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FINITE DEFORMATION OF ELASTIC MEMBRANES WITH APPLICATION TO THE STABILITY OF AN INFLATED SPHERE

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I. INTRODUCTION

The theory of finite deformations of elastic isotropic bodies has been considered by different authors. Rivlin has published a number of papers of which [1], [8], and [18] are listed in the bibliography. The general theory of finite deformations is developed in tensor notation by Green and Zerna [11] and Green and Shield [9] using a general system of coordinates. Two books, one by Green and Zerna [10], and one by Green and Adkins [7], also treat this subject in considerable detail. A brief review is presented in chapter II of the theory of finite deformations in a general curvilinear coordinate system.

The theory of finite deformations of elastic bodies is difficult to apply to specific problems because of the non-linearity of the equations involved. For thin shells, the problem may be simplified by considering only the motion of the middle surface of the shell. This approach simplifies the problem by removing the dependence of the displacements on one of the spatial variables; it has been applied to specific problems in [10] and [11]. A further simplification is achieved if the thin shell is considered as a thin membrane. In membrane theory, the stress couples in the membrane and shearing forces perpendicular to the membrane are neglected in comparison with the stress resultants in the membrane. This assumption reduces the complexity of the equations of motion but also neglects the variation of stress over the thickness of the shell. The general theory of finite deformations of a thin membrane has been treated by Rivlin [6] and Adkins and Rivlin [1]. Corneliussen and Shield [2] re-formulate this theory, an outline of which is presented in chapter III. Even with the membrane simplifications the resulting equations are non-linear. If in a particular problem the boundary of the body has a simple geometry, the problem may be solved in a semi-inverse manner. Specifically, the problem is to solve for the displacements when the body and surface forces are given. The semi-inverse method is used in V.A. to solve the problem of the homogeneous inflation of a spherical shell. The extended radius is given and the stress resultants and internal pressure necessary to maintain this inflation are determined from the equations of motion.

Another method which may be used to overcome the difficulty of solving a system of non-linear equations is to consider a perturbation theory in which a small deformation is superposed on a known finite deformation. This technique was developed in general by Green, Rivlin and Shield [8] and was reduced by Corneliussen and Shield [2] to the case of a thin membrane. The method for a thin membrane is re-formulated in IV and used in V.B. to develop the equations of motion for a thin spherical shell which has undergone a finite deformation by inflation. The motion is described by three second order homogeneous partial differential equations coupled in the unknown displacements.

In V.C. the solution of the equations of motion is obtained in closed form for motion independent of the circumferential angle. This general solution is obtained in terms of associated Legendre functions of the first and second kinds.

One of the proposed methods of placing a large satellite in orbit uses an inflatable sphere. This can be launched while deflated to reduce weight and air resistance and inflated after leaving the earth's atmosphere. The mode shapes for the small free vibrations of the satellite are needed before any dynamic analysis can be made as to the type of load or equipment the satellite can carry. Also, the stability of the inflated form of the sphere under small disturbances must be determined if it is to usefully serve as a satellite. The mode shapes are obtained from the solutions to the equations of motion. A dynamic stability analysis determines criteria that would allow the displacements to remain finite for all time. That is, the criteria determine values of physical and geometrical parameters for which small disturbances give rise only to bounded displacements. In some problems the equations are such that a direct dynamic stability analysis is not practical. A static analysis of stability can be used to predict the dynamic regions of stability.

This is much easier to apply and has been shown by Zeigler [3] to be equivalent to a dynamic analysis for holonomic systems. A body is said to be statically stable for a given finite deformation and given constraints if no non-trivial equilibrium configurations exist neighboring to the finite deformation and satisfying the constraints. If such configurations do exist, the body is statically unstable. The regions of stability of the body are thus determined by investigating the regions where a static instability will not occur. An investigation is made, in V. D., into the determination of the stability of the sphere and the frequencies of vibration for a neo-Hookian material.

II. FINITE DEFORMATION THEORY

Consider a homogeneous elastic body at rest, isotropic in its undeformed state. Referred to a rectangular Cartesian coordinate system x_i at $t = 0$, the coordinates of a generic point are (x_1, x_2, x_3). If the same body is referred to a general curvilinear coordinate system y^i, then

$$x_i = x_i(y^1, y^2, y^3). \tag{2.1}$$

The convention used is that Latin indices take on the values 1, 2, 3 and Greek indices the values 1, 2. If in some expression an index occurs twice, the expression is to be summed with respect to that index over its range of values.

If $\bar{r}$ is the position vector from the origin of the rectangular Cartesian system x_i to a generic point of the body, the covariant base vectors of the system y^i in the undeformed body are

$$\bar{g}_i = \frac{\partial \bar{r}}{\partial y^i} = \bar{r}_{,i}. \tag{2.2}$$

In (2.2) and hereafter, a comma followed by an index, say i, denotes differentiation with respect to y^i.

For convenience, the following definitions from tensor analysis are given.

A set of three quantities $\{A_i(x_1, x_2, x_3)\}$ associated with the coordinate system x_i represents components of a covariant tensor of rank one (a vector) if the corresponding set of three quantities $\{B_j(y^1, y^2, y^3)\}$ associated with the coordinate system y^j is given by

$$B_j(y^1, y^2, y^3) = \frac{\partial x_i}{\partial y^j} A_i(x_1, x_2, x_3). \tag{2.3}$$

Similarly, a set of three quantities $\{A^i(x_1, x_2, x_3)\}$ represents a contravariant vector if it transforms to $\{B^j(y^1, y^2, y^3)\}$ according to

$$B^j(y^1, y^2, y^3) = \frac{\partial y^j}{\partial x_i} A^i(x_1, x_2, x_3). \tag{2.4}$$

Similarly, a set of nine quantities $\{A_{ij}(x_1, x_2, x_3)\}$ associated with the coordinate system x_i represents components of a tensor of rank two if the corresponding set of nine quantities $\{B_{ij}(y^1, y^2, y^3)\}$ associated with the coordinate system y^i is given by

$$B_{ij} = \frac{\partial x_k}{\partial y^i} \frac{\partial x_m}{\partial y^j} A_{km} \cdot \tag{2.5}$$

Analogous to (2.4), there exists the transformation

$$B^{ij} = \frac{\partial y^i}{\partial x_k} \frac{\partial y^j}{\partial x_m} A^{km}. \tag{2.6}$$

A covariant tensor is defined by (2.5) and a contravariant tensor is defined by (2.6).

The line element ds is given by

$$ds^2 = dx_i\, dx_i = g_{ij}\, dy^i\, dy^j, \tag{2.7}$$

where the g_{ij}, covariant components of the metric tensor of the system y^i at $t = 0$, are given by

$$g_{ij} = \bar{g}_i \cdot \bar{g}_j \cdot \tag{2.8}$$

The contravariant components are calculated from the covariant components by

$$g^{ij} = \frac{c^{ij}}{g}, \quad g = |g_{ij}|, \tag{2.9}$$

where c^{ij} is the cofactor of g_{ij} in the determinant $|g_{ij}|$.

At time t the body has become deformed. The coordinates of a generic point of the deformed body in the coordinate system x_i are now (X_1, X_2, X_3). The curvilinear coordinates y^i are chosen so as to move with the body as it is deformed; thus

$$X_i = X_i(y^1, y^2, y^3, t). \tag{2.10}$$

If $\bar{R}$ is the position vector from the origin of the coordinate system x_i to a generic point of the deformed body at time t, the covariant base vectors of the system y^i at time t are

$$\bar{G}_i = \bar{R},_i \ . \tag{2.11}$$

The line element dS is given by

$$dS^2 = dX_i \, dX_i = G_{ij} \, dy^i \, dy^j \ , \tag{2.12}$$

where the G_{ij}, covariant components of the metric tensor of the system y^i at time t, are given by

$$G_{ij} = \bar{G}_i \cdot \bar{G}_j \ . \tag{2.13}$$

The contravariant components are calculated by

$$G^{ij} = \frac{C^{ij}}{G} , \qquad G = |G_{ij}| \ , \tag{2.14}$$

where C^{ij} is the cofactor of G_{ij} in the determinant $|G_{ij}|$. Consider the difference of the squares of the line elements,

$$dS^2 - ds^2 = (G_{ij} - g_{ij}) \, dy^i \, dy^j \ .$$

The covariant components of the strain tensor are defined as

$$2\epsilon_{ij} = G_{ij} - g_{ij} \ . \tag{2.15}$$

Thus, the strain is a measure of the change in arc length.

Consider an element ΔS of a surface with unit normal $\bar{n} = n^i \bar{G}_i$ situated in the body. Let $\bar{T}_i$ be the average force exerted across ΔS. Define the stress vector associated with the surface having unit normal components n^i as

$$\bar{t}_i = \lim_{\Delta S \to 0} \frac{\bar{T}_i}{\Delta S} \ . \tag{2.16}$$

If the n^i represent components of the unit normal to the surface y^i = constant of the deformed body, the contravariant components of stress, τ^{ij}, are given by

$$\bar{t}_i = \frac{\tau^{ij}}{\sqrt{G^{ii}}}\,\bar{G}_j \quad \text{(no sum on i)}. \tag{2.17}$$

If the body is perfectly elastic, a strain energy function Z is assumed to exist; for an isotropic body Z is given as a function of the three strain invariants, I_1, I_2 and I_3. These invariants are calculated by

$$\begin{aligned} I_1 &= g^{ij}\,G_{ij} \\ I_2 &= g_{ij}\,G^{ij}\,I_3 \\ I_3 &= \frac{G}{g}. \end{aligned} \tag{2.18}$$

The stress-strain law may be given in terms of the strain energy function as, see page 28 [7],

$$\tau^{ij} = \Phi\, g^{ij} + \psi\, D^{ij} + P\, G^{ij}, \tag{2.19}$$

where

$$\begin{aligned} \Phi &= 2\, I_3^{-1/2}\,\frac{\partial Z}{\partial I_1}, \\ \psi &= 2\, I_3^{-1/2}\,\frac{\partial Z}{\partial I_2}, \\ P &= 2\, I_3^{1/2}\,\frac{\partial Z}{\partial I_3}, \end{aligned} \tag{2.20}$$

$$D^{ij} = I_1\, g^{ij} - g^{ik}\, g^{jm}\, G_{km}.$$

For an incompressible material $I_3 = 1$ and

$$Z = Z(I_1, I_2). \tag{2.21}$$

The stress-strain law for an incompressible material is (2.19), where P is no longer calculated from (2.20) but is an invariant scalar function of position and time.

Considerations of moment equilibrium of an infinitesimal parallelepiped show that the stress tensor is symmetric, [10]. Also in [10], considerations of force equilibrium lead to the conclusion that

$$\tau^{ij}|_i + \rho F^j = \rho f^j . \tag{2.22}$$

In (2.22),

$$\tau^{ij}|_i = \frac{\partial \tau^{ij}}{\partial y^i} + \Gamma^i_{ik} \tau^{kj} + \Gamma^j_{ik} \tau^{ik} \tag{2.23}$$

is the covariant derivative of τ^{ij}, the Γ^i_{jk} are Christoffel symbols to be defined later, ρ is the density of the deformed body, the F^j represent components of the body forces and the f^j are components of the acceleration vector in the coordinate system y^i. If surface forces $\bar{p} = p^j \bar{G}_j$ are given on the surface of the deformed body and if the unit normal to the surface is expressed as $\bar{n} = n_i \bar{G}^i$, the boundary condition on the surface is that

$$\tau^{ij} n_i = p^j . \tag{2.24}$$

The elastic problem is to solve for the displacements, given the body and surface forces. The g_{ij} are given by the curvilinear coordinate system. The G_{ij}, I_i, D^{ij}, and Z are all given in terms of the displacements X_i, thus from (2.19) τ^{ij} is a function of the displacements. The differential tensor equation to solve for the displacements is (2.22) with boundary conditions (2.24).

