Variable elasticity of substitution production functions, technical change and factor shares

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VARIABLE ELASTICITY OF SUBSTITUTION PRODUCTION
FUNCTIONS, TECHNICAL CHANGE AND FACTOR SHARES

by

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A Dissertation Submitted to the
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I. INTRODUCTION

The behavioral relations among technical change, factor shares, the elasticity of substitution between factors of production, and economic growth have been spelled out theoretically by Hicks in his well-known Theory of Wages (21). Recently, Bruton (10) and Fellner (14) have revived his ideas.

Technical change is defined by Solow (57) as shifts in the production function. It enables a firm to produce a given level of output with less inputs or to produce a greater level of output with a given set of inputs. The increment of total output which is not attributable to the factors which contribute to the production, i.e., the residual after the distribution of income, is due to technical change. Then, how will the residual be allocated when technical change occurs? Will labor's share increase or decrease after technical change? These questions may be answered by following Hick's propositions.

Assume that the state of technique is constant. When the stock of capital is growing faster than the labor force, the prices of capital relative to that of labor declines, and the substitution of capital for labor takes place along a given isoquant. More goods which require much capital will be produced, and more known capitalistic methods which did not pay previously will come into use. But as capital continues to grow relative to labor the more advantageous applications will
be used up. Eventually, the elasticity of substitution will fall below unity, the share as well as the marginal productivity of capital will be dragged down, and those of labor will be pushed up. Ultimately, the profit rate will decline, and investment will dry up. The economy will, then, lead to the "stationary state."

Bruton (10) and Fellner (14) have asserted that in order to prevent the elasticity of substitution falling below unity, to keep the share and the marginal productivity of capital from declining, and to check the sweeping rise of labor's share, the rate of technical change must be not only sufficiently great enough to shift the production function so as to keep the diminishing returns to capital from setting in, it must also be labor-saving in the Hicksian sense to utilize the increased capital and to raise the productivity of capital.

In observing the sources of economic growth, many economists argued, especially in the late 1950's, that the major part of the increase in the productivity of labor in the United States manufacturing over the past half century has resulted from technical progress and only a small part of the increase in the labor productivity has been attributable to the increase in capital. Massell (32) even concluded that economists need to shift their emphasis from the theory of capital to the theory of technical progress as an explanation of economic growth. This view was later revised, and some weight
was given back to capital formation. Because most improvements in technique are likely to be embodied in the new capital stock, technical progress requires a positive rate of gross investment.

Given resources, then, how should a decision-maker allocate them between the improvements of technique and the expansion of capital stock? In answering this question, many attempts have been made to apportion the increase in total output per unit of labor due to the increase in capital and that due to technical progress. In order to measure the rate and types of technical change, to segregate the increment in total output between increase in capital and technical progress, and to investigate the behavior of the elasticity of substitution in response to changing relative supply of capital and labor, a specification of the form of production functions is indispensable. Although Solow (57, p. 317) maintains that a specific form of production functions is not used in his estimation of technical change in the U.S. economy from 1909 to 1949, Hogan (23, p. 411) later has pointed out that the equation Solow used is essentially of a Cobb-Douglas form.

Many algebraic forms can be used to represent production functions. In aggregate models the simplest form is the fixed coefficient production function, where the factors of production are combined in a fixed proportion. The elasticity
of substitution is zero.

Traditionally the Cobb-Douglas function has been widely used and has been considered the most appropriate form to represent production functions. Recently, however, the validity of the Cobb-Douglas function has been questioned; the underlying assumption that the elasticity of substitution is unity is too restrictive in the empirical as well as the theoretical applications.

Arrow, Chenery, Minhas, and Solow (hereafter abbreviated SMAC) have derived a CES (constant elasticity of substitution) production function, which is less restrictive and a much more fruitful approach. It allows the elasticity of substitution to assume any values ranging from zero to infinity. However, the theoretical basis of the formulation of the CES function is rather weak. Moreover, the CES production function is also subject to the restriction or limitation that the value of the elasticity of substitution is constant, although not necessarily unity.

The three forms of production functions are most widely used in the aggregate models. But the underlying assumption that the elasticity of substitution is constant is too limited in the applications of these functions on the study of the behavior of the elasticity of substitution in response to the changing relative supply of capital and labor. We need a more general form of production functions which allows the elasticity
of substitution to vary as the relative supply of capital and labor changes.
II. PRODUCTION FUNCTIONS

A production function expresses the relation between the maximum level of output and the inputs required to produce that level of output. It is defined as a mathematical relation expressing the maximum level of output attainable from any given sets of inputs, given the state of technique.

A production function is a convenient and useful tool in many fields of economic analysis. It serves as a basis for the analysis of economic growth and the determination of optimum patterns of international or interregional trade and provides a framework for the measurement of technical change and the study of the behavior of relative factor shares.

There is a wide choice of algebraic forms which can be used to represent production functions. In aggregate models the simplest form is the fixed coefficient production function, which is represented by rectangular isoquants with constant returns to scale. In this model, there is no opportunity for changing relative factor inputs; the elasticity of substitution between factors of production is zero. This is an extremely specialized form, but its simplicity explains its extensive use in many models, such as the well-known Harrod-Domar model.

Probably the most popular production function is the Cobb-Douglas function, which is of the form

\( (2.1) \quad V = \gamma K^\beta L^\alpha \)

where \( V \): output
Since its introduction in 1928, no single form of production functions has ever enjoyed quite the same popularity. It is simple to explain and easy to fit the logarithmic form of the function by the regular regression method. Moreover, it has the following attractive theoretical properties:

1. The exponents of labor ($\alpha$) and capital ($\beta$) inputs represent the elasticity of production with respect to labor and capital, respectively.

2. The function is homogeneous of degree $\alpha+\beta$. If $\alpha+\beta$ is greater than unity, there are increasing returns to scale; $\alpha+\beta = 1$ indicates constant returns to scale; and $\alpha+\beta$ less than unity represents decreasing returns to scale.

3. The marginal physical productivity of an input is positive but declines, if the elasticity of production of that factor is less than unity, as the input is increased. For example, the second partial derivatives of $V$ with respect to $L$ is

$$\frac{\partial^2 V}{\partial L^2} = \alpha(\alpha-1) \frac{V}{L^2}$$

which is negative if $\alpha<1$.

4. The marginal rate of substitution of labor for capital is $\alpha K/\beta L$, and thus the elasticity of substitution between capital and labor is unity (64).
The Cobb-Douglas function has had a long and successful life without serious rivals. But recently the validity of the Cobb-Douglas function has been questioned.

Empirically much evidence has shown that the technical constraints on the production allow considerable freedom of choice. Capital and labor can be substituted for each other in varying degrees. It is unlikely that the substitutability is uniform in different sectors and in different industries. The value of the elasticity of substitution is not necessarily restricted to unity. SMAC (2) estimate that the elasticity of substitution is 0.569 for the U.S. non-farm sectors, Kendrick and Sato (27) estimate 0.58, and Kravis (29) 0.64 for the entire U.S. economy.

Theoretically, the Cobb-Douglas function has been strongly challenged by a new production function—a CES production function—introduced by SMAC.

The CES function is derived on the basis of the good fit of a linear logarithmic function of the following type to observations on value added per unit of labor and the wage rates (2, pp. 225-28):

\[
(2.1) \quad \log V/L = \log a_1 + b_1 \log W + e_1
\]

where \( W \): wage rate

\( a_1, b_1 \): parameters

\( e_1 \): a random variable.

Starting with this empirical relation between labor productiv-
ity and the wage rate, SMAC have derived their function:\(^1\)

\[
V = \gamma [\delta K^{-\rho} + (1-\delta) L^{-\rho}]^{\frac{-1}{\rho}}
\]

where \( \rho \): the substitution parameter  
\( \delta \): the distribution parameter  
\( \gamma \): the efficiency parameter.

This function is linear and homogeneous, i.e., there are constant returns to scale. The efficiency parameter \( \gamma \) changes output for given quantities of inputs; the distribution parameter \( \delta (0 \leq \delta \leq 1) \) determines the division of factor income.

The substitution parameter \( \rho \) is a transform of the elasticity of substitution \( (\sigma) \), thus;

\[
\sigma = \frac{1}{1+\rho}.
\]

By choosing appropriate values for \( \sigma \) the CES function can be specialized to the fixed coefficient and Cobb-Douglas forms. As \( \sigma \) approaches unity (i.e., \( \rho \) approaches to zero), the CES function approaches the Cobb-Douglas form. When \( \sigma \) approaches zero (i.e., \( \rho \) approaches to infinity), the CES function approaches the fixed coefficient function. Therefore, the CES production function includes the Cobb-Douglas and fixed coefficient functions as special cases. Since the introduction of the CES function in 1961, many theoretical as well as empirical studies in production, distribution, technical change, international comparison of efficiency, and growth have

\(^1\)Brown and de Cani (8) later derived the same function directly from the definition of the elasticity of substitution.
used the function as derived originally. Examples include the contributions by Brown (5,6), Brown and de Cani (7,8), Brown and Popkin (9), Kendrick and Sato (27), Kmenta (28), McCarthy (34,35), McFadden (36), McGuire (37), Minhas (41), Pitchford (46), Resek (48), Sato (51), Thornber (59), and Uzawa (63).

Paroush (45) has extended the CES function to homogeneous of degree n, where n is not necessarily equal to unity. This function allows increasing and decreasing returns to scales. It is of the form

\[ V = \gamma \left[ (1-\delta) L^{-\rho} + \delta K^{-\rho} \right] \left( \frac{n}{\rho} \right) \]

where n is the degree of homogeneity.

The CES function also has been extended to more than two factors of production. Uzawa (63) has extended the CES function to the n factor case, which is of the following form:

\[ V = f(x_1, \ldots, x_n) \]

\[ = \gamma (a_1 x_1^{-\rho} + \cdots + a_n x_n^{-\rho}) \frac{1}{\rho} \]

where \( a_i \) (i=1,2,...,n) is a positive constant and the parameter \( \rho \) is greater than -1. This function is homogeneous of degree one and has partial derivatives of any order. The elasticity of substitution of any pair of factors of production is constant, identical, and independent of factor prices. Since identical partial elasticity of substitution might not repre-
sent a realistic situation, Dhrymes and Kurz (12) and Mukerji (42) have proposed a more general version of the CES function, which is of the form

\[ v = \gamma \left( \sum_{i=1}^{n} \alpha_i x_i^{-\rho_i} \right)^{-\frac{1}{\rho}}. \]

It has been shown that the function has a property of constant ratios of elasticity of substitution. Dhrymes and Kurz have used this function to isolate the impact of technical progress on electric generation.

Possibly the weakest point of the CES function is that the SMAC formulation assumes "the existence of a relationship between \( V/L \) and \( W \), independent of the stock of capital" (2, p. 231). If we take capital per unit of labor \( (K/L) \) as the capital variable, the SMAC assumption implies that the partial regression coefficient of \( \log K/L \) is zero. If this assumption does not hold, the regression coefficient of \( \log W \) in equation (2.1) may not represent the true value of the elasticity of substitution.

The CES function is also subject to the restriction or limitation that the value of the elasticity of substitution is constant, although not necessarily unity. However, when the capital/labor ratio varies due to changes in the factor price ratio, it is possible that the elasticity of substitution does not remain constant. It would be more desirable for the production function to possess a property such that the
elasticity of substitution could vary as the capital/labor ratio varied.

After the author derived a new form of production functions, Revankar (49) proposed a variable elasticity of substitution production function. He started with the hypothesis that the elasticity of substitution is a linear function of capital and labor; thus

$$\sigma = \sigma(K, L)$$

$$= 1 + \frac{\rho - 1}{1 - \delta} \frac{K}{L}.$$ 

Based on this hypothesis, he proposed a production function of the form

$$V = \gamma K^{\alpha(1-\delta \rho)} [L + (\rho-1)K]^{\alpha \delta \rho}$$

where $\alpha, \delta, \rho,$ and $\gamma$ are parameters. This function includes the fixed-coefficient function and the Cobb-Douglas function as special cases, but, unfortunately, it does not include the CES function as its special case. There is no smooth transition between this function and the CES function.

Although much effort has been made to generalize the CES function, its fundamental weakness and limitation—the basic assumption made in the derivation of the CES function and the constancy of the elasticity of substitution—still remain intact.

The objectives of this study are:

1. To attempt to derive a more general form of production
functions which (a) does not depend upon the SMAC assumption of independence in its formulation, (b) includes the CES function as a special case, and (c) has the property of variable elasticity of substitution:

2. To examine the theoretical properties of the new production function;

3. To show under what conditions the elasticity of substitution estimated by the CES function is the same as that of the new function;

4. To test the desirability of using the new function;

5. To investigate the behavior of the elasticity of substitution in response to the changing relative supply of capital and labor;

6. To formulate different alternative methods of estimating the parameters of the new function;

7. To measure technical change in seven three-digit U.S. Food and Kindred Products Industries with the new function;

8. To apply the new production function to the study of the behavior of relative factor shares in the Food and Kindred Products Industries.
III. A GENERALIZATION OF THE CES PRODUCTION FUNCTION

A. Derivation of the New Production Function

Hildebrand and Liu (22, p. 35) have suggested that "If one relies upon the goodness of fit of an empirical relationship as the initial basis for deriving a theoretical one, as Arrow, Chenery, Minhas, and Solow did, one probably would have to consider the three-variable relationship \((V/L, W, \text{and } K/L)\) as better established than the two-variable one \((V/L \text{ and } W)\)." Therefore, we begin with the following relationship:

\[
\log V/L = \log a + b \log W + c \log K/L + e
\]

where \(c\) is a constant and the other notation is the same as before. When the production function

\[
V = F(K,L)
\]

is homogeneous of degree one, we may rewrite (3.2) in

\[
V/L = F(K/L, 1).
\]

Set \(V/L = Y\) and \(K/L = X\).

Then, we have

\[
Y = f(X)
\]
or

\[
V = Lf(X).
\]

Let \(W\) be the wage rate with output as the numeraire. If both labor and product markets are competitive, then (2, p. 228)

\[
W = f(X) - Xf'(X)
\]
or

\[
W = Y - X \frac{dV}{dX}
\]
and \( r = f'(X) \)

where \( f'(X) \) is the marginal product of capital, \( f(X) - Xf'(X) \) the marginal product of labor, and \( r \) the returns to capital.

