Indeed, by the definition of the mean type in (2.2.6), then using (4.4.4), the mean type of \(\frac{E^*_o(z)}{E_o(z)}\) in the upper half plane is given by
\[ \text{mt}(E_o/E_o) = \limsup_{y \to +\infty} \frac{\log |E_o^*(iy)/E_o(iy)|}{y} \leq \limsup_{y \to +\infty} \frac{\log(1)}{y} = 0 \]

Now, since
\[ \frac{f(z)}{E(z)} = \frac{F(z)}{e^{-iaz}E_o(z)}E_o(z) = \frac{F(z)}{e^{-iaz}} \]
and
\[ \frac{f^*(z)}{E(z)} = \frac{F^*(z)}{e^{-iaz}E_o(z)}E_o^*(z) = \frac{F^*(z)}{e^{-iaz}} \frac{E^*_o(z)}{E_o(z)} \]
then, using the fact that product of two functions of bounded type is of bounded type, and the mean type of the product does not exceed the sum of the mean types of the two functions,
then $f(z)E(z)$ and $f^*(z)E(z)$ are of bounded type and nonpositive mean type in $\mathbb{C}^+$. Also,

$$\int_{\mathbb{R}} \left| f(t) \right|^2 dt = \int_{\mathbb{R}} \left| \frac{F(t)}{e^{-iat}} \right|^2 dt < \infty.$$ 

Thus, by Definition 3.2.2, $f \in \mathcal{H}(E)$, and $\|f\|_E = \|F\|_{L^2(\mathbb{R})}$. 

Theorem 4.4.4. Let $E \in \mathcal{HB}$ be of exponential type with $a = -\frac{1}{2} \text{mt} \frac{E^*}{E} \neq 0$. Let $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$. If $D^+(\mathcal{M}) < \frac{a}{\pi}$ then the set $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is incomplete.

Proof. Since $E \in \mathcal{HB}$ and is of exponential type then $E(z) = e^{-iaz}E_o(z)$, where $E_o(z)$ as defined in Definition 4.4.1 and $a = -\frac{1}{2} \text{mt} \frac{E^*}{E}$. Since $E(z)$ is of exponential type $\tau_E$ and $e^{-iaz}$ is of exponential $a$, then $E_o(z)$ is an entire function of exponential type, say $\tau_o$, such that $\tau_E \leq a + \tau_o$. It follows that $|E_o(\bar{z})| \leq |E_o(z)|$ for $z \in \mathbb{C}^+$ by Lemma 4.4.2.

Since $D^+(\mathcal{M}) < \frac{a}{\pi}$, then by Theorem 3.1.4 there exist $F \in PW_a = \mathcal{H}(e^{-iaz})$ such that $F(\mu_n) = 0$ for all $n \in \mathbb{Z}$, and $F \not\equiv 0$. Define the entire function $f(z) := F(z)E_o(z)$, then $f \not\equiv 0$, and $f(\mu_n) = 0$ for all $n \in \mathbb{Z}$, moreover, $f \in \mathcal{H}(E)$ by Lemma 4.4.3. Hence, the set $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ is incomplete in $\mathcal{H}(E)$. 

Theorem 4.4.5. Let $E \in \mathcal{HB}$ be of exponential type, with $a = -\frac{1}{2} \text{mt} \frac{E^*}{E} \neq 0$, and $\frac{E'}{E} \in L^\infty(\mathbb{R})$. Let $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$. Then, $\mathcal{M}$ is a Plancherel-Pólya sequence in $\mathcal{H}(E)$ if and only if $D^+(\mathcal{M}) < \infty$.

Proof. Since $E(z)$ is of exponential type then $E(z) = e^{-iaz}E_o(z)$, where $E_o(z)$ as defined in Definition 4.4.1. Also, since $a \neq 0$ then $\varphi'(x) \geq a > 0$. Therefore, by Theorem 4.1.6, if $D^+(\mathcal{M}) < \infty$ then $\mathcal{M}$ is a Plancherel-Pólya sequence in $\mathcal{H}(E)$.