III. FINITE DEFORMATION OF A THIN MEMBRANE

In this section the equations are simplified by considering the motion of the middle surface of the shell, and using stress resultants instead of stresses.

Let (y^1, y^2) be the coordinates of a generic point when referred to a general curvilinear coordinate system defined on a surface in space. At $t = 0$, let $\bar{a}$ denote the position vector from the origin of the x_i system to the point (y^1, y^2) on the surface. The covariant base vectors are given on the surface as

$$\bar{a}_\alpha = \bar{a}_{,\alpha} \, . \tag{3.1}$$

The unit normal to the surface is given by

$$\bar{a}_3 = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \quad . \tag{3.2}$$

Let y^3 be the distance from (y^1, y^2) on the surface to a point along the normal to the surface. y^3 is positive in the direction of $\bar{a}_3$. With these definitions a shell of thickness $h(y^1, y^2)$ may be described by the position vectors

$$\bar{r} = \bar{a}(y^1, y^2) + y^3 \, \bar{a}_3(y^1, y^2), \qquad |y^3| \le \frac{1}{2} h(y^1, y^2) \, . \tag{3.3}$$

The middle surface of this shell is given by $y^3 = 0$. Identify the coordinates (y^1, y^2, y^3) as the general curvilinear system given in II. The covariant base vectors will be given as

$$\bar{g}_\alpha = \bar{a}_{,\alpha} + y^3 \, \bar{a}_{3,\alpha} \, , \qquad \bar{g}_3 = \bar{a}_3 \, . \tag{3.4}$$

The covariant and contravariant components of the surface metric tensor are

$$a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta \, , \qquad a^{\alpha\beta} = \frac{C^{\alpha\beta}}{a} \, , \qquad a = |a_{\alpha\beta}| \, , \tag{3.5}$$

where $C^{\alpha\beta}$ is the cofactor of $a_{\alpha\beta}$ in $|a_{\alpha\beta}|$. Assume that the shell is thin so the variation of the quantities in (3.6) in the y_3 direction is negligible. Therefore, neglect terms in y^3 to obtain

$$g_{\alpha\beta} = a_{\alpha\beta}, \qquad g_{\alpha 3} = 0, \qquad g_{33} = 1,$$

$$g^{\alpha\beta} = a^{\alpha\beta}, \qquad g^{\alpha 3} = 0, \qquad g^{33} = 1, \tag{3.6}$$

$$g = a .$$

At time t, let $\bar{A}$ denote the position vector from the origin of the x_i system to the point $(y^1, y^2, 0)$ on the deformed surface. The base vectors are given on the deformed surface by

$$\bar{A}_\alpha = \bar{A},_\alpha \tag{3.7}$$

and the unit normal by

$$\bar{A}_3 = \frac{\bar{A}_1 \times \bar{A}_2}{|\bar{A}_1 \times \bar{A}_2|}. \tag{3.8}$$

The membrane assumption is made that the shearing stresses in the direction of $\bar{A}_3$ in planes perpendicular to the middle surface are negligibly small in comparison to the remaining stresses. Thus a principal direction of stress is given by $\bar{A}_3$. If elements which were normal to the middle surface before deformation are transferred into normal elements by the deformation,

$$\bar{R} = \bar{A}(y^1, y^2) + \lambda_3(y^1, y^2)\, y^3\, \bar{A}_3(y^1, y^2), \qquad |y^3| \le \frac{1}{2} h , \tag{3.9}$$

defines the position vector of the points on the deformed membrane. In (3.9), λ_3 is the ratio of the thickness of the shell after deformation to the thickness of the shell before deformation in the direction normal to the surface. From (2.11), assuming y^3 is small,

$$\bar{G}_\alpha = \bar{A},_\alpha , \qquad \bar{G}_3 = \lambda_3\, \bar{A}_3 . \tag{3.10}$$

The covariant and contravariant components of the surface metric tensor at time t are

$$A_{\alpha\beta} = \bar{A}_\alpha \cdot \bar{A}_\beta , \qquad A^{\alpha\beta} = \frac{C^{\alpha\beta}}{A}, \qquad A = |A_{\alpha\beta}| , \tag{3.11}$$

where $C^{\alpha\beta}$ is the cofactor of $A_{\alpha\beta}$ in $|A_{\alpha\beta}|$. Thus from (2.13), (2.14), (3.10) and (3.11),

$$G_{\alpha\beta} = A_{\alpha\beta}, \qquad G_{\alpha 3} = 0, \qquad G_{33} = \lambda_3^2,$$
$$G^{\alpha\beta} = A^{\alpha\beta}, \qquad G^{\alpha 3} = 0, \qquad G^{33} = \frac{1}{\lambda_3^2}, \tag{3.12}$$
$$G = \lambda_3^2 A .$$

Define the physical stress resultant, $\bar{N}_\alpha$, as the force, per unit length of the middle surface, acting on the surface y^α = constant in the deformed membrane. In terms of the stress vector, (2.16)

$$\bar{N}_\alpha = \lambda_3 h \bar{t}_\alpha . \tag{3.13}$$

If the contravariant components, $n^{\alpha\beta}$, of the physical stress resultant are defined by

$$\bar{N}_\alpha \sqrt{A^{\alpha\alpha}} = n^{\alpha\beta} \bar{A}_\beta \quad \text{(no sum on } \alpha\text{)}, \tag{3.14}$$

then from (2.17), (3.12) and (3.13),

$$\bar{N}_\alpha \sqrt{A^{\alpha\alpha}} = \lambda_3 h \tau^{\alpha\beta} \bar{A}_\beta , \qquad n^{\alpha\beta} = \lambda_3 h \tau^{\alpha\beta} . \tag{3.15}$$

Use (2.18), (2.19), (2.20) and (3.12) to write

$$n^{\alpha\beta} = \lambda_3 h [\Phi a^{\alpha\beta} + \psi D^{\alpha\beta} + P A^{\alpha\beta}], \tag{3.16}$$

where

$$D^{\alpha\beta} = \lambda_3^2 a^{\alpha\beta} + [a^{\lambda\mu} a^{\alpha\beta} - a^{\alpha\lambda} a^{\beta\mu}] A_{\lambda\mu} . \tag{3.17}$$

The strain invariants are given in terms of the surface quantities and λ_3 as

$$I_1 = a^{\alpha\beta} A_{\alpha\beta} + \lambda_3^2 ,$$
$$I_2 = I_3 [a_{\alpha\beta} A^{\alpha\beta} + \lambda_3^{-2}] , \tag{3.18}$$
$$I_3 = \lambda_3^2 \frac{A}{a} .$$

If the normal components of the tractions applied to the surface of the membrane are small in comparison to the stresses acting on the surface y^α = constant,

$$\tau^{33} = 0. \tag{3.19}$$

Thus from (2.19)

$$\Phi + \psi D^{33} + \frac{P}{\lambda_3^2} = 0 , \tag{3.20}$$

where

$$D^{33} = a^{\alpha\beta} A_{\alpha\beta} . \tag{3.21}$$

Substitute the value of P from (3.20) into (3.16) to obtain

$$n^{\alpha\beta} = \lambda_3 h [\Phi(a^{\alpha\beta} - \lambda_3^2 A^{\alpha\beta}) + \psi(D^{\alpha\beta} - \lambda_3^2 D^{33} A^{\alpha\beta})] . \tag{3.22}$$

For an incompressible material, $I_3 = 1$ and $\lambda_3^2 = \frac{a}{A}$. Thus,

$$I_1 = a^{\alpha\beta} A_{\alpha\beta} + \frac{a}{A} ,$$
$$I_2 = a_{\alpha\beta} A^{\alpha\beta} + \frac{A}{a} , \tag{3.23}$$

and

$$n^{\alpha\beta} = \sqrt{\frac{a}{A^3}}\, h [\Phi(A a^{\alpha\beta} - a A^{\alpha\beta}) + \psi(A D^{\alpha\beta} - a D^{33} A^{\alpha\beta})] , \tag{3.24}$$

where from (3.17),

$$D^{\alpha\beta} = \frac{a}{A} a^{\alpha\beta} + [a^{\lambda\mu} a^{\alpha\beta} - a^{\alpha\lambda} a^{\beta\mu}] A_{\lambda\mu} . \tag{3.25}$$

Consider an element of the deformed membrane at time t which is bounded by the coordinate curves y^α = constant and $y^\alpha + dy^\alpha$ = constant. Let the acceleration vector be given by

$$\frac{d^2 \bar{A}}{d t^2} = f^i \bar{A}_i \tag{3.26}$$

and let

$$\bar{p} = p^j \bar{A}_j \tag{3.27}$$

represent the resultant force acting on a surface of the element. The equations of motion are derived by consideration of the equilibrium of this element [10] as

$$\begin{aligned} p^\beta + n^{\alpha\beta}\big|_\alpha &= \sqrt{\frac{a}{A}}\, h\, \rho\, f^\beta \\ p^3 + n^{\alpha\beta} B_{\alpha\beta} &= \sqrt{\frac{a}{A}}\, h\, \rho\, f^3 \, . \end{aligned} \tag{3.28}$$

For (3.28) recall (2.23) where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} A^{\alpha\mu} \left[A_{\mu\beta,\gamma} + A_{\mu\gamma,\beta} - A_{\beta\gamma,\mu}\right] \tag{3.29}$$

are the Christoffel symbols on the surface and

$$B_{\alpha\beta} = \bar{A}_3 \cdot \bar{A}_{\alpha,\beta} \, . \tag{3.30}$$

If $\bar{e}_i$ are unit vectors along the fixed coordinate axes x_i, from (2.10)

$$\bar{A} = X_i \, \bar{e}_i \, . \tag{3.31}$$

Define A^i_j by

$$\bar{A}_j = A^i_j \, \bar{e}_i \, . \tag{3.32}$$

Combine (3.26), (3.31) and (3.32) to obtain

$$\frac{d^2 X_i}{d t^2} = f^j A^i_j \, . \tag{3.33}$$

Substitute the value of f^j from (3.33) into (3.28) to obtain three differential equations, in general non-linear, for the determination of the X_i. The $n^{\alpha\beta}$, I_i, $B^{\alpha\beta}$ and $\Gamma^\alpha_{\beta\gamma}$ are all given in terms of X_i and its derivatives because $\bar{A} = X_i \, \bar{e}_i$. Thus if the p^i are given, the only unknowns left in the three differential equations are the X_i.

IV. PERTURBATION TECHNIQUE

Consider a deformation of the body which is such that at any time the state of stress, strain and displacements differs only slightly from the state for a known finite deformation. Thus at time t, let $\bar{A} + \epsilon \bar{A}'$, ϵ assumed small, represent the position vector from the origin of the x_i system to a point on the perturbed middle surface. $\bar{A}$ is the position vector for the known finite deformation and ϵ is a constant, where higher powers of ϵ are neglected in comparison with ϵ in keeping with the above assumption.

