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Richard Ewers Horton
Iowa State University

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USE OF CONTINUOUS MEASUREMENTS IN A DISCRETE KALMAN FILTER

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Richard Ewers Horton

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I. INTRODUCTION

Classical linear filtering theory was founded by Wiener (4) in 1949 and Bode and Shannon (1) in 1950. Problems in optimal prediction and smoothing of random or stochastic signals were considered where additive measurement noise might also be present. The optimal filter is specified by its impulse response or transfer function.

In 1960 R. E. Kalman (3) introduced the "state-transition" method of linear filtering. Dynamics of the system are described by state-variable equations which are a set of simultaneous first-order differential equations often written in matrix form. Dependent variables are referred to as states of the system. Random signals are represented as the output of linear dynamic systems excited with white (uncorrelated) noise. The optimal filter is specified by a matrix of coefficients relating available discrete measurements to a priori estimates of the states.

In some applications the measurements are in fact continuous functions of time or a combination of discrete and continuous functions. Continuous measurements can be sampled and incorporated into the filter with other discrete measurements. However, if additive measurement noise is present, it might be reasonable to assume that a better discrete form or value of the continuous measurement can be obtained. If so, better discrete estimates of the random signals or states should result.

This dissertation investigates one method of presmoothing continuous measurements within discrete time intervals before incorporating them into a discrete Kalman filter.

II. DISCRETE MEASUREMENTS AND DISCRETE FILTERS

A. System Description

State variable equations will describe system dynamics in matrix form. The general solution for state response is also a matrix equation. This form is especially convenient for discrete time problems because a digital computer can be used for the numerous calculations involved.

The following set of equations will be used to describe system dynamics.

$$\dot{x}(t) = A(t)x(t) + G(t)u(t) \qquad 1$$

$$y(t) = M(t)x(t) + n(t) \qquad 2$$

$X(t)$ is the n-vector of state variables; $u(t)$ is the m-vector of unity white noise driving functions; $y(t)$ is the p-vector of measurements; $n(t)$ is the p-vector of additive measurement noise; and $A(t)$, $G(t)$, and $M(t)$ are $n \times n$, $n \times m$, and $p \times n$ matrices respectively. These matrices will be assumed to be constant during the discrete time intervals.

The general solution of Equation 1 is

$$x(t) = \varphi(t,t_0)x(t_0) + \int_{t_0}^{t} \varphi(t-\tau)G(\tau)u(\tau)d\tau \qquad 3$$

where $\varphi(t,t_0)$ is the state transition matrix. The state at time t consists of the projection of the state at t_0

through the transition matrix plus a contribution due to the random driving function. Therefore, an element of uncertainty exists about the true value of the state at time t.

If a perfect measurement of each of the states were available at time t, any uncertainty about the true value of the states would be eliminated. However, in many cases a direct measurement of each state is physically impossible. Furthermore, perfect measurements generally cannot be instrumented. Filters are therefore devised to utilize any information which is available from existing measurements. This information is used to reduce the uncertainty in the estimate of the true value of the state variables.

B. Kalman Filter Equations

Kalman's discrete filter can be described as

$$\hat{x}_n = \hat{x}'_n + b_n(y_n - \hat{y}'_n) \qquad 4$$

where

$\hat{x}_n$ = a posteriori estimate of x at time t_n

$\hat{x}'_n$ = a priori estimate of x at time t_n

b_n = weighting matrix at time t_n

y_n = measurement at time t_n

$\hat{y}'_n$ = a priori estimate of y at time t_n

The measurement is described by

$$y_n = M_n x_n + n_n \qquad 5$$

where

M_n = output matrix at time t_n

n_n = measurement noise (white) at time t_n

Since the expected values of the driving functions and measurement noise are zero,

$$\hat{x}'_n = \varphi(t_n - t_{n-1})\hat{x}_{n-1}$$

$$= \varphi_{n-1}\ \hat{x}_{n-1} \qquad 6$$

and

$$\hat{y}'_n = M_n \hat{x}'_n \qquad 7$$

The method of analysis and results will be useful for comparison with other filters. Equation 4 specifies the form of the filter, and analysis should yield a measure of the accuracy of the estimate x. The optimum weighting matrix should also be determined. Accuracy will be measured by the error-covariance matrix, and optimum will be defined as minimum mean square error.

The a posteriori estimation error at time t_n is defined as

$$e_n = \hat{x}_n - x_n$$

$$= \hat{x}'_n - x_n + b_n(y_n - \hat{y}'_n) \qquad 8$$

Substitution of Equation 5 into Equation 8 yields

$$e_n = (I - b_n M_n)e'_n + b_n n_n \qquad 9$$

where e'_n is the a priori estimation error, and I is the identity matrix.

The a posteriori error-covariance matrix P_n and loss function L are defined to be

$$P_n = E(e_n e_n^T) \qquad 10$$

and

$$L = E[Tr(e_n e_n^T)]$$

$$= Tr(P_n) \qquad 11$$

where $Tr(P_n)$ is the trace of the matrix P_n, and E is the expectation operator. When Equations 9 and 10 are used in Equation 11,

$$L = E\{Tr[(I-b_n M_n)e'_n + b_n n_n][(I-b_n M_n)e'_n + b_n n_n]^T\}$$

$$= Tr[(I-b_n M_n)E(e'_n e'^T_n)(I-b_n M_n)^T$$

$$+ (I-b_n M_n)E(e'_n n_n^T)b_n^T$$

$$+ b_n E(n_n e'^T_n)(I-b_n M_n)^T$$

$$+ b_n E(n_n n_n^T)b_n^T] \qquad 12$$

The a priori error-covariance matrix P_n^* and noise covariance matrix V_n will be defined as

$$P_n^* = E(e_n' e_n'^T) \tag{13}$$

and

$$V_n = E(n_n n_n^T) \tag{14}$$

Because the measurement noise is assumed to be white, no correlation exists between n_n and e_n'. Thus,

$$L = Tr[(I-b_n M_n)P_n^*(I-b_n M_n)^T + b_n V_n b_n^T] \tag{15}$$

The optimum weighting matrix b_n is found by differentiating the loss function L with respect to b_n.

$$\frac{\partial L}{\partial b_n} = 2(b_n M_n - I)P_n^* M_n^T + 2b_n V_n = 0 \tag{16}$$

Solving Equation 16 for b_n yields

$$b_n = P_n^* M_n^T (M_n P_n^* M_n + V_n)^{-1} \tag{17}$$

The a priori error-covariance matrix P_n^* is related to the previous a posteriori error-covariance matrix P_{n-1}.

$$\begin{aligned} e_n' &= \hat{x}_n' - x_n \\ &= \varphi_{n-1}\hat{x}_{n-1} - (\varphi_{n-1}x_{n-1} + \int_{t_{n-1}}^{t_n} \varphi(t_n-\tau)G_n u(\tau)d\tau) \\ &= \varphi_{n-1}e_{n-1} - \int_{t_{n-1}}^{t_n} \varphi(t_n-\tau)G_n u(\tau)d\tau \end{aligned} \tag{18}$$

Noting that no correlation exists between the error e_{n-1} and the white noise driving functions in the interval (t_{n-1}, t_n), Equation 18 leads to

$$P_n^* = \varphi_{n-1} P_{n-1} \varphi_{n-1}^T + H_{n-1} \qquad 19$$

where

$$H_{n-1} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \varphi(t_n - \tau_1) G_n E[u(\tau_1) u(\tau_2)^T] G_n^T \varphi^T (t_n - \tau_2) d\tau_1 d\tau_2 \qquad 20$$

The a posteriori error-covariance matrix P_n is found from Equation 15.

$$P_n = (I - b_n M_n) P_n^* (I - b_n M_n)^T + b_n V_n b_n^T \qquad 21$$

In summary, the Kalman recursive filter equations are

$$\hat{x}_n = \hat{x}'_n + b_n (y_n - \hat{y}'_n) \qquad 4$$

$$b_n = P_n^* M_n^T (M_n P_n^* M_n^T + V_n)^{-1} \qquad 17$$

$$P_n = (I - b_n M_n) P_n^* (I - b_n M_n)^T + b_n V_n b_n^T \qquad 21$$

where

$$\hat{x}'_n = \varphi_{n-1} \hat{x}_{n-1} \qquad 5$$

$$P_n^* = \varphi_{n-1} P_{n-1} \varphi_{n-1}^T + H_{n-1} \qquad 19$$

Equation 21 can also be expressed in other forms such as

$$P_n = P_n^* - b_n(M_n P_n^* M_n^T + V_n)b_n^T \qquad \text{21a}$$

This form is convenient because the term $(M_n P_n^* M_n^T + V_n)$ must first be calculated to determine b_n. Recalculation is not necessary.

III. CONTINUOUS MEASUREMENTS AND DISCRETE FILTERS

The physical measurement process can be continuous, discrete, or combination of both types. The discrete filter designed for discrete measurements could be applied to any of these situations by sampling the continuous measurements to obtain discrete values. However, some of the available information in continuous measurements may be lost by sampling. A continuous filter designed for continuous measurements would have no provisions for incorporating available discrete measurements. Information may also be lost in this case.

If the filter is constrained to be discrete and some or all of the measurements are continuous, more information might be recovered by presmoothing rather than by sampling. Any discrete measurements can still be utilized as shown in Chapter II.

