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On random coefficient INAR(1) processes

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ABSTRACT

We consider an integer-valued time-series model defined by the following random linear recursion:

$$X_{n+1} = \phi_n \circ X_n + Z_n,$$

where $\phi = (\phi_n \circ)_{n \in \mathbb{Z}}$ is a sequence of random binomial thinning operators with i.i.d. parameters and $Z = (Z_n)_{n \in \mathbb{Z}}$ is a sequence of non-negative integer-valued i.i.d. random variables. We assume that the sequences ϕ and Z are independent of each other.

The random sequence $(X_n)_{n \geq 0}$ is called RCINAR(1) (*random coefficient integer-valued autoregressive model of order one)* model. The model was proposed by by Zheng, Basawa, and Datta [69] as an extension of INAR(1) (*integer-valued autoregressive of order one)* model. In the latter, the parameters of $\phi_n \circ$ are identical for all $n \in \mathbb{Z}$. The RCINAR(1) model is a counterpart of the classical AR(1) (*first-order autoregressive)* model for real-valued data, with the binomial thinning operator replacing the usual multiplication by a random coefficient.

The model serves for forecasting and monitoring of counts data and has numerous applications in finance, control theory, and other applied fields. Formally, RCINAR(1) is a branching process with immigration formed by an infinite sequence of independent "unitary" Galton-Watson subcritical processes.

In this work we study the asymptotic behavior of this model (in particular, the distribution tails of its stationary solution, weak limits of extreme values, and the growth rate of partial sums) in the case where the additive term in the underlying random linear recursion belongs to the domain of attraction of a stable law.

CHAPTER 1. INTRODUCTION

In this work we consider a *first-order random coefficient integer-valued autoregressive* (abbreviated as RCINAR(1)) process that was introduced by Zheng, Basawa, and Datta in [69]. While the article [69] as well as a subsequent work have been focused mostly on direct statistical applications of the model, the primary goal of this work is to contribute to the understanding of its probabilistic structure.

1.1 The model

Let $\Phi := (\phi_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence of reals, each one taking values in the closed interval $[0,1]$. Further, let $\mathcal{Z} := (Z_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. integer-valued non-negative random variables, independent of Φ. The pair $(\Phi,\mathcal{Z})$ is referred to in [69] as a sequence of random coefficients associated with the model.

Let $\mathbb{Z}_+$ denote the set of non-negative integers $\{n\in\mathbb{Z} : n\geq 0\}$. The RCINAR(1) process $\mathcal{X} := (X_n)_{n\in\mathbb{Z}_+}$ is then defined as follows. Let $B := (B_{n,k})_{n\in\mathbb{Z},k\in\mathbb{Z}}$ be a collection of Bernoulli random variables independent of $\mathcal{Z}$ and such that, given a realization of Φ, the variables $B_{n,k}$ are independent and

$$P_\Phi(B_{n,k}=1)=\phi_n \qquad \text{and} \qquad P_\Phi(B_{n,k}=0)=1-\phi_n, \qquad \forall\, k\in\mathbb{N},$$

where P_Φ stands for the underlying probability measure conditional on Φ. Let $X_0=0$ and consider the following linear recursion:

$$X_n=\sum_{k=1}^{X_{n-1}} B_{n,k}+Z_n, \qquad n\in\mathbb{N}, \tag{1.1}$$

where we make the usual convention that an empty sum is equal to zero. To emphasize the formal dependence on the initial condition, we will denote the underlying probability measure (i.e., the joint law of Φ, $\mathcal{Z}$, B, and $\mathcal{X}$) conditional on $\{X_0 = 0\}$ by P_0 and denote the corresponding expectation by E_0. For the most of the thesis we will consider a natural initial assumption $X_0 = 0$ and hence consistently state our results for the measure P_0. We remark however that all our results (stated below in Section 2) are robust with respect to the initial condition X_0.

1.2 Background/previous literature

The RCINAR(1) process $\mathcal{X}$ defined by (1.1) is a generalization of the *integer-valued autoregressive of order one* (abbreviated as INAR(1)) model, in which the parameters ϕ_n are deterministic and identical for all $n \in \mathbb{Z}$. The model introduced in [69] has been further extended in [22, 62, 65, 66, 67, 68, 69]. We refer the reader to [31, 43, 45, 63] for a general review of integer-valued (data counting) time series models and their applications.

Formally, RCINAR(1) can be classified as a special kind of branching processes with immigration in the random environment Φ, cf. [34]. In particular, the process can be rigorously constructed on the state space of "genealogical trees" (see [26, Chapter VI]). The random variable X_n is then interpreted as the total number of individuals present at generation n. At the beginning of the n-th period of time, Z_n immigrants enter the system. Simultaneously and independently of it, each particle from the previous generation exits the system, producing in the next generation either one child (with probability ϕ_n) or none (with the complementary probability $1 - \phi_n$) [1]. The branching processes interpretation is a useful point of view on RCINAR(1) which provides powerful tools for the asymptotic analysis of the model.

In this work we focus on the case where production and immigration mechanisms are both defined by an i.i.d. environment and, furthermore, are independent each of other. More general type of branching process with immigration in random environment is considered, for instance, in [34, 55] and [28]. Assuming suitable moment conditions and ergodic/mixing properties of the environment, a law of large numbers and a central limit theorem for such processes are obtained in [55]. It would be interesting to carry over to a more general setting the results of this thesis which rest on the regular variation property of the coefficients when the moment conditions of

[1] Alternatively, one can think that each particle either survives to the next generation (with probability ϕ_n) or dies out (with probability $1 - \phi_n$).

[55] are not satisfied. It is plausible to assume and we leave this as a topic for future research that such an extension can be obtained by an adaptation of the techniques exploited in this work for the case of Markovian coefficients with a possible correlation between production and immigration mechanisms. We remark that a bottleneck for such a generalization of our results appears to be a suitable extension to a more general setup of the identity (3.2) and Lemma 3.1 below.

1.3 Analogy with real-valued $AR(1)$ process and branching processes

Let $\mathcal{N}_+$ denote the set of non-negative integer-valued random variables in the underlying probability space. The first term on the right-hand side of (1.1) can be thought of as the result of applying to X_n a *binomial thinning operator* which is associated with ϕ_n. More precisely, using the following operator notation introduced by Steutel and van Harn in [60]:

$$\phi_n \circ X := \sum_{n=1}^{X} B_{n,k}, \qquad X \in \mathcal{N}_+,$$

equation (1.1) can be written as

$$X_n = \phi_n \circ X_{n-1} + Z_n, \qquad n \in \mathbb{N}. \tag{1.2}$$

This form of the recursion indicates that an insight into the probabilistic structure of the RCINAR(1) process can be gained by comparing it to the classical AR(1) (*first-order autoregressive)* model for real-valued data. The latter is defined by means of i.i.d. pairs $(\phi_n, Z_n)_{n\in\mathbb{Z}}$ of real-valued random coefficients, through the following linear recursion:

$$Y_n = \phi_n Y_{n-1} + Z_n, \qquad n \in \mathbb{N}. \tag{1.3}$$

In this work we explore one of the aspects of the similarity between the RCINAR(1) and AR(1) processes. Namely, we show in Theorem 2.5 below that if Z_n are in the domain of attraction of a stable law so is the limiting distribution of X_n, and then consider some implications of this result for the asymptotic behavior of the sequence X_n. A prototype of our Theorem 2.5 for AR(1) processes has been obtained in [23, 25]. Our proof of Theorem 2.5 relies on an adaptation of the technique which has been developed in [23].