Define w_i by

$$\bar{A}' = w_i \bar{A}^i , \tag{4.1}$$

where $\bar{A}^\alpha$ are the covariant base vectors of the deformed middle surface and

$$\bar{A}^3 = \bar{A}_3 . \tag{4.2}$$

The covariant components of the deformed surface metric tensor are to first order terms in ϵ,

$$A_{\alpha\beta} + \epsilon A'_{\alpha\beta} = \bar{A}_\alpha \cdot \bar{A}_\beta + \epsilon(\bar{A}_\alpha \cdot \bar{A}'_\beta + \bar{A}'_\alpha \cdot \bar{A}_\beta) , \tag{4.3}$$

where

$$\bar{A}'_\beta = \bar{A}'_{,\beta} . \tag{4.4}$$

From (4.1) through (4.4)

$$A'_{\alpha\beta} = \bar{A}_\alpha \cdot (w_\gamma \bar{A}^\gamma)_{,\beta} + \bar{A}_\alpha \cdot (w_3 \bar{A}^3)_{,\beta} + (w_\gamma \bar{A}^\gamma)_{,\alpha} \cdot \bar{A}_\beta + (w_3 \bar{A}^3)_{,\alpha} \cdot \bar{A}_\beta . \tag{4.5}$$

Use the properties of the base vectors to reduce this to (see Appendix A for the details)

$$A'_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha} - 2\,\Gamma^\gamma_{\alpha\beta}\, w_\gamma - 2\,B_{\alpha\beta}\, w_3 . \tag{4.6}$$

The determinant of $A_{\alpha\beta} + \epsilon A'_{\alpha\beta}$ is, to first order terms in ϵ,

$$A + \epsilon A' = |A_{\alpha\beta} + \epsilon A'_{\alpha\beta}| = A_{11} A_{22} - A_{12}^2 + \epsilon [A'_{11} A_{22} + A_{11} A'_{22} - 2 A_{12} A'_{12}] . \tag{4.7}$$

Write the contravariant components as $A^{\alpha\beta} + \epsilon A'^{\alpha\beta}$.

$$A^{11} + \epsilon A'^{11} = \frac{(A_{22} + \epsilon A'_{22})}{A} \left[\frac{1}{1 + \epsilon A'/A}\right] = \frac{A_{22}}{A} + \frac{\epsilon}{A}\left[A'_{22} - \frac{A_{22} A'}{A}\right] . \tag{4.8}$$

Similarly

$$A^{12} + \epsilon A'^{12} = \frac{-A_{12}}{A} + \frac{\epsilon}{A}\left[\frac{A_{12} A'}{A} - A'_{12}\right] \tag{4.9}$$

and

$$A^{22} + \epsilon A'^{22} = \frac{A_{11}}{A} + \frac{\epsilon}{A}\left[A'_{11} - \frac{A_{11} A'}{A}\right] . \tag{4.10}$$

If the extension ratio perpendicular to the middle surface is $\lambda_3 + \epsilon \lambda'_3$, write the first invariant as

$$I_1 + \epsilon I'_1 = a^{\alpha\beta} (A_{\alpha\beta} + \epsilon A'_{\alpha\beta}) + (\lambda_3 + \epsilon \lambda'_3)^2 = a^{\alpha\beta} A_{\alpha\beta} + \lambda_3^2 + \epsilon (a^{\alpha\beta} A'_{\alpha\beta} + 2 \lambda_3 \lambda'_3) . \tag{4.11}$$

Similarly

$$I_3 + \epsilon I'_3 = \frac{A\lambda_3^2}{a} + \frac{\epsilon}{a}(A' \lambda_3^2 + 2 \lambda_3 \lambda'_3 A) \tag{4.12}$$

and

$$I_2 + \epsilon I'_2 = I_3 (a_{\alpha\beta} A^{\alpha\beta} + \lambda_3^{-2}) + \epsilon\left[I_3 (a_{\alpha\beta} A'^{\alpha\beta} - 2 \lambda'_3 \lambda_3^{-3}) + I'_3 (a_{\alpha\beta} A^{\alpha\beta} + \lambda_3^{-2})\right] . \tag{4.13}$$

For the strained body the strain-energy function is

$$Z = Z\,(I_1 + \epsilon\, I_1',\ I_2 + \epsilon\, I_2',\ I_3 + \epsilon\, I_3') \tag{4.14}$$

and Φ, ψ and P, (2.20), are all functions of $I_i + \epsilon\, I_i'$. Use Taylor's expansion, [13], to first order terms in ϵ, to obtain

$$\begin{aligned}
\Phi + \epsilon\, \Phi' &= \Phi\,(I_1, I_2, I_3) + \epsilon\, I_i' \frac{\partial \Phi}{\partial I_i}\,, \\
\psi + \epsilon\, \psi' &= \psi\,(I_1, I_2, I_3) + \epsilon\, I_i' \frac{\partial \psi}{\partial I_i}\,, \\
P + \epsilon\, P' &= P\,(I_1, I_2, I_3) + \epsilon\, I_i' \frac{\partial P}{\partial I_i}\,.
\end{aligned} \tag{4.15}$$

Thus use the original definitions (2.20) to write

$$\begin{aligned}
\Phi' &= 2\, I_3^{-1/2}\, I_i' \frac{\partial^2 Z\,(I_1, I_2, I_3)}{\partial I_1\, \partial I_i} - (2\, I_3)^{-1}\, I_3'\, \Phi\,, \\
\psi' &= 2\, I_3^{-1/2}\, I_i' \frac{\partial^2 Z\,(I_1, I_2, I_3)}{\partial I_2\, \partial I_i} - (2\, I_3)^{-1}\, I_3'\, \psi\,, \\
P' &= 2\, I_3^{1/2}\, I_i' \frac{\partial^2 Z\,(I_1, I_2, I_3)}{\partial I_3\, \partial I_i} + (2\, I_3)^{-1}\, I_3'\, P\,.
\end{aligned} \tag{4.16}$$

In a similar manner use (2.19), (2.20) and (3.18) to obtain

$$\tau'^{\alpha\beta} = \Phi'\, a^{\alpha\beta} + \psi'\, D^{\alpha\beta} + P'\, A^{\alpha\beta} + \psi\, D'^{\alpha\beta} + P\, A'^{\alpha\beta}\,, \tag{4.17}$$

$$D'^{\alpha\beta} = 2\, \lambda_3\, \lambda_3'\, a^{\alpha\beta} + (a^{\alpha\beta}\, a^{\mu\gamma} - a^{\alpha\mu}\, a^{\beta\gamma})\, A'_{\mu\gamma}\,, \tag{4.18}$$

$$\tau'^{33} = \Phi' + \psi'\,(I_1 - \lambda_3^2) + \psi\,(I_1' - 2\, \lambda_3\, \lambda_3') + \lambda_3^{-2}\, P' - 2\, \lambda_3'\, \lambda_3^{-3}\, P\,. \tag{4.19}$$

From (3.15),

$$n^{\alpha\beta} + \epsilon\, n'^{\alpha\beta} = \lambda_3\, h\, \tau^{\alpha\beta} + \epsilon\, h\, [\lambda_3'\, \tau^{\alpha\beta} + \lambda_3\, \tau'^{\alpha\beta}]\,. \tag{4.20}$$

If the material is incompressible, $I_3 = 1$ and $I_3' = 0$. Thus

$$\lambda_3' = -\frac{1}{2}\sqrt{\frac{a}{A^3}}\; A' ,$$
$$\lambda_3 = \sqrt{\frac{a}{A}} , \tag{4.21}$$

and

$$n'^{\alpha\beta} = h\sqrt{\frac{a}{A}}\left[\tau'^{\alpha\beta} - \frac{A'}{2A}\tau^{\alpha\beta}\right] . \tag{4.22}$$

If the normal components of tractions applied to the surface are small in comparison to the stresses acting on the surface y^α = constant, set $\tau^{33} + \epsilon\,\tau'^{33} = 0$. Substitute the values of P and P′ from (3.20) and (4.19) and the values of λ_3 and λ_3' from (4.21) into (4.17) to obtain (see Appendix B)

$$\begin{aligned}\tau'^{\alpha\beta} &= \Phi'\left[a^{\alpha\beta} - a\,A^{-1}\,A^{\alpha\beta}\right] + \psi'\left[D^{\alpha\beta} - a\,A^{-1}\,A^{\alpha\beta}\,a^{\mu\gamma}\,A_{\mu\gamma}\right] \\ &+ a\,A^{-1}\left[\Phi + a^{\mu\gamma}\,A_{\mu\gamma}\,\psi\right]\left[A'\,A^{-1}\,A^{\alpha\beta} - A'^{\alpha\beta}\right] \\ &+ \psi\left[D'^{\alpha\beta} - a\,A^{-1}\,A^{\alpha\beta}\,a^{\mu\gamma}\,A'_{\mu\gamma}\right],\end{aligned} \tag{4.23}$$

where Φ' and ψ' are given by (4.16) with $I_3' = 0$, $I_3 = 1$ and $Z = Z\,(I_1, I_2)$.

The force and acceleration vectors are

$$(p^k + \epsilon\, p'^k)(\bar{A}_k + \epsilon\,\bar{A}_k') = p^k\,\bar{A}_k + \epsilon\,(p'^k\,\bar{A}_k + p^k\,\bar{A}_k') \tag{4.24}$$

and

$$(f^k + \epsilon\, f'^k)(\bar{A}_k + \epsilon\,\bar{A}_k') = f^k\,\bar{A}_k + \epsilon\,(f'^k\,\bar{A}_k + f^k\,\bar{A}_k') . \tag{4.25}$$

Since

$$\bar{A} + \epsilon\,\bar{A}' = (X_i + \epsilon\,X_i')\,\bar{e}_i \tag{4.26}$$

and

$$\bar{A}_k' = A_k'^{\,i}\,\bar{e}_i , \tag{4.27}$$

$$\frac{d^2 X_i}{d\,t^2} + \epsilon\,\frac{d^2 X_i'}{d\,t^2} = f^k A_k^i + \epsilon\left[f'^k A_k^i + f^k A_k'^i\right]. \tag{4.28}$$

From (3.8) and (4.4),

$$\begin{aligned}\bar{A}_3 + \epsilon\,\bar{A}_3' &= \frac{\bar{A}_1 \times \bar{A}_2 + \epsilon\,(\bar{A}_1' \times \bar{A}_2 + \bar{A}_1 \times \bar{A}_2')}{\sqrt{A + \epsilon A'}} \\ &= \bar{A}_3 + \epsilon\left[\frac{\bar{A}_1' \times \bar{A}_2 + \bar{A}_1 \times \bar{A}_2'}{\sqrt{A}} - \frac{A'\,\bar{A}_3}{2\,A}\right].\end{aligned} \tag{4.29}$$

The equations of motion are, from (3.28),

$$\begin{aligned}&n'^{\alpha\beta}_{,\alpha} + \Gamma^{\alpha}_{\alpha\lambda}\, n'^{\lambda\beta} + \Gamma^{\beta}_{\alpha\lambda}\, n'^{\alpha\lambda} + \Gamma'^{\alpha}_{\alpha\lambda}\, n^{\lambda\beta} + \Gamma'^{\beta}_{\alpha\lambda}\, n^{\alpha\lambda} + p'^{\beta} \\ &= h\,\rho\sqrt{\frac{a}{A}}\left[f'^{\beta} - \frac{A'\,f^{\beta}}{2\,A}\right],\end{aligned} \tag{4.30}$$

$$n^{\alpha\beta}\, B'_{\alpha\beta} + n'^{\alpha\beta}\, B_{\alpha\beta} + p'^{3} = h\rho\sqrt{\frac{a}{A}}\left[f'^{3} - \frac{A'\,f^3}{2\,A}\right]. \tag{4.31}$$

From (3.29) and (3.30),

$$\begin{aligned}\Gamma'^{\alpha}_{\beta\gamma} &= \frac{1}{2}\,A^{\alpha\mu}\left[A'_{\mu\beta,\gamma} + A'_{\mu\gamma,\beta} - A'_{\beta\gamma,\mu}\right] \\ &\quad + \frac{1}{2}\,A'^{\alpha\mu}\left[A_{\mu\beta,\gamma} + A_{\mu\gamma,\beta} - A_{\beta\gamma,\mu}\right]\end{aligned} \tag{4.32}$$

$$B'_{\alpha\beta} = \bar{A}_3' \cdot \bar{A}_{\alpha,\beta} + \bar{A}_3 \cdot \bar{A}'_{\alpha,\beta}\,. \tag{4.33}$$

Thus solve for f'^k and f^k from (4.28) and substitute these values into (4.30) and (4.31) to obtain three differential equations for the determination of the X_i'.