In a manner similar to that of James, Nichols, and Phillips (2) a particular filter may be chosen and selected parameters optimized. This method may not yield the absolute optimum filter, but it might provide a substantial improvement. Before a reasonable choice can be made, several factors must be considered and related to the discrete Kalman filter. Not only must continuous measurements be smoothed to reduce unwanted measurement noise, but a discrete value must be formed before use in the estimation equation. Thus, instrumentation will differ from that required to sample the measurements. Because the basic filter design will be altered, new filter equations must be derived so that the accuracy of resultant estimations can be calculated. Finally, the filter must be capable of producing improved results to be of any practical use. Improved results must, however, be balanced against additional cost of instrumentation and complexity of the filter equations.

The discrete Kalman filter described by Equation 4 weights the discrete a priori measurement error $y_n - \hat{y}'_n$ at time t_n. By weighting the a priori measurement error rather than the measurement itself, a relatively simple relationship is developed between the a posteriori estimation error and the a priori estimation error as shown in Equation 9. For the same reason it is convenient to choose a filter which smooths the continuous a priori measurement error rather than only the measurement.. Furthermore, one of the simplest methods of smoothing a signal over a time interval is to average it. A discrete value or form is produced which represents the average continuous measurement error over the time interval.

Therefore, a reasonable filter to choose which presmooths the continuous measurement error is described by

$$\hat{x}_n = \hat{x}'_n + \frac{b_n}{\Delta t} \int_{t_{n-1}}^{t_n} [y(t) - \hat{y}'(t)]dt \qquad 22$$

This measurement error consists of the a priori estimation error e'(t) and the additive measurement noise n(t). Averaging is performed to smooth the measurement noise, not the estimation error. However, the filter described by Equation 22 smooths both. Thus, the correlation time of the estimation error or state response is related to the length of the discrete time interval which may be used. Results in Chapter V provide insight to this basic problem.

The assumed statistical character of the noise n(t) is an important factor. White noise implies no time correlation. This may be realistic or practical for a discrete measurement process where considerable time elapses between measurements. It is not a very realistic assumption, however, for a continuous measurement process. Even though the correlation time may be small, a certain amount is still more likely to occur than not. Thus, any continuous measurement noise will be assumed to be colored or Markov in character.

A. Optimal Filter Equations

The filter described by Equation 22 will be analyzed in a manner similar to that in Chapter II. The a posteriori estimation error at time t_n is

$$e_n = \hat{x}_n - x_n$$
$$= \hat{x}'_n - x_n + \frac{b_n}{\Delta t} \int_{t_{n-1}}^{t_n} [y(t) - \hat{y}'(t)]dt \qquad 23$$

The measurement in the interval (t_{n-1}, t_n) becomes

$$y(t) = M_n x(t) + n(t) \qquad 24$$

where M_n is the output matrix during the interval (t_{n-1}, t_n). Substitution of Equations 3 and 24 into Equation 23 yields

$$e_n = [\varphi_{n-1} - \frac{b_n M_n}{\Delta t} \int_{t_{n-1}}^{t_n} \varphi(t - t_{n-1})dt]e_{n-1}$$
$$- \int_{t_{n-1}}^{t_n} \varphi(t_n - \tau)G_n u(\tau)d\tau$$
$$+ \frac{b_n M_n}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \varphi(t-\tau)G_n u(\tau)d\tau dt$$
$$+ \frac{b_n}{\Delta t} \int_{t_{n-1}}^{t_n} n(t)dt \qquad 25$$

The loss function L is obtained from Equations 11 and 25.

$$L = Tr[\{\varphi_{n-1} - \frac{b_n M_n}{\Delta t} \int_{t_{n-1}}^{t_n} \varphi(t_1 - t_{n-1})dt_1\} P_{n-1}$$

$$\{\varphi_{n-1} - \frac{b_n M_n}{\Delta t} \int_{t_{n-1}}^{t_n} \varphi(t_2 - t_{n-1}) dt_2\}^T$$

$$+ \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \varphi(t_n - \tau_1) G_n E\{u(\tau_1) u^T(\tau_2)\} G_n^T \varphi^T(t_n - \tau_2) d\tau_1 d\tau_2$$

$$- \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \varphi(t_n - \tau_2) G_n E\{u(\tau_2) u^T(\tau_1)\} G_n^T \varphi^T(t - \tau_1) d\tau_1 dt d\tau_2 M_n^T b_n^T$$

$$- \frac{b_n M_n}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \varphi(t - \tau_1) G_n E\{u(\tau_1) u^T(\tau_2)\} G_n^T \varphi^T(t_n - \tau_2) d\tau_1 dt d\tau_2$$

$$+ \frac{b_n M_n}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_2} \int_{t_{n-1}}^{t_1} \varphi(t_1 - \tau_1) G_n E\{u(\tau_1) u^T(\tau_2)\}$$

$$G_n^T \varphi^T(t_2 - \tau_2) d\tau_1 d\tau_2 dt_1 dt_2 M_n^T b_n^T$$

$$+ \frac{b_n}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} E\{n(t_1) n^T(t_2)\} dt_1 dt_2 b_n^T$$

$$+ \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \varphi_{n-1} E\{e_{n-1} n^T(t)\} dt \; b_n^T$$

$$+ \frac{b_n}{\Delta t} \int_{t_{n-1}}^{t_n} E\{n(t) e_{n-1}^T\} \; \varphi_{n-1}^T dt$$

$$- \frac{b_n M_n}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \varphi(t_1 - t_{n-1}) E\{e_{n-1} n^T(t_2)\} dt_2 dt_1 b_n^T$$

$$- \frac{b_n}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} E\{n(t_2) e_{n-1}^T\} \varphi^T(t_1 - t_{n-1}) dt_2 dt_1 M_n^T b_n^T] \qquad 26$$

Equation 26 can be rewritten more conveniently as

$$L = Tr[\{\varphi_{n-1} - b_n M_n \Phi_{n-1}\} P_{n-1} \{\varphi_{n-1} - b_n M_n \Phi_{n-1}\}^T$$

$$+ H_{n-1} - b_n (M_n U_{n-1} - Q_{n-1}) - (M_n U_{n-1} - Q_{n-1})^T b_n^T$$

$$+ b_n (M_n R_{n-1} M_n^T + V_{n-1} - M_n W_{n-1} - W_{n-1}^T M_n^T) b_n^T] \qquad 27$$

by defining

$$\Phi_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \varphi(t - t_{n-1}) dt \qquad 28$$

$$H_{n-1} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \varphi(t_n - \tau_1) G_n E\{u(\tau_1) u^T(\tau_2)\}$$

$$G_n^T \varphi^T(t_n - \tau_2) d\tau_1 d\tau_2 \qquad 20$$

$$U_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \varphi(t - \tau_1) G_n E\{u(\tau_1) u^T(\tau_2)\}$$

$$G_n^T \varphi^T(t_n - \tau_2) d\tau_1 dt d\tau_2 \qquad 29$$

$$R_{n-1} = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_2} \int_{t_{n-1}}^{t_1} \varphi(t_1-\tau_1)G_n$$

$$E\{u(\tau_1)u^T(\tau_2)\}G_n^T\varphi^T(t_2-\tau_2)d\tau_1 d\tau_2 dt_1 dt_2 \qquad 30$$

$$V_{n-1} = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} E\{n(t_1)n^T(t_2)\}dt_2 dt_1 \qquad 31$$

$$Q_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} E\{n(t)e_{n-1}^T\}\ \varphi_{n-1}^T dt \qquad 32$$

$$W_{n-1} = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \varphi(t_1-t_{n-1})E\{e_{n-1}n^T(t_2)\}dt_1 dt_2$$

$$= \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \Phi_{n-1}E\{e_{n-1}n^T(t_2)\}dt_2 \qquad 33$$

The optimum weighting matrix b_n is again found by differentiating the loss function L with respect to b_n.

$$\frac{\partial L}{\partial b_n} = -\ 2\varphi_{n-1}P_{n-1}\Phi_{n-1}M_n^T + 2b_n M_n \Phi_{n-1}P_{n-1}\Phi_{n-1}^T M_n^T$$

$$-\ 2(U_{n-1}^T M_n^T - Q_{n-1}^T) + 2b_n(M_n R_{n-1}M_n^T + V_{n-1} - Z_{n-1}) = 0$$

34

where

$$Z_{n-1} = M_n W_{n-1} + W_{n-1}^T M_n^T \qquad 35$$

which is a symmetric matrix. Solving Equation 34 for b_n yields

$$b_n = [(\varphi_{n-1}P_{n-1}\Phi_{n-1}^T + U_{n-1}^T)M_n^T - Q_{n-1}^T]$$

$$[M_n(\Phi_{n-1}P_{n-1}\Phi_{n-1}^T + R_{n-1})M_n^T + V_{n-1} - Z_{n-1}]^{-1} \quad 36$$

The a posteriori error-covariance P_n is obtained from Equation 27.

$$P_n = (\varphi_{n-1} - b_nM_n\Phi_{n-1})P_{n-1}(\varphi_{n-1} - b_nM_n\Phi_{n-1})^T$$

$$+ H_{n-1} - b_n(M_nU_{n-1} - Q_{n-1}) - (M_nU_{n-1} - Q_{n-1})^Tb_n^T$$

$$+ b_n(M_nR_{n-1}M_n^T + V_{n-1} - Z_{n-1})b_n^T \quad 37$$

B. Suboptimal Filter Equations

The filter described by Equation 22 shows that the a posteriori estimate of x is composed of an a priori estimate of x and a weighted average of the measurement error. Accuracy of this filter is specified by the error-covariance matrix in Equation 37. Both of these equations are valid regardless of how the weighting matrix b_n is determined.

