

MATCHED ASYMPTOTIC EXPANSIONS  
APPLIED TO SCATTERING CALCULATIONS\*

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ABSTRACT

The diffraction of elastic waves by inclusions, cavities and cracks has been the subject of numerous studies in recent years. Because of analytical difficulties, however, most of these studies have dealt with spherical or circular inclusions and straight or circular cracks. In a series of recent papers we have shown that for long wavelengths it is possible to obtain an asymptotic expansion of the scattered field by the method of matched asymptotic expansions. These papers have dealt with ellipsoidal and elliptic inclusions. We also have been able to use this technique to solve scattering problems in a half-space. In this paper we present briefly the results that we have obtained in the course of this investigation. The results include scattering by ellipsoidal inclusions in three dimensions and elliptic cylinders in two dimensions, by buried circular cavities and elliptical inclusions in a half-space, and by an edge crack. In the context of the last problem it is shown that MAE together with analytic function techniques can be used to solve many (not necessarily straight) crack problems in two dimensions.

Introduction and Summary of the Results

The equation governing the displacement field  $\underline{u}$  in a homogeneous isotropic linearly elastic solid is

$$c_1 \nabla \nabla \cdot \underline{u} - c_2 \nabla \nabla \wedge \underline{u} = -\omega^2 \underline{u} \quad (1)$$

where harmonic time dependence has been assumed.  $c_1, c_2$  are the longitudinal and shear wave speeds in the medium. If an inclusion of different material properties is assumed to be embedded in this medium then the problem is to analyze the scattered displacement field, denoted by  $\underline{u}^{(s)}$ , when an incident wave, denoted by  $\underline{u}^{(i)}$ , impinges on the inclusion. We shall assume that the inclusion occupies a finite volume  $V$  (finite area in two dimensions) bounded by a closed surface  $S$  (closed curve in two dimensions). Let  $L$  be a characteristic linear dimension of the inclusion. Then it has been shown in References [1-6] that if  $\epsilon = \omega L / c_1$  is small, the problem can be solved by a method of matched asymptotic expansions. In brief the technique is the following.

Let us introduce non-dimensional inner variables

$$\bar{x} = x/L \quad (2)$$

Then equation (1) can be rewritten as

$$\bar{\nabla} \bar{\nabla} \cdot \underline{u} - \tau^{-2} \bar{\nabla} \bar{\nabla} \wedge \underline{u} = -\epsilon^2 \underline{u} \quad (3)$$

when

$$\underline{u} = \underline{u}^{(s)} + \underline{u}^{(i)}, \quad \bar{x} \notin V$$

The boundary conditions for this inner problem are provided by the continuity of displacement and

tractions across  $S$ . The solution to this inner problem is assumed to be of the form

$$\underline{u} = \underline{u}_0 + \mu_1(\epsilon) \underline{u}_1 + \mu_2(\epsilon) \underline{u}_2 + \dots \quad (4)$$

where  $\mu_{n+1}/\mu_n \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Substitution of (4) in (3) and equating the terms of the same order from both sides leads to a set of equations for  $\underline{u}_n$ . Since the solutions to these equations would not satisfy the appropriate radiation conditions at infinity, the  $\underline{u}_n$ 's are not completely determined.

In order to complete the solution we formulate the outer problem in terms of the outer variables

$$x' = \epsilon \bar{x} \quad (5)$$

and an assumed outer expansion

$$\underline{u} = \underline{u}^{(i)} + \nu_1(\epsilon) \underline{u}' + \dots \quad (6)$$

where  $\nu_{n+1}/\nu_n \rightarrow 0$  as  $n \rightarrow \infty$ . The equations governed by  $\underline{u}'_n$  are

$$\bar{\nabla}' \bar{\nabla}' \cdot \underline{u}'_n - \tau^{-2} \bar{\nabla}' \bar{\nabla}' \wedge \underline{u}'_n = -\underline{u}'_n \quad (7)$$

and  $\underline{u}'_n$  satisfies the appropriate radiation condition. The solution is now completed by matching the  $n$ th order outer expansion of the  $m$ th order inner expansion with the  $m$ th order inner expansion of the  $n$ th order outer expansion.\* It is found that in three dimensions  $\mu_n(\epsilon)$  and  $\nu_n(\epsilon)$  are integer powers of  $\epsilon$ . However, in two dimensions they are of the form  $\epsilon^{p/q} \ln^q \epsilon$  ( $q < p$ ). Using this approach we solved in Reference [1] the diffraction

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\*The matching can also be done by introducing an intermediate variable as was done in Reference [2-6].