The reason for using the theory of a small displacement superposed on a known finite deformation is that the resulting differential equations are simpler. The X_i, f^i, p^i, $B_{\alpha\beta}$, $n^{\alpha\beta}$, $\Gamma^{\alpha}_{\beta\gamma}$, A and a are quantities determined from the known finite deformation. The p'^i are the additional components of the resultant force acting on the surface, not taken into account in the known finite deformation. The problem to be solved is as follows: Given a known finite deformation and the p'^{i}, use the equations of motion to solve for the additional displacement X'_i. The quantities $n'^{\alpha\beta}$, $\Gamma'^{\alpha}_{\beta\gamma}$, $B'_{\alpha\beta}$ and f'^{k} are all given in terms of the X'_i. Note that in (4.30) and (4.31) there are no products of two primed quantities. Thus, equations (4.30) and (4.31) will be linear differential equations.

V. ANALYSIS FOR A SPHERE

A. Finite Deformation

Consider a spherical shell of thickness h made of an incompressible elastic material. Let c be the radius of the middle surface and assume the shell behaves as a membrane. Let $\frac{\theta^1}{c}$ denote the azimuth angle and $\frac{\theta^2}{c}$ the circumferential angle. The position vector, $\bar{a}$, of the undeformed middle surface is given as

$$\bar{a} = c \sin\frac{\theta^1}{c} \cos\frac{\theta^2}{c}\, \bar{e}_1 + c \sin\frac{\theta^1}{c} \sin\frac{\theta^2}{c}\, \bar{e}_2 + c \cos\frac{\theta^1}{c}\, \bar{e}_3 , \tag{5.1}$$

where the $\bar{e}_i$ are unit vectors in the direction of the rectangular Cartesian coordinates x_i. From (3.1) and (3.5), it follows that

$$\begin{aligned} &a_{11} = 1, \qquad a_{12} = 0, \qquad a_{22} = \sin^2\frac{\theta^1}{c}, \\ &a^{11} = 1, \qquad a^{12} = 0, \qquad a^{22} = \csc^2\frac{\theta^1}{c}, \\ &a = \sin^2\frac{\theta^1}{c} . \end{aligned} \tag{5.2}$$

Let the sphere be inflated in a manner so as to produce a quasi-static homogeneous finite deformation. Let λ represent the ratio of the radius of the middle surface after inflation to the initial radius of the sphere. The position vector of the deformed middle surface is given as

$$\bar{A} = \lambda\, \bar{a} . \tag{5.3}$$

Thus from (3.7), (3.11), (5.2) and (5.3) it follows that

$$\begin{aligned} &A_{11} = \lambda^2, \qquad A_{12} = 0, \qquad A_{22} = \lambda^2 \sin^2\frac{\theta^1}{c}, \\ &A^{11} = \frac{1}{\lambda^2}, \qquad A^{12} = 0, \qquad A^{22} = \frac{\csc^2\frac{\theta^1}{c}}{\lambda^2}, \\ &A = \lambda^4 \sin^2\frac{\theta^1}{c} . \end{aligned} \tag{5.4}$$

The strain invariants, as calculated from (3.18), are

$$I_1 = 2\,\lambda^2 + \lambda_3^2,$$

$$I_2 = 2\,\lambda_3^2\,\lambda^2 + \lambda^4, \qquad (5.5)$$

$$I_3 = \lambda_3^2\,\lambda^4 .$$

From (3.17) and (3.21)

$$D^{11} = \lambda^2 + \lambda_3^2, \qquad D^{12} = 0, \qquad D^{22} = (\lambda^2 + \lambda_3^2)\csc^2\frac{\theta^1}{c},$$

$$D^{33} = 2\,\lambda^2. \qquad (5.6)$$

Use (3.22) to write the stress resultants as

$$n^{11} = \lambda_3\, h\left[(1 - \lambda_3^2\,\lambda^{-2})\,\Phi + (\lambda^2 - \lambda_3^2)\,\psi\right],$$

$$n^{12} = 0, \qquad (5.7)$$

$$n^{22} = \lambda_3\, h\csc^2\frac{\theta^1}{c}\left[(1 - \lambda_3^2\,\lambda^{-2})\,\Phi + (\lambda^2 - \lambda_3^2)\,\psi\right].$$

The non-zero Christoffel symbols on the surface are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{c}\,\mathrm{ctn}\,\frac{\theta^1}{c},$$

$$\Gamma^1_{22} = -\frac{1}{c}\sin\frac{\theta^1}{c}\cos\frac{\theta^1}{c} \qquad (5.8)$$

and

$$B_{11} = -\frac{\lambda}{c}, \qquad B_{12} = 0, \qquad B_{22} = -\frac{\lambda}{c}\sin^2\frac{\theta^1}{c}. \qquad (5.9)$$

Since the sphere has undergone a quasi-static homogeneous finite deformation to a new radius, λc, there are no acceleration effects, i.e. from (3.33) $f^i = 0$. Therefore, from the equations of motions (3.28),

$$p^\beta = 0, \qquad (5.10)$$

$$p^3 = \frac{2\lambda\lambda_3 h}{c}\left[(1-\lambda_3^2\lambda^{-2})\,\Phi + (\lambda^2-\lambda_3^2)\,\psi\right]. \tag{5.11}$$

For an incompressible material, the pressure needed to support the inflated membrane is

$$p^3 = \frac{2h}{\lambda c}\left[1-\frac{1}{\lambda^6}\right]\left[\Phi + \lambda^2\psi\right]. \tag{5.12}$$

Also from (5.7),

$$n^{11} = \frac{h}{\lambda^2}\left[1-\frac{1}{\lambda^6}\right]\left[\Phi + \lambda^2\psi\right], \tag{5.13}$$

$$n^{22} = \frac{h}{\lambda^2}\csc^2\frac{\theta^1}{c}\left[1-\frac{1}{\lambda^6}\right]\left[\Phi + \lambda^2\psi\right]. \tag{5.14}$$

The components of the physical stress resultant, $\bar{N}_\alpha$, are, from (3.14) as

$$\begin{aligned}\bar{N}_1 &= h\left(1-\frac{1}{\lambda^6}\right)(\Phi+\lambda^2\psi)\,\frac{\bar{a}_1}{|\bar{a}_1|},\\ \bar{N}_2 &= h\left(1-\frac{1}{\lambda^6}\right)(\Phi+\lambda^2\psi)\,\frac{\bar{a}_2}{|\bar{a}_2|}.\end{aligned} \tag{5.15}$$

Thus for a spherical shell composed of incompressible material with an arbitrary strain-energy function the pressure, (5.12), needed to maintain a finite deformation to a radius λc gives rise to stresses given by (5.15).

B. Perturbation Equations

The position vector of a point on the deformed middle surface is given in Chapter IV as $\bar{A} + \epsilon\,\bar{A}'$. The $\bar{A}^i$ from (4.1) are given by

$$\bar{A}^1 = \frac{1}{\lambda}\left[\cos\frac{\theta^1}{c}\cos\frac{\theta^2}{c}\bar{e}_1 + \cos\frac{\theta^1}{c}\sin\frac{\theta^2}{c}\bar{e}_2 - \sin\frac{\theta^1}{c}\bar{e}_3\right],$$

$$\bar{A}^2 = \frac{\csc\frac{\theta^1}{c}}{\lambda}\left[-\sin\frac{\theta^2}{c}\bar{e}_1 + \cos\frac{\theta^2}{c}\bar{e}_2\right], \tag{5.16}$$

$$\bar{A}^3 = \sin\frac{\theta^1}{c}\cos\frac{\theta^2}{c}\bar{e}_1 + \sin\frac{\theta^1}{c}\sin\frac{\theta^2}{c}\bar{e}_2 + \cos\frac{\theta^1}{c}\bar{e}_3 .$$

If a spherical coordinate system is defined with respect to the rectangular system as usual, the physical components of displacement of the small additional deformation are given by ϵu, ϵv, ϵw, where u, v, and w are the components in the radial, circumferential and azimuthal directions respectively. Thus

$$\bar{A}' = u\,\bar{e}_r + v\,\bar{e}_{\theta^2} + w\,\bar{e}_{\theta^1}, \tag{5.17}$$

where

$$\bar{e}_r = \bar{A}^3$$

$$\bar{e}_{\theta^2} = -\sin\frac{\theta^2}{c}\bar{e}_1 + \cos\frac{\theta^2}{c}\bar{e}_2 \tag{5.18}$$

$$\bar{e}_{\theta^1} = \cos\frac{\theta^1}{c}\cos\frac{\theta^2}{c}\bar{e}_1 + \cos\frac{\theta^1}{c}\sin\frac{\theta^2}{c}\bar{e}_2 - \sin\frac{\theta^1}{c}\bar{e}_3$$

Compare (4.1) with (5.17) to obtain the relations

$$w_1 = \lambda w,$$

$$w_2 = \lambda v \sin\frac{\theta^1}{c}, \tag{5.19}$$

$$w_3 = u .$$

Use these values in (4.6) to obtain

$$A'_{11} = 2\lambda w,_1 + \frac{2\lambda}{c}u,$$

$$A'_{12} = \lambda w,_2 + \lambda v,_1 \sin\frac{\theta^1}{c} - \frac{\lambda}{c} v\cos\frac{\theta^1}{c}, \tag{5.20}$$

$$A'_{22} = 2\,\lambda\, v,_2 \sin\frac{\theta^1}{c} + \frac{2\,\lambda}{c}\, w \sin\frac{\theta^1}{c}\cos\frac{\theta^1}{c} + \frac{2\,\lambda}{c}\, u \sin^2\frac{\theta^1}{c},$$

and from (4.7)

$$A' = 2\,\lambda^3 \sin\frac{\theta^1}{c}\left[w,_1 \sin\frac{\theta^1}{c} + \frac{2}{c}\, u \sin\frac{\theta^1}{c} + v,_2 + \frac{w}{c}\cos\frac{\theta^1}{c}\right]. \tag{5.21}$$

From (4.8), (4.9) and (4.10) the contravariant components are

$$A'^{11} = \frac{-2\,u}{\lambda^3\, c} - \frac{2\, w,_1}{\lambda^3},$$

$$A'^{12} = \frac{1}{\lambda^3}\left[\frac{v}{c}\cos\frac{\theta^1}{c}\csc^2\frac{\theta^1}{c} - w,_2 \csc^2\frac{\theta^1}{c} - v,_1 \csc\frac{\theta^1}{c}\right], \tag{5.22}$$

$$A'^{22} = \frac{-2}{\lambda^3}\csc^2\frac{\theta^1}{c}\left[v,_2 \csc\frac{\theta^1}{c} + \frac{w}{c}\operatorname{ctn}\frac{\theta^1}{c} + \frac{u}{c}\right].$$

The values of I'_i are obtained from (4.11), (4.12) and (4.13) as

$$I'_1 = \frac{4\,\lambda}{c}\, u + 2\,\lambda\, w,_1 + \frac{2\,\lambda}{c}\, w \operatorname{ctn}\frac{\theta^1}{c} + 2\,\lambda\, v,_2 \csc\frac{\theta^1}{c} + 2\,\lambda_3\,\lambda'_3,$$

$$I'_2 = \left[2\,\lambda_3^2\,\lambda + 2\,\lambda^3\right]\left[w,_1 + \frac{2}{c}\, u + v,_2 \csc\frac{\theta^1}{c} + \frac{w}{c}\operatorname{ctn}\frac{\theta^1}{c}\right]$$
$$+ 2\,\lambda_3\,\lambda'_3\,\lambda^4\left[\frac{2}{\lambda^2} + \frac{1}{\lambda_3^2}\right], \tag{5.23}$$

$$I'_3 = 2\,\lambda_3^2\,\lambda^3\left[w,_1 + \frac{2\,u}{c} + v,_2 \csc\frac{\theta^1}{c} + \frac{w}{c}\operatorname{ctn}\frac{\theta^1}{c}\right] + 2\,\lambda_3\,\lambda'_3\,\lambda^4.$$