By substituting (3.7) into (3.1), we get the following differential equation:

\[
\log Y = \log a + b \log (Y - X \frac{dy}{dx}) + c \log x. 
\]

Solving for \( \frac{dy}{dx} \) results in

\[
\frac{dy}{dx} = \frac{y}{x} - \alpha x - \frac{c}{b} - 1 \frac{1}{y^b}
\]

where \( \alpha = a \frac{1}{b} \).

Since the differential equation is non-linear, it is difficult to solve. But by letting

\[
z = Y^{1-b}
\]

we can transform the non-linear differential equation into a linear equation:

\[
\frac{dz}{dx} + \frac{1-b}{b} \frac{1}{x} z = \alpha \frac{1-b}{b} x - \frac{c}{b} - 1.
\]

The solution of (3.10) gives

\[
\frac{1}{x^b} - l \frac{1}{b} z = \alpha n x^b + \beta
\]

or

\[
z = \alpha n x^b + \beta x^{1-b}
\]

where \( n = \frac{1-b}{1-b-c} \)

and \( \beta \) is the constant of integration. By transforming \( z \) back to \( Y \), we obtain the new production function
(3.13) \( Y = \left[ \alpha x \frac{c}{b} + \beta x^{\frac{b-1}{b}} \right]^{\frac{b}{b-1}} \)

or (3.14) \( V = \left[ \beta k^{\frac{b-1}{b}} + \alpha \eta \left( \frac{k}{L} \right)^{\frac{c}{b}} \right]^{\frac{b}{b-1}} \).

Written in the SMAC notation, (3.14) becomes

(3.15) \( V = \left[ \beta k^{\frac{b-1}{b}} + \alpha \eta \left( \frac{k}{L} \right)^{\frac{c}{b}} \left( 1 + \rho \right)^{\frac{1}{\rho}} \right] \)

where \( \rho = \frac{1}{b} - 1 \).

By setting \( \alpha = \left( 1 - \delta \right) \gamma^{-\rho} \)

\( \beta = \delta \gamma^{-\rho} \)

we obtain

(3.16) \( V = \gamma \left[ \delta k^{\frac{b-1}{b}} + (1 - \delta) \eta \left( \frac{k}{L} \right)^{\frac{c}{b}} \left( 1 + \rho \right)^{\frac{1}{\rho}} \right] \).

This production function has the same form as the CES function except that \( L^{-\rho} \) is multiplied by

(3.17) \( \eta \left( \frac{k}{L} \right)^{\frac{c}{b}} \left( 1 + \rho \right)^{\frac{1}{\rho}} \).

Obviously, if \( c \) is equal to zero, (3.17) becomes unity and the new function reduces to the CES function; therefore, the new function is a more general form which includes the CES function as a special case.
B. Theoretical Properties of the New Production Function

The new production function has the following theoretical properties: (i) positive marginal products, (ii) downward sloping marginal product curves over the relevant ranges of the inputs, (iii) linear homogeneity (i.e., constant returns to scale), and (iv) variable elasticity of substitution. Moreover, it includes the CES function as a special case.

The marginal products of capital and labor can be obtained from (3.7) and (3.9):

\[
\frac{\partial V}{\partial K} = \frac{\partial V}{\partial X} = \frac{1}{X} \left( Y - ax \right) \frac{1}{b \cdot y^b}
\]

\[
= \frac{1}{K} \left( V - \frac{\partial V}{\partial L} \cdot L \right)
\]

\[
(3.19) \quad \frac{\partial V}{\partial L} = Y - X \frac{\partial V}{\partial Y}
\]

\[
= ax \frac{1}{b \cdot y^b}.
\]

It is obvious that the marginal product of labor is positive. Since \( V \) and \( \frac{\partial V}{\partial L} \cdot L \) are positive, and the total product \( V \) is always greater than labor's share \( \frac{\partial V}{\partial L} \cdot L \), the marginal product of capital is also positive.

By taking the partial derivative of (3.19) with respect to \( L \), we get

\[
\frac{\partial^2 V}{\partial L^2} = -\frac{a}{b \cdot L} x \frac{c}{b} \frac{1}{y^b} - 1 \left( Y \cdot \frac{\partial V}{\partial X} - c \cdot Y \right).
\]
It will be shown later in (3.26) that
\[ x \frac{dy}{dx} - c Y > 0. \]

Hence, we get \( \frac{\partial^2 Y}{\partial L^2} < 0. \)

Therefore, the marginal product of labor slopes downward.

The addition of \( \eta \left( \frac{K}{L} \right)^{-c(1+\rho)} \) in the new function does not disturb the property of homogeneity. It can be readily shown that the new production function has the property of first degree homogeneity such that

\[ F(tK, tL) = tF(K, L) \]

where \( t \) is a constant.\(^2\)

\(^2\)When the production function (3.2) is homogeneous of degree \( n \), then

\[ \frac{V}{L^n} = F\left( \frac{K}{L}, 1 \right). \]

By setting \( \frac{V}{L^n} = Y \)

and \( \frac{K}{L} = X \)

the production function can again be rewritten in the form of (3.4). If both product and labor markets are competitive, then

\[ W = L^{n-1} (nf - Xf'). \]

Set \( W* = \frac{W}{L^{n-1}}. \)

Then we have

\[ W* = nf - Xf', \]

or \( W* = nY - x \frac{dy}{dx} \).  

(footnote continued on following page)
The elasticity of substitution ($\sigma$), according to Hicks (21, p. 117) is a measure of the ease at which the varying factor can be substituted for others. It is defined as the proportional change in the factor ratio in response to a

(footnote continued from preceding page)

Let us start with the following empirical relationship between $Y$, $W^*$ and $X$:

$$\log Y = \log a + b \log W^* + c \log X.$$  

Then inserting the value of $W^*$ into the above relationship results in the following nonlinear differential equation:

$$\log Y = \log a + b \log (nY - X) + c \log X.$$  

Following the same procedure as in Section A to solve the differential equation, we obtain

$$V = \left[ \delta K^\rho + \eta^* \left( \frac{K}{L} \right)^{-\mu} \rho^* \right] \frac{\rho^*}{\rho^*}$$

where

$$\eta^* = \frac{1-b}{n(1-b)-c} \quad \rho^* = n\left(\frac{1}{b} - 1\right)$$

and

$$\mu = \frac{c(\rho^* + n)}{n}$$

By letting $\beta = \delta \cdot \gamma - \frac{\rho^*}{\rho^*}$

and $\alpha = (1-\delta) \gamma - \frac{\rho^*}{\rho^*}$

we get

$$V = \gamma \left[ \delta K^\rho + (1-\delta) \eta^* \left( \frac{K}{L} \right)^{-\mu} \rho^* \right] \frac{n}{\rho^*}.$$  

It can be readily shown that this production function possesses the property of $n$-th degree homogeneity such that

$$F(tK,tL) = t^n \cdot F(K,L).$$

Since the degree of homogeneity is characterized by the parameter $n$, which can assume any value whatsoever, the function may represent any degree of homogeneity.

This function reduces to the new function homogeneous of degree one when $n$ is unity. When $c = 0$ and $n = 1$, it leads to the CES function.
proportional change in the marginal rate of substitution between the two factors. Symbolically, it is written as

\[(3.20) \sigma = \frac{\frac{dX}{X}}{\frac{dR}{R}}\]

where \( R \) is the marginal rate of substitution of labor for capital:

\[ R = - \frac{dK}{dL} = \frac{F_L}{F_K}. \]

It has been shown by Allen (1, pp. 341-43) that

\[(3.21) \sigma = \frac{V_K V_L (KV_K + LV_L)}{K L T}\]

where \( T = -(V_{kk} V_L^2 - 2 V_{kL} V_K V_L + V_{LL} V_K^2) \)

and the variable with subscript(s) denotes the partial derivative(s) of that variable with respect to the subscripted variable(s). When the production function is linear and homogeneous, \( \sigma \) reduces to the following simpler form:

\[(3.22) \sigma = \frac{V_K V_L}{V \cdot V_{LK}}. \]

The cross second-order partial derivative can be obtained by taking the partial derivative of (3.19) with respect to \( K \):

\[(3.23) \frac{\partial^2 V}{\partial K \partial L} = \frac{a}{b \cdot L} x^c - \frac{1}{y^b - 1} \frac{dY}{dx} (x \frac{dy}{dx} - cY). \]

Substituting the results of (3.18), (3.19), and (3.23) into (3.22) yields

\[(3.24) \sigma = \frac{b}{1 - \frac{c R}{X}}, \]

or \[(3.25) \sigma = \frac{b}{1-c(1+\frac{R}{X})}. \]
where \( b \) and \( c \) are assumed to be non-negative. Since \( \sigma \) is non-negative in the empirically relevant range of the capital/labor ratio, it follows that

\[
1 - \frac{cf}{xf'} \geq 0
\]

or (3.26) \( xf' - cf > 0 \)

and \( c \leq \frac{xf'}{x} = \frac{3V}{3k} \frac{K}{V} \).

The relative share of capital is less than unity, and thus \( c \) is less than unity.

Since \( R \) is a function of \( X \), moving along an isoquant, the elasticity of substitution varies with changes in capital/labor ratios. Therefore, the elasticity of substitution is not a constant, but a function of the capital/labor ratios. Hence, we term the newly-derived function the variable elasticity of substitution production function (hereafter abbreviated as the VES production function).

The first-order condition for minimum cost under pure competition is

\[
R = W/r.
\]

---

3. The explicit form of \( R \) as a function of \( X \) will be discussed in the following paragraphs.

4. Revankar (49) has proposed a VES function with the elasticity of substitution as a linear function of the capital/labor ratio. But, his VES function does not include the CES function as a special case. Sato (51,52) also has proposed a production function with linear elasticity of substitution.
Substituting the value of $R$ into (3.25) gives

\[
(3.27) \quad \sigma = \frac{b}{1 - c(1 + \frac{WL}{rK})}
\]

which can be used to estimate $\sigma$ empirically.

It is probably easier to grasp the concept of the elasticity of substitution by considering the substitution function,\(^5\) which relates the capital/labor ratio and the marginal rate of substitution. The elasticity of the substitution function measures the elasticity of substitution, and thus the elasticity of substitution can be read off the substitution curve in the same way that the elasticity of supply is read off a supply curve.

From the definition of the elasticity of substitution, we have

\[
(3.28) \quad \frac{dR}{R} = \frac{1}{\sigma} \frac{dx}{x}
\]

where $\sigma$ is a function of $x$. Inserting the value of $\sigma$ in (3.25) into (3.28) yields

\[
\frac{dR}{dx} = \frac{1-c}{b} \cdot \frac{R}{X} - \frac{c}{b} \cdot \frac{R^2}{X^2}.
\]

\(^5\)The substitution function was first introduced by Lerner (31, pp. 68-71). But he proposed that the capital/labor ratio is a decreasing function of the marginal rate of substitution, which is contrary to our proposition. If we had defined the elasticity of substitution as

\[
\sigma = -\frac{dx/dR}{X/R}
\]

we would have derived the same result. Ferguson (16, pp. 145-46) also suggests that "there is an inverse or negative relationship between the marginal rate of technical substitution and the capital-labor ratio." He defines the elasticity of substitution as shown above with a negative sign in the text, but in the footnote he defines the elasticity of substitution as shown in (3.20) without a negative sign.
The shapes of the substitution functions are shown in Figures 1, 2 and 3. Figure 1 shows the substitution functions of the CES case with the Cobb-Douglas as a special case. The substitution function in the CES case is an exponential function; the curve increases at an increasing or decreasing rate with an increased $X$, depending upon whether the value of $\sigma$ is less than or greater than unity. Given a curve every point on that curve has the same elasticity. When $\sigma = 1$, i.e., the Cobb-Douglas case, the substitution function is a straight line through the origin, indicating unitary elasticity of substitution throughout the line. In the case of the VES function, every point on a given curve has a different elasticity; the elasticity varies with $X$.

By inserting the value of $R$ in (3.32) into (3.25), we get

\begin{equation}
(3.34) \quad \sigma = \frac{b}{1-c(1 + \frac{1}{\frac{\beta}{\alpha} X^{A-B}})}
\end{equation}

where \[ A = \frac{b+c-1}{b} \]

\[ B = \frac{c}{b+c-1} \]

and \[ \frac{\beta}{\alpha} X^{A} - B = \frac{X}{R} \text{ is positive.} \]

Equation (3.34) indicates that $\sigma$ is an explicit function of $X$.

By taking the derivative of $\sigma$ with respect to $X$, we obtain
The capital/labor ratio

The marginal rate of substitution

$\sigma < 1$

$\sigma = 1$

$\sigma > 1$

The capital/labor ratio

Figure 1. The substitution functions of the CES function
Figure 2. The substitution function of the VES function with $b + c > 1$
The capital/labor ratio

Figure 3. The substitution function of the VES function with $b + c < 1$
\[
\frac{d\sigma}{dx} = -\frac{A\beta b c x^{A-3} R^2}{\alpha G^2}
\]

where \( G = 1 - c (1 + \frac{R}{X}) \).

If \( A \frac{b}{c} < 0 \), then \( \frac{d\sigma}{dx} \leq 0 \). Therefore, the effect of changes in \( X \) on \( \sigma \) depends on \( A \), or equivalently the parameters \( b \) and \( c \). If \( b + c > 1 \), the value of \( \sigma \) declines with increased \( X \) and approaches \( \frac{b}{1-c} \), which is greater than unity, as \( X \) increases without limit. The value of \( \sigma \) is independent of \( X \) when \( b + c = 1 \). If \( b + c < 1 \), the value of \( \sigma \) increases from \( \frac{b}{1-c} \), which is less than unity, to unity as \( X \) increases from zero to infinity. Figures 2 and 3 represent, respectively, the cases of \( b + c > 1 \) and \( b + c < 1 \).

C. Comparison of the Elasticity of Substitution Estimated by the CES Function and by the VES Function

As mentioned above, the crucial assumption SMAC made in deriving the CES function is that the partial regression coefficient of \( \log \frac{K}{L} \) is equal to zero. If the assumption does not hold, then the regression coefficient of \( \log W \) (i.e., \( b_1 \)) obtained by fitting equation (2.1) may not represent the true elasticity of substitution. In other words, the estimated elasticity of substitution from the SMAC function is biased.

