APPLICATION OF GEOMETRICAL DIFFRACTION THEORY TO SCATTERING BY CRACKS

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ABSTRACT

At high frequencies, the geometrical theory of diffraction provides useful and relatively simple approximations to diffraction phenomena. In this paper the theory is applied to the diffraction of time-harmonic longitudinal waves by a penny-shaped crack in an elastic solid.

Geometrical Theory of Diffraction-Basic Ideas

For diffraction phenomena which are governed by a single wave equation, geometrical diffraction theory for planar obstacles was developed by Keller. The formulation of Ref. 4, and subsequent papers, e.g., Ref. 5, is, however, not applicable to diffraction by cracks in solids, since wave motions in solids are governed by two wave equations, which are coupled by the boundary conditions on the diffraction plane. Thus, although in first approximation it is useful to neglect this coupling, and as done in Refs. 6 and 7, it is appropriate to consider the problem in a mathematically rigorous manner by taking the coupling into account.

For plane longitudinal and transverse waves, which are under arbitrary angles of incidence with a traction-free semi-infinite crack, the fields on the diffracted rays can be obtained by asymptotic considerations, as shown in Ref. 3. The results can be expressed in terms of diffraction coefficients which relate the diffracted fields to the incident fields. Geometrical diffraction theory provides modifications to the semi-infinite crack results, to account for curvature of incident wave-fronts and curvature of crack edges, and finite dimensions of the crack. In the usual terminology the results for diffraction of plane waves by a semi-infinite crack are the canonical solutions.

Geometrical diffraction theory is applicable if \( \omega d/c_l \gg 1 \), where \( \omega \) is a characteristic dimension of the crack and \( c_l \) is the velocity of longitudinal waves. For some special problems where comparisons with mathematically exact solutions were possible, we found good agreement for \( \omega d/c_l > 2.0 \). For a crack with a cross-sectional dimension of 1 mm in steel \( (c_l \sim 6.3 \times 10^3 \text{ mm/sec}) \), the frequency should then be larger than about 1 MHz. This seems to be within experimental capability, see e.g. Ref. 6 and 7.

It should be noted that a slightly different interpretation renders the results of geometrical diffraction theory useful as wavefront results for propagating pulses.

Geometrical Theory of Diffraction-Basic Ideas

The basic ideas of geometrical diffraction can best be explained by means of an example. Figure 1 shows a planar crack with a smoothly curving edge in an elastic solid. The origin of a cartesian coordinate system \( (x, y, z) \) is in the plane of the crack, and the \( z \)-axis is normal to that plane. A time harmonic wave of the form

\[
u_z = A_0 e^{i(\omega t/c_l - \zeta)}
\]

is incident on the crack. This is a case of normal incidence, i.e., the wavefronts are parallel...
The diffraction problem described above belongs to a class of elastodynamic diffraction problems whose complete solutions are very difficult to obtain. The one problem in that class for which analytical results can be derived in a relatively simple manner is the diffraction of a plane incident wave by a traction-free semi-infinite crack. In particular, for large values of \( \omega r/c_l \), the fields on the diffracted rays emanating from the tip of the semi-infinite crack can be obtained by asymptotic considerations. These results can be expressed in terms of diffraction coefficients which relate the diffracted fields to the incident field. Geometrical diffraction theory for cracks of arbitrary shape is based on the semi-infinite crack results, in that it immediately provides first-order corrections to the semi-infinite crack results, to account for a curvature of incident wavefronts and curvature of crack edges. In the usual terminology the results for diffraction of plane waves by a semi-infinite crack are the canonical solutions. With some additional effort corrections for the finite dimensions of a crack can now be obtained, as described in this paper.

The total fields at point Q are, indeed, not just comprised of the fields on the "primary diffracted body wave rays" \( P_1Q \) and \( P_2Q \). At the edge of the crack there are also rays of crackface motion generated. These rays intersect the crack edges again and generate additional diffracted body wave rays. Some of these "secondary diffracted rays" will pass through point Q. This system of reflected and diffracted rays can become rather extensive. Fortunately, on the faces of the crack the main contributions to the diffracted fields are not coming from the diffracted rays of longitudinal and transverse motion, but rather from rays of surface waves. These rays, which have not been studied before, are important, because in the first approximation the diffraction coefficients for the body wave motions vanish on the crack faces, except for diffracted horizontally polarized transverse wave motions. In addition, surface wave motions suffer less geometrical decay than body wave motions. In a recent paper Gautesen, Achenbach and Mcmaken have presented a theory for surface waves which are generated by the diffraction of body wave rays.

When a surface wave ray intersects the edge of a crack, a ray of reflected surface wave motion is generated, as well as cones of diffracted rays of longitudinal and transverse motions. The reflection coefficients can be computed. The cones of diffracted body rays can also be analyzed, and the associated diffraction coefficients can be obtained. With the aid of these results the stepwise radiation of energy which is temporarily trapped by the crack in the form of surface motion can be analyzed, and the scattered field can be computed as the sum of primary diffractions and a system of secondary diffractions.