To prove the other direction, suppose that $D^+(\mathcal{M}) = \infty$, we will show that the sequence $\mathcal{M}$ is not Plancherel-Pólya sequence in $\mathcal{H}(E)$.

Note that the function $F(z) := \frac{\sin az}{\pi z}$ belongs to the Paley-Wiener space $PW_a$, where $a = -\frac{1}{2} \text{mt} \frac{E'}{E}$. Furthermore, the translation of this function also belongs to the same space, i.e.,
\( F_\beta(z) := \frac{\sin (a(z-\beta))}{\pi(z-\beta)} \in PW_a \) for all \( \beta \in \mathbb{R} \), and in fact, \( F_\beta(z) = K_a(\beta, z) \), the reproducing kernel of \( PW_a \) at \( w = \beta \), see (3.2.6). For the given \( \beta \), define the entire function \( f_\beta(z) := F_\beta(z)E_0(z) \).

Then, \( f_\beta(z) \in \mathcal{H}(E) \) by Lemma 4.4.3 for all \( \beta \), and

\[
\|f_\beta\|_E^2 = \|F_\beta\|_{PW_a}^2 = \|F\|_{PW_a}^2 = K_a(\beta, \beta) = a/\pi, \quad \text{for all} \; \beta \in \mathbb{R}.
\]

We will prove that given \( N \), there exists \( \beta \) such that

\[
\sum_{n \in \mathbb{Z}} \left| f_\beta(\mu_n) \right|^2 \geq L.N \|f_\beta\|^2,
\]

where \( L \) is a constant independent of \( N \). Hence, since \( N > 0 \) is arbitrary, it will follows that the sequence \( \mathcal{M} \) is not a Plancherel-Pólya sequence in \( \mathcal{H}(E) \), by Definition 2.5.1.

First note that the function \( F(x) \) is continuous on \( \mathbb{R} \) and nonzero at \( x = 0 \), so there exists \( h > 0 \) such that

\[
C := \inf_{x \in (0,h)} |F(x)| > 0
\]

Now, since we assume that \( D^+(\mathcal{M}) = \infty \), then by part (d) of Lemma 2.5.3, for the given \( h > 0 \) above, and any \( N \in \mathbb{N} \), we have

\[
\sup_{n \in \mathbb{Z}} \sharp(\mathcal{M} \cap (nh - h, nh + h)) \geq N,
\]

hence, there exists \( n_o = n_o(N) \in \mathbb{Z} \) such that

\[
\sharp(\mathcal{M} \cap (\beta-h, \beta + h)) \geq N,
\]

where \( \beta = n_o h \). Define the index set \( I_N := \{ n \in \mathbb{Z} : \mu_n \in (-h+h+\beta) \} = \{ n \in \mathbb{Z} : \mu_n - \beta \in (-h, h) \} \). Since \( \frac{\varphi'}{\varphi} \in L^\infty(\mathbb{R}) \) then by Lemma 3.5.4 there exist \( M > 0 \) such that \( \varphi'(x) \leq M \) for all \( x \in \mathbb{R} \). Let \( K_E(w, z) \) be the corresponding reproducing kernel of \( \mathcal{H}(E) \), then

\[
\sum_{n \in \mathbb{Z}} \left| \frac{f_\beta(t)}{K_E(\mu_n, t)} \right|^2 = \sum_{n \in \mathbb{Z}} \left| f_\beta(\mu_n) \right|^2 \frac{\pi}{\varphi'(\mu_n)}, \quad \text{by property (3.2.4)}
\]

\[
= \sum_{n \in \mathbb{Z}} \left| \frac{f_\beta(\mu_n)}{E(\mu_n)} \right|^2 \frac{\pi}{\varphi'(\mu_n)}, \quad \text{by (3.4.2)}
\]