The mean square error or loss function will be minimum if the optimum weighting matrix described in Equation 36 is used. However, another weighting matrix which is simpler in form can yield comparable results under certain conditions.

The correlation between e_{n-1} and n_n in the discrete Kalman filter was zero because n_n was considered to be white noise. But the noise n(t) in the continuous case is assumed to be Markov. Therefore, some correlation does exist between e_{n-1} and n(t) in the interval (t_{n-1},t_n). Equation 25 shows that the error e at a particular point in time is composed of terms involving the error at the previous point in time, the white noise driving functions occurring between the points in time, and the measurement noise occurring between the points in time. The white noise driving functions in the interval (t_{n-2},t_{n-1}) are not correlated with n(t) in any interval. However, n(t) in the interval (t_{n-2}, t_{n-1}) is correlated with n(t) in the interval (t_{n-1},t_n) because n(t) is Markov noise. If the length of the time intervals are large relative to the correlation time of n(t), any correlation between e_{n-1} and n(t) in the interval (t_{n-1}, t_n) may be insignificant. If this is true, then the correlation between e_{n-2} and n(t) in the interval (t_{n-1},t_n) will be even less significant.

If the assumption is made that $\int_{t_{n-1}}^{t_n} E[e_{n-1}n^T(t)]dt$ is zero, another weighting matrix can be calculated by

$$b_n = [\varphi_{n-1} P_{n-1} \Phi_{n-1}^T + U_{n-1}^T] M_n^T [M_n(\Phi_{n-1} P_{n-1} \Phi_{n-1}^T$$

$$+ R_{n-1}) M_n^T + V_{n-1}]^{-1} \qquad 38$$

where Q_{n-1}, W_{n-1}, and Z_{n-1} are assumed to be zero. Since this is not the optimum weighting matrix, it must be considered as suboptimum.

IV. EXAMPLE

The example system in Figure 1 described in standard block-diagram terminology was chosen to illustrate how the filters previously discussed are applied to a problem. It will also provide a basis for comparison of results. The system is continuous so the measurement can be considered to be sampled or averaged. First-order dynamics were chosen so that evaluation and interpretation of results would not be unnecessarily complicated. Finally, the continuous Wiener filter can be found with relative ease. This will provide an optimum continuous result to compare with optimum and suboptimum discrete results.

A. Discrete Kalman Filter

The state variable $x(t)$ is related to the input $u_1(t)$ by

$$\dot{x}(t) = u_1(t) \qquad 39$$

Since the measurement noise $n(t)$ is time-stationary Markov noise rather than white noise, it can be thought of as the result of passing white noise through a linear dynamic system as shown in Figure 2. The differential equation relating $n(t)$ to $u_2(t)$ is

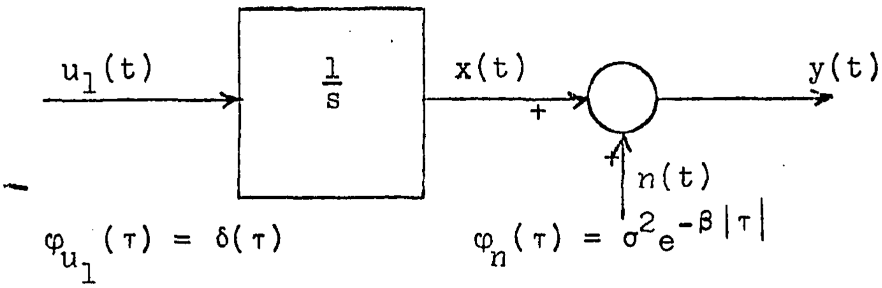


Figure 1. Example system

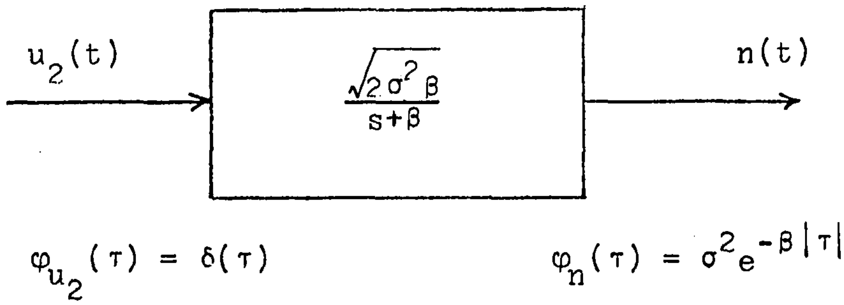


Figure 2. Linear dynamic system

$$\dot{n}(t) = -\beta n(t) + \sqrt{2\sigma^2\beta}\, u_2(t) \qquad 40$$

The noise n(t) is now considered an additional state variable to be incorporated into the dynamics of the system. By defining

$$x_1(t) = x(t) \qquad 41$$

and

$$x_2(t) = n(t) \qquad 42$$

Equations 1 and 2 become

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2\sigma^2\beta} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \qquad 43$$

and

$$y(t) = (1\ 1) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + 0 \qquad 44$$

Therefore,

$$\varphi_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta\Delta t} \end{pmatrix} \qquad 45$$

and

$$V_n = 0 \qquad 46$$

Term by term evaluation of H_{n-1} yields

$$H(1,1)_{n-1} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2 = \Delta t \qquad 47$$

$$H(1,2)_{n-1} = H(2,1)_{n-1} = 0 \qquad 48$$

$$H(2,2)_{n-1} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} 2\sigma^2 \beta e^{-\beta|t_n - \tau_1|} e^{-\beta|t_n - \tau_2|}$$

$$\delta(\tau_1 - \tau_2) d\tau_1 d\tau_2 = \int_{t_{n-1}}^{t_n} 2\sigma^2 \beta e^{-2\beta(t_n - \tau_2)} d\tau_2$$

$$= \sigma^2(1 - e^{-2\beta\Delta t}) \qquad 49$$

Thus

$$H_{n-1} = \begin{pmatrix} \Delta t & 0 \\ 0 & \sigma^2(1 - e^{-2\beta\Delta t}) \end{pmatrix} \qquad 50$$

$H(1,1)_{n-1}$ shows that the a priori mean square error of state x_1 $[P^*(1,1)_n]$ is increased each time by the length of the discrete time interval Δt. This is analogous to the random-walk problem of probability theory. The a priori mean square error of state x_2 is increased by $\sigma^2(1-e^{-2\beta\Delta t})$. As Δt approaches zero, both of these terms also approach zero. The measurement is considered to be perfect since the noise $n(t)$ is treated as a state variable. Thus, as Δt approaches zero, the discrete Kalman filter should converge to the continuous Wiener filter. However, as Δt increases, $H(2,2)_{n-1}$ approaches σ^2 which is constant, but $H(1,1)_{n-1}$ increases as Δt. Since measurements occur less frequently in this situation and the a priori mean square error of state x_1 increases, the discrete Kalman filter diverges from the continuous Wiener filter. Computed results in Chapter V illustrate these concepts.

B. Optimal Averaging Filter

The appropriate system equations for the averaging discrete filter defined by Equation 22 are

$$\dot{x}(t) = u(t) \tag{51}$$

$$y(t) = x(t) + n(t) \tag{52}$$

where $A(t) = 0$, $G(t) = 1$, $M(t) = 1$, and $\varphi(t) = 1$. Note that $n(t)$ is not considered to be a state variable but to be additive measurement noise.

Several terms must first be evaluated. From Equation 28

$$\Phi_{n-1} = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} dt = 1 \tag{53}$$

From Equation 20

$$H_{n-1} = \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \delta(\tau_1-\tau_2)d\tau_1 d\tau_2 = \Delta t \tag{47}$$

From Equation 29

$$U_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \delta(\tau_1 - \tau_2) d\tau_1 dt d\tau_2$$

$$= \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} dt \, d\tau_2$$

$$= \int_{t_{n-1}}^{t_n} d\tau_2$$

$$= \Delta t \qquad 54$$

From Equation 30

$$R_{n-1} = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_2} \int_{t_{n-1}}^{t_1} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2 dt_1 dt_2$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_2} \int_{t_{n-1}}^{t_1} d\tau_1 dt_1 dt_2$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_2}^{t_n} \int_{t_{n-1}}^{t_1} d\tau_2 dt_1 dt_2$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_2} (t_1 - t_{n-1}) dt_1 dt_2$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_2}^{t_n} (t_2 - t_{n-1}) dt_1 dt_2$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \left(\frac{t_2^2}{2} - t_{n-1}t_2 - \frac{t_{n-1}^2}{2} + t_{n-1}^2\right)dt_2$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} (t_2 t_n - t_{n-1}t_n - t_2^2 + t_{n-1}t_2)dt_2$$

$$= \frac{\Delta t}{3} \qquad 55$$

From Equation 31

$$V_{n-1} = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \sigma^2 e^{-\beta|\tau_1-\tau_2|} d\tau_2 d\tau_1$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{\tau_1} \sigma^2 e^{-\beta(\tau_1-\tau_2)} d\tau_2 d\tau_1$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{\tau_1}^{t_n} \sigma^2 e^{\beta(\tau_1-\tau_2)} d\tau_2 d\tau_1$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \frac{\sigma^2}{\beta} \left(2 - e^{-\beta\tau_1} e^{\beta t_{n-1}} - e^{\beta\tau_1} e^{-\beta t_n}\right) d\tau_1$$

$$= 2\sigma^2 \left[\frac{1}{\beta\Delta t} - \frac{1-e^{-\beta\Delta t}}{(\beta\Delta t)^2}\right] \qquad 56$$

Additional consideration must be given to the evaluation of Q_{n-1} and W_{n-1}. The noise $n(t)$ occurring in the interval (t_{n-1}, t_n) is correlated with the measurement noise which has occurred in each of the previous time intervals. However, the degree of correlation is different in each case as is the weighting factor. It is possible at each point in time to determine the cumulative effect of this correlation and weighting by summing each calculated contribution. But this method leads to a growing memory problem because information must be saved from each of the previous time intervals.