We conclude the introduction with the following remarks on the motivation for our study. Although it appears that most of our results (stated in Section 2 below) could be extended to a more general type of processes than is considered here, we prefer to focus on one important model. It is well-known that certain *quenched* characteristics of branching processes in random environment satisfy the linear difference equation (1.3). In two different settings, both yielding stationary solutions to (1.3) with regularly varying tails, this observation has been used to obtain the asymptotic behavior of the extinction probabilities in a branching processes in random environment [23, 24] and the cumulative population for branching processes in random environment with immigration [33, 34]. These studies make it appealing to consider a model like (1.2) which evidently combines features of both branching processes in random environment (with immigration) and AR(1) time series.

1.4 Applications of the model

In general, probabilistic analysis of the future behavior of average and extreme value characteristics of the underlying system might be handy for typical real-world applications of a counting data model. Our results thus constitute a natural complement to the statistical inference tools developed for the RCINAR(1) processes in [69]. For the sake of example, consider

1) maximal number of unemployed per month in an economy, according to the model discussed in [69, Section 1];

2) a variation of the model for city size distributions studied in [20, 21] where the underlying AR(1) equation is replaced by its suitable integer-valued analogue. More precisely, while it is argued in [20, 21] that the evolution of the normalized (to the total size of the population) size of a city Y_n obeys (1.3), we propose (1.2) as a possible alternative model for non-normalized size of the city population X_n, where ϕ_n is an average proportion of the population which will continue to live in the city in the observation epoch $n+1$ and Z_n is the factor accumulating both the natural population growth and migration;

3) total number of arrivals in the random coefficient variation of the queueing system proposed in [2, Section 3.2].

On the technical side, in contrary to [69], we do not restrict ourselves to a setup with $E[Z_0^2] < \infty$. This finite variance condition apparently does not pose a real limitation on the possibility of applications of RCINAR(1) to, say, the unemployment rate and the cities growth models mentioned above. In both the cases, it is reasonable to assume that the innovations Z_n are typically relatively small comparing to X_n and, furthermore, large fluctuations of their values are not very likely to occur. However, the situation seems to be quite different if one wishes to apply the theory of RCINAR(1) processes to the models of queueing theory (as it has been done in [2]) when the latter are assumed to operate under a heavy traffic regime. See, for instance, [6, 8, 14, 17, 48, 72] and [10, 49, 57] for queueing network models where it is assumed that the network input has sub-exponential or, more specifically, regularly varying distribution tails (typically resulting from the distribution of the length of ON/OFF periods). We remark that the extensive literature on queueing networks in a heavy traffic regime is partially motivated by the research on the Internet network activity where it has been shown that in many instances a web traffic is well-described by heavy-tailed random patterns; see, for instance, [15, 41, 50, 64].

CHAPTER 2. MAIN RESULTS

This chapter contains the statement of our main results, and is structured as follows. We start with a formulation of our specific assumptions on the coefficients $(\Phi, \mathcal{Z})$ of the model (see Assumptions 2.1 and 2.2 below). Proposition 2.3 then ensures the existence of the limiting distribution of X_n and also states formally some related basic properties of this Markov chain. Theorem 2.5 is concerned with the asymptotic of the tail of the limiting distribution in the case where the additive coefficients Z_n belong to the domain of attraction of a stable law. The theorem shows that in this case, the tails of the limiting distribution inherit the structure of the tails of Z_0. This observation leads us to Theorem 2.6, which is an extreme value limit theorem for the sequence $(X_n)_{n\in\mathbb{Z}_+}$. Weak convergence of suitably normalized partial sums of X_n is the content of Theorems 2.8 and 2.9. The proofs of main theorems stated below in this chapter are given in Chapter 3 while the proofs of an auxiliary proposition is deferred to the Appendix.

2.1 Specific assumptions on the random coefficients.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha \in \mathbb{R}$ and a function L such that $\lim_{t\to\infty} L(\lambda t)/L(t) = 1$ for all $\lambda > 0$. The parameter α is called the index of the regular variation. If $\alpha = 0$, then f is said to be slowly varying. We will denote by $\mathcal{R}_\alpha$ the class of all regularly varying real-valued functions with index α. We will impose the following assumption on the coefficients of the model defined by (1.1).

Assumption 2.1.

(A1) $P(\phi_0 = 1) < 1$.

(A2) *For some* $\alpha > 0$, *there exists* $h \in \mathcal{R}_\alpha$ *such that* $\lim_{t\to\infty} h(t) \cdot P(Z_n > t) = 1$.

Throughout the rest of this work we will assume (actually, without loss of generality in view of *(A2)* and Theorem 1.5.4 in [7] which ensures the existence of a non-decreasing equivalent for h) that the the following condition is included in Assumption 2.1:

(A3) Let $h : (0, \infty) \to \mathbb{R}$ be as in *(A2)*. Then $\sup_{t>0} h(t) \cdot P(Z_n > t) < \infty$.

The assumption of heavy-tailed innovations (noise terms) in autoregressive models is quite common in the applied probability literature. See for instance [23, 25], more recent articles [12, 13, 27, 38, 47, 58, 59], and references therein. It is a well-known paradigm that such an assumption yields a rich probabilistic structure of the stationary solution and allows for a great flexibility in the modeling of its asymptotic behavior [1, 39, 51, 52, 54].

An important topic in the study of integer-valued time-series is design of stationary process with a distribution within a given parametric class. See, for instance, recent articles [4, 9, 45, 46, 63, 70, 71] and references therein for a review of the area. A large amount of literature is devoted to integer-valued autoregressive processes with geometrically stable and, more generally, heavy-tailed marginal distributions with Pareto-like tails (see, for instance, [4, 9, 30, 39, 45, 71] and references therein). $DAR(p)$ stationary sequences (cf. [29]) with regularly varying marginal distributions were considered as an input governing arrivals to a single server queueing system in [35, 36].

In this dissertation we will consider the model defined in Section 1.1 in a "heavy traffic" regime, namely when the distribution tails of the innovations Z_n are regularly varying and, furthermore, they dominate the tails of $\xi_{n,k}$. Theorem 2.5 below shows that in this case the distribution tails of the stationary sequence X_n inherit the asymptotic structure of the tails of Z_n. Thus, in contrast to the majority of the literature on the $INAR(p)$-type, we focus on the propagation of the tail behavior of the input variables Z_n into the corresponding feature of X_n, rather than on designing a stationary model with a marginal distribution within a given parametric class.

In a few occasions (including a central limit theorem stated below in Theorem 2.10) we will use the following weaker version of Assumption 2.1:

Assumption 2.2. *Condition (A1) of Assumption 2.1 is satisfied and, in addition, the following holds:*

(A4) $E[Z_0^\beta] < \infty$ *for some* $\beta > 0$.