\[
\geq \frac{\pi}{M} \sum_{n \in \mathbb{Z}} \left| \frac{f_\beta(\mu_n)}{E(\mu_n)} \right|^2
\]

\[
= \frac{\pi}{M} \sum_{n \in \mathbb{Z}} \left| \frac{F_\beta(\mu_n)E_0(\mu_n)}{e^{-ia\mu_n}E_0(\mu_n)} \right|^2
\]
\[
\frac{\pi}{M} \sum_{n \in \mathbb{Z}} |F_\beta(\mu_n)|^2 \\
= \frac{\pi}{M} \sum_{n \in \mathbb{Z}} |\langle F_\beta(t), K_\alpha(\mu_n, t) \rangle_{PW_a}|^2, \quad \text{by property (3.2.4)} \\
\geq \frac{\pi}{M} \sum_{n \in I_N} |\langle F_\beta(t), K_\alpha(\mu_n, t) \rangle_{PW_a}|^2 \\
= \frac{\pi}{M} \sum_{n \in I_N} |F_\beta(\mu_n)|^2 \\
\geq \frac{\pi}{M} \inf_{n \in I_N} |F(\mu_n - \beta)|^2 \\
= \frac{\pi}{M} N \inf_{x \in (-h, h)} |F(x)|^2 \\
\geq \frac{\pi}{M} N C_2 \frac{\pi}{a} \|F_\beta\|^2 \\
= \frac{\pi^2 C_2 N}{aM} \|f_\beta\|^2
\]

Thus,
\[
\sum_{n \in \mathbb{Z}} |\langle f_\beta(t), \frac{K_E(\mu_n, t)}{\|K_E(\mu_n, \cdot)\|_E} \rangle_E|^2 \geq L \cdot N \|f_\beta\|^2
\]
where \( L = \frac{\pi^2 C_2}{aM} \) and \( \|f_\beta\|^2 = a/\pi \).

Recall that a sequence \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \) is an interpolating sequence in \( \mathcal{H}(E) \) if for every sequence of scalars \( \{c_n\}_{n \in \mathbb{Z}} \) there exist \( f \in \mathcal{H}(E) \) such that
\[
f(\gamma_n) = c_n, \quad \text{whenever} \quad \sum_{n} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} < \infty
\]
for all \( n \in \mathbb{Z} \), where \( K(w, z) \) is the corresponding reproducing kernel of \( \mathcal{H}(E) \). We have already proved a necessary conditions of interpolating sequences in the space \( \mathcal{H}(E) \) in Theorem 4.3.2 using the Comparison Theorem approach. We now describe interpolating sequences for some de Branges spaces of exponential type. The core of our approach is to turn our problem into one about interpolating sequences in Paley-Wiener spaces \( PW_a \).

**Theorem 4.4.6.** Let \( E \in \mathcal{HB} \) be of exponential type with \( a = -\frac{1}{2} \text{int} \frac{E'}{E} \neq 0 \) and \( \frac{E'}{E} \in L^\infty(\mathbb{R}) \). Let \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be a uniformly separated sequence. If \( D^+(\Gamma) < \frac{a}{\pi} \) then \( \Gamma \) is an interpolating sequence in \( \mathcal{H}(E) \).

**Proof.** Let \( \{c_n\}_{n \in \mathbb{Z}} \) be any sequence of scalars such that
\[
\sum_{n} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} < \infty.
\]
We will find a function $f \in \mathcal{H}(E)$ which solves the interpolation problem $f(\gamma_n) = c_n$ for all $n$. Since $E \in \mathcal{H}B$ and of exponential type then $E(z)$ has the representation in (4.4.1); $E(z) = e^{-iaz}E_o(z)$, where $E_o(z)$ as defined in Definition 4.4.1 and $a = -\frac{1}{2} \text{mt} \frac{E^*}{E}$. By Lemma 4.1.11 we may assume without loss of generality that $E(z)$ has no real zeros. Define a sequence $a_n := \frac{c_n}{E_o(\gamma_n)}$, for all $n \in \mathbb{Z}$.