For convenience, let a logarithm of a variable be denoted by a prime of the variable. Then, (2.1) and (3.1) can be re-written in
(3.35) \[ Y' = a'_l + b'_l W' + e'_l \]
(3.36) \[ Y' = a' + b W' + c X' + e . \]
The least squares estimates of \( b'_l \), \( b \) and \( c \) give

(3.37) \[ \hat{b}_l = \frac{\Sigma w y}{\Sigma w^2} \]

and (3.38) \[ \left( \frac{\hat{b}}{c} \right) = \left[ \frac{\Sigma w^2}{\Sigma w x} \right]^{-1} \cdot \left[ \frac{\Sigma w y}{\Sigma x y} \right] \]

where the lower case letter denotes the deviation of the logarithmic variable from its mean of the respective variable.

Substituting (3.37) into (3.38), we have

(3.39) \[ \hat{b} = \frac{\hat{b}_l - r_{w x} r_{x y} \Sigma w^2}{1 - r_{w x}^2} \]

and

(3.40) \[ \hat{c} = \sqrt{\frac{\Sigma y^2}{\Sigma x^2} \left( r_{x y} - r_{w x} r_{w y} \right)} \cdot \frac{1}{1 - r_{w x}^2} \]

where the \( r \)'s are correlation coefficients between the subscripted variables.

If \( b = b'_l \), then, from (3.39), we get

\[ b'_l r_{w x} = r_{xy} \sqrt{\Sigma w^2} . \]

By substituting \( b'_l = \Sigma w y / \Sigma w^2 \) into the above equation, we get

\[ r_{xy} = r_{w x} \cdot r_{w y} . \]

On the other hand, if

\[ r_{xy} = r_{w x} \cdot r_{w y} \]
then substituting this relationship into (3.39) results in
\[
b = \frac{b_1 - r_{wx}^2 r_{wy} \sqrt{\sum_{w}^2}}{1 - r_{wx}^2} = \frac{b_1 (1 - r_{wx}^2)}{1 - r_{wx}^2} = b_1.
\]

Therefore, we obtain the result that \( b_1 = b \) if and only if \( r_{xy} = r_{wx} \cdot r_{wy} \). It is apparent from (3.40) that \( c = 0 \) if and only if \( r_{xy} = r_{wx} \cdot r_{wy} \). Consequently, we conclude that \( c = 0 \) if and only if \( b_1 = b \).

From (3.27) or (3.34) it is obvious that \( \sigma \) will be equal to \( b \) if \( c \) is equal to zero. It follows that \( c = 0 \) implies that \( \sigma = b_1 \). That is, the elasticity of substitution estimated by the VES function and the CES function will be identically equal under the condition that the partial regression coefficient of \( \log \frac{K}{L} \) turns out to be zero.

D. Tests of Hypothesis

1. The data

To test the null hypothesis that \( c = 0 \), U.S. census data for 1957 on 17 two-digit manufacturing industries were used for the regression analysis. The data were obtained from Appendix I in *Manufacturing Production Functions in the United States, 1957*, by Hildebrand and Liu (22). They cover output, labor, and capital variables as follows:

\[V: \text{Value added in dollars in 1957.}\]

\[K: \text{Gross book value (owned) of plant and equipment at the beginning of 1957 in dollars.}\]
\( K_g \): Gross book value (owned plus rented) of plant and equipment at the beginning of 1957 in dollars.

\( K_n \): Gross book value minus accumulated depreciation and depletion.

\( L \): Employment (production workers and nonproduction employees).

\( L_p \): Employment of production workers.

\( W \): Average wage and salary rate for production workers and non-production employees in dollars per man-hour.

\( W_p \): Average wage rate for production workers in dollars per man-hour.

2. Methods and procedures

The above-mentioned data were used to fit the following regressions for each of the 17 industries:

\[(3.41) \log \frac{V}{L} = \log a_1 + b_1 \log W + e_1\]

\[(3.42) \log \frac{V}{L} = \log a + b \log W + c \log \frac{K_g}{L} + e_2\]

\[(3.43) \log \frac{V}{L} = \log a + b \log W + c \log \frac{K_g'}{L} + e_3\]

\[(3.44) \log \frac{V}{L} = \log a + b \log W + c \log \frac{K_n}{L} + e_4\]

\[(3.45) \log \frac{V}{L_p} = \log a_1 + b_1 \log W_p + e_5\]

\[(3.46) \log \frac{V}{L_p} = \log a + b \log W_p + c \log \frac{K_g}{L_p} + e_6\]

\[(3.47) \log \frac{V}{L_p} = \log a + b \log W_p + c \log \frac{K_g'}{L_p} + e_7\]

\[(3.48) \log \frac{V}{L_p} = \log a + b \log W_p + c \log \frac{K_n}{L_p} + e_8\]

The results of the regression analysis are presented in Tables 1 through 8.\(^6\)

Under the assumption that the error term has a normal dis-

\(--\)

\(^6\)Our estimates of the regression coefficients and the coefficients of determination are slightly different from those of Hildebrand and Liu. However, the general conclusions concerning the significance of the regression coefficient of \( \log K/L \) are the same.
tribution, the null hypothesis that the coefficient of regression of \( \log K/L \), \( c \), equals zero can be tested by use of \( t \). In the regressions of (3.42) and (3.46) (Tables 2 and 6) 7 out of 17 \( c \) values are significantly different from zero at the 5 percent level; and in the regressions of (3.43), (3.47) and (3.48) (Tables 3, 7 and 8) 8 out of 17 \( c \) values are different from zero at the 5 percent significant level; and 9 out of 17 \( c \) values in the regression of (3.44) (Table 4) differ significantly from zero at the 5 percent level.

To look at the results in total, we performed the following test. Set 5 percent as the level of significance. Then, under the null hypothesis that \( c = 0 \), the probability (p) of getting one \( c \) value significantly different from zero is 5 percent. The expected mean frequencies of occurrence of significant \( c \) values in 17 observations (N) will be

\[
Np = 17 \times 0.05 = 0.85.
\]

Since \( Np \) is large relative to \( p \), and \( N \) is large relative to \( Np \), we may assume that the frequencies of occurrence of \( c \) values significantly different from zero follow a Poisson distribution. The probability of getting 7 or more significant \( c \) values out of 17 observations is

\[
p(x \geq 7) = 1 - \sum_{x=0}^{6} \frac{e^{-0.85}}{x!} \frac{0.85^x}{x!} = 0.0000.
\]

Of course the probabilities of getting 8 or more and 9 or more significant \( c \) values out of 17 observations are also nearly
zero. This evidence strongly suggests that the samples were not likely drawn from the population in which the c value is zero. Therefore, the null hypothesis that c = 0 is rejected, and we conclude that c is not equal to zero.

3. Estimation of the elasticity of substitution

The elasticity of substitution at any point on an isoquant can be calculated by use of the formulas (3.27) or (3.34). We used the mean value of the observed $\frac{W}{F} \cdot \frac{L}{K}$ to compute the elasticity of substitution for each industry by (3.27). The results are shown in the last column of each of Tables 2, 3, 4, 6, 7, and 8.
<table>
<thead>
<tr>
<th>Industry</th>
<th>Number of observations (states)</th>
<th>$b_1$</th>
<th>$S_{b_1}$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food and kindred products</td>
<td>35</td>
<td>1.667</td>
<td>0.385</td>
<td>0.363</td>
</tr>
<tr>
<td>Textile mill products</td>
<td>18</td>
<td>1.355</td>
<td>0.238</td>
<td>0.669</td>
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<td>1.961</td>
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<td>0.531</td>
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<tr>
<td>Furniture and fixtures</td>
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<td>1.925</td>
<td>0.192</td>
<td>0.855</td>
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<tr>
<td>Pulp, paper and products</td>
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<td>3.026</td>
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<td>Chemicals and products</td>
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<td>1.083</td>
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<td>0.231</td>
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<tr>
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<tr>
<td>Machinery except electrical</td>
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<tr>
<td>Electrical machinery</td>
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<td>.501</td>
<td>.557</td>
<td>.039</td>
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<tr>
<td>Transportation equipment</td>
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</table>
### Table 2. Regressions of $\log \frac{V}{L}$ on $\log W$ and $\log \frac{K_F}{L}$

<table>
<thead>
<tr>
<th>Industry</th>
<th>Number of observations (states)</th>
<th>$b$</th>
<th>$c$</th>
<th>$S_b$</th>
<th>$S_c$</th>
<th>$t_c$</th>
<th>$R^2$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food and kindred products</td>
<td>35</td>
<td>0.930</td>
<td>0.715</td>
<td>0.408</td>
<td>0.220</td>
<td>3.247**</td>
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<tr>
<td>Textile mill products</td>
<td>18</td>
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<td>0.178</td>
<td>0.274</td>
<td>0.127</td>
<td>1.402</td>
<td>0.707</td>
<td>3.305</td>
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<td>0.377</td>
<td>1.573</td>
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<td>0.569</td>
<td>0.947</td>
<td>1.610</td>
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<td>2.758**</td>
<td>0.246</td>
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<td>1.996</td>
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<td>1.274</td>
<td>0.129</td>
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**indicates $P \leq 0.01$.

*indicates $P \leq 0.05$. 
Table 3. Regressions of $\log \frac{V}{L}$ on $\log W$ and $\log \frac{Kg'}{L}$

<table>
<thead>
<tr>
<th>Industry</th>
<th>Number of observations (states)</th>
<th>b</th>
<th>c</th>
<th>$S_b$</th>
<th>$S_c$</th>
<th>$t_c$</th>
<th>$R^2$</th>
<th>c</th>
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<tr>
<td>Food and kindred products</td>
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<td>.397</td>
<td>.199</td>
<td>3.083**</td>
<td>.509</td>
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<td>.946</td>
<td>1.495</td>
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<td>.304</td>
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<td>.298</td>
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**indicates P < 0.01.
* indicates P < 0.05.
Table 4. Regressions of $\log \frac{V}{L}$ on $\log W$ and $\log \frac{Kn}{L}$

<table>
<thead>
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**indicates $P < 0.01$.  
*indicates $P \leq 0.05$. 
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Table 6. Regressions of \( \log \frac{V}{L_p} \) on \( \log W_p \) and \( \log \frac{K_q}{L_p} \)

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**indicates \( P < 0.01 \).

*indicates \( P < 0.05 \).
Table 7. Regressions of \( \log \frac{V}{L_P} \) on \( \log W_p \) and \( \log \frac{Kg}{L_P} \)

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*indicates \( P \leq 0.05 \).
Table 8. Regressions of \( \log \frac{V}{L_p} \) on \( \log W_p \) and \( \log \frac{Kn}{L_p} \)

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</tr>
<tr>
<td>Fabricated metal products</td>
<td>32</td>
<td>1.734</td>
<td>.376</td>
<td>2.881</td>
<td>.336</td>
<td>1.119</td>
<td>.069</td>
<td>5.819</td>
</tr>
<tr>
<td>Machinery except electrical</td>
<td>25</td>
<td>.843</td>
<td>.653</td>
<td>.432</td>
<td>.143</td>
<td>4.549**</td>
<td>.542</td>
<td>9.065</td>
</tr>
<tr>
<td>Electrical machinery</td>
<td>22</td>
<td>1.875</td>
<td>1.034</td>
<td>.599</td>
<td>.312</td>
<td>3.316**</td>
<td>.556</td>
<td>-2.967</td>
</tr>
<tr>
<td>Transportation equipment</td>
<td>26</td>
<td>1.379</td>
<td>1.062</td>
<td>1.433</td>
<td>.449</td>
<td>2.363*</td>
<td>.328</td>
<td>-5.75</td>
</tr>
<tr>
<td>Instruments and related</td>
<td>12</td>
<td>2.149</td>
<td>1.061</td>
<td>.680</td>
<td>.489</td>
<td>2.167*</td>
<td>.790</td>
<td>-2.908</td>
</tr>
</tbody>
</table>

**indicates \( P < 0.01 \).
*indicates \( P \leq 0.05 \).
IV. ESTIMATION OF THE PARAMETERS
OF THE VES FUNCTION

The VES functions, like the CES functions, are nonlinear in parameters, and thus a simple least squares procedure cannot be used to obtain the parameters.

Kmenta (28) has derived an approximation method to estimate the parameters of the CES functions, using Taylor's formula for expansion around a fixed value of \( \rho = \rho_0 \). Consider the logarithmic transform of the CES function

\[
\log V = \log \gamma - \frac{n}{\rho} f(\rho) + u,
\]

where \( f(\rho) = \log \left[ \delta K + (1-\delta) L \right] \) and \( n \) is a return to scale parameter. Expanding \( f(\rho) \) by Taylor's formula around zero gives

\[
f(\rho) = -\rho \left[ \delta \log K + (1-\delta) \log L \right] \\
+ \frac{1}{2} \rho^2 \delta (1-\delta) \left[ \log K - \log L \right]^2 \\
- \frac{1}{6} \rho^3 \delta (1-\delta)(1-2\delta) \left[ \log K - \log L \right]^3 + ...
\]

If the true value of \( \rho \) is close to zero, the terms of the third and higher order may be disregarded. Substituting (4.2) into (4.1) leads to

\[
\log V = \log \gamma - n \delta \log K + n(1-\delta) \log L \\
- \frac{1}{2} \rho n \delta (1-\delta) \left[ \log K - \log L \right]^2.
\]

The error of approximating the CES function by (4.3) depends on the extent to which \( \rho \) departs from zero, on the ratio of the two inputs, and on the values of the remaining parameters.
Kmenta has indicated that the error of approximation is relatively small for $\rho<1$.

Using the approximation equation (4.3), the parameters of the CES function can easily be estimated by the least squares method applied to

$$\log V = a_0 + a_1 \log K + a_2 \log L + a_3 (\log K/L)^2 + u.$$  

Given the usual assumptions about the disturbance term $u$, consistent estimates of the CES function can be obtained from the least squares estimates $a_r$ ($r = 0, 1, 2, 3$) as follows:

$$\log \hat{y} = \hat{a}_0,$$

$$\hat{n} = \hat{a}_1 + \hat{a}_2,$$

$$\hat{\delta} = \hat{a}_1 / (\hat{a}_1 + \hat{a}_2),$$

$$\hat{\beta} = -2\hat{a}_3 (\hat{a}_1 + \hat{a}_2) / (\hat{a}_1 \hat{a}_2).$$

But the addition of the parameter $c$ in the VES function makes the application of this method impossible. Attempts have been made to linearize (3.33) by expanding around $(\rho, c) = (0, 0)$, using Taylor's formula, but there are more parameters in the approximation equation than the numbers of parameters in the original equation. The restricted least squares method is also difficult to use, since the relationship between the new parameters is nonlinear. Expanding (3.30) around $(\rho, c) = (0, 0)$ results in less parameters in the approximation equation than the numbers of parameters in the original equation.