of elastic waves by a rigid spheroidal inclusion. For simplicity this analysis was confined to a plain longitudinal wave incident along the axis of symmetry. Results were presented for both the motion of the spheroid and the scattered far field. Figures 1-2 show the amplitudes of oscillation of different spheroids in comparison with the exact solutions obtained in References [7,8]. It is seen that the agreement is rather remarkable. In Figs. 3-10 we show the angular dependence of the far-field displacement components for scattering from rigid spheroids of different shapes. Here

$$U_{\xi}^S(T) = iU_{\xi}\xi' e^{-i\xi'} / \tau a_1, \quad U_{\eta}^S(T) = iU_{\eta}\xi' e^{-i\tau\xi'} / \tau b_1 \quad (8)$$

for prolate spheroids, and

$$U_{\xi}^S(T) = -U_{\xi}\xi' e^{-i\xi'} / \tau a_1, \quad U_{\eta}^S(T) = -U_{\eta}\xi' e^{-i\tau\xi'} / \tau b_1 \quad (9)$$

The superscripts S or T stand for the cases when the spheroid is held fixed or not.  $a_1$  is a normalization constant and

$$b_1 = a_1 / \tau^2, \quad \epsilon_1 = \epsilon \xi_0 / \pi$$

The Poisson's ratio has been taken to be 1/4. Later Sangster [9] considered the more general case of oblique incidence.

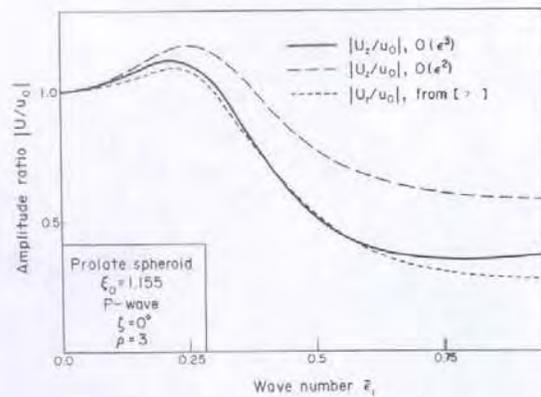


Figure 1. Amplitude of oscillation of a prolate spheroid  $\epsilon_1 = \epsilon \xi_0 / \pi$ .

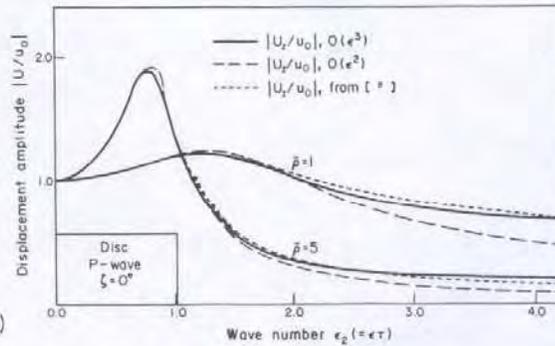


Figure 2. Amplitude of oscillation of a rigid disc.

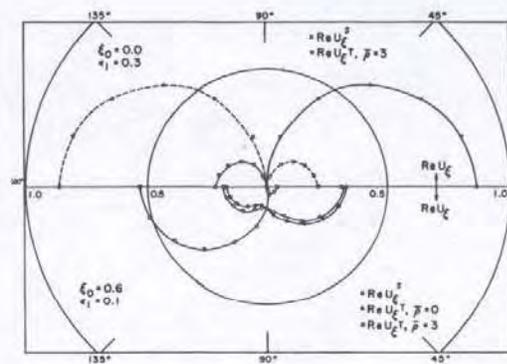


Figure 3. Far field displacement amplitudes for scattering of longitudinal wave by rigid spheroids.

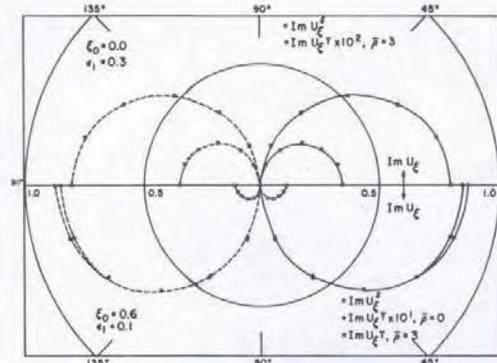


Figure 4. Far field displacement amplitudes for scattering of longitudinal wave by rigid spheroids.

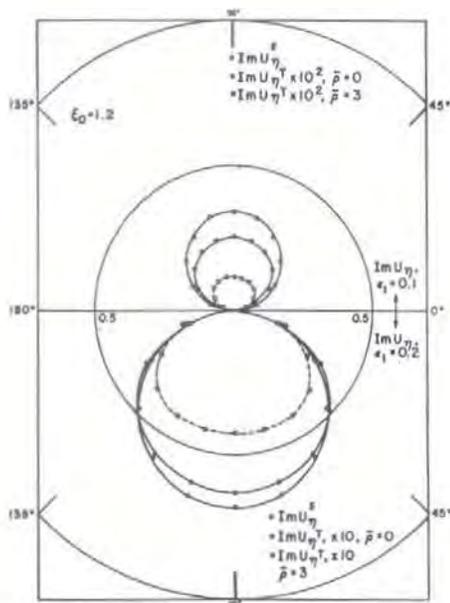


Figure 10. Far field displacement amplitudes for scattering of longitudinal wave by rigid spheroids.