Then using the fact that $\varphi'(x) \leq M$ by Lemma 3.5.4, with $|E(x)| = |e^{-iax}E_o(x)| = |E_o(x)|$, and $K(x, x) = \frac{1}{\pi} \varphi'(x)|E(x)|^2$ for all $x \in \mathbb{R}$, we get

$$
\sum_{n \in \mathbb{Z}} |a_n|^2 = \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{|E_o(\gamma_n)|^2} \leq \frac{M}{\pi} \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{|E(\gamma_n)|^2 \varphi'(\gamma_n)} = \frac{M}{\pi} \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{K(\gamma_n, \gamma_n)} < \infty,
$$

hence, $\{a_n\}_{n \in \mathbb{Z}} \in l^2$. Since $\Gamma$ is uniformly separated with $D^+(\Gamma) < \frac{a}{\pi}$ then, by Theorem 3.1.3, $\Gamma$ is an interpolating sequence in $\mathcal{H}(e^{-iaz})$. Therefore, since $\{a_n\}_{n \in \mathbb{Z}} \in l^2$ there exist $G \in \mathcal{H}(e^{-iaz})$ such that

$$
G(\gamma_n) = a_n, \text{ for all } n \in \mathbb{Z}.
$$

Define the function $f(z) := G(z)E_o(z)$, then $f \in \mathcal{H}(E)$ by Lemma 4.4.3. Moreover, note that

$$
f(\gamma_n) = G(\gamma_n)E_o(\gamma_n) = a_nE_o(\gamma_n) = c_n
$$

for all $n \in \mathbb{Z}$, where the sequence $\{c_n\}$ satisfying (4.4.5) above. Therefore, the sequence $\Gamma$ is an interpolating sequence in $\mathcal{H}(E)$.

We are now in a position to connect our results to the so called Feichtinger conjecture. The Feichtinger conjecture originated in harmonic analysis and currently is a topic of high interest as it has been shown to be equivalent to the celebrated Kadison-Singer problem [7].
The Feichtinger conjecture asks whether every bounded Bessel sequence \( \{f_n\}_{n \in I} \) can be written as the union of finitely many Riesz sequences [8]. Note that any Bessel sequence \( \{f_n\}_{n \in I} \) is uniformly bounded above in norm by its Bessel bound, this follows directly from (2.1.10). We say that it is bounded if it is bounded away from zero, that is there exist a constant \( C > 0 \) such that \( \|f_n\| \geq C \) for every \( n \in I \).

As we have seen, the sequences of interest are sequences of normalized reproducing kernels. Given a reproducing kernel Hilbert space \( \mathcal{H} \) with reproducing kernel function \( K(x, y) \) on a set \( X \), the normalized reproducing kernel at \( x \) is the function \( K(x, \cdot)/\sqrt{K(x,x)} \). So, given a sequence of points \( \{\mu_n\}_{n \in I} \) we obtain a sequence of unit norm functions \( \{K(\mu_n, \cdot)/\sqrt{K(\mu_n, \mu_n)}\}_{n \in I} \) in \( \mathcal{H} \), and the Feichtinger conjecture is equivalent to the following statement:

**Feichtinger Conjecture:** *Every Bessel sequence of unit functions in a Hilbert space can be partitioned into finitely many Riesz sequences.*

As an example, Nikolski [38] proved that the Hardy space \( H^2(\mathbb{D}) \) on the unit disc \( \mathbb{D} \), with reproducing kernel given in (2.4.3) satisfies the Feichtinger Conjecture. Now, since de Branges spaces are reproducing kernel Hilbert spaces it should be natural that we ask if this conjecture is true in such spaces. We will show that the Feichtinger conjecture is true if the underlying space is of exponential type where the structure function \( E(z) \) satisfies certain conditions.