There are, however, several alternative methods of fitting the VES function.
A. A First Method of Estimation

A method specifying side conditions is most widely used in fitting CES functions. SMAC (2) use the marginal product and marginal rate of substitution side relations, Minasian (39) and McKinnon (38) use the marginal product side relation, and Brown and de Cani (8) use the marginal rate of substitution side relation.

Under perfect competition the profit maximization condition equates the value of the marginal product of labor with the wage rate, i.e.,

\[ P \cdot \frac{\partial V}{\partial L} = W. \]

Since the output is measured by value added, its price (P) is unity. From (3.19) we have

\[ (4.5) \quad \frac{\partial V}{\partial L} = \alpha X^C Y^B = W. \]

Solving for \( Y \) yields

\[ (4.6) \quad Y = \alpha^{-b} W^b X^c. \]

By substituting \( \alpha = (1-\delta) y^\gamma \) into (4.6) we can derive

\[ \frac{W L}{V} = (1-\delta)^b y^{-\rho} W l-b X^{-c}. \]

Assume that technical change is neutral and proceeds at a constant geometric rate so that \( y(t) = y_0 e^{\lambda t} \). Then, we can obtain\(^7\)

\[ (4.7) \quad \log \left( \frac{W L}{V} \right) = [b \log (1-\delta) + (b-1) \log y_0] \]

\(^7\)All logs refer to natural logarithms.
\[ + (1-b) \log W + \lambda (b-1) t - c \log \frac{K}{L} \]

or

\[ (4.8) \log \left( \frac{WL}{V} \right) = \beta_0 + \beta_1 \log W + \beta_2 t + \beta_3 \log \frac{K}{L} \]

where

\[ (4.9) \beta_0 = b \log (1-\delta) + (b-1) \log \gamma_0 \]

\[ \beta_1 = (1-b) \]

\[ \beta_2 = \lambda (b-1) \]

\[ \beta_3 = -c \]

Given the observations of labor's share relative total value added \( \left( \frac{WL}{V} \right) \), the wage rate, and the capital/labor ratio at each time period, the parameters \( \beta \)'s can be estimated by use of the least squares method. From these estimates of \( \beta \)'s, it is easy to solve for estimates of \( b \), \( \lambda \), and \( c \), thus:

\[ \hat{b} = 1 - \hat{\beta}_1 \]

\[ \hat{\lambda} = - \frac{\hat{\beta}_2}{\hat{\beta}_1} \]

\[ \hat{c} = - \hat{\beta}_3 \]

But there is only one relation (4.9) between \( \gamma_0 \) and \( \delta \). Therefore, we use the following relations to obtain separate estimates of these parameters.

From (4.9) we obtain

\[ \frac{\beta_0}{b} = \log (1-\delta) + \frac{b-1}{b} \log \gamma_0 \]

or

\[ \frac{\beta_0}{b} = \log \left( \frac{1-\delta}{\gamma_0^\rho} \right) \]
Let
\[(4.10) \frac{1}{q} = \frac{1-\delta}{\gamma_0^\rho} = \text{antilog} \frac{\beta_0}{\beta} = e^{\frac{\beta_0}{\beta}}.\]

Substituting \(\gamma(t) = \gamma_0 e^{\lambda t}\)
into (3.16) gives
\[y = \gamma_0 e^{\lambda t} \frac{c}{\beta} \frac{1}{\delta x^{-\rho}} + \frac{1}{\rho}.\]

or
\[y^{-\rho} \gamma_0^\rho e^{\lambda \rho t} = (1-\delta) \eta x^{-\rho}.\]

Dividing through both sides of the equation by \((1-\delta)x^{-\rho}\) yields
\[(4.11) \frac{\delta}{1-\delta} = \frac{x}{\gamma_0^\rho} \gamma_0^\rho e^{\lambda \rho t} - \frac{c}{\beta} + \rho\]

or
\[\frac{\delta}{1-\delta} = q \frac{1}{\gamma_0^\rho} \frac{1}{\beta} 1-\delta \frac{1}{\rho} \frac{1}{\lambda (1-b) t} - \frac{1-b-c}{b} \frac{1}{\eta (\frac{x}{\rho})}.\]

We may estimate \(\delta\) from (4.11) by using the average values of \(\frac{\delta}{1-\delta}\). The value of \(\gamma_0\), then, can be computed from (4.10) by the relation
\[(4.12) \gamma_0 = \left[q(1-\delta)\right]^\rho.\]

An alternative method of obtaining the estimates of \(\delta\) and \(\gamma_0\) is by using the estimates of \(b\) and \(c\) to fit a second regression. From (3.11) we have
\[(4.13) \frac{V}{\rho} \frac{1-b}{b} = \beta + \alpha \eta X\]

or
\[(4.14) z_1 = \beta + \beta_1 z_2.\]

The new variables \(z_1 = \frac{V}{\rho} 1 - \frac{1}{b}\)
and \(z_2 = X \frac{1-b-c}{b}\)
can be computed by using the values of b and c estimated by the first regression. A second regression is fitted to (4.14), and \( \alpha \) and \( \beta \) can be solved from \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \); thus
\[
\hat{\alpha} = \frac{\hat{\beta}_1}{\eta} \\
\hat{\beta} = \hat{\beta}_0
\]
The parameter \( \delta \), then, can be obtained from \( \alpha \) and \( \beta \) by the relation
\[
\hat{\delta} = \frac{1}{1 + \hat{\delta}/\hat{\beta}}
\]
Substituting the value of \( \hat{\delta} \) into (4.12) gives the estimates of \( \gamma_0 \).

The weaknesses of this estimation method is the specification of the side relations, which imposes additional assumptions on the estimation. In addition, the independent variable, the wage rate, enters into the dependent variable. If there are errors in the measurement of the wage rate, the wage rate will be correlated with the error term of the equation. Hence, unbiased and consistent estimates of \( \beta \)'s can not be estimated by the classical least squares method.

B. A Second Method of Estimation

This method is derived from the original function, based on which the VES function is derived. By substituting
\[
a = a^{-b} \\
= (1-\delta)^{-b} \gamma_0^{1-b} e^{\lambda(1-b)t}
\]
into (3.1) we obtain

\[(4.15) \quad \log \frac{K}{L} = -b[\log (1-\delta) - \rho \log \gamma_0] + \lambda(1-b) t + b \log W + c \log X.\]

Fitting regression to

\[
\log \frac{V}{L} = \beta_0 + \beta_1 t + \beta_2 \log W + \beta_3 \log \frac{K}{L}
\]

gives the following estimation of the parameters:

\[
\hat{\lambda} = \frac{\hat{\beta}_1}{1-\hat{\beta}_2} \\
\hat{b} = \hat{\beta}_2 \\
\hat{c} = \hat{\beta}_3.
\]

Again, there is one relation between \(\delta\) and \(\gamma_0\). Following the same procedure as in the first method, we get

\[(4.16) \quad q = \gamma_0^\rho \frac{\beta_0}{1-\delta} = e^{\frac{\beta_0}{1-\delta}}.\]

Substituting (4.16) into (4.11) yields the estimate of \(\delta\).

The value of \(\gamma_0\) can be obtained from (4.12).

Given the estimates of \(b\) and \(c\), we may also follow the same procedure as in the first method to obtain the remaining parameters \(\delta\) and \(\gamma_0\) by fitting a regression to (4.14).

C. A Third Method of Estimation

The third method of fitting the VES function is that of using the method of successive approximation, for which computer programs are now available [e.g., Atkinson (3)]. This method estimates the parameters of the production function directly,
i.e., without the specification and estimation of the side relations.

Hartley (19) has modified the well known 'Gauss-Newton' method of iterative solution. His method, while sharing the advantages of the Gauss-Newton method, has an additional merit of guaranteed convergence. It requires a starting value for each parameter to be estimated. These starting values, which may be obtained from other estimation methods such as the first and second methods, are then improved through minimization of the residual sum of squares.

Consider the VES function (3.13)

\[ Y = [\theta_1 X^2 + \theta_3 X^4]^{1/4} \]

which can be rewritten in a general form as

\[ (4.17) \quad Y = f(X; \theta) = f(X; \theta_1, \theta_2, \theta_3, \theta_4) \]

where

\[ \begin{align*}
\theta_1 &= \alpha, \\
\theta_2 &= \frac{c}{b}, \\
\theta_3 &= \beta, \\
\theta_4 &= \frac{b-1}{b}.
\end{align*} \]

Given \( n \) sets of observed values of \((Y_h, X_h)\), the classical least squares problem is to minimize

\[ (4.18) \quad Q(\theta) = \sum_{h=1}^{n} (Y_h - f(X_h; \theta))^2. \]

Linearizing \( f(X_h; \theta) \) by use of a multiple first-order Taylor expansion at \( \theta = \theta_0 \) results in
(4.19) \( f(x^n; \theta) = f(x^n; \theta_0) + \sum_{i=1}^{4} f_i(x^n; \theta_0)(\theta_i - \theta_{i0}) \)

where the zero subscript indicates the starting value, and
\( \theta_i - \theta_{i0} \) is the familiar Newton coefficient \( D_i \). Thus, \( Q(\theta) \) can be expressed as a linear function of the unknown Newton coefficients. By substituting (4.19) into (4.18), we have

(4.20) \( Q(\theta) = \sum_{h=1}^{n} [Y_h - f(x_h; \theta_0) - \sum_{i=1}^{4} f_i(x_h; \theta_0)D_i]^2 \).

To minimize the sum of squares \( Q(\theta) \), take the partial derivatives of \( Q(\theta) \) with respect to \( D_i \) (i=1,2,3,4),

\( \frac{\partial Q(\theta)}{\partial D_i} = -2\sum_{h} [Y_h - f(x_h; \theta_0) - \sum_{i=1}^{4} f_i(x_h; \theta_0)D_i] f_i \).

Equating to zero gives the following system of normal equations:

\[
\begin{bmatrix}
\Sigma f_1^2 & \Sigma f_1 f_2 & \cdots & \Sigma f_1 f_n \\
\Sigma f_2 f_1 & \Sigma f_2^2 & \cdots & \Sigma f_2 f_n \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma f_m f_1 & \Sigma f_m f_2 & \cdots & \Sigma f_m f_n \\
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2 \\
\vdots \\
D_n \\
\end{bmatrix}
=
\begin{bmatrix}
\Sigma [Y_h - f(x_h; \theta_0)] f_1 \\
\Sigma [Y_h - f(x_h; \theta_0)] f_2 \\
\vdots \\
\Sigma [Y_h - f(x_h; \theta_0)] f_m \\
\end{bmatrix}.
\]

By assumption the matrix \( (\Sigma f_i f_j) \) is non-singular and thus \( D \) can be solved by inverting the matrix \( (\Sigma f_i f_j) \). At this point Hartley has modified the Gauss-Newton procedure. Let

\( \theta_{i1} = \theta_{i0} + vD_i \), for \( 0 < v < 1 \).

Then, \( Q \) is also a function of \( v \), i.e.,

\( Q(v) = Q(X; \theta + v D) \).

Find the value of \( v \) for which \( Q \) is a minimum on the interval
0 \leq v \leq 1 and denote it by \( v' \). Replacing \( \theta_{i0} \) in (4.20) by

\[
\theta_{i1} = \theta_{i0} + v'D_i
\]

we have

\[
Q(X; \theta) \leq Q(X; \theta_0).
\]

The above procedure is repeated until the value of the estimated parameter \( \theta_{ij} \) converges to its true value.
V. TECHNICAL CHANGE

So far we have discussed the properties of the VES production function under the assumption that technique is given and unchanging over the period of investigation. That is, given the state of art, as one or both inputs of the factor of production are varied, productivity will behave accordingly. We will, now, proceed to investigate the effect of technical change on productivity if the underlying production function is of the VES form.

A. Definition and Classification of Technical Change

A flow of new technique leads to continuous changes in the production function. The shifts in the production function are defined by Solow (57, p. 312) as technical change. Technical progress enables a given level of output to be produced with less inputs or a given combination of inputs to produce a greater level of output. In general there are two types of technical change: neutral and non-neutral. Hicks (21, pp. 121-124) defines a neutral technical change as a pure scale change, leaving the marginal rates of substitution unchanged at given capital/labor ratios. Shifts in the production function are defined as non-neutral if they alter the marginal rates of substitution at given capital/labor ratios. The change can be either labor-saving or capital-saving. If the shifts in the pro-
duction function are such that the marginal product of capital rises relative to the marginal product of labor for each combination of capital and labor, the change is labor-saving, because the increase in the marginal product of capital will motivate the entrepreneur to use more capital than labor. If technical change increases the marginal product of labor relative to that of capital at given capital/labor ratios, labor-saving technical change occurs.

Salter (50, pp. 31-33) criticizes that Hick's definitions do not, and cannot, provide any guide to the effects of technical change on productivity, and he proposes another set of definitions. He defines technical change as being neutral if the capital/labor ratios remains unchanged when relative factor prices are constant. If technical change increases the capital/labor ratios at constant relative factor prices, the change is labor-saving. On the other hand, if the shifts in the production function are such that the capital/labor ratios decline at given relative factor prices, capital-saving technical change occurs.

Since the marginal rate of substitution is equated with the factor price ratio at the equilibrium under perfect competition, these two definitions are equivalent.

Graphically, technical change can be viewed as the movement of a series of successive dated production functions toward the origin as shown in Figure 4. In appearance the produc-
Figure 4. Neutral technical change
Figure 5. Capital-saving technical change
Figure 6. Labor-saving technical change
tion functions look like a family of isoquants in production theory. But each production function represents the same level of output attainable by different levels of inputs. Each new production function is superior to its predecessor in the sense that less of one or more inputs is required to produce a given level of output while other inputs remain unchanged.