Since the a posteriori error e_{n-1} contains the weighted effect of the measurement noise which has occurred in each of the previous time intervals, a recursive relationship should exist for

$$\int_{t_{n-1}}^{t_n} E[e_{n-1}n^T(t)]dt \ .$$

If so, the most desirable feature of the discrete Kalman filter, recursive filter equations would be preserved.

The unweighted contribution of the measurement noise in different time intervals is determined first. Defining V^* as

$$V_1^* = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-2}}^{t_{n-1}} E\{n(t_1)n^T(t_2)\}dt_2dt_1$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-2}}^{t_{n-1}} \sigma^2 e^{-\beta|t_1-t_2|} dt_2dt_1$$

$$= \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \frac{\sigma^2}{\beta} e^{-\beta t_1} (e^{\beta t_{n-1}} - e^{\beta t_{n-2}})dt_1$$

$$= \sigma^2 \left(\frac{1-e^{-\beta \Delta t}}{\beta \Delta t}\right)^2 \qquad 57$$

and V_2^* as

$$V_2^* = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-3}}^{t_{n-2}} E\{n(t_1)n^T(t_2)\}dt_2 dt_1$$

$$= e^{-\beta \Delta t} \sigma^2 \left(\frac{1-e^{-\beta \Delta t}}{\beta \Delta t}\right)^2 \qquad 58$$

the general expression for V_n^* is seen to be

$$V_n^* = e^{-(n-1)\beta \Delta t} \sigma^2 \left(\frac{1-e^{-\beta \Delta t}}{\beta \Delta t}\right)^2$$

$$= e^{-\beta \Delta t} V_{n-1}^* \qquad 59$$

The weighting factors can be determined by using Equation 23 to express e_{n-1} in Equation 32 in terms of the previous error e_{n-2}, the driving function $u(t)$ in the interval (t_{n-2}, t_{n-1}), and the measurement noise $n(t)$ in the interval (t_{n-2}, t_{n-1}). Thus, Equation 32 becomes

$$Q_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} E\{n(t)e_{n-1}^T\}dt \, \varphi_{n-1}^T$$

$$= \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} E\{n(t)e_{n-2}^T\}dt(\varphi_{n-2} - b_{n-1}M_{n-1}\Phi_{n-2})^T \varphi_{n-1}^T$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-2}}^{t_{n-1}} E\{n(t_1)n^T(t_2)\}dt_2 dt_1 b_{n-1}^T \, {}_{n-1}^T \qquad 60$$

where no correlation exists between n(t) and u(t). In a similar manner

$$Q_{n-1} = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} E\{n(t)e_{n-3}^T\}(\varphi_{n-3} - b_{n-2}M_{n-2}\Phi_{n-3})^T$$

$$(\varphi_{n-2} - b_{n-1}M_{n-1}\Phi_{n-2})^T \varphi_{n-1}^T$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-3}}^{t_{n-2}} E\{n(t_1)n^T(t_2)\}dt_2 dt_1 b_{n-2}^T$$

$$(\varphi_{n-2} - b_{n-1}M_{n-1}\Phi_{n-2})^T \varphi_{n-1}^T$$

$$+ \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-2}}^{t_{n-1}} E\{n(t_1)n^T(t_2)\}dt_2 dt_1 b_{n-1}^T \varphi_{n-1}^T \qquad 61$$

This process could be carried further, but the recursive relationship can be detected from inspection of Equation 61. If S_{n-1} is defined as

$$S_{n-1} = V^* b_{n-1}^T + e^{-\beta \Delta t} S_{n-2}(\varphi_{n-2} - b_{n-1}M_{n-1}\Phi_{n-2})^T \qquad 62$$

where

$$S_0 = 0 \qquad 63$$

and

$$V^* = \sigma^2 \left(\frac{1-e^{-\beta \Delta t}}{\beta \Delta t} \right)^2 \qquad 64$$

then

$$Q_{n-1} = S_{n-1}\varphi_{n-1}^{T} \tag{65}$$

For the example in Figure 1

$$S_{n-1} = V^{*}b_{n-1}^{T} + e^{-\beta\Delta t}\, S_{n-2}(1-b_{n-1})^{T} \tag{66}$$

and

$$Q_{n-1} = S_{n-1} \tag{67}$$

Comparison of Equations 32 and 33 shows that

$$W_{n-1} = \Phi_{n-1}S_{n-1}^{T} \tag{68}$$

Thus, for the example in Figure 1

$$W_{n-1} = S_{n-1}^{T} \tag{69}$$

and

$$Z_{n-1} = 2W_{n-1} \tag{70}$$

C. Suboptimal Averaging Filter

Since the only difference between the optimal and suboptimal averaging filter involves neglecting Q_{n-1} and Z_{n-1} in the calculation of b_n, all necessary quantities have been determined.

D. Continuous Wiener Filter

Because the example in Figure 1 is a continuous system, it is possible to determine the optimal continuous Wiener filter. The resultant error-covariance should represent a lower bound which can only be approached by any discrete filter.

The Wiener problem is formulated as shown in Figure 3. The response g(t) can be written as

$$g(t) = \int_0^t y(\tau,t)[n(t-\tau) + x(t-\tau)]d\tau \qquad 71$$

where

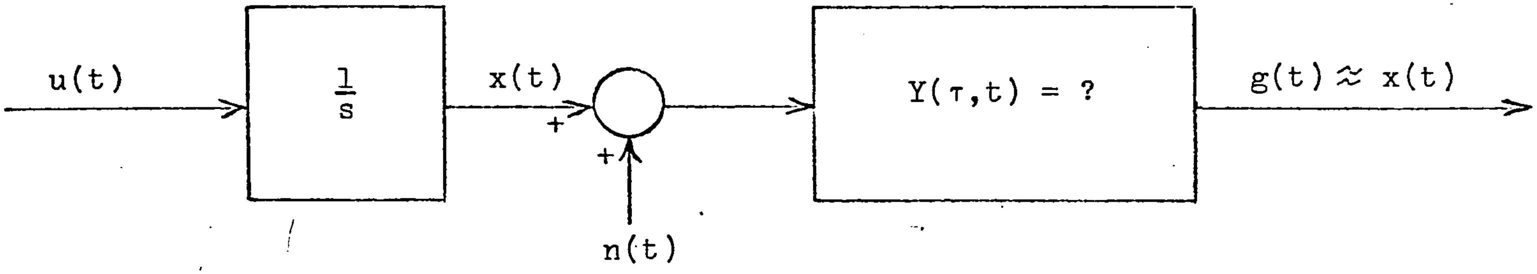
$$x(t) = \int_0^t u(t-\tau)d\tau \qquad 72$$

The mean square error is averaged in an ensemble sense because x(t) is a nontime-stationary process.

$$\overline{e^2(t)} = \int_0^t \int_0^t y(u,t)y(v,t)[\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}]dudv - 2\int_0^t y(u,t)\,\overline{x(t-u)x(t)}du + \overline{x^2(t)} \qquad 73$$

No correlation exists between u(t) and n(t).