For curved wavefronts and for curved edges of diffraction, the cones of diffracted rays have envelopes, at which the rays coalesce and the fields become singular. The envelopes are called caustics. The results of the geometrical theory of diffraction are not valid near caustics. It is, however, possible to extend the theory to the caustics. In the next section this is shown for the case of a penny-shaped crack which is under the normal incidence of a plane longitudinal wave. For this case the normal axis through the center point of the crack is a caustic. Uniformly valid expressions for the field on the body wave rays are obtained.

**Some General Results for Diffraction of Longitudinal Waves**

In this section we briefly summarize some pertinent expressions. The details can be found in a paper by Gautesen, Achenbach and Mcmaken.

Primary diffracted body wave rays. For an incident longitudinal wave the displacement fields on the diffracted body wave rays are

\[
\mathbf{u}_d = e^{i \omega s / c_l} \left[ s_p \left( 1 + s_p / s_d \right) \right]^{-i} \mathbf{u}_d
\]

(2)
Here $\psi^\phi$ defines the incident wave at the point of diffraction. In Eq. (2), the superscript $\phi$ denotes the nature of the wave motion on the diffracted rays. Thus we will use $\phi = L$ and $\phi = T$ for longitudinal and transverse waves, respectively. The distances $S_\phi$ are along the diffracted rays from the point of Diffraction $O$ to the point of observation. The unit vectors $\mathbf{\hat{n}}_\phi$ relate the displacement directions of the diffracted fields to those of the incident fields. The symbols $\rho_\phi$ define, the distance from $O$ to the caustics and $D_\phi$ are the diffraction coefficients. For an incident longitudinal wave we have

$$\rho_L^L = a \sin^2 \phi_L \left[ a \left( d\phi_L / ds \right) \sin \phi_L + \cos \phi_L \right]^{-1} \tag{3}$$

where $a$ is the radius of curvature of the edge at the point of diffraction, $s$ is the distance measured along the edge, and $\phi_L$, $\phi_T$ are the angles between the relevant diffracted rays and the normal to the crack. The angles $\phi_L$ and $\phi_T$ are related by

$$c_T \cos \phi_T = c_L \cos \phi_L \tag{4}$$

The diffraction coefficients are obtained from the canonical problem of diffraction of a plane longitudinal wave by a semi-infinite crack.

**Diffraction surface wave rays.** For normal incidence, only symmetric surface wave motions are generated on the faces of the crack. The displacements on the diffracted surface wave rays then are

$$u_d^R = e^{i\omega S^L/c_L} \left[ L S^L / R^L \right]^{-\frac{1}{2}} \left( \frac{\partial}{\partial L} \right) L^L \tag{5}$$

The principal difference between Eqs. (5) and (2) is the additional term $S^L / R^L$ in Eq. (2). This term reflects three dimensional (spherical) growth and decay in Eq. (2) versus two dimensional (cylindrical) growth and decay in Eq. (5). In Eq. (5) we have

$$\rho_R^L = a \sin \phi_R \left( a \left( d\phi_R / ds \right) + 1 \right)^{-1} \tag{6}$$

where $\phi_R$ is related to $\phi_L$ by

$$c_L \cos \phi_L = c_R \cos \phi_R \tag{7}$$

Reflection of surface wave rays. A surface wave ray which intersects the edge of a crack gives rise to a ray of reflected surface waves, and to two cones of diffracted body rays. For a surface wave incident on the edge of a semi-infinite crack these reflection and diffraction processes were studied by Freund. In the spirit of geometrical diffraction theory, we can immediately introduce the appropriate corrections for curvature of the incident wavefront and for curvature of the edge of the crack.

A surface wave ray is reflected such that the angle between the incident ray and the tangent to the edge is just the same as the angle of incidence, $\phi_\theta$, between the incident ray and the tangent to the edge. Moreover, rays of symmetric (antisymmetric) surface waves are reflected as rays of symmetric (antisymmetric) surface waves.

The incident field is defined by Eq. (5). Quantitatively, the fields on the reflected surface rays are given by

$$u_r^R = e^{i\omega S^L/c_L} \left[ L S^L / R^L \right]^{-\frac{1}{2}} \left( \frac{\partial}{\partial R} \right) R^L \tag{8}$$

The nature of the motions on the incident rays is the same as on the reflected rays. In Eq. (6) $S_\phi$ is the distance from the point of reflection to the point of observation, $\phi$ is the reflection coefficient, and $\rho_\phi$ is the distance to the caustic where

$$\rho_R^L = a \sin \phi_R \left( a \left( d\phi_R / ds \right) + 1 \right)^{-1} \tag{9}$$

This is the same formula as given by Eq. (6), but here $\phi_\theta$ is the given angle of incidence, while in Eq. (6), $\phi_\theta$ was computed from Eq. (7).