Figures 5 and 6 are constructed with the same notation as Figure 4. Consider the production of only two periods. Before technical change occurs, the equilibrium combination of capital and labor is $P_1$, where the marginal rate of substitution of labor for capital is equated with its price ratio. Technical change causes the production function to move from $V_1$ to $V_2$, where the new equilibrium point $P_2$ is attained when the factor price ratio is constant. Figure 4 shows that the marginal rate of substitution remains the same at given capital/labor ratios after technical change. The new equilibrium point $P_2$ falls on the line $oP_1$ which connects $P_1$ and the origin. It represents a neutral technical change in the sense that both factors are saved in the same proportion.

Figures 5 and 6 show that the new equilibrium points do not fall on the line $oP_1$, illustrating that one of the factors of production is saved more than the other. In Figure 5 technical change increases the marginal rate of substitution of labor for capital, i.e., the marginal product of labor increases relative to that of capital. The entrepreneur, then,
has an incentive to use more labor than capital. Therefore, the change is capital-saving, and the new equilibrium point $p_2$ is shown on the right of $o_1$. It is apparent from the figure that more capital is saved than labor. Using the same reasoning, Figure 6 represents labor-saving technical change.

B. The Measurement of Technical Change

Many attempts have been made to segregate the portion of total increase in output attributed to the growth in the capital stock from that attributed to technical change. A quantitative estimate of the relative importance of these two forces is of important to decision-makers in order to allocate resources between improving technique and expanding capital stock.

The first important recent paper attempting to measure technical change has been presented by Solow (57). He has tried to estimate technical change from 1909 to 1947 for the private, non-farm sector, using the equation

\[
\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - w_k \frac{\Delta X}{X}
\]

where $A$ measures the effect of shifts in the production function, $Y$ refers to output per unit of labor, $X$ denotes the capital/labor ratio, and

\[
w_k = \frac{\partial Y}{\partial K} \cdot \frac{K}{V}
\]

which is capital's share relative to total output. He has concluded that technical change is neutral on the average and
that about 87-1/2 percent of the increase in gross output per man-hour is attributable to technical change, and the remaining 12-1/2 percent to increased use of capital.

Massell (32) has used Solow's method to estimate the proportion of total increase in output due to increases in capital input per man-hour and that due to technical change in the manufacturing sector. He has estimated that 90 percent of the increase in output per man-hour is attributable to technical change and given greatest importance to technical change. He says that "In view of these findings, policymakers may wish to concern themselves more with the variables which govern the rate at which innovations are injected into the economic system than with the variables which determine the rate at which additions are made to the capital stock. Such issues as expenditure by business on research and the policies of firms regarding the replacement of obsolescent equipment will perhaps be deemed more important than the rate of net investment," (32, p. 188) and concludes that "The present paper offers evidence to support the view that technological change is of overriding importance in bringing about increased labor productivity over time and that there is a need for economists to shift emphasis from the theory of capital to the theory of technical progress, as an explanation of the growth in aggregate output." (32, p. 187) These papers represent a general view held by many researchers in the late 1950's that
the larger part of the observed increase in output per head is a consequence of technical progress rather than of increased capital per head.

In his second paper, Solow has gone some way toward attributing greater importance back to capital. He says that "Improvements in technology affect output only to the extent that they are carried into practice either by net capital formation or by the replacement of old-fashioned equipment by the latest models, with a consequent shift in the distribution of equipment by date of birth." (54, p. 91) That is, technical progress occurs only when capital used embodies the new technology. Hence, the capital formation is important in order to make use of new methods.

Several different methods have been used to estimate the effects of different types of technical change on output. One group of economists, e.g., Schultz (53), Mincer (40), Becker (4) and others, reason that labor and physical inputs are not the only factors which determine output. They argue, in addition that the growth of output also depends upon various types of investments in human beings, i.e., investments in human capital. Such investments may take the form of expenditures for such things as health programs, formal education, or on-the-job training. Denison has used a simple approach to estimate the contribution of increased inputs, including education, on the growth rate of real national income. He treats
a change in the average quality of labor in exactly the same way as an increase in its quantity. It is here education comes into his study. He concludes that "From 1929 to 1957 the amount of education the average worker had received was increasing almost 2 percent a year, and this was raising the average quality of labor by 0.97 percent a year, and contributing 0.67 percentage points to the growth rate of real national income.» (11, p. 127)

Other group of economists, e.g., Solow (55), Massell (33), Johansen (25), Nelson (43), and McCarthy (34,35), retain the two-factor production function framework and make use of the assumption that improvements which are embodied in physical equipment are "quantity augmenting." That is, improvements in the quality of capital can be treated as increases in the quantity of capital of a given quality. The increased efficiency due to such improvements is referred to as "embodied technical progress." One of the first of the quantity augmenting models is developed by Solow (55). He uses the Cobb-Douglas function and assumes that all technical progress takes place through improvements which are embodied in new physical equipment. He does not consider disembodied technical progress, which includes such things as improvements in the quality of the labor force and reorganization of the existing and unchanged factors of production. McCarthy (34) incorporates both embodied and disembodied technical change in the Cobb-
Douglas and CES models, and attempts to distinguish between the effects of embodied and disembodied technical progress on output. In such a model, according to McCarthy, embodied technical progress is in general non-neutral, and embodied as well as disembodied technical progress results in a neutral shift in the aggregate production function.

Brown (6) uses an epochal method to measure a non-neutral technical change, based on the hypothesis that technical change is occurring in a discrete manner. Brown defines a technological epoch as a period of time within which the short- and long-run production functions are stable. In other words, during the technological epoch only a neutral technical change takes place. These epochs are distinguished by a uniform technique where there are non-neutral shifts of the production function from one to another. During the end of an epoch the new technology is gradually going into use just as, during the first part of the next epoch, the old technology is gradually going out of use.

To search for epochs the Cobb-Douglas function and the expansion path function of the CES function are fitted to an arbitrarily small period of time, say $n$ years, at the beginning of the overall period. This gives one set of estimates. Another arbitrarily small period of time, say $m$ years, continuous to the first set of observations is used for a second fit of the equations, yielding a second set of esti-
mates. A third set of estimates is obtained by fitting the combined observation n+m of the two continuous time periods. An F test is used to examine whether a structure change has taken place between the first and second periods. If none has occurred the procedure is repeated on successive continuous time period. If there is a significant change in the parameters, a structure break is uncovered and the procedure is repeated, beginning with the observation right after the structure break.

C. Technical Change in the VES Model

Technical change shifts aggregate production functions, and shifts in aggregate production functions can be viewed as changes in the parameters of the production functions. There are four parameters in the VES production function: \( \gamma, \delta, b, \) and \( c \). We will proceed to investigate the nature of technical change in terms of changes in these parameters.

An increase in the parameter \( \gamma \) augments output but does not alter the marginal rate of substitution of labor for capital. It corresponds to a neutral technical change and is graphically represented by the movement of successive production functions toward the origin. In an industry where technical change is rapid there will be large differences in the position of consecutive production functions, and in a technically stagnant industry the production functions will be bunched together.
From (3.32) we have

\[
(5.2) \quad R = \frac{1}{\delta^{-\frac{c-1}{b}} \frac{X}{1-\delta^X} + \frac{c}{(1-b-c)X}}
\]

or

\[
R = f(X; \delta, b, c).
\]

At given capital/labor ratios the marginal rate of substitution varies with changes in \(\delta, b, \) and \(c\). Hence, a non-neutral technical change is associated with changes in these parameters. Whether a change in a parameter is capital-saving, neutral, or labor-saving depends upon the partial derivative of \(R\) with respect to the parameter, as being greater, equal to, or less than zero. If the net effect of changes in these parameters is such that the marginal rate of substitution falls, remains constant, or rises, the underlying technical change is labor-saving, neutral, or capital-saving.

Suppose that a non-neutral technical change occurs such that only \(\delta\) is increased, then taking the partial derivative of \(R\) in (5.2) with respect to \(\delta\) yields

\[
\frac{\partial R}{\partial \delta} = -\frac{\frac{c-1}{b} \frac{X}{D^2(1-\delta)^2}}
\]

where

\[
D = \delta^{-\frac{c-1}{b}} \frac{X}{1-\delta^X} + \frac{c}{(1-b-c)X}.
\]

The marginal product of capital rises relative to that of labor; therefore, the change is labor-saving.

The other sources of a non-neutral technical change are
changes in the parameters $b$ and $c$. By taking the partial derivative of $R$ in (5.2) with respect to $b$, we have

$$\frac{\partial R}{\partial b} = -\frac{1}{b^2} \left[ \frac{\delta}{1-\delta} X^b (\log X)^{\frac{1-c}{b^2}} + \frac{c}{(1-b-c)^2 X} \right]$$

which is less than zero if $\log X$ is greater than zero. In other words, if the capital stock grows faster than the labor force, the non-neutral technical change which increases the value of the parameter $b$ is labor-saving. If $\log X$ is less than zero, the change may or may not be capital-saving, depending upon the relative magnitude of the two terms in the bracket.

The partial derivative of $R$ with respect to $c$ is

$$\frac{\partial R}{\partial c} = -\frac{1}{b^2} \left[ \frac{\delta}{1-\delta} X^b (\log X)^{-\frac{1-b}{b^2}} \right]$$

which is less than zero if $\log X$ is greater than zero and $b$ is less than unity. In this case, the non-neutral technical change which increases the value of $c$ is labor-saving. If $\log X$ is less than zero and $b$ is greater than unity, the change is capital-saving.

From the above analysis we may conclude that a non-neutral technical change which increases the value of $\delta$ is always labor-saving. If $\log X$ is greater than zero and $b$ is less than unity, 

---

8 The inequality $\log X > 0$ implies that $\log K - \log L > 0$. The derivative of $\log X$ with respect to time $(t)$ gives $\dot{X}/X = \dot{L}/L > 0$. Therefore, $\log X > 0$ implies that capital grows faster than labor.
a non-neutral technical change which increases the value of b and/or c is always labor-saving. This proposition can be reasoned as follows by noting that increases in b and c imply a rise in the elasticity of substitution and that a rise in the elasticity of substitution implies that it is easier to substitute capital for labor at given capital/labor ratios. In other words, the non-neutral technical change permits a larger amount of capital to be substituted for labor. If capital grows faster than labor, capital will be substituted for labor at the margin in the production process. Therefore, in this case, the rise in b and c is capital-using, i.e., labor-saving.

D. Empirical Investigation of Technical Change in the Food and Kindred Products Industries

1. The data

We will now apply the VES production function to measure technical change in this section and to study the behavior of relative factor shares in the last section of the next chapter. For these purposes seven three-digit U.S. Food and Kindred Products Industries, Industries 201 to 207, were included in this study. Industry 208 was excluded due to lack of capital data.

In the empirical specification of production functions the collection of data is an important part of empirical work.
The fitted production functions can be correct to the extent that the data behind it reliably represents the relationship between inputs and output. Incomplete data will result in incomplete implications drawn from the fitted production functions.

Construction of a series of data for fitting aggregate production functions is a difficult and intricate task, because the source of data which we rely on is always less than desired for our purpose. It is necessary to make adjustments in the data which can with reasonable confidence be expected to improve them for the problem being investigated. But often the costs of adjustment may be too high relative to the benefits derived.

Data concerning value added, payrolls, and employees were obtained from the Census of Manufactures (61) and the Annual Survey of Manufactures (60). Capital data were obtained from Appendix A of Stigler's Capital and Rates of Return in Manufacturing Industries (58) and Source Book of Statistics of Income (62). The variables included in this study were as follows:

a. \( V \): Value added Value added by manufacture is defined by the United States of Commerce as the value of shipment minus the cost of materials, supplies, fuel, electric energy, cost of resales and miscellaneous receipts.

b. \( L \): All employees The term all employees, according
to the United States Department of Commerce, consists of all full-time and part-time employees on the payrolls of operating manufacturing establishments who worked or received pay for any part of the pay period which included the 12th and ended nearest the 15th of the month specified on the report form. The employees of central offices and auxiliaries and officers of corporations are included, but proprietors and partners of unincorporated firms are excluded.

c. $L_p$: Man-hour of production workers

Man-hours of production workers comprises all plant man-hours of production and related workers. It represents all man-hours worked or paid for at the plant including actual overtime hours, excluding hours paid for vacations, holidays, or sick leaves.

d. $W$: The average wage rate for all employees

The average wage rate for all employees is obtained by dividing payrolls by the numbers of all employees, where payrolls includes the gross earnings paid to all employees on the payroll of manufacturing establishments. Payrolls include salaries of officers of corporations but exclude payments to the proprietors and partners.

e. $W_p$: The average wage rate for production workers

The average wage rate for production workers is derived by dividing the total wage bills by the total man-hours of production workers.

f. $K$: The capital input

The capital input is measured by book value of total assets as defined in Source Book of
Statistics of Income by the Internal Revenue Service. The main problem encountered in using the capital data is that the capital figures reflect changing price levels. It would be more desirable that the capital data be deflated to approximate capital in stable prices. Stigler (58, pp. 107-202) has calculated a series of capital values in 1947 prices from 1938 to 1957 for two-digit manufacturing industries. But no sufficient data are available for the calculation of capital values in constant prices for three-digit industries. Therefore, the capital data in book values were used in this study.

2. Methods and procedures

To estimate the parameters of the VES production functions the above two sets of data, i.e., total employees data and production workers data, were used to fit the equations (4.8) and (4.15), which are the variation forms of the VES function. All together the following four regressions were fitted for each industry:

a. \[ \log \frac{V}{L} = \beta_0 + \beta_1 t + \beta_2 \log W + \beta_3 \log \frac{K}{L} + e \]

b. \[ \log \frac{V}{L_p} = \beta_0 + \beta_1 t + \beta_2 \log W_p + \beta_3 \log \frac{K}{L_p} + e \]

c. \[ \log \frac{W}{L/V} = \beta_0 + \beta_1 t + \beta_2 \log W + \beta_3 \log \frac{K}{L} + e \]

d. \[ \log \frac{W_p}{L_p/V} = \beta_0 + \beta_1 t + \beta_2 \log W_p + \beta_3 \log \frac{K}{L_p} + e \]

where all logs refer to natural logarithms.

It can be shown by algebraic transformations that (4.8) and (4.15) are identically the same. Therefore, the estimated parameters of the VES functions, including their standard errors,
of the first two regressions were exactly the same as those of the last two regressions as we expected, but the $R^2$'s were quite different. The $R^2$'s of the first two regressions were very high, ranging from 0.951 to 0.997, whereas those of the last two regressions were relatively low, ranging from 0.383 to 0.927. However, they are equally good in term of fitting the VES functions, since they give identically the same residual sums of squares.