Assumption 2.2 is stronger than the usual $E(\log^+ |Z_0|) < +\infty$, where $x^+ := \max\{x, 0\}$ for $x \in \mathbb{R}$, which is essentially required for the existence and uniqueness of the stationary solution to (1.2). It can be seen through the formula $E[Z_0^\beta] = \int_0^\infty \beta x^{\beta-1} P(Z_0 > x)dx$ (recall that $Z_0 \geq 0$) that (A4) is basically equivalent to the assumption that the distribution tails of Z_0 are "not too thick".

2.2 Limiting distribution of X_n.

Let $Y_n \Rightarrow Y_\infty$ stand for the convergence in distribution of a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ to a random variable Y_∞ (we will usually omit the indication "as $n \to \infty$"). We will use the notation $X =_D Y$ to indicate that the distributions of random variables X and Y coincide under the law P_0. For $X \in \mathcal{N}_+$ define $\Pi_0 \circ X := X$ and, recursively, $\Pi_{k+1} \circ X := \phi_{k+1} \circ \big(\Pi_k \circ X\big)$. This defines a sequence of random operators acting in $\mathcal{N}_+$ as follows:

$$\Pi_k \circ X = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1 \circ X, \qquad X \in \mathcal{N}_+. \tag{2.1}$$

The existence of the stationary distribution for the sequence $\mathcal{X} = (X_n)_{n\geq 0}$ introduced in (1.1) is the content of the following proposition.

Proposition 2.3. *Let Assumption 2.2 hold. Then,*

(a) The following series converges to a finite limit with probability one:

$$X_\infty := \sum_{k=0}^{\infty} X_{0,k}, \tag{2.2}$$

where the random variables $(X_{0,k})_{k\in\mathbb{Z}_+}$ *are independent, and* $X_{0,k} =_D \Pi_k \circ Z_0$ *for any* $k \in \mathbb{N}$.

(b) $X_n \Rightarrow X_\infty$ *for any* $X_0 \in \mathcal{N}_+$. *Here* $(X_n)_{n\in\mathbb{Z}_+}$ *is understood as the sequence produced by the recursion rule (1.1) with an arbitrary initial value* X_0.

(c) The distribution of X_∞ *is the unique distribution of* X_0 *which makes* $(X_n)_{n\in\mathbb{Z}_+}$ *into a stationary sequence.*

We remark that the proposition can be in principle derived from the results of [34] and [55]. For the sake of completeness we provide here a proof of the proposition, which is deferred to the appendix. We remark that if $E[Z_0^2] < \infty$ is assumed, the above statement is essentially Proposition 2.2 in [69]. For a counterpart of this result for AR(1) processes see, for instance, Theorem 1 in [11]. It is not hard to deduce from the above proposition the following corollary, whose proof is omitted:

Corollary 2.4. *Suppose that Assumption 2.2 holds, and let $\mathcal{X} = (X_n)_{n\in\mathbb{Z}_+}$ be a random sequence defined by (1.1). Then $\mathcal{X}$ is an irreducible, aperiodic, and positive-recurrent Markov chain whose stationary measure is supported on a set of integers $\{k \in \mathbb{Z}_+ : k \geq k_{\min}\}$, where $k_{\min} := \min\{k \in \mathbb{Z}_+ : P(Z_0 = k) > 0\}$. In particular, $\mathcal{X}$ is an ergodic sequence.*

It follows from the above proposition that X_∞ is the unique solution to the distributional fixed point equation $X =_D \phi_0 \circ X + Z_0$ which is independent of (ϕ_0, B_0, Z_0), where B_0 denotes the sequence $(B_{0,k})_{k\in\mathbb{N}}$. In fact, the explicit form (2.2) of the stationary distribution along with the identity $(\phi_n, Z_n)_{n\in\mathbb{Z}} =_D (\phi_{-n}, Z_{-n})_{n\in\mathbb{Z}}$, implies that the unique stationary solution to (1.1) is given by the following infinite series:

$$X_n = \sum_{k=-\infty}^{n} X_{k,n}, \tag{2.3}$$

where the random variables $(X_{k,n})_{k\in\mathbb{Z}}$ are independent, and

$$X_{k,n} =_P \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_{k+1} \circ Z_k, \qquad k \leq n.$$

By means of the branching process interpretation,

$$X_{k,n} = \#\{\text{progeny alive at time } n \text{ of all the immigrants who arrived at time } k\}, \tag{2.4}$$

with the convention that $X_{n,n} = Z_n$ and $X_{k,n} = 0$ for $k > n$. Thus (2.3) states that the stationary solution to (1.1) is formally obtained by letting the zero generation to be formed as a union of the following two groups of individuals:

1. Z_0 immigrants arriving at time zero, and

2. descendants, present in the population at time zero, of all "demo-immigrants" who has entered the system at the negative times $k = -1, -2, \ldots$

The random variables $X_{k,n}$ can be defined rigorously on the natural state space of the branching process, which is a space of family trees describing the "genealogy" of the individuals (see [26, Chapter VI]). To distinguish between the branching process starting at time zero with $X_0 = 0$ and its stationary version "starting at time $-\infty$", we will denote by P the distribution of the latter, while continuing to use P_0 for the probability law of the former. We will denote by E the expectation operator associated with the probability measure P. We will use the notation $X =_P Y$ to indicate that the distributions of random variables X and Y coincide under the stationary law P. As it has been mentioned earlier, we will consistently state our results for the underlying process under the law P_0 and thus will consider measure P as an auxiliary tool rather than a primary object of interest.

In the case when the additive term in the underlying random linear recursion belongs to the domain of attraction of a stable law we have the following

Theorem 2.5. *Let Assumption 2.1 hold. Then,*

$$\lim_{t\to\infty} h(t) \cdot P(X_\infty > t) = \big(1 - E[\phi_0^\alpha]\big)^{-1} \in (0, \infty).$$

A prototype of this result for AR(1) processes has been obtained in [23, 25]. The proof of Theorem 2.5 given in Section 3.1 relies on an adaptation to our setup of a technique which has been developed in [23].

2.3 Extreme values of $\mathcal{X}$.

We next show that the running maximum of the sequence $\mathcal{X}$ exhibits the same asymptotic behavior as that of $\mathcal{Z} = (Z_n)_{n\in\mathbb{Z}_+}$. Let

$$M_n = \max\{X_1, \ldots, X_n\}, \qquad n \in \mathbb{N}, \tag{2.5}$$

and

$$b_n = \inf\{t > 0 : h(t) \geq n\}, \tag{2.6}$$

where $h(t)$ is the function introduced in Assumption 2.1.

The proof of the following theorem is given in Section 3.2 below.

Theorem 2.6. *Let Assumption 2.1 hold. Then, under the law P_0,*

$$M_n/b_n \Rightarrow M_\infty,$$

where M_∞ is a proper random variable with the following distribution function:

$$P_0(M_\infty > x) = e^{-x^{-1/\alpha}}, \qquad x > 0,$$

where $\alpha > 0$ is the constant introduced in Assumption 2.1.