It follows from (4.18) and (5.20) that

$$D'^{11} = 2\,\lambda\, v,_2 \csc\frac{\theta^1}{c} + \frac{2\,\lambda}{c}\, w \operatorname{ctn}\frac{\theta^1}{c} + \frac{2\,\lambda}{c}\, u + 2\,\lambda_3\,\lambda'_3,$$

$$D'^{12} = \frac{\lambda}{c}\, v \cos\frac{\theta^1}{c}\csc^2\frac{\theta^1}{c} - \lambda\, v,_1 \csc\frac{\theta^1}{c} - \lambda\, w,_2 \csc^2\frac{\theta^1}{c}, \tag{5.24}$$

$$D'^{22} = 2\,\lambda_3\,\lambda'_3 \csc^2\frac{\theta^1}{c} + 2\,\lambda\left(w,_1 + \frac{u}{c}\right)\csc^2\frac{\theta^1}{c}.$$

For an incompressible material, $I_3 = 1$ and $I_3' = 0$. Therefore from (5.5) and (5.23),

$$\lambda_3 = \frac{1}{\lambda^2}, \qquad \lambda_3' = \frac{-1}{\lambda^3}\left[w,_1 + \frac{2}{c}u + v,_2 \csc\frac{\theta^1}{c} + \frac{w}{c}\operatorname{ctn}\frac{\theta^1}{c}\right]. \tag{5.25}$$

The perturbed contravariant components of the physical stress resultant are, from (3.15), (4.22), (4.23), (5.13), (5.14),

$$\begin{aligned} n'^{11} \doteq \frac{h}{\lambda^2}\Bigg[& \frac{2}{c}\left|K_1 + \frac{3}{\lambda^5}\psi + \left(\frac{4}{\lambda^7} - \frac{1}{\lambda}\right)\Phi\right| u \\ & + \frac{1}{c}\left|K_1 + \left(\frac{1}{\lambda^5} + \lambda\right)\psi + \left(\frac{3}{\lambda^7} - \frac{1}{\lambda}\right)\Phi\right| w \operatorname{ctn}\frac{\theta^1}{c} \\ & + \left|K_1 + \left(\frac{5}{\lambda^7} - \frac{1}{\lambda}\right)\left(\Phi + \lambda^2\psi\right)\right| w,_1 \\ & + \left|K_1 + \left(\frac{1}{\lambda^5} + \lambda\right)\psi + \left(\frac{3}{\lambda^7} - \frac{1}{\lambda}\right)\Phi\right| v,_2 \csc\frac{\theta^1}{c}\Bigg], \end{aligned} \tag{5.26}$$

$$\begin{aligned} n'^{12} = \frac{h}{\lambda^2}\Bigg[& \frac{1}{c}\left|\lambda\psi - \frac{1}{\lambda^7}\left(\Phi + 2\lambda^2\psi\right)\right| v \csc\frac{\theta^1}{c}\operatorname{ctn}\frac{\theta^1}{c} \\ & + \left|-\lambda\psi + \frac{1}{\lambda^7}\left(\Phi + 2\lambda^2\psi\right)\right| v,_1 \csc\frac{\theta^1}{c} \\ & + \left|-\lambda\psi + \frac{1}{\lambda^7}\left(\Phi + 2\lambda^2\psi\right)\right| w,_2 \csc^2\frac{\theta^1}{c}\Bigg], \end{aligned} \tag{5.27}$$

$$\begin{aligned} n'^{22} = \frac{h}{\lambda^2}\csc^2\frac{\theta^1}{c}\Bigg[& \frac{2}{c}\left|K_1 + \left(\frac{4}{\lambda^7} - \frac{1}{\lambda}\right)\Phi + \frac{3}{\lambda^5}\psi\right| u \\ & + \left|K_1 + \left(\frac{3}{\lambda^7} - \frac{1}{\lambda}\right)\Phi + \left(\frac{1}{\lambda^5} + \lambda\right)\psi\right| w,_1 \end{aligned} \tag{5.28}$$

$$+ \frac{1}{c} \left\{ K_1 + \left(\frac{5}{\lambda^7} - \frac{1}{\lambda} \right) \left(\Phi + \lambda^2 \psi \right) \right\} w \operatorname{ctn} \frac{\theta^1}{c}$$

$$+ \left\{ K_1 + \left(\frac{5}{\lambda^7} - \frac{1}{\lambda} \right) \left(\Phi + \lambda^2 \psi \right) \right\} v_{,2} \csc \frac{\theta^1}{c} \Bigg],$$

where

$$K_1 = 2 \left\{ 1 - \frac{1}{\lambda^6} \right\} \left\{ \lambda \left(1 - \frac{1}{\lambda^6} \right) \frac{\partial \Phi}{\partial I_1} + \lambda^3 \left(1 - \frac{2}{\lambda^6} \right) \frac{\partial \Phi}{\partial I_2} - \frac{1}{\lambda} \frac{\partial \psi}{\partial I_2} \right\}. \tag{5.29}$$

From section A,

$$f^i = 0. \tag{5.30}$$

Also,

$$\begin{aligned} \bar{A}' &= X'_i \bar{e}_i \\ &= \left[u \sin \frac{\theta^1}{c} \cos \frac{\theta^2}{c} - v \sin \frac{\theta^2}{c} + w \cos \frac{\theta^1}{c} \cos \frac{\theta^2}{c} \right] \bar{e}_1 \\ &+ \left[u \sin \frac{\theta^1}{c} \sin \frac{\theta^2}{c} + v \cos \frac{\theta^2}{c} + w \cos \frac{\theta^1}{c} \sin \frac{\theta^2}{c} \right] \bar{e}_2 \\ &+ \left[u \cos \frac{\theta^1}{c} - w \sin \frac{\theta^1}{c} \right] \bar{e}_3 , \end{aligned} \tag{5.31}$$

and

$$\bar{A}_k = A^i_k \bar{e}_i , \tag{5.32}$$

where

$$\begin{aligned} \bar{A}_1 &= \lambda \cos \frac{\theta^1}{c} \cos \frac{\theta^2}{c} \bar{e}_1 + \lambda \cos \frac{\theta^1}{c} \sin \frac{\theta^2}{c} \bar{e}_2 - \lambda \sin \frac{\theta^1}{c} \bar{e}_3 , \\ \bar{A}_2 &= - \lambda \sin \frac{\theta^1}{c} \sin \frac{\theta^2}{c} \bar{e}_1 + \lambda \sin \frac{\theta^1}{c} \cos \frac{\theta^2}{c} \bar{e}_2 , \\ \bar{A}_3 &= \sin \frac{\theta^1}{c} \cos \frac{\theta^2}{c} \bar{e}_1 + \sin \frac{\theta^1}{c} \sin \frac{\theta^2}{c} \bar{e}_2 + \cos \frac{\theta^1}{c} \bar{e}_3 . \end{aligned} \tag{5.33}$$

Thus from the tensor equation (4.28), use (4.27), (5.30), (5.31), (5.32) and (5.33) to obtain

$$
\begin{aligned}
&\ddot{u} \sin\frac{\theta^1}{c} \cos\frac{\theta^2}{c} - \ddot{v} \sin\frac{\theta^2}{c} + \ddot{w} \cos\frac{\theta^1}{c} \cos\frac{\theta^2}{c} = \\
&f'^1 \lambda \cos\frac{\theta^1}{c} \cos\frac{\theta^2}{c} - f'^2 \lambda \sin\frac{\theta^1}{c} \sin\frac{\theta^2}{c} + f'^3 \sin\frac{\theta^1}{c} \cos\frac{\theta^2}{c} , \\
&\ddot{u} \sin\frac{\theta^1}{c} \sin\frac{\theta^2}{c} + \ddot{v} \cos\frac{\theta^2}{c} + \ddot{w} \cos\frac{\theta^1}{c} \sin\frac{\theta^2}{c} = \\
&f'^1 \lambda \cos\frac{\theta^1}{c} \sin\frac{\theta^2}{c} + f'^2 \lambda \sin\frac{\theta^1}{c} \cos\frac{\theta^2}{c} + f'^3 \sin\frac{\theta^1}{c} \sin\frac{\theta^2}{c} , \\
&\ddot{u} \cos\frac{\theta^1}{c} - \ddot{w} \sin\frac{\theta^1}{c} = - f'^1 \lambda \sin\frac{\theta^1}{c} + f'^3 \cos\frac{\theta^1}{c} .
\end{aligned}
\tag{5.34}
$$

(5.34) has the solution

$$
\begin{aligned}
&f'^1 = \frac{1}{\lambda}\ddot{w} , \\
&f'^2 = \frac{1}{\lambda} \ddot{v} \csc\frac{\theta^1}{c} , \\
&f'^3 = \ddot{u} ,
\end{aligned}
\tag{5.35}
$$

where the dots denote differentiation with respect to time. The remaining quantities needed in the equations of motion, (4.30) and (4.31), are the $B'_{\alpha\beta}$ and $\Gamma'^{\alpha}_{\beta\gamma}$ which are obtained from (4.32) and (4.33) as

$$
\begin{aligned}
&B'_{11} = - \frac{1}{c^2} u - \frac{2}{c} w_{,1} + u_{,11} , \\
&B'_{12} = \frac{1}{c^2} v \cos\frac{\theta^1}{c} - \frac{1}{c} v_{,1} \sin\frac{\theta^1}{c} - \frac{1}{c} u_{,2} \operatorname{ctn}\frac{\theta^1}{c} - \frac{1}{c} w_{,2} + u_{,12} , \\
&B'_{22} = -\frac{1}{c^2} u \sin^2\frac{\theta^1}{c} - \frac{2}{c^2} w \sin\frac{\theta^1}{c} \cos\frac{\theta^1}{c} + \frac{1}{c} u_{,1} \sin\frac{\theta^1}{c} \cos\frac{\theta^1}{c} - \frac{2}{c} v_{,2} \sin\frac{\theta^1}{c} + u_{,22} ;
\end{aligned}
$$

$$\Gamma'^{\,1}_{11} = \frac{1}{\lambda} w_{,11} + \frac{1}{\lambda c} u_{,1} , \tag{5.36}$$

$$\Gamma'^{\,1}_{12} = \Gamma'^{\,1}_{21} = \frac{1}{\lambda} w_{,12} + \frac{1}{\lambda c} u_{,2} + \frac{1}{\lambda c^2} v \cos^2 \frac{\theta^1}{c} \csc \frac{\theta^1}{c}$$

$$- \frac{1}{\lambda c} v_{,1} \cos \frac{\theta^1}{c} - \frac{1}{\lambda c} w_{,2} \operatorname{ctn} \frac{\theta^1}{c} ,$$

$$\Gamma'^{\,1}_{22} = \frac{1}{\lambda c} w_{,1} \sin \frac{\theta^1}{c} \cos \frac{\theta^1}{c} - \frac{2}{\lambda c} v_{,2} \cos \frac{\theta^1}{c} + \frac{1}{\lambda} w_{,22}$$

$$- \frac{1}{\lambda c} u_{,1} \sin^2 \frac{\theta^1}{c} - \frac{w}{\lambda c^2} \left[\cos^2 \frac{\theta^1}{c} - \sin^2 \frac{\theta^1}{c} \right],$$

$$\Gamma'^{\,2}_{11} = \frac{1}{\lambda} v_{,11} \csc \frac{\theta^1}{c} - \frac{1}{\lambda c} u_{,2} \csc^2 \frac{\theta^1}{c} + \frac{1}{\lambda c^2} v \csc \frac{\theta^1}{c} ,$$

$$\Gamma'^{\,2}_{12} = \Gamma'^{\,2}_{21} = \frac{1}{\lambda} v_{,21} \csc \frac{\theta^1}{c} + \frac{1}{\lambda c} u_{,1} + \frac{1}{\lambda c} w_{,1} \operatorname{ctn} \frac{\theta^1}{c}$$

$$- \frac{1}{\lambda c} v_{,2} \cos \frac{\theta^1}{c} \csc^2 \frac{\theta^1}{c} - \frac{1}{\lambda c^2} w \csc^2 \frac{\theta^1}{c} ,$$

$$\Gamma'^{\,2}_{22} = \frac{1}{\lambda} v_{,22} \csc \frac{\theta^1}{c} + \frac{2}{\lambda c} w_{,2} \operatorname{ctn} \frac{\theta^1}{c} + \frac{1}{\lambda c} v_{,1} \cos \frac{\theta^1}{c}$$

$$+ \frac{1}{\lambda c} u_{,2} - \frac{1}{\lambda c^2} v \cos^2 \frac{\theta^1}{c} \csc \frac{\theta^1}{c} .$$