Minimization of the mean square error using variational calculus leads to the integral equation



$\varphi_u(\tau) = k\delta(\tau)$ $\qquad\qquad$ $\varphi_n(\tau) = \sigma^2 e^{-\beta|\tau|}$

Figure 3. Wiener filter problem

$$\int_0^t y(v,t)[\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}]dv$$

$$- \overline{x(t-u)x(t)} = 0 \qquad 74$$

where $0 < u, v < t$. For the example in Figure 1, Equation 74 becomes

$$\sigma^2 e^{-\beta u} \int_0^u y(v,t)e^{\beta v}dv + \sigma^2 e^{\beta u} \int_u^t y(v,t)e^{-\beta v}dv$$

$$+ k(t-u) \int_0^u y(v,t)dv + kt \int_u^t y(v,t)dv$$

$$- k \int_u^t vy(v,t)dv - k(t-u) = 0 \qquad 75$$

A lead-lag filter of the form

$$y(v,t) = c_1 \delta(v) + c_2 e^{-\alpha v} + c_3 e^{\alpha v} \qquad 76$$

is a solution for Equation 75. Results of substituting Equation 76 into Equation 75 and solving for the unknown parameters are

$$\alpha^2 = \frac{k\beta^2}{2\sigma^2\beta+k} \qquad 77$$

$$C_1 = \frac{\alpha[\alpha^2+\beta^2-\beta]\sinh \alpha t+\alpha^2[1-2\beta]\cosh \alpha t}{\alpha\beta[2\beta-1]\sinh \alpha t + \beta[\beta-\beta^2-\alpha^2]\cosh \alpha t} \qquad 78$$

$$C_2 = \frac{\alpha[\alpha^3+\alpha^2\beta-\alpha\beta^2-\beta^3+\beta^2-\alpha^2]e^{\alpha t}}{\alpha\beta[2\beta-1]\sinh\,\alpha t+\beta[\beta-\beta^2-\alpha^2]\cosh\,\alpha t} \qquad 79$$

$$C_3 = \frac{\alpha(\alpha^2-\beta^2)(1+\alpha-\beta)e^{-\alpha t}}{\alpha\beta[2\beta-1]\sinh\,\alpha t+\beta[\beta-\beta^2-\alpha^2]\cosh\,\alpha t} \qquad 80$$

The steady-state values of these parameters are

$$C_{1_{ss}} = \frac{\alpha}{\beta} \qquad 81$$

$$C_{2_{ss}} = \frac{\alpha(\beta-\alpha)}{\beta} \qquad 82$$

$$C_{3_{ss}} = 0 \qquad 83$$

The resultant steady-state error can be written as

$$e^2_{ss} = \frac{\gamma^{1/2}-k}{\beta} \qquad 84$$

where

$$\gamma = 2k\sigma^2\beta \qquad 85$$

The Appendix contains a more detailed derivation of these equations.

V. RESULTS

The recursive filter equations developed in Chapter IV for the example system were programmed for calculation on an IBM Operating System/360 digital computer. With the initial error-covariance matrix set to zero in each case, the recursive calculations were repeated until the a posteriori error-covariance reached a steady-state value. Various discrete time intervals were chosen for each of five sets of Markov measurement noise parameters. Figures 4, 5, 6, 7, and 8 illustrate the calculated steady-state a posteriori error-covariance of state x_1 for each filter discussed in Chapter IV with one exception. Suboptimum calculations were not attempted for the case illustrated in Figure 7 because optimum results showed that no improvement was possible by averaging.

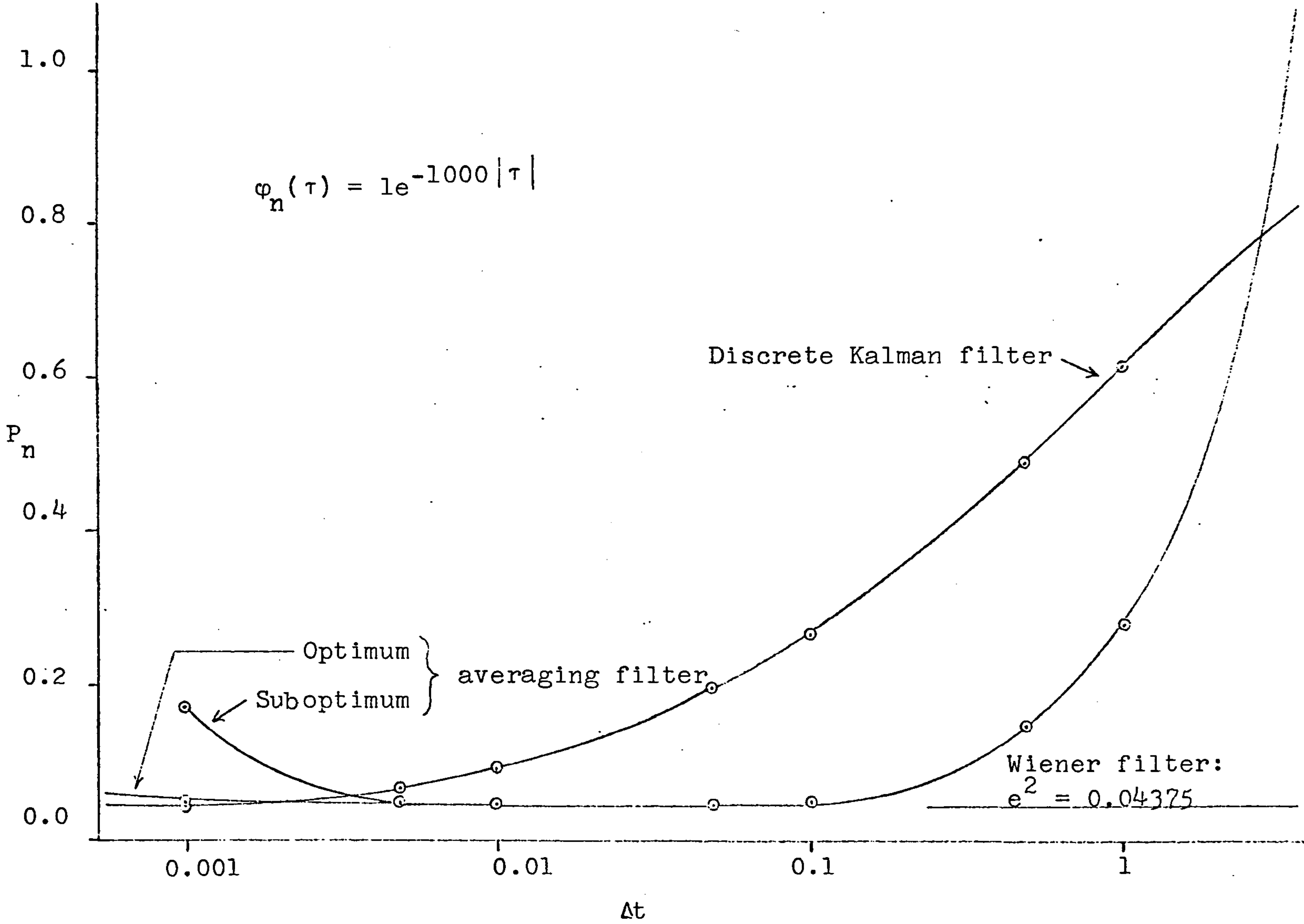
In Figures 4, 5, 6, and 7 the only noise parameter varied is β which is inversely proportional to correlation time. Figure 8 illustrates the results obtained with a variation in the noise parameter σ^2 which is proportional to magnitude.

As the time interval approaches zero in every case, discrete Kalman filter results become comparable to those of the continuous Wiener filter as expected. However, the averaging filter does not behave in the same manner. If improvement is possible by averaging the measurement error, it only exists for a finite range of discrete time intervals. When the interval is too large or too small the averaging filter produces less accurate results than the discrete Kalman filter. Different reasons exist for each of these extreme cases.

Since the measurement noise is not considered as a state variable, an effective perfect measurement is not possible. The contamination is represented by the terms V_{n-1}, Q_{n-1}, and W_{n-1}. The most significant of these is V_{n-1} which tends to σ^2 which is constant as Δt approaches zero. Thus, at this extreme the averaging filter diverges from both the discrete Kalman filter and the continuous Wiener filter. The suboptimum averaging filter which neglects Q_{n-1} and W_{n-1} in the calculation of b_n is more sensitive to this condition than the optimum averaging filter.

At the other extreme, V_{n-1}, Q_{n-1}, and W_{n-1} all approach zero as Δt becomes large. Thus, the effect of the measurement noise tends to become negligible. But the averaging process was purposely chosen to produce this desired effect. Therefore, some other reason must exist to explain why the averaging filter does not produce improved results for large Δt. In Chapter III a decision was made to average the a priori measurement error rather than only the measurement because resultant filter equations were much simpler in form. The measurement error contains the measurement noise and estimation error, and both are averaged. Because the estimation error of the state x_1 does change proportional to Δt, its average does not represent the true estimation error at the time of interest t_n, and weight is being given to this average. The problem increases as Δt becomes larger. Thus, for large values of Δt, requirements for improved results by averaging the measurement error are incompatible with existing system dynamics. Since the average measurement noise does approach zero, the optimum and suboptimum averaging filters behave identically at this extreme.