The distribution of M_∞ belongs to the class of the so called Fréchet extreme value distributions and in fact (see, for instance, [18, Section 3.3]),

$$P_0(M_\infty > x) = \lim_{n\to\infty} P\big(\max_{1\le k\le n} Z_k > xb_n\big), \qquad x > 0.$$

It is quite remarkable that the distribution of ϕ_0 does not play any role in the result of Theorem 2.6. An intuitive explanation for this phenomenon, which can be derived from the proof, is as follows. Due to the basic property of regular variation, two independent terms $\phi_n \circ X_{n-1}$ and Z_n are unlikely to "help" each other in creating a large value of the sum $X_{n+1} = \phi_n \circ X_{n-1} + Z_n$. Moreover, the law of large numbers ensures that the ratio $\phi_n \circ X_{n-1}/X_{n-1}$ is bounded away from one with an overwhelming probability whenever $\phi_n \circ X_{n-1}$ is large. Therefore, the asymptotic of the extreme value of the sequence X_n follows that of Z_n.

2.4 Growth rate and fluctuations of the partial sums of $\mathcal{X}$.

The results in this section are quoted from [56]. Their proofs are outside of the scope of this thesis and therefore are omitted.

For $n \in \mathbb{N}$, let

$$S_n = \sum_{k=1}^{n} X_k.$$

The following law of large numbers is a direct consequence of Corollary 2.4.

Proposition 2.7. *Let Assumption 2.2 hold with $\beta = 1$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_0] = \frac{E[Z_0]}{1 - E[\phi_0]}, \qquad P_0 - \text{a. s.}$$

The next theorem is concerned with the rate of the growth of the partial sums when Z_0 has infinite mean. For $\alpha \in (0,2]$ and $b > 0$ denote by $\mathcal{L}_{\alpha,b}$ the strictly asymmetric stable law of index α with the characteristic function

$$\log \widehat{\mathcal{L}}_{\alpha,b}(t) = -b|t|^\alpha \left(1 + i\frac{t}{|t|} f_\alpha(t)\right), \tag{2.7}$$

where $f_\alpha(t) = -\tan \frac{\pi}{2}\alpha$ if $\alpha \neq 1$, $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\alpha < 1$, $\mathcal{L}_{\alpha,b}$ is supported on the positive reals, and if $\alpha \in (1,2]$, it has zero mean [18, Section 2.2].

Recall b_n from (2.6).

Theorem 2.8. *Let Assumption 2.1 hold with $\alpha \in (0,1)$. Then $b_n^{-1} S_n \Rightarrow \mathcal{L}_{\alpha,b}$.*

We next study the fluctuations of the partial sums in the case where non-trivial centering of X_n is required to obtain a proper weak limit for the partial sums.

Theorem 2.9. *Let Assumption 2.1 hold with $\alpha \in [1,2]$. For $n \in \mathbb{N}$, define*

$$a_n = \begin{cases} b_n, \text{ where } b_n \text{ is defined in (2.6)}, & \text{if } \alpha < 2, \\ \inf\{t > 0 : nt^{-2} \cdot E\big[X_0^2;\, X_0 \le t\big] \le 1\} & \text{if } \alpha = 2. \end{cases}$$

Denote $\mu := E[X_0]$. Then the following holds for some $b > 0$:

(i) If $\alpha = 1$, then $a_n^{-1}(S_n - c_n) \Rightarrow \mathcal{L}_{1,b}$ with $c_n = nE[X_0; X_0 \le a_n]$.

(ii) If $\alpha \in (1,2)$, then $a_n^{-1}(S_n - n\mu) \Rightarrow \mathcal{L}_{\alpha,b}$.

(iii) If $\alpha = 2$ and $E[Z_0^2] = \infty$, then $a_n^{-1}(S_n - n\mu) \Rightarrow \mathcal{L}_{2,b}$.

If an appropriate second moment condition is assumed, one can establish the following functional limit theorem for normalized partial sums of $\mathcal{X}$. Let $D(\mathbb{R}_+, \mathbb{R})$ denote the set of real-valued càdlàg functions on $\mathbb{R}_+ := [0,\infty)$, endowed with the Skorokhod J_1-topology. Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. We have:

Theorem 2.10. *Let Assumption 2.2 hold with a constant $\beta > 2$. Then, as $n \to \infty$, the sequence of processes*

$$S_t^{(n)} = n^{-1/2}\big(S_{\lfloor nt \rfloor} - nt\mu\big), \qquad t \in [0,1].$$

in $D(\mathbb{R}_+, \mathbb{R})$ converges weakly to a non-degenerate Brownian motion W_t, $t \in [0,1]$.

Theorem 2.10 is a particular case of [55, Theorem 1.5]. Notice that the conditions of the theorem are satisfied if Assumption 2.1 holds with $\alpha > 2$.

CHAPTER 3. PROOFS

This section is devoted to the proof of our main results stated in Section 2, namely, Theorems 2.5, 2.6, 2.8 and 2.9. The chapter is divided into four sections correspondingly.

3.1 Proof of Theorem 2.5

First, we observe the following.

Lemma 3.1. *Let $X \in \mathcal{N}_+$ be a random variable in the underlying probability space such that*

(i) X is independent of $(\phi_n, Z_n, B_n)_{n\in\mathbb{Z}_+}$, where $B_n := (B_{n,k})_{k\in\mathbb{N}}$.

(ii) $\lim_{t\to\infty} h(t) \cdot P_\Phi(X > t) = 1$ for some $h \in \mathcal{R}_\alpha$, $\alpha > 0$.

Then $\lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t) = \phi_0^\alpha$.

Remark 3.2. *The lemma essentially says that*

$$\lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t) = \lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \cdot X > t). \tag{3.1}$$

Indeed, it follows immediately from the definition of the regular variation and Assumption (A2) that

$$\lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \cdot X > t) = \phi_0^\alpha.$$

Heuristically, identity (3.1) is the consequence of the law of large numbers which claims that $\phi_0 \circ X \approx \phi_0 \cdot X$ for large values of X and of the large deviation principle asserting that the deviation of $\phi_0 \circ X$ from $\phi_0 \cdot X$ by $\varepsilon \cdot X$ are exponentially unlikely for any $\varepsilon > 0$. The application of the law of large numbers and of the large deviation principle are intuitively justified by the simple observation that $\phi_0 \cdot X$ can be large only X itself is large.