**Proposition 4.4.7.** If \( E \in \mathcal{HB} \) is of exponential type with \( a = -\frac{1}{2} \liminf \frac{E^*}{z} \neq 0 \) and \( \frac{E^*}{z} \in L^\infty(\mathbb{R}) \), then every Bessel sequence of normalized reproducing kernels in \( \mathcal{H}(E) \) can be partitioned into finitely many Riesz sequences.

**Proof.** Let \( \mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \), and \( \{k_{\mu_n}(z)\}_{n \in \mathbb{Z}} \) be a Bessel sequence in \( \mathcal{H}(E) \). Then by Theorem 4.4.5 \( D^+(\mathcal{M}) < \infty \). It follows by Lemma 2.5.3 that the sequence \( \mathcal{M} \) can be partitioned into finitely many disjoint separated sequences \( \mathcal{M}_k, k = 1, 2, \ldots, N \), each with density \( D^+(\mathcal{M}_k) < \infty \). Without loss of generality, we may assume that \( D^+(\mathcal{M}_k) < \frac{a}{\pi} \) for all \( k = 1, 2, \ldots, N \) (for if \( D^+(\mathcal{M}_{k_0}) \geq \frac{a}{\pi} \) for some \( k_0 \in \{1, 2, \ldots, N\} \) we can partition the sequence \( \mathcal{M}_{k_0} \) into \( N_{k_0} \) disjoint separated sequences each with density less than \( \frac{a}{\pi} \), since the density is finite we can do this finitely many times). Theorem 4.4.6 now implies that the se-
quence of the corresponding normalized reproducing kernels of $\mathcal{M}_k$ is a Riesz sequence for all $k = 1, 2, \ldots, N$.

It should be noted that, in our context, Proposition 4.4.7 means that every Plancherel-Pólya sequence in $\mathcal{H}(E)$ can be written as a finite union of interpolating sequences. Similar proof as above shows that in de Branges spaces with the same conditions in Proposition 4.4.7, every Bessel sequence of normalized reproducing kernels can be partitioned into finitely many incomplete sequences.

**Proposition 4.4.8.** If $E \in \mathcal{HB}$ is of exponential type with $a = -\frac{1}{2} \text{mt} \frac{E^*}{E} \neq 0$ and $\frac{E'}{E} \in L^\infty(\mathbb{R})$, then every Bessel sequence of normalized reproducing kernels in $\mathcal{H}(E)$ can be partitioned into finitely many incomplete sequences.

**Proof.** Let $\mathcal{M} = \{\mu_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$, and $\{k_{\mu_n}(z)\}_{n \in \mathbb{Z}}$ be a Bessel sequence in $\mathcal{H}(E)$, then by Theorem 4.4.5 $D^+ (\mathcal{M}) < \infty$. It follows by Lemma 2.5.3 that the sequence $\mathcal{M}$ can be partitioned into finitely many disjoint separated sequences $\mathcal{M}_k, k = 1, 2, \ldots, N$, each with density $D^+(\mathcal{M}_k) < \infty$. Without loss of generality, we may assume that $D^+(\mathcal{M}_k) < \frac{a}{\pi}$ for all $k = 1, 2, \ldots, N$, for the same reason above. Theorem 4.4.4 now implies that the sequence of the corresponding normalized reproducing kernels of $\mathcal{M}_k$ is an incomplete sequence for all $k = 1, 2, \ldots, N$.

**Remark 4.4.1.** It should be noted that in all results in this chapter where $\varphi'(x)$ was required to be bounded away from zero in the proofs, the constant $\delta$ where $\varphi'(x) \geq \delta > 0$, can be replaced by the constant $a$ where $a = -\frac{1}{2} \text{mt}(E^*/E) \neq 0$ whenever the structure function $E(z)$ is of exponential type. This fact follows from the representation of $\varphi'(x)$ in (3.4.4).
BIBLIOGRAPHY