The estimated values of the parameters of the VES functions are shown in Tables 9 and 10. The figure in the parentheses denotes the standard error of the estimate above it. The code numbers refer to the Standard Industrial Classification (SIC) numbers used by the U.S. Department of Commerce. The corresponding industry names are as follows:

<table>
<thead>
<tr>
<th>Code Number</th>
<th>Industry Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>201</td>
<td>Meat products</td>
</tr>
<tr>
<td>202</td>
<td>Dairies</td>
</tr>
<tr>
<td>203</td>
<td>Canned and frozen foods</td>
</tr>
<tr>
<td>204</td>
<td>Grain mills</td>
</tr>
<tr>
<td>205</td>
<td>Bakery products</td>
</tr>
<tr>
<td>206</td>
<td>Sugar</td>
</tr>
<tr>
<td>207</td>
<td>Candy and related products</td>
</tr>
</tbody>
</table>

The parameters $\delta$ and $\gamma_0$ based on total employees data were estimated by (4.11) and (4.12). They are shown in the fifth and the sixth columns of Table 9. But no reasonable estimates based
Table 9. Regressions of $\frac{V}{L}$ on $t$, $W$, and $\frac{K}{L}$

<table>
<thead>
<tr>
<th>Industry</th>
<th>No. of Observations</th>
<th>$b$</th>
<th>$c$</th>
<th>$\lambda$</th>
<th>$\gamma_0$</th>
<th>$\delta$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>201</td>
<td>18</td>
<td>0.7972</td>
<td>0.0028</td>
<td>0.0601</td>
<td>1.5113</td>
<td>0.5376</td>
<td>0.9883</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.2255)</td>
<td>(0.1928)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>202</td>
<td>18</td>
<td>0.5805</td>
<td>0.1326</td>
<td>0.336</td>
<td>1.3824</td>
<td>0.7538</td>
<td>0.9815</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.3108)</td>
<td>(0.0352)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>203</td>
<td>18</td>
<td>0.7802</td>
<td>0.0677</td>
<td>0.1126</td>
<td>1.0662</td>
<td>0.6001</td>
<td>0.9843</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.4707)</td>
<td>(0.2277)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>204</td>
<td>18</td>
<td>0.4011</td>
<td>0.6078</td>
<td>0.0032</td>
<td>0.0843</td>
<td>0.9851</td>
<td>0.9900</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.2014)</td>
<td>(0.1309)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>205</td>
<td>18</td>
<td>0.5740</td>
<td>-0.0196</td>
<td>0.0619</td>
<td>1.5391</td>
<td>0.7589</td>
<td>0.9971</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.2112)</td>
<td>(0.0795)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>206</td>
<td>18</td>
<td>0.9837</td>
<td>0.9097</td>
<td>-0.8859</td>
<td>-</td>
<td>0.0917</td>
<td>0.9509</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.6259)</td>
<td>(0.5090)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>207</td>
<td>18</td>
<td>-1.2986</td>
<td>0.3603</td>
<td>0.0373</td>
<td>2.6141</td>
<td>0.1021</td>
<td>0.9718</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.7869)</td>
<td>(0.2026)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10. Regressions of $\frac{V}{L_p}$ on $t$, $W_p$, and $\frac{K}{L_p}$

<table>
<thead>
<tr>
<th>Industry</th>
<th>No. of Observations</th>
<th>$b$</th>
<th>$c$</th>
<th>$\lambda$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>201</td>
<td>18</td>
<td>0.8726</td>
<td>0.0038</td>
<td>0.0790</td>
<td>0.9907</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1689)</td>
<td>(0.1687)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>202</td>
<td>18</td>
<td>1.0221</td>
<td>-0.0298</td>
<td>-1.2101</td>
<td>0.9929</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0539)</td>
<td>(0.0340)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>203</td>
<td>18</td>
<td>0.3803</td>
<td>0.2188</td>
<td>0.0585</td>
<td>0.9853</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.4918)</td>
<td>(0.1905)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>204</td>
<td>18</td>
<td>0.4829</td>
<td>0.4714</td>
<td>0.0184</td>
<td>0.9937</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.2328)</td>
<td>(0.1289)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>205</td>
<td>18</td>
<td>1.0431</td>
<td>-0.0692</td>
<td>-0.3358</td>
<td>0.9933</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.2214)</td>
<td>(0.1913)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>206</td>
<td>18</td>
<td>0.3412</td>
<td>0.4641</td>
<td>0.0520</td>
<td>0.9532</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.6027)</td>
<td>(0.4430)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>207</td>
<td>18</td>
<td>-1.6439</td>
<td>0.3939</td>
<td>0.0401</td>
<td>0.9880</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5362)</td>
<td>(0.1398)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
on production workers data were obtained, and, hence, they were not listed.

The elasticity of substitution for each time period was computed by (3.34) using the values of the parameters in Table 9. It can be seen from Table 11 that if \( b + c \) is less than unity, e.g., Industries 201, 202, and 203, an increase in the capital/labor ratios increases the elasticity of substitution. In Industry 204, \( b + c \) is greater than unity. Hence, the increased capital/labor ratios result in a decline in the elasticity of substitution over time.

3. **Estimation of the proportion of the increase in output per unit of labor attributable to technical change and that attributable to increased capital intensity**

We have assumed that technical change is neutral and proceeds at a geometric rate such that

\[ Y = Y_0 e^{\lambda t} \]

Under this assumption the annual rate of technical change can be obtained by taking the derivative of \( \log Y \) with respect to \( t \),

\[ \frac{\dot{Y}}{Y} = \lambda \]

which is shown in column 5 of Tables 9 and 10. In Industry 201, the annual rate of technical change estimated by the VES function is 0.0601 based on total employees data. The estimate based on production workers data is 0.079. During this period the annual average growth rate of output per unit of labor based on total employees data is 4.85 percent and that based on
Table 11. Changes in the elasticity of substitution

<table>
<thead>
<tr>
<th>Year</th>
<th>201</th>
<th>202</th>
<th>203</th>
<th>204</th>
<th>205</th>
<th>206</th>
<th>207</th>
</tr>
</thead>
<tbody>
<tr>
<td>1964</td>
<td>0.8029</td>
<td>0.7612</td>
<td>0.8932</td>
<td>2.1240</td>
<td>0.5461</td>
<td>-3.1388</td>
<td>-2.3046</td>
</tr>
<tr>
<td>1963</td>
<td>0.8029</td>
<td>0.7596</td>
<td>0.8931</td>
<td>2.2355</td>
<td>0.5474</td>
<td>-2.7479</td>
<td>-2.3069</td>
</tr>
<tr>
<td>1962</td>
<td>0.8029</td>
<td>0.7564</td>
<td>0.8932</td>
<td>2.3890</td>
<td>0.5493</td>
<td>-2.4586</td>
<td>-2.2961</td>
</tr>
<tr>
<td>1961</td>
<td>0.8028</td>
<td>0.7546</td>
<td>0.8933</td>
<td>2.6832</td>
<td>0.5497</td>
<td>-2.3937</td>
<td>-2.3052</td>
</tr>
<tr>
<td>1960</td>
<td>0.8028</td>
<td>0.7534</td>
<td>0.8931</td>
<td>2.6968</td>
<td>0.5504</td>
<td>-2.2537</td>
<td>-2.3222</td>
</tr>
<tr>
<td>1959</td>
<td>0.8028</td>
<td>0.7518</td>
<td>0.8930</td>
<td>2.7599</td>
<td>0.5507</td>
<td>-2.0387</td>
<td>-2.3452</td>
</tr>
<tr>
<td>1958</td>
<td>0.8028</td>
<td>0.7492</td>
<td>0.8924</td>
<td>2.9447</td>
<td>0.5509</td>
<td>-1.9130</td>
<td>-2.3874</td>
</tr>
<tr>
<td>1957</td>
<td>0.8027</td>
<td>0.7361</td>
<td>0.8926</td>
<td>3.5549</td>
<td>0.5515</td>
<td>-2.0114</td>
<td>-2.3070</td>
</tr>
<tr>
<td>1956</td>
<td>0.8027</td>
<td>0.7381</td>
<td>0.8926</td>
<td>4.0297</td>
<td>0.5519</td>
<td>-2.0020</td>
<td>-2.4820</td>
</tr>
<tr>
<td>1955</td>
<td>0.8027</td>
<td>0.7354</td>
<td>0.8920</td>
<td>4.7143</td>
<td>0.5516</td>
<td>-1.8549</td>
<td>-2.4885</td>
</tr>
<tr>
<td>1954</td>
<td>0.8027</td>
<td>0.7336</td>
<td>0.8919</td>
<td>5.4219</td>
<td>0.5525</td>
<td>-1.7315</td>
<td>-2.5053</td>
</tr>
<tr>
<td>1953</td>
<td>0.8027</td>
<td>0.7642</td>
<td>0.8915</td>
<td>6.5644</td>
<td>0.5523</td>
<td>-1.5074</td>
<td>-2.5101</td>
</tr>
<tr>
<td>1952</td>
<td>0.8027</td>
<td>0.7667</td>
<td>0.8913</td>
<td>11.2683</td>
<td>0.5534</td>
<td>-1.5401</td>
<td>-2.5125</td>
</tr>
<tr>
<td>1951</td>
<td>0.8027</td>
<td>0.7672</td>
<td>0.8910</td>
<td>7.9984</td>
<td>0.5534</td>
<td>-1.3753</td>
<td>-2.4905</td>
</tr>
<tr>
<td>1950</td>
<td>0.8027</td>
<td>0.7619</td>
<td>0.8908</td>
<td>12.9880</td>
<td>0.5534</td>
<td>-1.2203</td>
<td>-2.7018</td>
</tr>
<tr>
<td>1949</td>
<td>0.8026</td>
<td>0.7590</td>
<td>0.8902</td>
<td>-9.2142</td>
<td>0.5540</td>
<td>-0.9673</td>
<td>-2.7854</td>
</tr>
<tr>
<td>1948</td>
<td>0.8026</td>
<td>0.7590</td>
<td>0.8903</td>
<td>-6.1026</td>
<td>0.5542</td>
<td>-0.9718</td>
<td>-2.7941</td>
</tr>
<tr>
<td>1947</td>
<td>0.8026</td>
<td>0.7568</td>
<td>0.8897</td>
<td>37.9244</td>
<td>0.5546</td>
<td>-1.1839</td>
<td>-2.7553</td>
</tr>
</tbody>
</table>
production workers data is 5.15 percent. Apparently, the rate of technical change has been overestimated.

In Industry 202, the annual rate of technical change estimated by the VES function is 3.36 percent for total employees data. The rate based on production workers data is -121.01 percent, which is not realistic.

If we consider that the annual rate of technical change is 3.36 percent and that the technical change index in 1947 is

\[ \gamma_{47} = \gamma_{0} \]

then, in 1964 the technical change index is

\[ \gamma_{64} = \gamma_{0} (1 + 0.0336)^{17} \]

Let \( A_t \) be the ratio of \( \gamma_{64} \) to \( \gamma_{47} \), then,

\[ A_t = 1.772 \]

which is the full shift factor of the production function over 18 years. During the same period output per man-year has increased 2.1146 times from 6.4209 to 13.5799 based on total employees data, and output per man-hour based on production workers data has increased 3.7299 times from 3.7166 in 1947 to 13.8609 in 1964 as shown in Table 13. Following Solow's (57) reasoning, we may segregate the effects of technical change and increased capital intensity on increase in output per unit of labor. Dividing the 1964 output 13.5799 by the shift factor 1.772 gives 7.6779, which would be the 1964 output per man-year if the state of technique has remained con-
stant. The increase in output per man-year of 1.2570 is attributable to the increased use of capital. The ratio of 1.2570 to 7.1590, which is the total increase in output per man-year contributed jointed by technical change and the increase in capital, gives 17.36 percent of total increase in output per man-year attributable to the increase in capital and the remaining 82.64 percent to technical change. This reasoning follows directly from Solow's definition of technical change, which is considered as a residual.

Massell (32) has employed Solow's model to estimate the rate of technical change, but he has used a different method to apportion the proportion of total increase in output per unit of labor due to technical change and that due to capital formation. Since technical change has shifted the production function by a factor of 1.772, he argues that with the same capital and labor combination as in 1947, 1.772 times of 1947 output per man-year, i.e., 11.3778, can be produced in 1964. In other words, technical progress has increased the output per man-year from 6.4207 in 1947 to 11.3778 in 1964. Dividing 11.3778 by the 1964 output per man-year 13.5799 yields 83.78 percent, which indicates the proportion of the increase in output per man-year due to technical change. The remainder, which is not attributable to technical change, must be due to the increased capital intensity.