Proof of Lemma 3.1. Fix a constant $\varepsilon \in (0,1)$. For $t > 0$ define the following three events:

$$\begin{aligned} A_{t,\varepsilon} &= \big\{X > t\cdot(\phi_0^{-1}+\varepsilon)\big\}, \\ B_{t,\varepsilon} &= \big\{t\cdot(\phi_0^{-1}-\varepsilon) < X \le t\cdot(\phi_0^{-1}+\varepsilon)\big\}, \\ C_{t,\varepsilon} &= \big\{X \le t\cdot(\phi_0^{-1}-\varepsilon)\big\}. \end{aligned}$$

We will use the following splitting formula:

$$P_\Phi\big(\phi_0\circ X > t\big) = P_\Phi\big(\phi_0\circ X > t;\, A_{t,\varepsilon}\big) + P_\Phi\big(\phi_0\circ X > t;\, B_{t,\varepsilon}\big) + P_\Phi\big(\phi_0\circ X > t;\, C_{t,\varepsilon}\big).$$

By the law of large numbers,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} B_{1,k} = \phi_0, \qquad P-\text{a. s.}$$

Since $h(t)$ is regularly varying, Chernoff's bound (Cramér's large deviation theorem for coin flipping, see [16]) applied to the partial sums $\sum_{k=1}^{n} B_k$ implies that

$$0 \le \limsup_{t\to\infty} h(t)\cdot P_\Phi\big(\phi_0\circ X > t;\, C_{t,\varepsilon}\big) \le \limsup_{t\to\infty} h(t)\cdot P_\Phi\Big(\sum_{k=1}^{\lfloor t(\phi_0^{-1}-\varepsilon)\rfloor} B_k > t\Big) = 0.$$

Next, by the conditions of the lemma,

$$\begin{aligned} \lim_{t\to\infty} h(t)\cdot P_\Phi\big(\phi_0\circ X > t;\, B_{t,\varepsilon}\big) &\le \lim_{t\to\infty} h(t)\cdot P_\Phi(B_{t,\varepsilon}) \\ &= \big[(\phi_0^{-1}-\varepsilon)^{-\alpha} - (\phi_0^{-1}+\varepsilon)^{-\alpha}\big] \to_{\varepsilon\to 0} 0. \end{aligned}$$

Finally, using again the large deviation principle for $\sum_{k=1}^{n} B_k$,

$$\begin{aligned} \liminf_{t\to\infty}\ h(t)\cdot P_\Phi\big(\phi_0\circ X > t;\, A_{t,\varepsilon}\big) &= \liminf_{t\to\infty}\ h(t)\cdot\Big[P_\Phi\big(A_{t,\varepsilon}\big) - P_\Phi\big(\phi_0\circ X \le t;\, A_{t,\varepsilon}\big)\Big] \\ &\ge \liminf_{t\to\infty}\ h(t)\cdot P_\Phi\big(A_{t,\varepsilon}\big) = (\phi_0^{-1}+\varepsilon)^{-\alpha}. \end{aligned}$$

On the other hand, clearly,

$$\liminf_{t\to\infty}\ h(t)\cdot P_\Phi\big(\phi_0\circ X > t;\, A_{t,\varepsilon}\big) \le \liminf_{t\to\infty}\ h(t)\cdot P_\Phi\big(A_{t,\varepsilon}\big) = (\phi_0^{-1}+\varepsilon)^{-\alpha}.$$

Since $\varepsilon > 0$ is arbitrary and $(\phi_0^{-1}+\varepsilon)^{-\alpha} \to \phi_0^{\alpha}$ as ε goes to zero, this completes the proof of the lemma. $\square$

Remark 3.3. *The above proof of Lemma 3.1 can be adopted without modification for a more general type of sums* $\sum_{k=1}^{X} B_k$*, where* $X \in \mathcal{N}_+$ *has regularly varying distribution tails and* $(B_k)_{k\in\mathbb{N}}$ *are independent of* X*. In fact, the only property of the sequence* B_k *required by the proof is the availability of a non-trivial large deviations upper bound for its partial sums. Note that if* $f(\lambda) := E_\Phi\big[e^{\lambda B_1}\big]$ *is finite in a neighborhood of zero, such a bound in the form*

$$P_\Phi\Big(\Big|\frac{1}{n}\sum_{k=1}^{n} B_k - E_\Phi[B_1]\Big| > x\Big) \le c(x)e^{-nI(x)}$$

with suitable constants $c(x), I(x) > 0$ *holds for any* $x > 0$ *(see, for instance, the first inequality in the proof of Lemma 2.2.20 in [16]).*

Recall (see, for instance, [18, Lemma 1.3.1]) that if X and Y are two independent random variables such that $\lim_{x\to\infty} h(x)\cdot P(X > x) = c_1 > 0$ and $\lim_{x\to\infty} h(x)\cdot P(Y > x) = c_2 > 0$ for some $h \in \mathcal{R}_\alpha$, $\alpha > 0$, then

$$\lim_{x\to\infty} h(x)\cdot P(X+Y > x) = c_1 + c_2. \tag{3.2}$$

Using this property and iterating (1.1), one can deduce from Lemma 3.1 the following corollary. Consider (in an enlarged probability space, if needed) a sequence $\widetilde{\mathcal{X}} = \big(\widetilde{X}_n\big)_{n\in\mathbb{Z}_+}$ which solves (1.1), that is a sequence such that

$$\widetilde{X}_n = \sum_{k=1}^{\widetilde{X}_{n-1}} B_{n,k} + Z_n, \qquad n \in \mathbb{N}, \tag{3.3}$$

for some initial (not necessarily equal to zero) random value $\widetilde{X}_0$.

Corollary 3.4. *Let Assumption 2.1 hold and suppose in addition that the following two conditions are satisfied:*

(i) $\widetilde{X}_0$ *is independent of* $(\phi_k, B_k, Z_k)_{k>0}$*, where* $B_k = (B_{k,j})_{j\in\mathbb{N}}$*.*

(ii) $\lim_{t\to\infty} h(t)\cdot P_\Phi\big(\widetilde{X}_0 > t\big) = c_0$ *for some random variable* $c_0 = c_0(\Phi)$*.*

Then $\lim_{t\to\infty} h(t)\cdot P_\Phi\big(\widetilde{X}_n > t\big) = c_n$ *for any* $n \in \mathbb{N}$*, where the random variables* $c_n = c_n(\Phi)$ *are defined recursively by*

$$c_{n+1} = c_n\phi_{n+1}^{\alpha} + 1, \qquad n \in \mathbb{Z}_+. \tag{3.4}$$

The recursive relation (3.4) implies that

$$c_n = \bar{c}_n + c_0 \prod_{j=1}^{n} \phi_j^{\alpha}, \qquad \text{where} \qquad \bar{c}_n = 1 + \sum_{k=2}^{n} \prod_{j=k}^{n} \phi_j^{\alpha}, \tag{3.5}$$

and hence (see, for instance, Theorem 1 in [11]) the random variables c_n converge in distribution, as $n \to \infty$, to

$$c_\infty := 1 + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \phi_{-i}^{\alpha}. \tag{3.6}$$

Furthermore, we have the following:

Corollary 3.5. *Suppose that the conditions of Corollary 3.4 are satisfied and, in addition, there exist a positive constant $C > 0$ and such that the following holds:*

$$P\Big(\sup_{t>0}\big\{h(t) \cdot P_\Phi\big(\widetilde{X}_0 > t\big)\big\} < C\Big) = 1.$$

Then the following limit exists and the identity holds:

$$\lim_{n\to\infty} h(t) \cdot P\big(\widetilde{X}_n > t\big) = E[c_n], \qquad n \in \mathbb{N}, \tag{3.7}$$

where c_n are random variables defined in (3.5).