Figure 4. Calculated steady-state a posteriori error-covariance of state x_1



1.0
0.8
0.6
0.4
0.2
0.0
P_n
$\varphi_n(\tau) = 1e^{-1000|\tau|}$
Discrete Kalman filter
Optimum
Suboptimum
averaging filter
Wiener filter:
$e^2 = 0.04375$
0.001
0.01
0.1
1
Δt

Figure 5. Calculated steady-state a posteriori error-covariance of state x_1

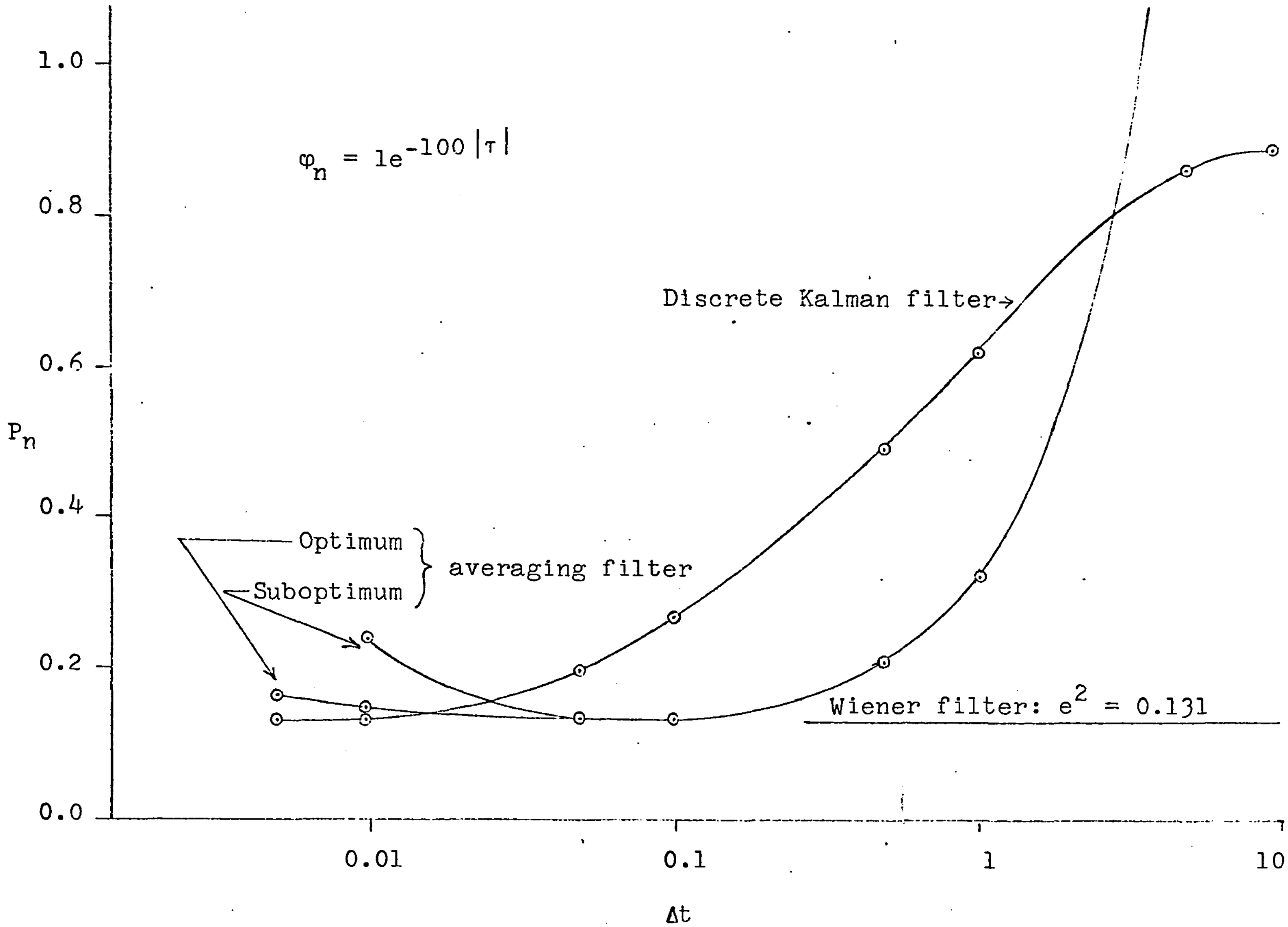
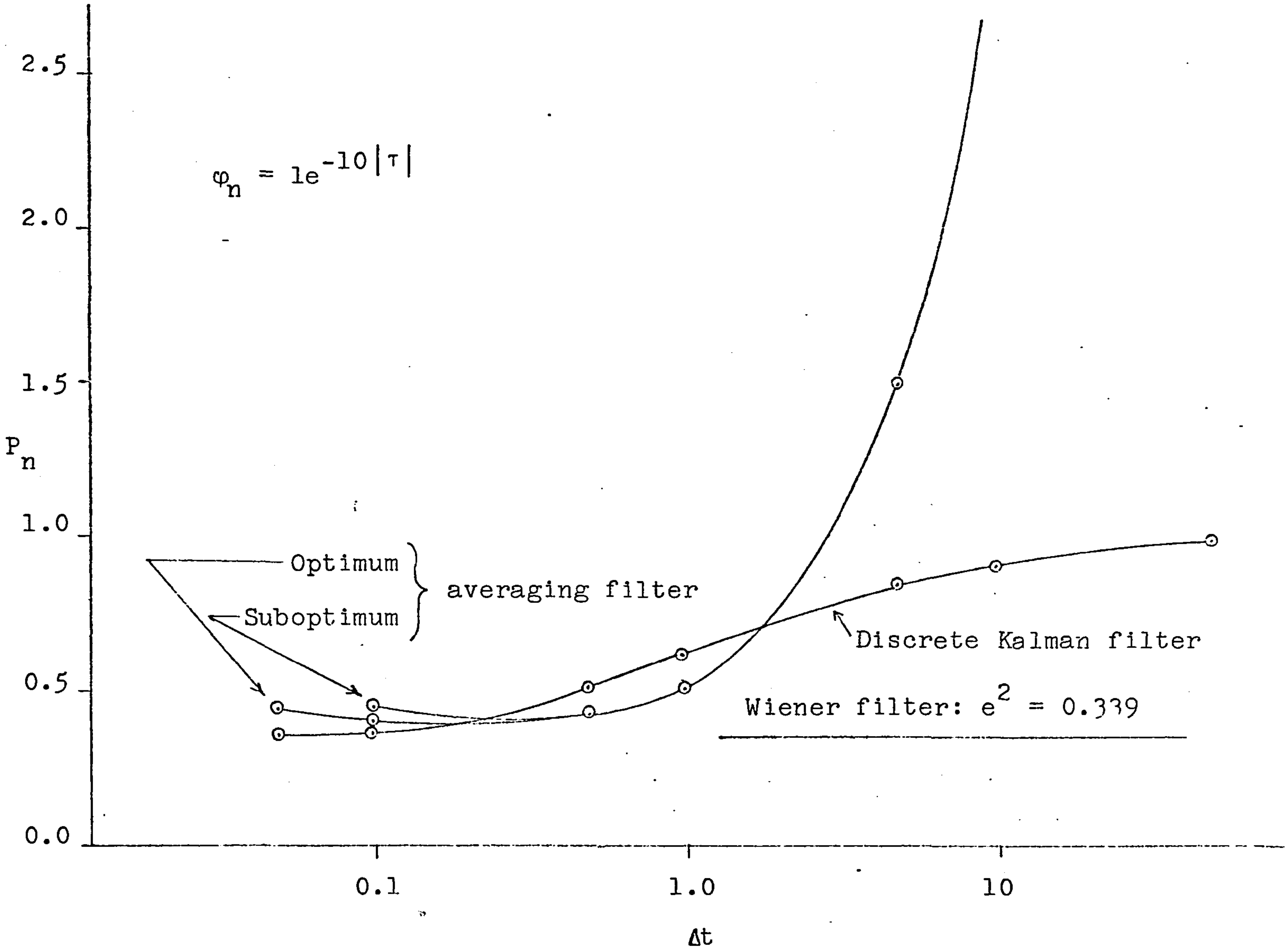
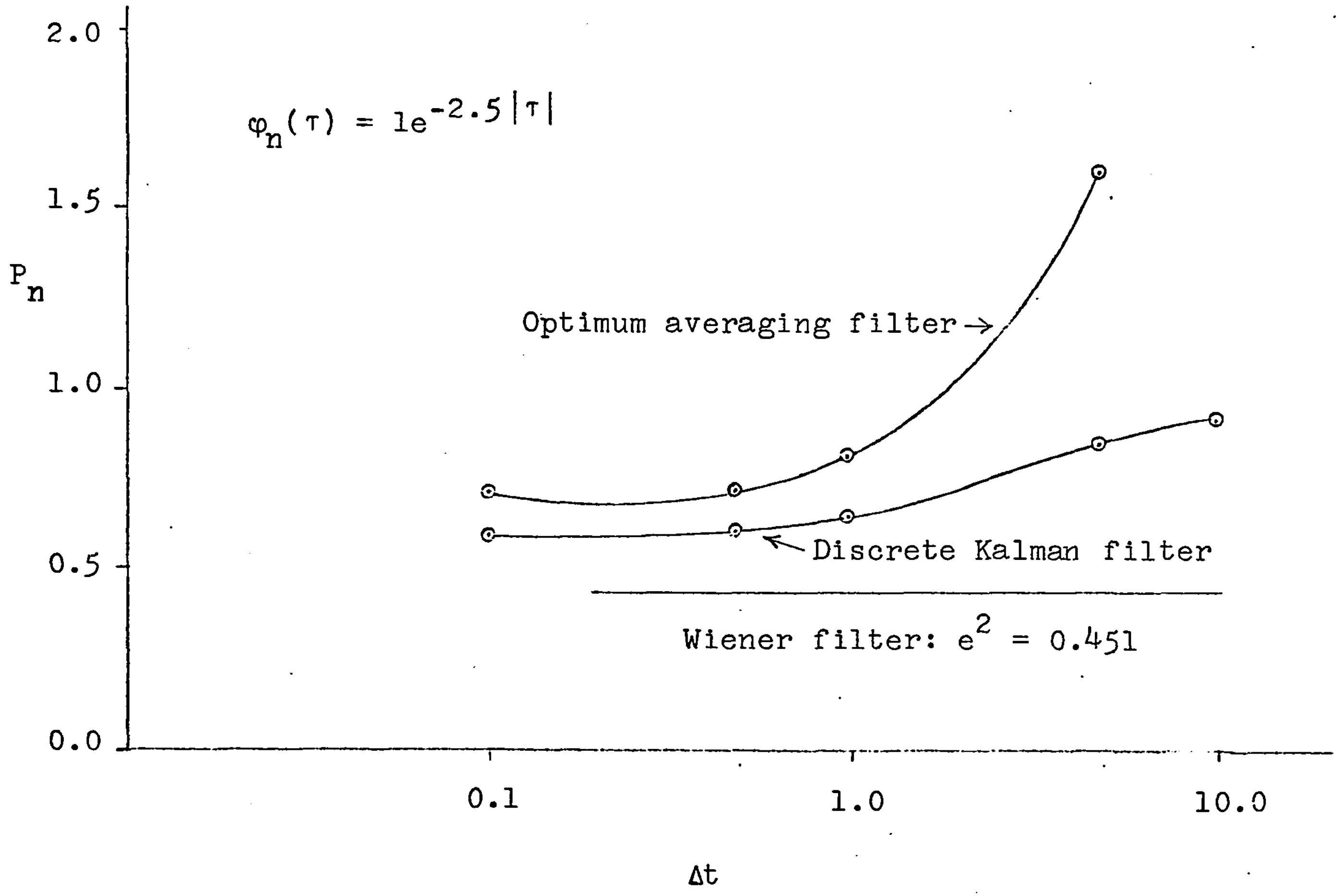