Substitute (5.26), (5.27), (5.28), (5.30), (5.35) and (5.36) into (4.30) and (4.31), combine and simplify the terms to reduce the equations of motion to

$$\rho \ddot{u} = \frac{\lambda^2 p'^3}{h} + \frac{2}{c^2} (K_2 + K_3) u + \frac{K_2}{c^2} w \operatorname{ctn} \frac{\theta^1}{c} + \frac{K_2}{c} w_{,1}$$

$$+ \frac{K_3}{c} u_{,1} \operatorname{ctn} \frac{\theta^1}{c} + \frac{K_2}{c} v_{,2} \csc \frac{\theta^1}{c} + K_3 u_{,11} + K_3 u_{,22} \csc^2 \frac{\theta^1}{c} , \tag{5.37}$$

$$\frac{\rho \ddot{w}}{\lambda} = \frac{\lambda^2 p'^1}{h} + \left[\frac{2 K_4}{c^2} - \frac{K_5}{c^2} \csc^2 \frac{\theta^1}{c} \right] w - \frac{K_2}{\lambda c} u_{,1} + \frac{K_5}{c} w_{,1} \operatorname{ctn} \frac{\theta^1}{c}$$

$$- \frac{K_6}{c} v_{,2} \cos \frac{\theta^1}{c} \csc^2 \frac{\theta^1}{c} + K_5 w_{,11} - \frac{K_2}{2\lambda} v_{,12} \csc \frac{\theta^1}{c} \qquad (5.38)$$

$$+ K_4 w_{,22} \csc^2 \frac{\theta^1}{c} ,$$

$$\frac{\rho \ddot{v}}{\lambda} = \frac{\lambda^2 p'^2}{h} \sin \frac{\theta^1}{c} + \frac{K_4}{c^2} \left(2 - \csc^2 \frac{\theta^1}{c} \right) v + \frac{K_4}{c} v_{,1} \operatorname{ctn} \frac{\theta^1}{c}$$

$$- \frac{K_2}{\lambda c} u_{,2} \csc \frac{\theta^1}{c} + \frac{K_6}{c} w_{,2} \cos \frac{\theta^1}{c} \csc^2 \frac{\theta^1}{c} - \frac{K_2}{2\lambda} w_{,12} \csc \frac{\theta^1}{c} \qquad (5.39)$$

$$+ K_4 v_{,11} + K_5 v_{,22} \csc^2 \frac{\theta^1}{c} ,$$

where

$$K_2 = -2\lambda \left[\frac{3}{\lambda^7} \Phi + \left(\lambda + \frac{2}{\lambda^5} \right) \psi + K_1 \right],$$

$$K_3 = \left(1 - \frac{1}{\lambda^6} \right) \left(\Phi + \lambda^2 \psi \right),$$

$$K_4 = \frac{1}{\lambda} \Phi + \frac{1}{\lambda^5} \psi , \qquad (5.40)$$

$$K_5 = \left(\frac{1}{\lambda} + \frac{3}{\lambda^7} \right) \left(\Phi + \lambda^2 \psi \right) + K_1 ,$$

$$K_6 = \left(\lambda + \frac{4}{\lambda^5} \right) \psi + \left(\frac{2}{\lambda} + \frac{3}{\lambda^7} \right) \Phi + K_1 .$$

C. Solutions of the Equations of Motion

For small displacements superposed on a known finite deformation of a sphere composed of an incompressible elastic material, the equations of motion are (5.37), (5.38) and (5.39). In these equations the p'^i are components of the resultant force acting on the surface and for free vibration are equated to zero. Consider motion which is independent of the circumferential angle, $\frac{\theta^2}{c}$. Let $y = \frac{\theta^1}{c}$ and let the displacements vary in time as

$$u = U(y)\, e^{i\omega t} ,$$

$$v = V(y)\, e^{i\omega t} , \qquad (5.41)$$

$$w = W(y)\, e^{i\omega t} .$$

The equations of motion will have the form

$$K_3 \left\{ \frac{d^2U}{dy^2} + \text{ctn } y \frac{dU}{dy} + \left[\frac{\lambda\beta + 2K_2 + 2K_3}{K_3} \right] U \right\} + K_2 \left\{ \frac{dW}{dy} + \text{ctn } y\, W \right\} = 0 , \qquad (5.42)$$

$$-\frac{K_2}{\lambda} \frac{dU}{dy} + K_5 \left\{ \frac{d^2W}{dy^2} + \text{ctn } y \frac{dW}{dy} - \csc^2 y\, W + \frac{2K_4}{K_5} W \right\} + \beta W = 0 , \qquad (5.43)$$

$$\beta\, V + K_4 \left\{ \frac{d^2V}{dy^2} + \text{ctn } y \frac{dV}{dy} + (2 - \csc^2 y)\, V \right\} = 0 , \qquad (5.44)$$

where

$$\beta = \frac{c^2 \rho\, \omega^2}{\lambda} .$$

1. <u>Static case</u>

For the static case, $\beta = 0$, the equations of motion can be put in the form

$$K_3 \left\{ \frac{d^2U}{dy^2} + \text{ctn } y \frac{dU}{dy} + 2\left(1 + \frac{K_2}{K_3}\right) U \right\} + K_2 \left\{ \frac{dW}{dy} + \text{ctn } y\, W \right\} = 0 , \qquad (5.45)$$

$$\frac{dU}{dy} = \frac{\lambda K_5}{K_2} \left\{ \frac{d^2W}{dy^2} + \text{ctn } y \frac{dW}{dy} - \left(\csc^2 y - \frac{2K_4}{K_5} \right) W \right\} , \qquad (5.46)$$

$$\frac{d^2V}{dy^2} + \text{ctn } y \frac{dV}{dy} + (2 - \csc^2 y)\, V = 0 . \qquad (5.47)$$

Equation (5.47) is immediately solvable and

$$V = C\, P_1^1 (\cos y) , \tag{5.48}$$

is the only solution which remains finite for $y = 0$ and π, C being an arbitrary constant, and P_1^1 the associated Legendre function of degree and order one.

In order to facilitate solution of (5.45) and (5.46) let the operator L be defined by

$$L (W) = \frac{d^2 W}{dy^2} + \text{ctn}\, y \frac{dW}{dy} - \csc^2 y\, W , \tag{5.49}$$

and let Q (y) be defined by

$$L (W) + a W = Q (y) , \tag{5.50}$$

where a will be specified later. Substitute (5.49) and (5.50) in (5.46), and substitute the resulting equation into the equation obtained by differentiating (5.45) to obtain

$$L (Q) + 2 \left(1 + \frac{K_2}{K_3}\right) Q + \left\{ \frac{K_2^2}{\lambda K_3 K_5} - a + \frac{2 K_4}{K_5} \right\} \left\{ L (W) + \frac{2 \left(1 + \frac{K_2}{K_3}\right) \left(a - \frac{2 K_4}{K_5}\right)}{a - \frac{2 K_4}{K_5} - \frac{K_2^2}{\lambda K_3 K_5}} W \right\} = 0 . \tag{5.51}$$

If a is defined as the solution to

$$a = \frac{2 \left(1 + \frac{K_2}{K_3}\right) \left(a - \frac{2 K_4}{K_5}\right)}{a - \frac{2 K_4}{K_5} - \frac{K_2^2}{\lambda K_3 K_5}} , \tag{5.52}$$

(5.51) reduces to

$$L(Q) + \left\{ 2 \left(1 + \frac{K_2}{K_3}\right) + \frac{K_2^2}{\lambda K_3 K_5} - a + \frac{2 K_4}{K_5} \right\} Q = 0 . \tag{5.53}$$

The solution of (5.53) is

$$Q = A P_n^1 (\cos y) + B Q_n^1 (\cos y) , \tag{5.54}$$

where

$$n(n + 1) = 2 \left(1 + \frac{K_2}{K_3} + \frac{K_4}{K_5}\right) + \frac{K_2^2}{\lambda K_3 K_5} - a , \tag{5.55}$$

n is not necessarily an integer. The $P_n^1 (\cos y)$ and $Q_n^1 (\cos y)$ are Ferrers' associated Legendre functions of degree n and order one of the first and second kinds respectively. Substitute the value of Q, given by (5.54) into (5.50) to obtain the differential equation

$$L(W) + a W = A P_n^1 (\cos y) + B Q_n^1 (\cos y) . \tag{5.56}$$

If $a \neq n(n + 1)$, the general solution of (5.56) is

$$W = C_1 P_m^1 (\cos y) + C_2 Q_m^1 (\cos y) + \frac{A P_n^1 (\cos y) + B Q_n^1 (\cos y)}{m(m + 1) - n(n + 1)} , \tag{5.57}$$

where, for convenience a is written as

$$a = m(m + 1) . \tag{5.58}$$

Use the variation of parameters technique to obtain a particular solution if $a = n(n + 1)$. It is convenient to let $x = \cos y$ and define $W(\cos^{-1} x) = W^*(x)$. This notation changes (5.56) to

$$(1 - x^2) \frac{d^2 W^*}{dx^2} - 2x \frac{dW^*}{dx} + \left[n(n + 1) - \frac{1}{1 - x^2} \right] W^* = A P_n^1 (x) + B Q_n^1 (x) . \tag{5.59}$$

Substitute

$$W_p^* = A\ (x)\ P_n^1\ (x) + B\ (x)\ Q_n^1\ (x)\ , \tag{5.60}$$

where A (x) and B (x) are differentiable functions, into (5.59) and obtain

$$P_n^1\ (x)\ \frac{d}{dx} A\ (x) + Q_n^1\ (x)\ \frac{d}{dx} B\ (x) = 0\ , \tag{5.61}$$

$$(1 - x^2) \left[\frac{d}{dx} P_n^1\ (x)\ \frac{d}{dx} A\ (x) + \frac{d}{dx} Q_n^1\ (x)\ \frac{d}{dx} B\ (x) \right] = A\ P_n^1\ (x) + B\ Q_n^1\ (x)\ . \tag{5.62}$$

Solve (5.61) and (5.62) simultaneously for $\frac{d}{dx} A\ (x)$ and $\frac{d}{dx} B\ (x)$. Use the relationship

$$(1 - x^2) \left[P_\mu^\nu\ (x)\ \frac{d}{dx} Q_\mu^\nu\ (x) - Q_\mu^\nu\ (x)\ \frac{d}{dx} P_\mu^\nu\ (x) \right] = \frac{2^{2\nu}\ \Gamma\left(1 + \frac{\mu}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{1}{2} + \frac{\mu}{2} + \frac{\nu}{2}\right)}{\Gamma\left(1 + \frac{\mu}{2} - \frac{\nu}{2}\right) \Gamma\left(\frac{1}{2} + \frac{\mu}{2} - \frac{\nu}{2}\right)} \tag{5.63}$$

from [4], page 46, and integrate to obtain

$$A\ (x) = \frac{-1}{n\ (n+1)} \left\{ A \int P_n^1\ (x)\ Q_n^1\ (x)\ dx + B \int \left[Q_n^1\ (x) \right]^2 dx \right\}, \tag{5.64}$$

$$B\ (x) = \frac{1}{n\ (n+1)} \left\{ A \int \left[P_n^1\ (x) \right]^2 dx + B \int Q_n^1\ (x)\ P_n^1\ (x)\ dx \right\}. \tag{5.65}$$