Proof of Corollary 3.5. Corollary 3.4 and the bounded convergence theorem imply that

$$\begin{aligned}\lim_{t\to\infty} h(t) \cdot P\big(\widetilde{X}_n > t\big) &= \lim_{t\to\infty} h(t) \cdot E\big[P_\Phi\big(\widetilde{X}_n > t\big)\big] \\ &= E\big[\lim_{t\to\infty} h(t) \cdot P_\Phi\big(\widetilde{X}_n > t\big)\big] = E[c_n]. \end{aligned} \tag{3.8}$$

To justify interchanging of the limit with the expectation, observe that $\widetilde{X}_n \leq \widetilde{X}_0 + \sum_{k=1}^n Z_k$ and hence, by virtue of assumption *(A3)*, the following inequalities hold with probability one for some positive constant $C_1 > 0$:

$$\begin{aligned} h(t) \cdot P_\Phi\big(\widetilde{X}_n > t\big) &\leq h(t) \cdot P_\Phi\big(\widetilde{X}_0 > t/2\big) + h(t) \cdot P_\Phi\Big(\sum_{k=1}^{n} Z_k > t/2\Big) \\ &\leq h(t) \cdot P_\Phi\big(\widetilde{X}_0 > t/2\big) + nh(t) \cdot P\big(Z_0 > t/(2n)\big) \\ &\leq C\frac{h(t)}{h(t/2)} + C_1 n \frac{h(t)}{h(t/(2n))}. \end{aligned}$$

It follows (see, for instance, [23, Lemma 1]) that there exists a constant $C_2 > 0$ such that

$$P\Big(\sup_{t>t_0}\big\{h(t) \cdot P_\Phi\big(\widetilde{X}_n > t\big)\big\} < C_2\Big) = 1.$$

This enables one to apply the bounded convergence theorem in (3.8) and thus completes the proof of the corollary. □

Notice that by virtue of (3.6) and (3.7), the result in Theorem 2.5 is the claim that

$$\lim_{t\to\infty}\lim_{n\to\infty} h(t)\cdot P\big(\widetilde{X}_n > t\big) = \lim_{n\to\infty}\lim_{t\to\infty} h(t)\cdot P\big(\widetilde{X}_n > t\big) = \lim_{n\to\infty} E[c_n] = E[c_\infty].$$

Instead of trying to justify exchange of the limits in the above identity directly, we will apply the method of Grey [23] and use two different initial conditions X_0 producing two special sequences of X_n with distribution tails dominating those of X_∞ and having the prescribed asymptotic behavior.

In what follows notations $X \le_D Y$ and $X \ge_D Y$ for random variables X and Y are used to indicate that $P(X > t) \le P(Y > t)$ or, respectively, $P(X > t) \ge P(Y > t)$ holds for all $t \in \mathbb{R}$. In order to exploit Corollary 3.5 in the proof of Theorem 2.5, we need the following:

Lemma 3.6. *Suppose that the conditions of Corollary 3.5 are satisfied. Then:*

(a) If $\widetilde{X}_0 \le_D \widetilde{X}_1$, then $\widetilde{X}_n \le_D \widetilde{X}_{n+1}$ for all $n \in \mathbb{N}$.

(b) If $\widetilde{X}_0 \ge_D \widetilde{X}_1$, then $\widetilde{X}_n \ge_D \widetilde{X}_{n+1}$ for all $n \in \mathbb{N}$.

Proof of Lemma 3.6. The proof is by induction. Suppose first that $\widetilde{X}_{n-1} \le_D \widetilde{X}_n$ for some $n \in \mathbb{N}$, $\widetilde{X}_{n-1}$ is independent of $(\phi_k, B_k, Z_k)_{k>n-1}$, and $\widetilde{X}_n$ is independent of $(\phi_k, B_k, Z_k)_{k>n}$. We will now use the following standard trick to construct an auxiliary random pair (V_{n-1}, V_n) such that

$$P(V_{n-1} \le V_n = 1), \qquad V_{n-1} =_P \widetilde{X}_{n-1}, \qquad \text{and} \qquad V_n =_P \widetilde{X}_n. \tag{3.9}$$

Let U be a uniform random variable on $[0,1]$, independent of the random coefficients sequence $(\Phi, \mathcal{Z})$. Denote by F_n and F_{n-1}, respectively, the distribution functions of X_n and X_{n-1}. Set $V_n = F_n^{-1}(U)$ and $V_{n-1} = F_{n-1}^{-1}(U)$, where $F^{-1}(y) := \inf\{x \in \mathbb{R} : F(x) \ge y\}$, $y \in [0,1]$, with the convention that $\inf \emptyset = \infty$.

Let $\widetilde{X}_{n+1} = \phi_{n+1} \circ \widetilde{X}_n + Z_{n+1}$. Then $\widetilde{X}_{n+1}$ is independent of $(\phi_k, B_k, Z_k)_{k>n+1}$. Furthermore, since (V_{n-1}, V_n) is independent of $(\Phi, \mathcal{Z})$, we obtain for any $t > 0$,

$$\begin{aligned} P\big(\widetilde{X}_{n+1} > t\big) &= P\big(\phi_{n+1} \circ \widetilde{X}_n + Z_{n+1} > t\big) = P(\phi_{n+1} \circ V_n + Z_{n+1} > t) \\ &\geq P(\phi_{n+1} \circ V_{n-1} + Z_{n+1} > t) = P(\phi_n \circ V_{n-1} + Z_n > t) \\ &= P\big(\phi_n \circ \widetilde{X}_{n-1} + Z_n > t\big) = P\big(\widetilde{X}_n > t\big). \end{aligned} \tag{3.10}$$

This shows that part (a) of the lemma holds true. The same argument, but with $\leq$ replaced by $\geq$ and vice versa in the base of induction, (3.9), and (3.10), yields part (b). □

We are now in a position to complete the proof of Theorem 2.5. First, we have:

Lemma 3.7. *There exists a random variable $\widetilde{X}_0 \geq 0$ satisfying the conditions of Corollary 3.5, such that $\widetilde{X}_1 \geq_D \widetilde{X}_0$.*

Proof of Lemma 3.7. Set $\widetilde{X}_0 = Z_{-1}$. □

In view of Lemma 3.6, this implies that we can find a sequence $\widetilde{X}_n$ that solves (1.1) and such that $\widetilde{X}_n \leq_D \widetilde{X}_\infty$, while $\widetilde{X}_0$ satisfies the conditions of Corollary 3.5. Combining this result with the conclusion of the corollary yields:

$$\liminf_{t\to\infty} h(t) \cdot P(X_\infty > t) \geq \lim_{t\to\infty} h(t) \cdot P\big(\widetilde{X}_n > t\big) = E[c_n], \qquad n \in \mathbb{N}.$$

Hence

$$\liminf_{t\to\infty} h(t) \cdot P(X_\infty > t) \geq \lim_{n\to\infty} E[c_n] = \frac{1}{1 - E[\phi_0^\alpha]}. \tag{3.11}$$

On other hand, we have

Lemma 3.8. *Let Assumption 2.1 hold. There exists a random variable $\widetilde{X}_0 \geq 0$ satisfying the conditions of Lemma 3.1, and such that $\widetilde{X}_1 \leq_D \widetilde{X}_0$.*