Algebraically, we may summarize Solow's and Massell's
Table 12. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

<table>
<thead>
<tr>
<th>Year</th>
<th>( \frac{W}{V} )</th>
<th>( \frac{K}{L} )</th>
<th>( W )</th>
<th>( \frac{W}{L} )</th>
<th>( \frac{V}{L} )</th>
<th>( \frac{K}{L} )</th>
<th>( W_p )</th>
<th>( \frac{V}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1964</td>
<td>0.5626</td>
<td>9.8632</td>
<td>5.7711</td>
<td>10.2580</td>
<td>0.4142</td>
<td>5.8390</td>
<td>2.5155</td>
<td>6.0727</td>
</tr>
<tr>
<td>1963</td>
<td>0.5692</td>
<td>9.8969</td>
<td>5.4769</td>
<td>9.6222</td>
<td>0.4205</td>
<td>6.1078</td>
<td>2.4971</td>
<td>5.9383</td>
</tr>
<tr>
<td>1962</td>
<td>0.5677</td>
<td>9.7524</td>
<td>5.4091</td>
<td>9.5284</td>
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Annual average rate of growth

\[-0.0041\] \[0.0248\] \[0.0426\] \[0.0485\] \[-0.0051\] \[0.0283\] \[0.0449\] \[0.0515\]

Estimated rate of changes in labor's share relative to total output

\[-0.0037\] \[-0.0045\]
Table 13. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

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<th>( \frac{V}{L} )</th>
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Annual average rate of growth

| 0.0064 | 0.0411 | 0.0511 | 0.0463 | -0.0235 | 0.1524 | 0.1481 | 0.1807 |

Estimated rate of changes in labor's share relative to total output

| 0.0019 | -0.0255 |
Table 14. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

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Annual average rate of growth
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Estimated rate of changes in labor' share relative to total output
-0.0182   -0.0200
Table 15. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

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Annual average rate of growth
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Estimated rate of changes in labor share relative to total output
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Annual average rate of growth
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Estimated rate of changes in labor's share relative to total output
-0.0070 -0.0122
Table 17. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

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<th>( \frac{K}{L} )</th>
<th>( W )</th>
<th>( \frac{V}{L} )</th>
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Annual average rate of growth

| 0.0062 | 0.0333 | 0.0514 | 0.0578     | 0.0029     | 0.0390 | 0.0532 | 0.0631 |

Estimated rate of changes in labor's share relative to total output

-0.9523

-0.0173
Table 18. Changes in labor's share, capital/labor ratios, wage rates, and labor productivity

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Annual average rate of growth: 0.0051 0.0386 0.0420 0.0418 0.0000 0.0433 0.0445 0.0525

Estimated rate of changes in labor's share relative to total output: -0.0043 -0.0054
methods as follows:

Solow's method:

\[ (5.3) \quad P_K = \frac{Y_t/A_t - Y_o}{Y_t - Y_o} \]

\[ (5.4) \quad P_T = 1 - P_K \]

Massell's method:

\[ (5.6) \quad P_T = \frac{Y_o A_t}{Y_t} \]

\[ (5.7) \quad P_K = 1 - P_T \]

where \( P_K \) denotes the proportion of total increase in output per unit of labor attributable to the increased use of capital per unit of labor and \( P_T \) that proportion attributable to technical change.

Although in the above example Massell's estimates are quite close to that of Solow, it seems that the former's method of estimation is misleading; it overestimates the proportion of the increase in output per unit of labor contributed by technical change. In (5.6) we may consider \( Y_o A_t \) as the 1964 output per man-year attainable by technical progress without changing the capital/labor ratio, but \( Y_o A_t \) includes the increased output per man-year due to technical progress and 1947 output per man-year \( (Y_o) \), which is the end product of some combination of technical change and capital formation in the past; \( Y_o A_t \) does not denote the increased output per man-year due to technical change only. The same is true for \( Y_t \), which includes \( Y_o \) and the total increased output per man-year
over the period of study. Therefore, dividing $A_t Y_o$ by $Y_t$ does not give the proportion of total increased output per man-year due to technical change. The 1947 output per man-year $Y_o$ should be subtracted from both $A_t Y_o$ and $Y_t$. Hence, we propose the following modified method to estimate the proportion of the increased output per man-year due to technical change and that due to the increased use of capital:

\[ P_T = \frac{A_t Y_o - Y_o}{Y_t - Y_o} \]

\[ = \frac{A_t - 1}{Y_t Y_o} \]

Using this formula, the estimated proportion of the increased output per man-year attributable to technical change is 69.24 percent, which is quite low compared with Solow's estimate. The reason is that Solow subtracts the portion of the increased output per man-year contributed by the increased use of capital and imputes the residual to technical change, whereas the modified Massell method subtracts the portion of the increased output per man-year contributed by technical change and imputes the residual to increased capital intensity. If the portion of the increased output per man-year contributed by increased capital intensity and that contributed by technical change do not add up to the total increase in output per man-year, the results obtained by the two methods will be different. In our example, the portion of the increased output per man-year
contributed by technical change is

\[ A_t Y_o - Y_o = 4.9569 \]

and that contributed by increased capital intensity is

\[ \frac{Y_t}{A_t} - Y_o = 1.2570 \]

Adding 4.9569 to 1.2570 results in 6.2139, which is not equal to the increased output per man-year of 7.1590. Solow imputes the residual after subtracting the portion of the increased output per unit of labor due to technical change and that due to increased use of capital, which is equal to 0.9451, to technical change, while in the modified Massell method the residual is imputed to increased capital intensity. Therefore, Solow's method overestimates the proportion of the increased output per man-year due to technical change, whereas the modified Massell method overestimates that due to increased capital intensity.

Both Solow's and the modified Massell methods can be used for the purpose of rough calculation. They are not satisfactory methods for the measurement of the effects of technical change and the increase in capital on the increased output per man-year, because their estimates are influenced by extreme or unusual values of the first and last years of the period under investigation. Therefore, the results obtained by using these methods may be biased. For example, in Industry 206 the output per man-hour based on production worker data has risen 2.5526 times from 3.3783 in 1947 to 8.6233 in 1964 as shown in Table
17. The proportion of the increased output per man-hour due to technical change is 91.59 percent estimated by the modified Massell method and 97.20 percent by Solow's method. If we leave out the observations of 1947 and 1964 and observe only the period 1948 to 1963, then the output per man-hour has increased 3.1416 times from 3.3176 in 1948 to 10.4427 in 1963, and the technical change index of 1963 is 2.577 times of that of 1948. The proportion of the increased output per man-hour due to technical change estimated by the modified Massell method is 73.64 percent and that estimated by Solow's method is 89.77 percent. The variations of the estimated proportions of the increased output per man-hour are very large with only a slight change in the period of observation.

In order to overcome this difficulty, we may use the annual average rate of growth of output per unit of labor to compute the ratio of $Y_t$ to $Y_o$. The annual average growth rate of output per unit of labor can be computed by

$$r_y = \frac{\sum_{i=1}^{N-1} \frac{Y_i - Y_{i-1}}{Y_i}}{N-1}$$

where $N$ denotes the number of years of observation. Let $Y_o$ be the output per unit of labor in 1947 and $Y_t$ that in 1964, then

$$Y_t = Y_o (1 + r_y)^{17}.$$ 

Therefore, the 1964 output per unit of labor is $(1 + r_y)^{17}$ times of that of 1947. In Industry 202, the annual average rate
of growth of output per man-year based on total employees
data is 0.0463 as shown in Table 13, and the rate of technical
progress is 0.0336. The proportion of the increased output
per man-year due to technical change is 69.24 percent esti-
mated by the modified Massell method and 82.64 percent esti-
mated by Solow's method.

Tables 19 and 20 summarize the proportion of the increased
output per unit of labor due to technical change and that due
to increased capital intensity. In Table 19 \( Y_0 \) and \( Y_t \) were
obtained directly from Tables 12 through 18, whereas in Table
20, the annual average rate of growth of output per unit of
labor \( (r_y) \), which value is shown in the second row from the
bottom in Tables 12 to 18, was used to compute \( Y_t \) using (5.10).

The industry with a dash in its entry indicates that in
that industry the rate of technical change was overestimated
or negative, and no reasonable proportion of the increased
output per unit of labor due to technical change and that due
to capital formation were obtained.

As pointed out before, if the residual is positive,
Solow's method overestimates the proportion of the increased
output per unit of labor due to technical change, whereas the
modified Massell method underestimates this proportion. If
the residual is negative, the situation is reversed. If there
are no better ways to impute the residual between technical
change and the increased use of capital, the average value of
Table 19. Proportion of the increased labor productivity due to technical change and that due to capital formation

<table>
<thead>
<tr>
<th>Industry</th>
<th>Data</th>
<th>$A_t$</th>
<th>$Y_t/Y_0$</th>
<th>Percent of increased labor productivity due to technical change</th>
<th>Capital formation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Solow's Modif. Massell</td>
<td>Solow's Modif. Massell</td>
</tr>
<tr>
<td>201</td>
<td>Total employees</td>
<td>2.778</td>
<td>2.281</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>5.847</td>
<td>2.400</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>202</td>
<td>Total employees</td>
<td>1.772</td>
<td>2.197</td>
<td>79.97</td>
<td>64.49</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>203</td>
<td>Total employees</td>
<td>6.781</td>
<td>2.778</td>
<td>95.33</td>
<td>87.38</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>2.703</td>
<td>2.949</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>204</td>
<td>Total employees</td>
<td>1.056</td>
<td>2.561</td>
<td>8.70</td>
<td>9.98</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>1.368</td>
<td>2.928</td>
<td>40.85</td>
<td>19.09</td>
</tr>
<tr>
<td>205</td>
<td>Total employees</td>
<td>2.866</td>
<td>2.359</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>206</td>
<td>Total employees</td>
<td>2.422</td>
<td>2.923</td>
<td>89.24</td>
<td>73.75</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>207</td>
<td>Total employees</td>
<td>1.900</td>
<td>2.035</td>
<td>93.13</td>
<td>86.96</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>1.976</td>
<td>2.441</td>
<td>83.67</td>
<td>67.73</td>
</tr>
</tbody>
</table>
Table 20. Proportion of the increased labor productivity due to technical change and that due to capital formation

<table>
<thead>
<tr>
<th>Industry</th>
<th>Data</th>
<th>$A_t$</th>
<th>$Y_t/Y_o$</th>
<th>Percent of increased labor productivity due to</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Technical change</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Solow's Modif.</td>
</tr>
<tr>
<td>201</td>
<td>Total employees</td>
<td>2.778</td>
<td>2.198</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>5.847</td>
<td>2.317</td>
<td>-</td>
</tr>
<tr>
<td>202</td>
<td>Total employees</td>
<td>1.772</td>
<td>2.115</td>
<td>82.64</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>203</td>
<td>Total employees</td>
<td>6.781</td>
<td>2.567</td>
<td>99.20</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>2.703</td>
<td>2.949</td>
<td>-</td>
</tr>
<tr>
<td>204</td>
<td>Total employees</td>
<td>1.056</td>
<td>2.561</td>
<td>9.02</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>1.368</td>
<td>2.784</td>
<td>41.98</td>
</tr>
<tr>
<td>205</td>
<td>Total employees</td>
<td>2.866</td>
<td>2.304</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>206</td>
<td>Total employees</td>
<td>2.422</td>
<td>2.552</td>
<td>97.20</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>2.422</td>
<td>2.552</td>
<td>-</td>
</tr>
<tr>
<td>207</td>
<td>Total employees</td>
<td>1.900</td>
<td>1.9704</td>
<td>96.18</td>
</tr>
<tr>
<td></td>
<td>Production workers</td>
<td>1.976</td>
<td>2.1385</td>
<td>92.77</td>
</tr>
</tbody>
</table>
Solow's and the modified Massell estimates may give better measurement of the proportion of the increased output per unit of labor due to technical change and that due to the increased capital intensity.
VI. RELATIVE FACTOR SHARES AND TECHNICAL CHANGE

A. Introduction

Relative factor shares represent the relative pay-offs of production factors associated with their relative contributions to the total product. They are the end results of the interactions of many forces, such as the elasticity of substitution, relative supply of the factors of production, wage rates, technical change, saving, government fiscal policy, imperfect competition, and business cycle (6).

It has been widely believed that labor's share relative to the total output was constant in the late nineteenth and early twentieth century. A number of attempts have been made to explore and to explain this remarkable fact. Of these attempts the most important contributions, according to Reder (47), are marginal productivity theories, "make-up" theories, and "widor's cruse" theories.

In using the marginal productivity theories to explain the behavior of relative factor shares, it is necessary to specify the form of the production function.

B. Relative Factor Shares in the Cobb-Douglas Model

The usual explanation of the constancy of the relative factor shares by the marginal productivity theories is based upon the hypothesis that the underlying production function is
of the Cobb-Douglas type:
\[ V = \gamma L^\alpha K^\beta. \]
Under perfect competition, the wage rate is equated with the marginal product of labor:
\[ \frac{\partial V}{\partial L} = \alpha V/L = w. \]
The relative share of labor in the total output is the ratio of its total reward (the wage rate times the amount of labor service used) to the total output, i.e.,
\[ \frac{WL}{V} = \frac{\partial V}{\partial L} \cdot \frac{L}{V} = \alpha, \]
which is a constant. By the same token we can obtain capital's share, which is equal to \( \beta \). On this model, changes in relative factor shares can be explained only by shifts in the production functions, i.e., changes in techniques.

Recently, the underlying assumption of the Cobb-Douglas function that the elasticity of substitution is unity has been attacked and the constancy of the relative factor shares has been questioned. Kravis (29) has examined U.S. data for the period from 1900 to 1957 and has concluded that the notion of long-run constancy in the relative factor shares is false; the labor's share has risen from 55 percent of national income at the beginning of the century to 67 percent in the 1930's and has remained at approximately that level in the ensuing decades. Johnson (26), studying the functional distribution of income in
the United States during the period 1850 to 1952, has observed that employee compensation increased from 55.0 percent of national income in 1900-1909 to 64.3 percent in 1947-1952. Solow (56) also has estimated that the share of compensation of employees rose from 58.2 percent of national income in 1929 to 68.9 percent in 1955.

As pointed out before, many empirical evidences have shown that the elasticity of substitution is not unity. Therefore, the Cobb-Douglas function fails to explain the changes in the relative factor shares.

C. Relative Factor Shares in the CES Model

Since the introduction of the CES production function in 1961, many attempts have been made to explain the relationship between the elasticity of substitution and the behavior of relative factor shares based on this production function [e.g., Brown (6,8)]. On this model, changes in relative factor shares need not be explained by shifts in the production function; changes in the relative combination of factor inputs alone may change the relative factor shares. From (3.31) we have

\[
(6.1) \quad \frac{WL}{rK} = \frac{\sigma}{\beta} \left( \frac{K}{L} \right)^{\frac{1}{\sigma} - 1} = \frac{1-\delta}{\delta} \left( \frac{K}{L} \right)^{\frac{1}{\sigma} - 1}.
\]

If \( \sigma \) is less than unity, an increase in the capital stock relative to the labor force will result in an increase in
labor's share relative to that of capital. If \( \sigma \) is equal to unity, which is the case of the Cobb-Douglas model, the relative shares will remain constant with changing capital/labor ratios. If \( \sigma \) is greater than unity, an increased capital/labor ratio will raise capital's share relative to that of labor.

As previously mentioned, the weakness of this model is that the formulation of the CES function has ignored the capital/labor ratio, which plays an important role in explaining the behavior of relative factor shares. Furthermore, if the parameter \( c \) is not equal to zero, the estimated value of \( \sigma \) in the CES model is biased.

D. Relative Factor Shares in the VES Model

The relationship between relative factor shares, the parameters of the VES function, and the capital/labor ratio can be derived from (3.32) as follows:

\[
(6.2) \quad \frac{W_L}{L_K} = \frac{R}{X} = \frac{1}{\frac{b}{\alpha} X^A - B},
\]

where \( A = \frac{b+c-1}{b} \) and \( B = \frac{c}{b+c-1} \).