Thus the general solution of (5.56) for a = n (n + 1) is

$$\begin{aligned} W = {} & C_1\ P_n^1\ (\cos y) + C_2\ Q_n^1\ (\cos y) \\ & + \frac{A}{n\ (n+1)} \left\{ Q_n^1\ (\cos y) \int \left[P_n^1\ (\cos y) \right]^2 \sin y\ dy \right. \\ & \left. - P_n^1\ (\cos y) \int P_n^1\ (\cos y)\ Q_n^1\ (\cos y)\ \sin y\ dy \right\} \\ & + \frac{B}{n\ (n+1)} \left\{ Q_n^1\ (\cos y) \int P_n^1\ (\cos y)\ Q_n^1\ (\cos y)\ \sin y\ dy \right. \end{aligned} \tag{5.66}$$

$$- P_n^1 (\cos y) \int \left[Q_n^1 (\cos y)\right]^2 \sin y \, dy \Big\} .$$

Once W is determined, U may be calculated from (5.46) and

$$U = \frac{\lambda K_5}{K_2} \left[A \int P_n^1 (\cos y) \, dy + B \int Q_n^1 (\cos y) - \left(a - \frac{2 K_4}{K_5} \right) \int W \, dy \right] , \quad (5.67)$$

where a and W are given from either (5.57) or (5.66). If the strain-energy function, Z, and extension ratio, λ, are such that

$$\left[2 \left(1 + \frac{K_2}{K_3} + \frac{K_4}{K_5} \right) + \frac{K_2^2}{\lambda K_3 K_5} \right]^2 - \frac{16 K_4}{K_5} \left(1 + \frac{K_2}{K_3} \right) = 0 , \quad (5.68)$$

$a = n (n + 1)$ and W is given by (5.66). If the quantity in (5.68) is not zero, $a = m (m + 1)$ and W is given by (5.57). If the quantity in (5.68) is negative, the associated Legendre functions are of complex degree. This fact will be used in the stability analysis in part D of this chapter.

2. Dynamic case

The method of solution of the equations of motion with $\beta \neq 0$ follows the same pattern as used for $\beta = 0$. Let Q (y) be defined by (5.50) and use this to simplify the form of (5.43). Substitute the value of $\frac{dU}{dy}$ from (5.43) into the equation obtained by differentiating (5.42). This will eliminate U and leave the differential equation

$$\frac{\lambda K_3 K_5}{K_2} \left\{ L (Q) + \left(\frac{\lambda \beta + 2 K_2 + 2 K_3}{K_3} \right) Q + \left[\frac{\beta + 2 K_4}{K_5} - a \right] \left[L (W) + \left(\frac{\lambda \beta + 2 K_2 + 2 K_3}{K_3} \right) W \right] \right\} + K_2 \, L (W) = 0 . \quad (5.69)$$

Rewrite (5.69) as

$$L (Q) + a_1 Q + a_3 L (W) + a_1 a_2 W = 0 , \quad (5.70)$$

where

$$a_1 = \frac{\lambda\beta + 2K_2 + 2K_3}{K_3},$$

$$a_2 = \frac{\beta + 2K_4}{K_5} - a,$$

$$a_3 = a_2 + \frac{K_2^2}{\lambda K_3 K_5}.$$

To simplify the form of (5.70) and reduce it to a differential equation in one dependent variable, let the a in (5.50) be defined as the solution to

$$a\, a_3 = a_1\, a_2 . \tag{5.71}$$

Thus, (5.70) reduces to

$$L(Q) + (a_1 + a_3)\, Q = 0 . \tag{5.72}$$

The general solution of (5.72) is

$$Q = A\, P_n^1 (\cos y) + B\, Q_n^1 (\cos y) , \tag{5.73}$$

where for convenience $a_3 + a_1$ is written as

$$a_3 + a_1 = n\,(n + 1) . \tag{5.74}$$

Substitute (5.73) into (5.50) to obtain

$$L(W) + a\, W = A\, P_n^1 (\cos y) + B\, Q_n^1 (\cos y) , \tag{5.75}$$

which is the same differential equation, (5.56), derived for the static case. The difference is that now n is given by (5.74) and the a, set equal to m (m + 1) for convenience, is given by the solution to (5.71). If $a \neq n\,(n + 1)$, the general solution of (5.75) is

$$W = C_1\, P_m^1 (\cos y) + C_2\, Q_m^1 (\cos y) + \frac{A\, P_n^1 (\cos y) + B\, Q_n^1 (\cos y)}{m\,(m + 1) - n\,(n + 1)} . \tag{5.76}$$

If $a = n(n+1)$, the general solution of (5.75) is

$$
\begin{aligned}
W = {} & C_1 P_n^1(\cos y) + C_2 Q_n^1(\cos y) \\
& + \frac{A}{n(n+1)} \left\{ Q_n^1(\cos y) \int \left[P_n^1(\cos y) \right]^2 \sin y \, dy \right. \\
& \left. - P_n^1(\cos y) \int P_n^1(\cos y) \, Q_n^1(\cos y) \sin y \, dy \right\} \qquad (5.77) \\
& + \frac{B}{n(n+1)} \left\{ Q_n^1(\cos y) \int P_n^1(\cos y) \, Q_n^1(\cos y) \sin y \, dy \right. \\
& \left. - P_n^1(\cos y) \int \left[Q_n^1(\cos y) \right]^2 \sin y \, dy \right\} .
\end{aligned}
$$

The condition $a = n(n+1)$ implies that

$$[a_1 + a_3 + a]^2 - 4 [a_2 + a] a_1 = 0 . \qquad (5.78)$$

Once W is determined, integrate (5.43) to obtain

$$U = \frac{\lambda K_5}{K_2} \int \left[L(W) + \left(\frac{\beta + 2K_4}{K_5} \right) W \right] dy . \qquad (5.79)$$

Use (5.75) to change (5.79) to

$$U = \frac{\lambda K_5}{K_2} \int \left[\left(\frac{\beta + 2K_4}{K_5} - a \right) W + A P_n^1(\cos y) + B Q_n^1(\cos y) \right] dy . \qquad (5.80)$$

The third displacement, V, is obtained from (5.44). The general solution of (5.44) is

$$V = C_3 P_k^1(\cos y) + C_4 Q_k^1(\cos y) , \qquad (5.81)$$

where

$$k(k+1) = 2 + \frac{\beta}{K_4} . \qquad (5.82)$$

Thus V, from (5.81), W, from (5.76) or (5.77), and U, from (5.80), constitute the general solution of the equations of motion for an arbitrary strain-energy function.

There are no "edge" conditions for the complete sphere which could be applied to the solution of the equations of motion. A reasonable condition to assume is that solution, i.e. the displacements, remain finite throughout the region. This implies that the Legendre functions be of integral degree, see [19] page 91 or [17] page 714. Since the degree is a function of the frequencies, the frequencies may be calculated from this requirement. This technique is used in [12] for the free vibrations of orthotropic shells.

The frequencies for circumferential motion are immediately obtained from (5.82) as

$$\beta = K_4 (K + 2)(K - 1) , \tag{5.83}$$

where K takes on integral values.

The motion of the inflated sphere in the radial direction is coupled with the motion in the azimuthal direction as seen in (5.42) and (5.43). The frequencies associated with this motion may be calculated from one of two equations, (5.71) or (5.74). If $a = M (M + 1)$, where M is an integer, (5.71) can be used to solve for β. If a is given in terms of β by the solution to (5.71), then (5.74), where $n = N$, an integer, can be used to solve for β. The two methods give the same quadratic equation for the determination of the frequencies,

$$\begin{aligned} &\beta^2 + \frac{\beta}{\lambda} \left[2 (\lambda K_4 + K_2 + K_3) - N (N + 1) (\lambda K_5 + K_3) \right] \\ &+ \frac{2}{\lambda} \left[K_2 + K_3 \right] \left[K_4 - N (N (N + 1) K_5 \right] \\ &- \frac{N (N + 1)}{\lambda} \left[2 K_3 K_4 + \frac{K_2^2}{\lambda} + N (N + 1) K_3 K_5 \right] = 0 . \end{aligned} \tag{5.84}$$

Use the quadratic formula and combine terms to obtain the solution of (5.84) as

$$\begin{aligned} 2 \lambda \beta &= N (N + 1) (\lambda K_5 + K_3) - 2 (\lambda K_4 + K_2 + K_3) \\ &\pm \sqrt{\left[N (N+1) (\lambda K_5 - K_3) + 2 (K_2 + K_3 - \lambda K_4) \right]^2 + 4 N (N+1) K_2^2} . \end{aligned} \tag{5.85}$$

Thus for an arbitrary strain-energy function, the frequencies for the circumferential modes are given by (5.83) and the frequencies for the coupled radial and azimuthal modes are given by (5.85). For the infinitesimal vibrations of a spherical shell, [12] and [14], and the free vibrations of a cylinder which has been inflated and extended [2], two sets of frequencies exist for each integer. The frequencies for free vibrations of the inflated sphere are expected to behave in somewhat the same manner. This behavior is shown in (5.85) where two sets of frequencies exist for each integer N.

D. Stability Analysis for a Neo-Hookian Material

The stability of a body having undergone a finite deformation is defined as follows: An elastic body in a known state of finite deformation is subjected to small additional dynamic displacements. If the resulting motion of the body remains finite for all time, the body is said to be stable. If the resulting motion is unbounded, the body is said to be unstable. Thus from (5.41), where $\lambda \beta = c^2 \rho \omega^2$, if β is negative or complex the displacements will not be bounded. Thus the dynamic stability criterion is that β have real positive values.

This dynamic criterion sometimes involves a difficult and lengthy process when applied to a specific problem. The concept of static instability is sometimes valuable as it will determine the values of certain parameters which characterize the stability of a system in a much simpler fashion. A system is said to be statically unstable if for a given finite deformation and certain constraints, a neighboring non-trivial equilibrium state exists which satisfies the constraints. If such a state does not exist, the system is said to be statically stable.

To illustrate the techniques used to determine the regions of stability for the case here, consider a strain-energy of the form suggested by Mooney [16],

$$Z = H \left[(I_1 - 3) + \Gamma (I_2 - 3) \right], \tag{5.86}$$

where I_1 and I_2 are the strain invariants and H and Γ are positive constants. For a neo-Hookian material, $\Gamma = 0$, and the constants given by (5.29) and (5.40) are

$$
\begin{aligned}
K_1 &= 0 , \\
K_2 &= \frac{-12}{\lambda^6} H , \\
K_3 &= 2 \left(1 - \frac{1}{\lambda^6}\right) H , \\
K_4 &= \frac{2}{\lambda} H , \\
K_5 &= 2\left(\frac{1}{\lambda} + \frac{3}{\lambda^7}\right) H , \\
K_6 &= 2\left(\frac{2}{\lambda} + \frac{3}{\lambda^7}\right) H .
\end{aligned}
\tag{5.87}
$$

For a neo-Hookian material K_4 is positive. Thus, from (5.83), β is never negative and no instabilities can result from the circumferential vibrations. The frequencies of the circumferential vibrations are given by

$$\lambda \beta = 2 H (K + 2) (K - 1) , \tag{5.88}$$

where

$$K = 1, 2, 3, ...$$

Since (5.85) is the equation from which the frequencies for non-circumferential motion are determined, the dynamic stability criterion is that the frequencies be non-negative for each integral value of N. Thus if λ is such that the frequencies are negative for some value of N, the state of finite deformation characterized by λ will not be a stable configuration. For a general strain-energy function, β, from (5.85), will be negative if either

$$N (N + 1) (\lambda K_5 + K_3) - 2 (\lambda K_4 + K_2 + K_3) < 0 , \tag{5.89}$$

or

$$N^2 (N+1)^2 K_3 K_5 - N(N+1) \left[2 K_3 K_4 + 2 K_5 (K_2 + K_3) + \frac{K_2^2}{\lambda} \right] + 4 K_4 (K_2 + K_3) < 0 . \tag{5.90}$$

For a neo-Hookian material (5.89) reduces to

$$N (N+1) (1 + \frac{1}{\lambda^6}) + (\frac{7}{\lambda^6} - 2) < 0 . \tag{5.91}$$

As N takes the values 1, 2, 3, . . . , obviously (5.91) will never be satisfied, i.e. no instabilities result from this condition. Also for a neo-Hookian material, (5.90) becomes

$$N^2 (N+1)^2 (\lambda^6 - 1)(\lambda^6 + 3) - 2 N (N+1)(2 \lambda^6 + 1)(\lambda^6 - 3) + 4 \lambda^6 (\lambda^6 - 7) < 0 . \tag{5.92}$$

As N takes the values 1, 2, 3, . . . , it is easily shown for $\lambda > 1$ that (5.92) is never satisfied. Thus the dynamic stability criterion predicts, for vibrations independent of the circumferential angle, that the inflated sphere will be stable for any extension ratio greater than one. In the stability analysis of an inflated and extended cylinder [2] an analogous result occurs. There a cylinder, composed of Mooney type material, is found to be stable for all extension ratios of the radius which exceed a certain value as long as the extended cylinder length lies within a specified range.