Proof of Lemma 3.8. Given a realization of the sequence Φ, choose a constant c_0 in such a way that

$$c_0 > \frac{1}{1 - E[\phi_0^\alpha]}.$$

Let $Y_0 = c_0^{1/\alpha} Z_{-1}$. Then $\lim_{t\to\infty} h(t)\cdot P(Y_0 > t) = c_0$. If we would choose $\widetilde{X}_0 = Y_0$, we would have $c_1 := \lim_{t\to\infty} h(t)\cdot P\big(\widetilde{X}_1 > t\big) < c_0$ by virtue of (3.4) and Corollary 3.5. This would imply that $P\big(\widetilde{X}_1 > t\big) < P\big(\widetilde{X}_0 > t\big)$ for $t > t_0$, where $t_0 > 0$ is a positive constant which depends on c_0. Consider now (in an enlarged probability space, if needed) a random variable $\widetilde{X}_0$ such that $\widetilde{X}_0$ is independent of $(\phi_k, B_k, Z_k)_{k\in\mathbb{Z}}$ and

$$P\big(\widetilde{X}_0 > t\big) = P(Y_0 > t | Y_0 > t_0).$$

Note that such $\widetilde{X}_0$ satisfies the conditions of Corollary 3.5 because $P_\Phi\big(\widetilde{X}_0 > t\big) = P\big(\widetilde{X}_0 > t\big)$ with probability one, and for $t > t_0$,

$$h(t)\cdot P\big(\widetilde{X}_0 > t\big) \leq \frac{1}{P(c_0^\alpha Z_0 > t_0)} \cdot \frac{h(t)}{h(tc_0^{-\alpha})}\Big(h(tc_0^{-\alpha})\cdot P(Z_0 > tc_0^{-\alpha})\Big),$$

and $\sup_{t>0} h(t)/h(tc_0^{-\alpha}) < \infty$ (see, for instance, Lemma 1 in [23]). Then, for $t > t_0$,

$$\begin{aligned} P\big(\phi_1\circ\widetilde{X}_0 + Z_1 > t\big) &= P(\phi_1\circ Y_0 + Z_1 > t | Y_0 > t_0) \\ &= \frac{P(\phi_1\circ Y_0 + Z_1 > t; Y_0 > t_0)}{P(Y_0 > t_0)} \leq \frac{P(\phi_1\circ Y_0 + Z_1 > t)}{P(Y_0 > t_0)} \\ &\leq \frac{P(Y_0 > t)}{P(Y_0 > t_0)} = P(Y_0 > t | Y_0 > t_0) = P\big(\widetilde{X}_0 > t\big). \end{aligned}$$

On the other hand, if $t \leq t_0$ then

$$P\big(\widetilde{X}_0 > t\big) = P\big(\widetilde{X}_0 > t \big| \widetilde{X}_0 > t_0\big) = 1.$$

Thus

$$P\big(\phi_1\circ\widetilde{X}_0 + Z_1 > t\big) \leq P\big(\widetilde{X}_0 > t\big)$$

for all $t > t_0$, and we can set $\widetilde{X}_0$ as the initial value for the recursion. □

Combining this result with Corollary 3.5 yields:

$$\limsup_{t\to\infty} h(t)\cdot P(X_\infty > t) \leq \lim_{t\to\infty} h(t)\cdot P_0(X_n > t) = E[c_n], \qquad n\in\mathbb{N}.$$

Hence,

$$\limsup_{t\to\infty} h(t)\cdot P(X_\infty > t) \leq \lim_{n\to\infty} E[c_n] = \frac{1}{1 - E[\phi_0^\alpha]}.$$

The proof of Theorem 2.5 is completed in view of (3.11).

3.2 Proof of Theorem 2.6

For $n \in \mathbb{N}$, denote $K_n = \max_{1\leq k\leq n} Z_k$. It follows from (1.1) that $M_n \geq_D K_n$. To conclude the proof of the theorem, it thus suffices to show that

$$\limsup_{n\to\infty} P_0(M_n > xb_n) \leq \lim_{n\to\infty} P_0(K_n > xb_n) = e^{-x^{-1/\alpha}}, \qquad x > 0.$$

Observe that, under the stationary law P, the branching process (without immigration) originated by the initial X_0 individuals will eventually die out. Therefore, the total number of progeny of the individuals in the zero generation is $P-$a. s. finite. Furthermore, the branching process $X_n - \sum_{k=-\infty}^{0} X_{k,n}$, $n \in \mathbb{N}$, obtained by excluding the contribution of these individuals from the original one, is distributed under P as X_n, $n \in \mathbb{N}$, under P_0. It thus suffices to show that

$$\limsup_{n\to\infty} P(M_n > xb_n) \leq \lim_{n\to\infty} P(K_n > xb_n) = e^{-x^{-1/\alpha}}, \qquad x > 0.$$

Toward this end, define the following events. For $x > 0$, $\delta > 0$, and $\varepsilon \in (0, 1/2)$, let

$$\begin{aligned}
A^{(n)}_{x,\delta} &= \{xb_n < M_n \leq x(1+\delta)b_n\}, \qquad n \in \mathbb{N},\\
B^{(n)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \bigcap \{x(1-\varepsilon)b_n < K_n \leq x(1+\delta)b_n\}, \qquad n \in \mathbb{N},\\
C^{(n,k)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \bigcap \{X_k > xb_n,\ \varepsilon x b_n < Z_k \leq x(1-\varepsilon)b_n\}, \qquad n \in \mathbb{N},\ k = 1, \ldots, n,\\
D^{(n,k)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \bigcap \{X_k > xb_n,\ Z_k \leq x\varepsilon b_n\}, \qquad n \in \mathbb{N},\ k = 1, \ldots, n.
\end{aligned}$$

Then

$$\begin{aligned}
P\big(A^{(n)}_{x,\delta}\big) &\leq P\big(B^{(n)}_{x,\delta,\varepsilon}\big) + P\Big(\bigcup_{k=1}^{n} C^{(n,k)}_{x,\delta,\varepsilon}\Big) + P\Big(\bigcup_{k=1}^{n} D^{(n,k)}_{x,\delta,\varepsilon}\Big)\\
&\leq P\Big(x(1-\varepsilon)b_n < K_n \leq x(1+\delta)b_n\Big) + n\,P\big(C^{(n,1)}_{x,\delta,\varepsilon}\big) + n\,P\big(D^{(n,1)}_{x,\delta,\varepsilon}\big). \qquad (3.12)
\end{aligned}$$

Taking into account the independence of the pair (ϕ_k, X_{k-1}) of Z_k, it follows from (1.2), Assumption 2.1, and Lemma 3.1 that for any positive constants $\delta, x, \varepsilon > 0$

$$\limsup_{n\to\infty} n\,P\big(C^{(n,1)}_{x,\delta,\varepsilon}\big) \leq \lim_{n\to\infty} n\,P\big(\phi_1 \circ X_0 > \varepsilon x b_n, Z_1 > \varepsilon x b_n\big) = 0. \qquad (3.13)$$