1.0
0.8
0.6
0.4
0.2
0.0
P_n
$\varphi_n = 1e^{-100|\tau|}$
Discrete Kalman filter
Optimum
Suboptimum
averaging filter
Wiener filter: $e^2 = 0.131$
0.01
0.1
1
10
Δt

Figure 6. Calculated steady-state a posteriori error-covariance of state x_1



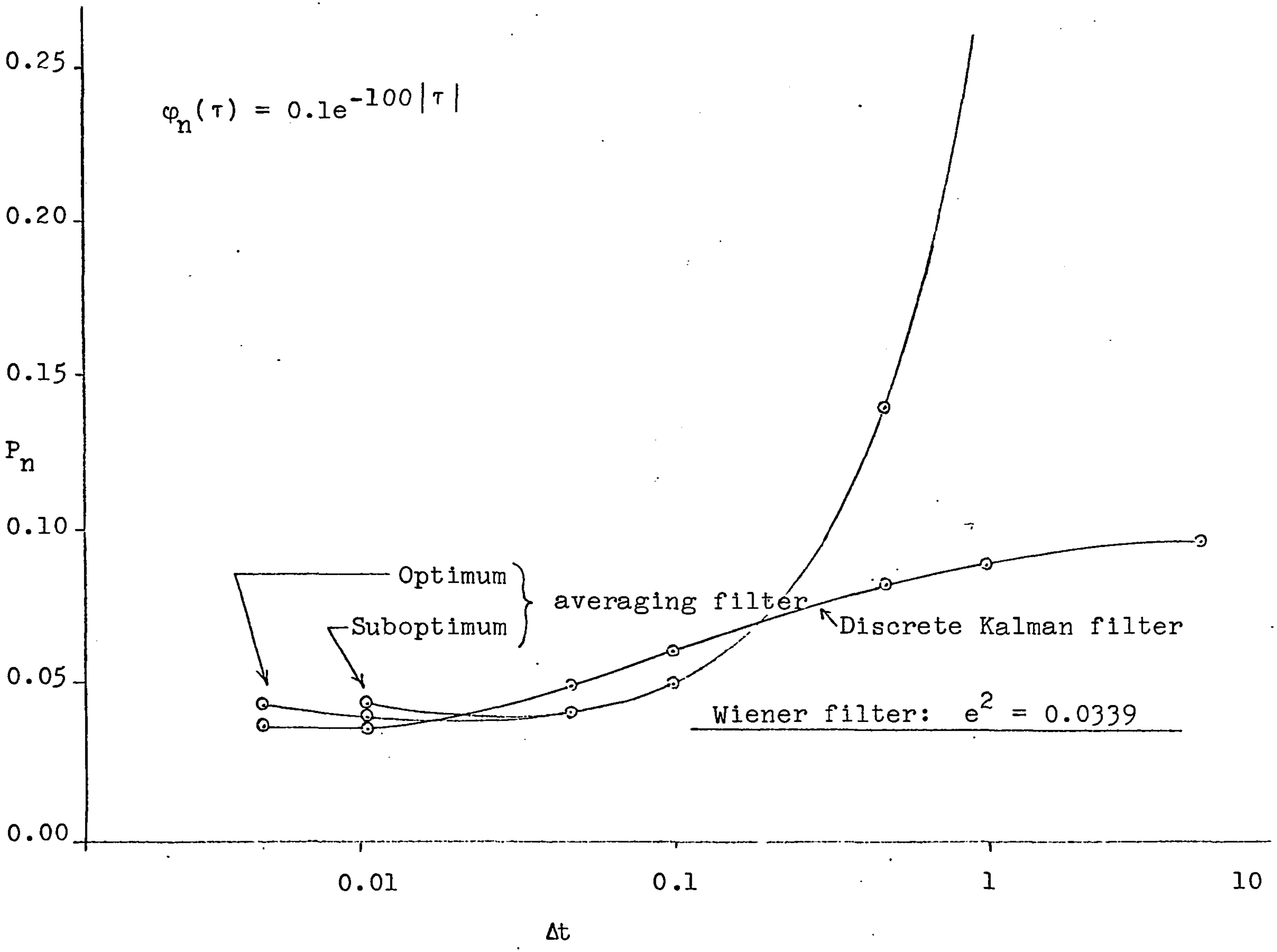
$\varphi_n = 1e^{-10|\tau|}$
P_n
2.5
2.0
1.5
1.0
0.5
0.0
0.1
1.0
10
Δt
Optimum
Suboptimum
averaging filter
Discrete Kalman filter
Wiener filter: $e^2 = 0.339$

Figure 7. Calculated steady-state a posteriori error-covariance of state x_1



2.0
1.5
1.0
0.5
0.0
P_n
$\varphi_n(\tau) = 1e^{-2.5|\tau|}$
Optimum averaging filter
Discrete Kalman filter
Wiener filter: $e^2 = 0.451$
0.1
1.0
10.0
Δt

Figure 8. Calculated steady-state a posteriori error-covariance of state x_1



φn(τ) = 0.1e-100|τ|
0.25
0.20
0.15
0.10
0.05
0.00
Pn
Optimum
Suboptimum
averaging filter
Discrete Kalman filter
Wiener filter: e2 = 0.0339
0.01
0.1
1
10
Δt

VI. CONCLUSIONS

The results in Chapter V illustrate how the correlation times of the measurement noise and state responses are related to the length of the discrete time interval.

For substantial improvement to be realized by presmoothing the length of the discrete time interval must be small relative to the correlation time of the state response and large relative to the correlation time of the measurement noise. A definite range of permissible discrete time intervals exists where improvement is possible by presmoothing. The effective measurement noise is significantly reduced in this range.

If the discrete time interval is small relative to the correlation time of the measurement noise no presmoothing is possible. In this case measurement noise cannot be reduced by the averaging filter. The discrete Kalman filter with measurement noise considered as an additional state variable has an effective perfect measurement. Thus, as the time interval approaches zero, the results approach those of the optimal continuous Wiener filter.

If the discrete time interval is large relative to the correlation time of the state responses, presmoothing reduces the effect of measurement noise. However, the state response portion of the measurement error is also significantly smoothed which is undesirable. Presmoothing is intended to reduce only the measurement noise so that a better estimate of the true state, not the averaged state, can be found.

Therefore, under the conditions described, improved results can be realized in discrete filtering by presmoothing available continuous measurements. A decrease in required computation time might also be realized for two reasons. The size of matrices are larger when continuous measurement noise is considered as a state variable, and comparable results can be obtained by averaging the measurement error with larger discrete time intervals.

VII. LITERATURE CITED

1. Bode, H. W. and Shannon, C. E. A simplified derivation of linear least-squares smoothing and prediction theory. Institute of Radio Engineers Proceedings 38: 417-425. 1950.

2. James, Hubert M., Nichols, Nathaniel B. and Phillips, Ralph S. Theory of servomechanisms. New York, N.Y., McGraw-Hill Book Co., Inc. 1957.

3. Kalman, R. E. A new approach to linear filtering and prediction problems. Journal of Basic Engineering (American Society of Mechanical Engineers Transactions, Series D) 82: 35-45. 1960.

4. Wiener, N. The extrapolation, interpolation and smoothing of stationary time series. New York, N.Y., John Wiley and Sons, Inc. 1949.