The region of stability may also be obtained from considerations of static instability. The static analysis has been shown to be equivalent to the dynamic analysis only for holonomic systems. Since there are no physical boundary conditions for the complete sphere, the constraints here are non-holonomic. Thus results which are derived from a static analysis must be viewed with reservation until the static analysis of stability is shown to be valid for non-holonomic systems.

It is interesting to investigate the regions of stability predicted from a static analysis. Use the results from V.C.1. to determine a range of values of λ which will not admit a solution to the static equations of motion. From (5.52), (5.55) and (5.58), a non-trivial solution for the static case will not exist if a is a complex number. For a neo-Hookian material a is complex if the discriminant of the quadratic equation (5.52) is negative, i.e.

$$f(\Psi) = 4\Psi^4 - 20\Psi^3 - 59\Psi^2 + 102\Psi + 9 < 0 , \tag{5.93}$$

where

$$\Psi = \lambda^6 .$$

From the Budan-Fourier theorem, $f(\Psi) = 0$ has two positive roots and $f'(\Psi) = 0$ has one real root for $\Psi \geq 1$. Since

$$\begin{aligned} f(1) &= 36 , \\ f(1.4) &= -3.3 , \\ f(6.7) &= -383 , \\ f(6.8) &= 228 , \\ f'(\Psi) &= 0 \text{ for } 4.5 < \Psi < 5.5 , \end{aligned} \tag{5.94}$$

the inflated form of the sphere will be statically stable if

$$1.4 \leq \Psi \leq 6.7 ,$$

or (5.95)

$$1.06 \leq \lambda \leq 1.37 .$$

The region of stability predicted by a static analysis is included in the region predicted by the dynamic analysis. This indicates that the equivalence of static and dynamic methods of predicting stability can not be shown for a non-holonomic system.

In order to determine the modes of vibration, it must be determined whether W is given by (5.76) or (5.77). The condition that W be given by (5.77) is that equation (5.78) be satisfied. For a neo-Hookian material this condition becomes

$$g(\lambda\beta) = \lambda^{14}\beta^{2} + 12\lambda^{7}(\lambda^{6} - 2)\beta + 9(9\lambda^{12} + 18\lambda^{6} + 1) = 0\,. \qquad (5.96)$$

The minimum value that $g(\lambda\beta)$ obtains is $5\lambda^{12} + 34\lambda^{6} - 15$. Since $\lambda > 1$, this implies that $g(\lambda\beta) > 0$ and (5.78) will not be satisfied. Thus for a neo-Hookian material the displacements are given from (5.76), (5.80) and (5.81). In these equations the terms involving the associated Legendre function of the second kind must be omitted from the solution because of their logarithmic singularity at the poles. Thus the modes associated with the natural frequencies of vibration are given by integrals of associated Legendre functions of the first kind of integral degree and order one and the functions themselves.

VI. CONCLUSION

The theory of small deformations superposed on a known finite deformation has been used to derive equations of motion for the free vibrations of an inflated sphere composed of an incompressible material with an arbitrary strain-energy function. For motion which is independent of the circumferential angle, these equations are solved and the displacements are obtained in closed form. For a neo-Hookian material the mode shapes are given by integrals of associated Legendre functions of the first kind with integral degree and order one and the functions themselves. These mode shapes are necessary for any dynamic analysis of a spherical satellite which includes loads or equipment.

The natural frequencies of the dynamic system are given for a material described by an arbitrary strain-energy function by (5.83) and (5.85).

The dynamic stability criterion is that the displacements remain bounded for all time. This in turn implies that the frequencies be non-negative, (5.89) and (5.90). From (5.83) the sphere will be unstable due to circumferential vibrations if $K_4 < 0$. For a sphere composed of a neo-Hookian material, the dynamic stability criterion predicts that the sphere will be stable for free vibration independent of the circumferential angle for all values of extension ratio.

A static analysis predicts that the sphere will be stable for a neo-Hookian material if the extension ratio, λ, is such that

$$1.06 \leq \lambda \leq 1.37 \; . \tag{6.1}$$

The static analysis has been shown [3] to be equivalent to a dynamic analysis for holonomic systems. However the above results seem to imply that this equivalence can not be shown for non-holonomic systems.

The material set forth in sections I. through V. can be used as the origin for other considerations.

The theory of sections III. and IV. could be used to derive the equations of motion of a toroid or a cone capped by a sphere. The technique of solution in V. might then be used to determine the stability of the body and the mode shapes.

The solution of the equations of motion for the sphere, V.C., might be analyzed for a material with a strain-energy function different from the one describing a neo-Hookian material.

Using the mode shapes derived here, a study could be made into the behavior of an inflated sphere with various loads attached.

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IX. APPENDIX A

$$A'_{\alpha\beta} = \bar{A}_{\alpha} \cdot \left[w_{\gamma,\beta}\, \bar{A}^{\gamma} + w_{\gamma}\, \bar{A}^{\gamma}_{,\beta} \right] + \bar{A}_{\alpha} \cdot w_{3,\beta}\, \bar{A}^{3} + w_{3}\, \bar{A}_{\alpha} \cdot \bar{A}_{3,\beta}$$
$$+ \bar{A}_{\beta} \cdot \left[w_{\gamma,\alpha}\, \bar{A}^{\gamma} + w_{\gamma}\, \bar{A}^{\gamma}_{,\alpha} \right] + \bar{A}_{\beta} \cdot w_{3,\alpha}\, \bar{A}^{3} + w_{3}\, \bar{A}_{\beta} \cdot \bar{A}_{3,\alpha} \, . \tag{9.1}$$

Since $\bar{A}_3 \cdot \bar{A}_{\alpha} = 0$ and $\bar{A}^3 = \bar{A}_3$,

$$\bar{A}_{3,\beta} \cdot \bar{A}_{\alpha} = - \bar{A}_3 \cdot \bar{A}_{\alpha,\beta} \, . \tag{9.2}$$

As $\bar{A}_{\alpha} \cdot \bar{A}^{\beta} = \delta_{\alpha\beta}$, (9.1) reduces to

$$A'_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha} - w_3 \left[\bar{A}_3 \cdot \bar{A}_{\alpha,\beta} + \bar{A}_3 \cdot \bar{A}_{\beta,\alpha} \right]$$
$$+ w_{\gamma} \left[\bar{A}_{\alpha} \cdot \bar{A}^{\gamma}_{,\beta} + \bar{A}_{\beta} \cdot \bar{A}^{\gamma}_{,\alpha} \right] . \tag{9.3}$$

Since $\bar{A}_{\alpha,\beta} = \bar{A}_{\beta,\alpha}$, from (3.29),

$$\bar{A}_3 \cdot \bar{A}_{\alpha,\beta} + \bar{A}_3 \cdot \bar{A}_{\beta,\alpha} = 2\, B_{\alpha\beta} \, . \tag{9.4}$$

$$\bar{A}_{\alpha} \cdot \bar{A}^{\gamma}_{,\beta} + \bar{A}_{\beta} \cdot \bar{A}^{\gamma}_{,\alpha} = \bar{A}_{\alpha} \cdot \left[A^{\gamma\lambda} \bar{A}_{\lambda} \right]_{,\beta} + \bar{A}_{\beta} \cdot \left[A^{\gamma\lambda} \bar{A}_{\lambda} \right]_{,\alpha}$$
$$= A^{\gamma\lambda} \left[\bar{A}_{\alpha} \cdot \bar{A}_{\lambda,\beta} + \bar{A}_{\beta} \cdot \bar{A}_{\lambda,\alpha} \right]$$
$$+ A_{\alpha\lambda} A^{\gamma\lambda}_{,\beta} + A_{\beta\lambda} A^{\gamma\lambda}_{,\alpha}$$
$$= A^{\gamma\lambda} \left[\bar{A}_{\alpha} \cdot \bar{A}_{\beta,\lambda} + \bar{A}_{\beta} \cdot \bar{A}_{\alpha,\lambda} \right]$$
$$- A^{\gamma\lambda} A_{\alpha\lambda,\beta} - A^{\gamma\lambda} A_{\beta\lambda,\alpha}$$
$$= A^{\gamma\lambda} \left[A_{\beta\alpha,\lambda} - A_{\lambda\alpha,\beta} - A_{\lambda\beta,\alpha} \right]$$
$$= -2\, \Gamma^{\gamma}_{\beta\alpha} \, .$$

X. APPENDIX B

$$
\begin{aligned}
\tau'^{\alpha\beta} &= \Phi' \, a^{\alpha\beta} + \psi' \, D^{\alpha\beta} + P' \, A^{\alpha\beta} + \psi \, D'^{\alpha\beta} + P \, A'^{\alpha\beta} \\
&= \Phi' \, a^{\alpha\beta} + \psi' \, D^{\alpha\beta} + \lambda_3^2 \, A^{\alpha\beta} \left\{ \frac{2 \, \lambda_3'}{\lambda_3^3} P - a^{\mu\nu} A'_{\mu\nu} \psi - a^{\mu\nu} A_{\mu\nu} \psi' - \Phi' \right\} \\
&\quad + \psi \, D'^{\alpha\beta} + P \, A'^{\alpha\beta} \\
&= \Phi' \left\{ a^{\alpha\beta} - \lambda_3^2 \, A^{\alpha\beta} \right\} + \psi' \left\{ D^{\alpha\beta} - \lambda_3^2 \, a^{\mu\nu} A_{\mu\nu} A^{\alpha\beta} \right\} \\
&\quad + \psi \left\{ D'^{\alpha\beta} - \lambda_3^2 \, a^{\mu\nu} A'_{\mu\nu} A^{\alpha\beta} \right\} + P \left\{ \frac{2 \, \lambda_3'}{\lambda_3} A^{\alpha\beta} + A'^{\alpha\beta} \right\} \\
&= \Phi' \left\{ a^{\alpha\beta} - a \, A^{-1} \, A^{\alpha\beta} \right\} + \psi' \left\{ D^{\alpha\beta} - a \, A^{-1} \, a^{\mu\nu} A_{\mu\nu} A^{\alpha\beta} \right\} \\
&\quad + \psi \left\{ D'^{\alpha\beta} - a \, A^{-1} \, a^{\mu\nu} A'_{\mu\nu} A^{\alpha\beta} \right\} \\
&\quad + a \, A^{-1} \left\{ \Phi + a^{\mu\nu} A_{\mu\nu} \psi \right\} \left\{ A' \, A^{-1} \, A^{\alpha\beta} - A'^{\alpha\beta} \right\} .
\end{aligned}
$$