**Body wave rays generated by diffraction of surface way rays.** For this case the displacement fields are of the general form

$$u_d^B = e^{i\omega S^L/c_L} \left[ L S^L / R^L \right]^{-\frac{1}{2}} \left( \frac{\partial}{\partial B} \right) B^L \tag{10}$$

where $\theta = L$ or $\theta = T$ for diffracted rays of longitudinal and transverse motion, respectively. Also $D_\phi$ is the pertinent diffraction coefficient which can be computed, see Ref. 8. The distances to the caustics are given by $\rho_L^L$ and $\rho_T^T$, respectively.

**Diffraction by a penny-shaped crack.** For normal incidence of a plane longitudinal wave of the kind given by Eq. (1), the incident and diffracted fields are axially symmetric with respect to the $z$-axis. Thus only the radial distance $r = (x^2 + y^2)^{1/2}$ and the axial distance $z$ enter in the results. The geometry is shown in Fig. 2. The diffracted field at any point $Q$ away from the crack can now easily be computed on the basis of the semi-infinite crack solution, as explained in the previous section. In cylindrical coordinates the radial and axial displacements for the diffracted field are obtained as

$$\left( u_{d_1}^B \right)_r = A_1 \sum_{j=1}^2 Q_j (-1)^j x_j \cos \left[ \chi \left( \theta_j \right) \cos \phi \right] e^{i\omega R_j / c_l \left( -\phi \theta_j \right)} \sin \theta_j \left[ \chi \left( \theta_j \right) \cos \phi \right] e^{i\omega R_j / c_l \left( -\phi \theta_j \right)} \tag{11}$$

$$\left( u_{d_2}^B \right)_z = A_1 \sum_{j=1}^2 Q_j \left( -1 \right)^j x_j \tag{12}$$
where \( j = 1 \) and \( j = 2 \) correspond to the contributions from \( P_1 \) (the point closer to \( Q \)) and \( P_2 \), respectively, while

\[
G_L(\theta_j) = e^{iwR_j/c_L} + G_T(\theta_j) e^{i\varphi_j R_j/c_T},
\]

\[
G_T(\theta_j) = e^{iwR_j/c_T} + G_T(\theta_j) e^{i\varphi_j R_j/c_T},
\]

(12) \( u_0 = A_0 \frac{(2\pi a)}{R} e^{i\theta/c_T} \left[ \varphi L_{1/2}(\theta_j) J_0(q_L) \cos \theta \right]

\[
+ G_T(\theta_j) J_1(q_T) \sin \theta e^{i\varphi R/c_T} \right]
\]

(13) \( R_j = a^2 + [a + (-1)^{j+1}] z^2 \),

(14) \( G_T(\theta_j) = b^2 + [a + (-1)^{j+1}] z^2 \),

(15) \( Q_j = R_j b \left[ 1 - (R_j/a) \cos \theta \right]^{-1} \),

(16) \( \cos \theta_j = a + (-1)^j R_j \),

(17) \( \cos \theta_j = a + (-1)^j R_j \).
A useful quantity to compute for diffraction of a longitudinal wave by a penny-shaped crack is the scattering cross section \( \sigma_{SC} \). This quantity is defined as the ratio of the time-averaged scattered power over the intensity of the incident wave, \( I \). For an incident wave of the form given by Eq. (1) we have:

\[
I = \frac{2}{w} p c A_0^2.
\]  

The far field may in general terms be expressed as:

\[
u_0(\zeta_0) = \frac{A_1(\zeta_0)}{4\pi |\zeta_0|} \exp[\text{iu} |\zeta_0|/w] \times
\]

\[
+ \frac{A^T(\zeta_0)}{4\pi |\zeta_0|} \exp[\text{iu} |\zeta_0|/c_T].
\]  

The interesting result, which was shown by Tan10, now is that only the far-field diffracted wave amplitude which is of the same type as the incident wave, and computed for \( \zeta_0 \) as a unit vector in the direction of wave incidence, occurs in the expression for \( \sigma_{SC} \). For the problem at hand the simple result is:

\[
\sigma_{SC} = \frac{c_L}{w} \frac{1}{A_0} \text{Im} \left[ A_1(\zeta_0) \right].
\]  

It is easy to compute the right-hand side from Eq. (19). For \( v = 0.25 \) the result has been plotted in Fig. 5 versus the dimensionless frequency \( wa/c_L \).

The result shown in Fig. 5 is valid for \( wa/c_L \) sufficiently larger than unity. With results available in the literature, it is also possible to compute the corresponding result for \( wa/c_L \ll 1 \). With the low frequency approximations available in the literature, and with the high frequency approximations of the work proposed here, it seems very likely that enough information will be available to cover the whole frequency range.

Finally, we present some computations for the amplitude of the axial displacement on the center axis of the crack for fixed \( z/a \), as a function of \( wa/c_L \). The results were computed from Eq. (19). The results are shown in Fig. 6.

References