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IX. APPENDIX

The error e(t) can be written as

$$e(t) = g(t) - x(t)$$

$$\int_0^t y(\tau,t)[n(t-\tau) + x(t-\tau)]d\tau - x(t) \qquad 86$$

Because the problem is nontime stationary, the mean square error must be averaged in an ensemble sense.

$$\overline{e^2(t)} = \int_0^t \int_0^t y(u,t)y(v,t)\overline{[n(t-u) + x(t-u)]}$$

$$\overline{[n(t-v) + x(t-v)]}dudv$$

$$- 2\int_0^t y(u,t)\overline{[n(t-u) + x(t-u)]x(t)}du + \overline{x^2(t)}$$

87

The driving function u(t) and the measurement noise n(t) are uncorrelated, so Equation 87 reduces to

$$\overline{e^2(t)} = \int_0^t \int_0^t y(u,t)y(v,t)[\overline{n(t-u)n(t-v)}$$

$$+ \overline{x(t-u)x(t-v)}]dudv$$

$$- 2\int_0^t y(u,t)\overline{x(t-u)x(t)}du + \overline{x^2(t)} \qquad 73$$

To minimize the mean square error $y(\tau,t)$ is replaced by $y(\tau,t) + \epsilon\lambda(\tau,t)$, and the resultant mean square error is differentiated with respect to ϵ.

$$\frac{\partial e^2(t)}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \Big\{ \int_0^t \int_0^t [y(u,t) + \epsilon\lambda(u,t] [y(v,t) + \epsilon\lambda(v,t)][\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}]du\,dv - 2\int_0^t [y(u,t) + \epsilon\lambda(u,t)]\overline{x(t-u)x(t)}du + \overline{x^2(t)}\Big\}$$

$$= \int_0^t \int_0^t [y(u,t)\lambda(v,t) + y(v,t)\lambda(u,t)] [\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}]du\,dv - 2\int_0^t \lambda(u,t)\overline{x(t-u)x(t)}\,du = 0 \qquad 88$$

Therefore,

$$2\int_0^t \int_0^t y(v,t)\lambda(u,t)[\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}]du\,dv - 2\int_0^t \lambda(u,t)\overline{x(t-u)x(t)}\,du = 0 \qquad 89$$

Equation 89 can be rewritten as

$$\int_0^t \lambda(u,t)\ [\int_0^t y(v,t)\{\overline{n(t-u)n(t-v)} + \overline{x(t-u)x(t-v)}\}dv$$

$$-\widetilde{x(t-u)x(t)}]du = 0 \qquad 90$$

Since the perturbation function $\lambda(u,t)$ is arbitrary, the square bracketed expression in Equation 90 must be zero.

$$\int_0^t y(v,t)[\widetilde{n(t-u)n(t-v)} + \widetilde{x(t-u)x(t-v)}]dv$$

$$- \widetilde{x(t-u)x(t)} = 0 \qquad 74$$

where $0 < u, v < t$.

As intermediate work, the ensemble averages can be computed.

$$\widetilde{x^2(t)} = \int_0^t \int_0^t u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^t \int_0^t k\delta(\tau_1-\tau_2)d\tau_1 d\tau_2$$

$$= kt \qquad 91$$

$$\widetilde{x(t-u)x(t)} = \int_0^{t-u} \int_0^t u(t-u-\tau_1)u(t-\tau_2)d\tau_2 d\tau_1$$

$$= \int_0^{t-u} \int_0^t k\delta(\tau_1-\tau_2+u)d\tau_2 d\tau_1$$

$$= \int_0^{t-u} kd\tau_1 = k(t-u) \qquad 92$$

Figure 9 illustrates the region of integration for Equation 92. Figure 10 illustrates the regions of integration

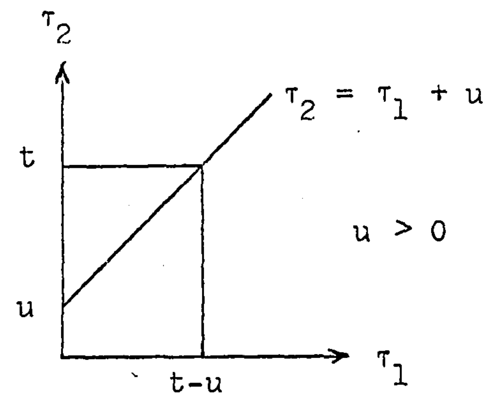

Figure 9. Region of integration

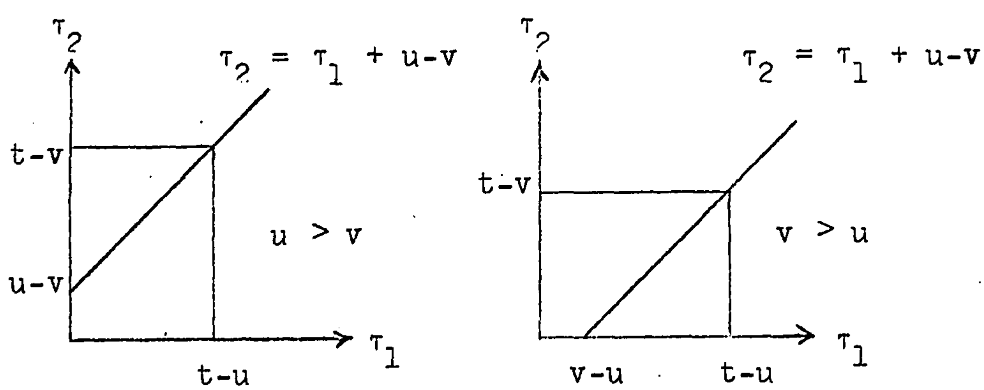

Figure 10. Regions of integration

for Equations 93.

$$\overline{x(t-u)x(t-v)} = \int_0^{t-u}\int_0^{t-v} u(t-u-\tau_1)u(t-v-\tau_2)d\tau_2 d\tau_1$$

$$= \int_0^{t-u}\int_0^{t-v} k\delta(\tau_1-\tau_2 + u-v)d\tau_2 d\tau_1$$

$$= \begin{cases} \int_0^{t-u} kd\tau_1 = k(t-u) & \text{for } u > v \\ \int_0^{t-v} kd\tau_2 = k(t-v) & \text{for } v > u \end{cases} \qquad 93$$

$$\overline{n(t-u)n(t-v)} = \sigma^2 e^{-\beta|u-v|} \qquad 94$$

Substitution of Equations 92, 93, and 94 into Equation 74 results in

$$\int_0^u y(v,t)\sigma^2 e^{-\beta(u-v)}\, dv + \int_u^t y(v,t)\sigma^2 e^{\beta(u-v)} dv$$

$$+ \int_0^u y(v,t)k(t-u)dv + \int_u^t y(v,t)k(t-v)dv - k(t-u) = 0 \qquad 75$$

The assumed solution

$$y(v,t) = C_1\delta(v) + C_2 e^{-\alpha v} + C_3 e^{\alpha v} \qquad 76$$

is then used in Equation 74. The results after integration can be written as

$$\left[\frac{\sigma^2}{-\alpha+\beta} + \frac{\sigma^2}{\alpha+\beta} - \frac{k}{\sigma^2}\right](C_2 e^{-\alpha u} + C_3 e^{\alpha u})$$

$$+ [C_1 - \frac{C_2}{-\alpha+\beta} - \frac{C_3}{\alpha+\beta}]\sigma^2 e^{-\beta u}$$

$$+ [C_2 e^{-\alpha t} + C_3 e^{\alpha t}] \frac{k}{\alpha^2}$$

$$+ [\frac{C_2 e^{-\alpha t}}{-\alpha-\beta} + \frac{C_3 e^{\alpha t}}{\alpha-\beta}] \sigma^2 e^{-\beta(t-u)}$$

$$+ [C_1 + \frac{C_2}{\alpha} - \frac{C_3}{\alpha} - 1] k(t-u) = 0 \qquad 95$$

The square bracketed terms are each set equal to zero.

$$\frac{\sigma^2}{-\alpha+\beta} + \frac{\sigma^2}{\alpha+\beta} - \frac{k}{\alpha^2} = 0 \qquad 96$$

$$C_1 - \frac{C_2}{-\alpha+\beta} - \frac{C_3}{\alpha+\beta} = 0 \qquad 97$$

$$C_2 e^{-\alpha t} + C_3 e^{\alpha t} = 0 \qquad 98$$

$$\frac{C_2 e^{-\alpha t}}{-\alpha-\beta} + \frac{C_3 e^{\alpha t}}{\alpha-\beta} = 0 \qquad 99$$

$$C_1 + \frac{C_2}{\alpha} - \frac{C_3}{\alpha} - 1 = 0 \qquad 100$$

Equation 96 reduces to

$$\alpha^2 = \frac{k\beta^2}{2\sigma^2\beta+k} \qquad 77$$

Equation 97 reduces to

$$C_1(-\alpha^2+\beta^2) - C_2(\alpha+\beta) + C_3(\alpha-\beta) = 0 \qquad 101$$

When Equations 98 and 99 are added, they reduce to

$$C_2(1+\alpha-\beta)e^{-\alpha t} + C_3(1-\alpha-\beta)e^{\alpha t} = 0 \qquad 102$$

Equation 100 reduces to

$$C_1\alpha + C_2 - C_3 = \alpha \qquad 103$$

Equations 101, 102, and 103 can now be solved simultaneously for C_1, C_2, and C_3. Results are shown in Equations 78, 79, and 80. The steady-state error in Equation 84 is found by evaluating Equation 73.