If technique is unchanging, (6.2) states that an increase in the capital/labor ratio will raise, hold constant, or reduce labor's share relative to that of capital, depending upon whether \( A \) (or \( b + c - 1 \)) is less than, equal to, or greater than zero.
From the foregoing study, we know that an increase in \( X \) (the capital/labor ratio) will increase, hold constant, or decrease the elasticity of substitution, depending upon the parameters \( b \) and \( c \). If \( b + c - 1 > 0 \), the value of \( \sigma \) declines with an increased \( X \) and approaches \( b/(1-c) \), which is greater than unity, as \( X \) increases without limit. On the other hand, if \( b + c - 1 \) is less than zero, the value of \( \sigma \) increases from \( b/(1-c) \), which is less than unity, to unity as \( X \) increases from zero to infinity.

The effect on the relative factor shares of changes in \( \sigma \) due to changes in \( X \) with given values of \( b \) and \( c \) can be shown by

\[
\frac{R}{X} = \frac{WL}{RK} = \left(\frac{1}{c} \right) - \frac{b}{c^2} \]

which is derived from (3.25). The partial derivative of \( R/X \) with respect to \( \sigma \) gives

\[
\frac{\partial (R/X)}{\partial \sigma} = \frac{b}{c^2} \]

which is positive if \( b \) and \( c \) have the same sign. Under this condition an increase in \( \sigma \) will always raise the share of labor relative to that of capital. Therefore, we may conclude that if \( b+c \) is greater than unity, \( \sigma \) also is greater than unity and an increase in \( X \) will decrease the value of \( \sigma \) as well as labor's share relative to capital's share. On the contrary, if \( b+c \) is less than unity, \( \sigma \) is less than unity, \( \sigma \). An increase in \( X \) will increase the value of \( \sigma \) as well as the relative share of labor
to capital.

E. Effects of Technical Change on Relative Factor Shares

The above discussion assumed that the state of technique is unchanged. The relative factor shares, however, are closely related to technical change. The obvious connection is that, under technical change, both marginal products of labor and capital increase. Furthermore, technical change may be non-neutral; the change may increase the marginal product of labor more than that of capital or vice versa. Thus, technical change may change the relative marginal products of factor inputs and hence change the relative factor shares.

The effects on the relative factor shares of the relative supply of capital and labor, changes in the elasticity of substitution, and growth of the wage rates can be seen by differentiating (4.7) totally with respect to \( t \) (time),

\[
\frac{\dot{S}}{S} = (1-b) \frac{\dot{W}}{W} - c \frac{\dot{X}}{X} + \lambda(b-1),
\]

where \( S \) refers to labor's share relative to total output and a dot on the variable denotes the derivative of that variable with respect to time. It is obvious from (6.3) that the direction and the rate of changes in labor's share relative to total output depend upon the parameters of the production function \( b \) and \( c \), the growth rate of the wage rates, the rate of changes in the capital/labor ratio, and the rate of tech-
nical change. If \( b < 1, c > 0, \) and \( \lambda > 0, \) as was true in most of the cases we observed in the Food and Kindred Products Industries, an increase in the wage rate will raise the share of labor, while an increase in the capital/labor ratios will reduce labor's share relative to the total product. If we assume that the state of technique is constant, i.e., \( \lambda = 0, \) an increase in the wage rates and the capital/labor ratios will raise, hold constant, or decrease labor's share relative to total output, depending upon the result of the interaction of two forces: labor's share raising force, \( (1-b) \frac{\dot{W}}{W}, \) and capital's share raising force, \( c \frac{\dot{X}}{X}. \)

Under perfect competition the marginal rate of substitution is equated with the factor price ratio, 
\[
R = \frac{W}{r}.
\]

Then, we may redefine the elasticity of substitution as 
\[
\sigma = \frac{\frac{dX}{X}}{\frac{d(W)}{W}} \frac{\dot{W}}{W} \frac{r}{r}
\]

When \( b + c \) is less than unity, so is \( \sigma. \) This implies that the factor price ratio grows at a rate faster than that of the capital/labor ratio, 
\[
\frac{dX}{X} < \frac{d(W)}{W} \frac{r}{r}
\]
or
\[
(6.4) \quad \frac{\dot{X}}{X} < \frac{(\frac{W}{r})}{\frac{W}{r}}
\]
where the right side term can be rewritten as

\[ \frac{\dot{W}}{W} = \frac{\dot{W}}{W} - \frac{r}{r} \cdot \]

If the rate of return to capital is not declining over time, i.e., \( \dot{r} \geq 0 \), then

\[ (6.5) \quad \frac{\dot{W}}{W} \geq \frac{\dot{W}}{W} \cdot \]

From the relation \( b + c < 1 \), we have

\[ (6.6) \quad 1 - b > c. \]

It follows from (6.4), (6.5), and (6.6) that if \( b + c < 1 \), then

\[ (1-b) \frac{\dot{W}}{W} > c \frac{\dot{x}}{x}. \]

Hence, an increase in the wage rate and the capital/labor ratio will raise labor's share. Following the same procedure, we may conclude that if \( b + c > 1 \), an increased wage rate and capital/labor ratio will decrease labor's share.

When technical change occurs, the increment of output due to technical change may be entirely allocated to capital or labor, depending upon the parameters of the production function. In the model of (6.3) the increment of output due to technical change will be allocated entirely to labor if \( b \) is greater than unity. If \( b \) is less than unity, technical change will favor capital's share.
F. Empirical Investigation of Factor Shares in the Food and Kindred Products Industries

Changes in labor's share relative to total output, wage rates, and capital/labor ratios of the seven industries over the period 1947 to 1964 for both total employees data and production workers data are shown in Tables 12 through 18. In all seven industries, the wage rates and the capital/labor ratios based on both total employees data and production workers data have increased over time. The average annual growth rate for each variable computed by (5.9) is listed in the second row from the bottom, and the estimated growth rate of labor's share relative to total output is listed in the last row.

In Industry 201, the values of the parameters of the production function based on total employees data are: b = 0.7972, c = 0.0028, and \( \lambda = 0.0604 \). The annual growth rate of the capital/labor ratio is 0.0248, and that of the wage rate is 0.0426. Since \( b + c \) is less than unity, the labor's share raising force is greater than that of capital. Hence, labor's share would have increased at a rate of 0.85 percent a year if technical change had not occurred. Apparently, this is not true. The observed labor's share relative to total output, in fact, has declined at a rate of 0.41 percent a year. During this period technical change has taken place at an average rate of 6.01 percent a year. Since \( b \) is less than unity, changes in technique will depress labor's share
relative to total output by an annual rate of
\[ \lambda(1-b) = 1.22 \text{ percent} \]
and the net effect of the three forces is -0.37 percent.
Therefore, we will expect that labor's share relative to total output would decline slightly over time. This estimation is quite close to the observed rate of changes in labor's share relative to total output, which is -0.41 percent.

In the same industry based on production workers data, the rate of changes of labor's share relative to total output is estimated to be -0.45 percent a year, while the observed rate of changes is -0.51 percent a year.

In Industry 204, based on total employees data, \( b + c = 1.0089 \), which is greater than unity. Since both capital/labor ratios and wage rates have increased over time, labor's share relative to total output would have declined at a rate of 0.55 percent a year in the absence of technical change as estimated by the model. Since \( b \) is less than unity, technical change will further reduce the rate of changes in labor's share relative to total output.
VII. SUMMARY AND CONCLUSIONS

This study has consisted of two general parts: the first part reviewed fixed coefficient, Cobb-Douglas, and CES production functions, and derived a VES production function which includes the other three forms of production functions as special cases; and the second part used the production function as a tool to the measurement of the rate of technical change, the estimation of the proportion of the increase in output per unit of labor attributable to technical change and that attributable to the capital formation, and the study of the behavior of factor shares.

A. Production Functions

In many fields of economic analysis, such as the determination of the optimum pattern of international, interregional, intersectoral, or intertemporal allocation of resources; the measurement of technical change; the study of the behavior of relative factor shares; the estimation of demand for factors of production and supply of products; and economic growth, it is necessary to specify explicitly the form of production functions. In aggregate models, fixed coefficient and Cobb-Douglas functions are most widely used, because they are simple to explain and easy to fit. But they are too restrictive in some economic applications. Recently, Arrow, Chenery, Minhas, and Solow have introduced the CES function, which includes fixed
coefficient and Cobb-Douglas functions as special cases. It is less restrictive in economic applications, but the main weakness of the CES production function seems to be its empirical starting point. The derivation of the CES function is based upon the assumption that the partial regression coefficient of \( \log \frac{K}{L} \) is equal to zero. The above assumption does not seem realistic. This position is supported by our empirical findings. It is also restrictive to assume that the elasticity of substitution between capital and labor is constant, especially in the study of the behavior of relative factor shares. Therefore, we have derived a more general form of the production function that does not depend upon SMAC assumption of independence and which includes the CES function as a special case. Our generalized production function has the property of variable elasticity of substitution.

We have shown that the elasticity of substitution estimated by the CES function will be the same as the estimate by the VES function if the partial regression coefficient of \( \log \frac{K}{L} \) turns out to be zero. To test the validity of this assumption, U.S. data for 1957 on 17 two-digit industries were used to fit the regressions of the logarithmic functions of value added per unit of labor on the wage rate, and value added per unit of labor on the wage rate and capital per unit of labor. The results from our regression analysis strongly suggested that the partial regression coefficient of \( \log \frac{K}{L} \) is not equal
to zero. Therefore, we cannot conclude that the elasticity of substitution estimated by the CES function and by the VES function are equal.

It has been shown that the VES function satisfies the following criteria: (i) the marginal products are positive, (ii) the marginal products slope downward over the relevant ranges of the inputs, and (iii) the function may characterize any degree of returns to scale.

Moving along an isoquant, the elasticity of substitution of the VES function varies with changes in the capital/labor ratios. The elasticity of substitution will increase, remain constant, or decrease with increased capital/labor ratio, depending upon whether the sum of the parameters, b and c, is less than, equal to, or greater than unity.

The VES function possesses the same limitations as the CES function. First, it is difficult to generalize the VES function to allow more than two inputs. Second, the VES function is non-linear in parameters; it is difficult to estimate the parameters. Third, in the case of non-constant returns to scale, the equation for the elasticity of substitution becomes very complicated.

However, the VES function has taken account of capital as a variable in its derivation. It has the property of variable elasticity of substitution and includes the CES function as a special case. In the absence of the knowledge about the c
value, the VES function may be used profitably.

B. Applications of Production Functions on the Study of Technical Change and Relative Factor Shares

1. Technical change and capital formation

Technical change and capital formation are widely recognized as the sources of economic growth. Output can be increased through allocation of resources on either the expansion of the existing capital stock or research and development for technical advance. In order to make optimum allocation of a given amount of resources between the expansion of the existing capital stock and research and development for technical advance, the quantitative estimate of the relative importance of technical change and capital formation in contributing to the increase in output per unit of labor is indispensable.

Two sets of data, i.e., total employees data and production workers data, from 1947 to 1964 in seven three-digit Food and Kindred Products Industries were used to fit regressions of (4.8) in order to obtain the rate of technical change and other parameters of the production function. Both Solow's and Massell's methods were used to estimate the proportion of the increase in output per unit of labor attributable to technical change and that attributable to capital formation. We have pointed out that Massell's method is biased, and we have modified his method.
Both Solow's and the modified Massell methods use the ratio of the last year's output per unit of labor to the first year's output per unit of labor as an index for the measurement of the increase in output per unit of labor over the period of investigation. We have pointed out that the index is not reliable, since it is influenced by extreme or unusual values of the last and the first year's output per unit of labor. Hence, we have suggested that the growth rate of output per unit of labor, which can be computed by (5.9), be used to estimate the increase in output per unit of labor.

The results of our analysis showed that the proportion of the increase in output per unit of labor due to technical change and that due to capital formation estimated by Solow's method are different from those by the modified Massell method. The reason is that Solow's method imputes the residual, which is the increased output per unit of labor subtracting the portion of the increased output per unit of labor contributed by technical change and that contributed by capital formation, to technical change; whereas the modified Massell method imputes the residual to capital formation. Therefore, if the residual is positive, as was true in most of the cases we observed in the Food and Kindred Products Industries, Solow's method overestimates the portion contributed by technical change, whereas the modified Massell method overestimates the portion contributed by capital formation. If the residual is
negative, this conclusion is reversed.

Our empirical results indicated that technical progress had played an important role in the increased output per unit of labor. In most industries we observed, more than sixty percent of the increased output per unit of labor was contributed by technical progress. But the role of capital formation still cannot be overridden, since most technical progress is embodied in the new capital stock. Capital formation is essential for technical progress.

2. Relative factor shares

Many factors affect changes in relative factor shares. In the VES model, changes in relative factor shares depend upon the parameters of the production function, the relative supply of capital and the labor force, the growth of wage rates, and technical change. An increase in the wage rate and the capital/labor ratio will raise labor's share relative to total output, if $b+c$ is less than unity. If $b+c$ is greater than unity, an increased wage rate and capital/labor ratio will result in a decline in labor's share relative to total output. The increased output due to technical progress will be allocated in favor of labor's share if $b$ is greater than unity. If $b$ is less than unity, technical progress will favor capital's share.

In all seven industries we observed, wage rates and capital/labor ratios had increased quite rapidly in the period of ob-
observation. In most industries b+c was less than unity, and so was b. Therefore, we estimated that labor's share relative to total output would decline slightly over time. This estimation was quite consistent with our empirical observations.

C. Suggestions for Further Research

1. Technical change is defined as shifts in the production function, and shifts in the production function imply changes in the parameters of the production function. There are four parameters in the VES function, but we have allowed only one parameter to vary. The other three parameters actually may have varied over time. If the combination of time series and cross-section data are available, we may postulate that all the parameters of the VES function are varied over time and examine the effects of changes in these parameters on the types of technical change and the behavior of relative factor shares.

2. Another important field of economic research closely related to our study is economic growth. It may be interesting to introduce the VES function into a macroeconomic model of economic growth. An important implication concerning the growth rate of capital/labor ratios, saving rates, and economic growth may be uncovered.
VIII. BIBLIOGRAPHY


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