Furthermore,

$$
\begin{aligned}
P\big(D^{(n,1)}_{x,\delta,\varepsilon}\big) &\le P\big(\phi_1\circ X_0 > (1-\varepsilon)xb_n, X_0\le x(1+\delta)b_n\big)\\
&\le P\big(\phi_1\circ X_0 > (1-\varepsilon)xb_n \big| X_0\le x(1+\delta)b_n\big) \le P\Big(\sum_{i=1}^{\lfloor x(1+\delta)b_n\rfloor} B_{0,i} > (1-\varepsilon)xb_n\Big)\\
&= E\Big[P_\Phi\Big(\frac{1}{x(1+\delta)b_n}\sum_{i=1}^{\lfloor x(1+\delta)b_n\rfloor} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\Big)\Big]. \qquad (3.14)
\end{aligned}
$$

Assume now that the constants $\delta>0$ and $\varepsilon>0$ are chosen so small that $\frac{1-\varepsilon}{1+\delta} > E[\phi_0]$, and hence

$$
\frac{1-\varepsilon}{1+\delta} > \eta E[\phi_0] \quad \text{for some} \quad \eta>1. \qquad (3.15)
$$

We next derive a simple large-deviations type upper bound for the right-most expression in (3.14). Denote $x_0=\frac{1-\varepsilon}{1+\delta}$. It follows from Chebyshev's inequality that for any $\lambda>0$,

$$
E\Big[P_\Phi\Big(\frac{1}{n}\sum_{i=1}^{n} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\Big)\Big] \le e^{-n\lambda x_0} E\big[(1-\phi_0+\phi_0 e^{\lambda})^n\big].
$$

Thus for all $\lambda>0$ small enough, namely for all $\lambda>0$ such that $e^{\lambda}<1+\eta\lambda$, we have

$$
\begin{aligned}
E\Big[P_\Phi\Big(\frac{1}{n}\sum_{i=1}^{n} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\Big)\Big] &\le e^{-n\lambda x_0} E\big[\big(1-\phi_0+\phi_0(1+\eta\lambda)\big)^n\big]\\
&= e^{-n\lambda x_0} E\big[(1+\phi_0\eta\lambda)^n\big] \le e^{-n\lambda x_0} E\big[e^{\phi_0\cdot n\eta\lambda}\big].
\end{aligned}
$$

Therefore, for all $\lambda>0$ small enough we have

$$
\limsup_{n\to\infty}\frac{1}{n}\log E\Big[P_\Phi\Big(\frac{1}{n}\sum_{i=1}^{n} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\Big)\Big] \le -\lambda x_0 + \log E\big[e^{\eta\lambda\phi_0}\big].
$$

Given η, let $f(\lambda)=\log E\big[e^{\eta\lambda\phi_0}\big]$. By the bounded convergence theorem, $f'(0)=\eta E[\phi_0]$. Hence, in view of (3.15),

$$
\limsup_{n\to\infty}\frac{1}{n}\log P\Big(\frac{1}{n}\sum_{i=1}^{n} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\Big) < 0.
$$

Since b_n is a regularly varying sequence, it follows from (3.14) that

$$
\lim_{n\to\infty} n\,P\big(D^{(n,1)}_{x,\delta,\varepsilon}\big) = 0. \qquad (3.16)
$$

Therefore, since $\varepsilon > 0$ above can be made arbitrary small (in particular, the left-hand side of (3.15) is an increasing function of ε), combining (3.16) together with (3.13) and (3.14) yields:

$$\limsup_{n\to\infty} P\big(A_{x,\delta}^{(n)}\big) \le P\big(xb_n < K_n \le x(1+\delta)b_n\big),$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} P(M_n > xb_n) &= \limsup_{n\to\infty} \sum_{k=0}^{\infty} P\big((1+k\delta)xb_n < M_n \le (1+k\delta+\delta)xb_n\big)\\ &\le \sum_{k=0}^{\infty} \limsup_{n\to\infty} P\big((1+k\delta)xb_n < M_n \le (1+k\delta+\delta)xb_n\big)\\ &\le \sum_{k=0}^{\infty} P\big((1+k\delta)xb_n < K_n \le (1+k\delta+\delta)xb_n\big) = P(K_n > xb_n).\end{aligned}$$

The proof of Theorem 2.6 is complete. $\square$

APPENDIX A. Proof of an auxiliary proposition

A.1 Proof of Proposition 2.3

(a) By Jensen's inequality, if $E[Z_0^\beta] < \infty$ for $\beta > 0$, then $E[Z_0^{\beta/m}] < \infty$ for any $m \in \mathbb{N}$. Therefore, without loss of generality we can assume that $\beta \in (0,1)$ in Assumption 2.2. Assuming from now on and throughout the proof of part (a) of Proposition 2.3 that $\beta \in (0,1)$, we obtain by virtue of Jensen's inequality for conditional expectations that

$$\begin{aligned} E_0\big[(\Pi_k \circ Z_k)^\beta\big] &= E_0\big[E_0\big[(\Pi_k \circ Z_k)^\beta | \Phi, \mathcal{Z}\big]\big] \le E_0\Big[\big(E_0\big[\Pi_k \circ Z_k | \Phi, \mathcal{Z}\big]\big)^\beta\Big] \\ &= E\Big[\Big(\prod_{j=1}^k \phi_j \cdot Z_k\Big)^\beta\Big] = E[Z_0^\beta] \cdot \big(E[\phi_0^\beta]\big)^k. \end{aligned} \tag{A.1}$$

Hence

$$E[X_\infty^\beta] = E\Big[\Big(\sum_{k=0}^\infty X_{0,k}\Big)^\beta\Big] \le \sum_{k=0}^\infty E\big[X_{0,k}^\beta\big] \le E[Z_0^\beta] \cdot \sum_{k=0}^\infty \big(E[\phi_0^\beta]\big)^k < \infty.$$

In particular, X_∞ is $P-$ a. s. finite.

(b) For $n \in \mathbb{N}$, we have

$$X_n = \sum_{k=1}^n X_{k,n} + X^{(0,n)},$$

where $X^{(0,n)} =_P \Pi_n \circ X_0$. Since $P\big(\lim_{n\to\infty} \Pi_n \circ X_0 = 0\big) = 1$ for any $X_0 \in \mathcal{N}_+$, the limiting distribution of X_n, if exists, is independent of X_0. Furthermore, if $X_0 = 0$, the i.i.d. structure of $(\Phi, \mathcal{Z})$ yields:

$$X_n =_P \sum_{k=-n+1}^{0} X_{k,0} =_P Z_0 + \sum_{k=1}^{n-1} \Pi_k \circ Z_k,$$

The claim of part (b) follows now from the almost sure convergence of the series on the right-hand side of the above identity to X_∞.

(c) To see that the stationary distribution is unique, consider two stationary solutions $\big(X_n^{(1)}\big)_{n\in\mathbb{Z}_+}$ and $\big(X_n^{(2)}\big)_{n\in\mathbb{Z}_+}$ to (1.1) corresponding to different initial values, $X_n^{(1)}$ and $X_n^{(2)}$, respectively. Then, since Π_n are "thinning" operators,

$$|X_n^{(1)} - X_n^{(2)}| \le \Pi_{n+1} \circ |X_0^{(1)} - X_0^{(2)}|,$$

and hence

$$\lim_{n\to\infty} \big(X_n^{(1)} - X_n^{(2)}\big) = 0, \qquad P - \text{a. s.}$$

The proof of the proposition is complete. $\square$

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