In References [2,3] the analysis was confined to the diffraction of an antiplane shear wave by an elliptic cylinder and the far field was computed to  $O(\epsilon^2)$  with  $\epsilon = \omega L/C_2$  in this case. More recently [10] we have extended the calculations to  $O(\epsilon^4)$  and the far field displacement amplitudes are shown in Figs. 11-12. In [11] we have analyzed the problem of a branched crack and have derived the results for an edge crack in a semi-infinite plane. Angular dependence of the amplitudes is shown in Fig. 13. It is shown in [11] that in two dimensions matched asymptotic expansions along with conformal mapping techniques can be used for arbitrarily shaped cavities. This is particularly useful for branched or curved cracks.

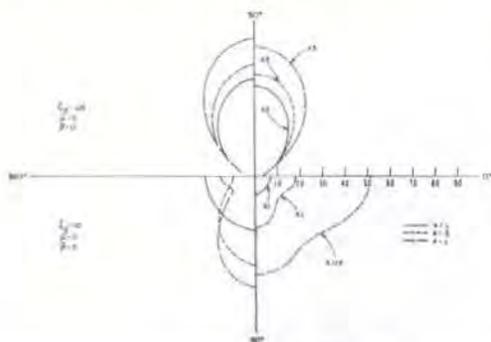


Figure 11. Back and forward scattered amplitudes for scattering of antiplane shear waves by an elliptic inclusion.

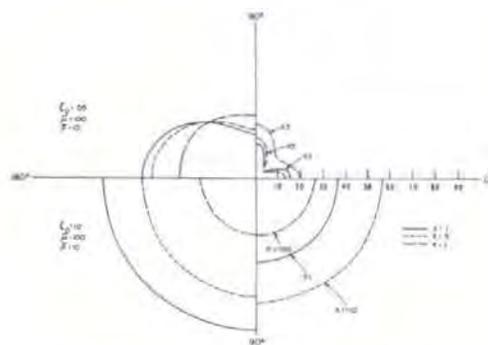


Figure 12. Back and forward scattered amplitudes for scattering of antiplane shear waves by an elliptic inclusion.

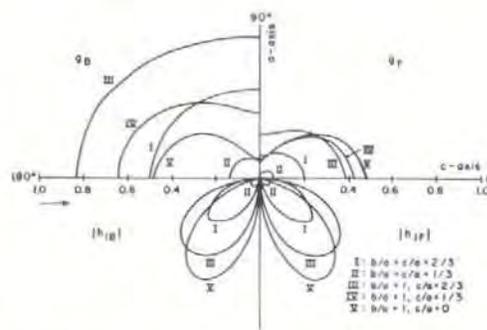


Figure 13. Far field displacement amplitude for scattering of antiplane shear waves by an edge crack.

Three dimensional diffraction by an elastic ellipsoidal inclusion has been solved in Reference [5]. The far field displacement amplitudes are shown in Figs. 14-17 for ellipsoidal cavities of various shapes. Since the calculations were confined to the lowest order in  $\epsilon$  the normalized amplitudes defined as

$$g(\theta, \omega) = \frac{r^i u_r(s)}{\epsilon^3 u_0} e^{-ir^i}, \quad h_1(\theta, \omega) = \frac{r r^i u_\theta(s)}{\epsilon^3 u_0} e^{-ir^i} \quad (10)$$

$$h_2(\theta, \omega) = \frac{u_\omega(s) r r^i}{\epsilon^3 u_0} e^{-ir^i}$$

do not depend on the frequency.

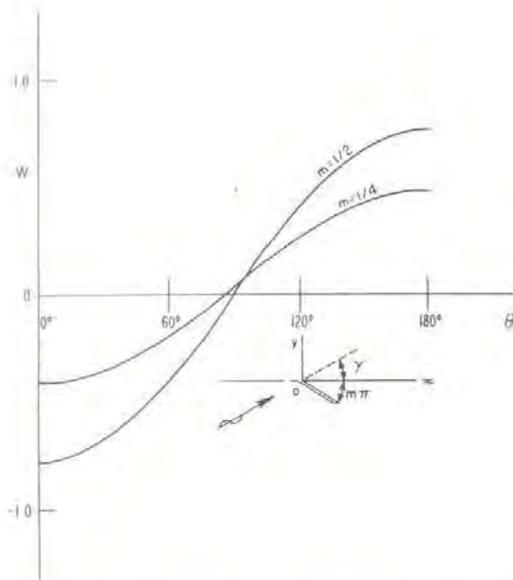


Figure 14. Back and forward scattered amplitudes of scattered longitudinal and shear waves for scattering by an ellipsoidal cavity (Incident longitudinal wave).

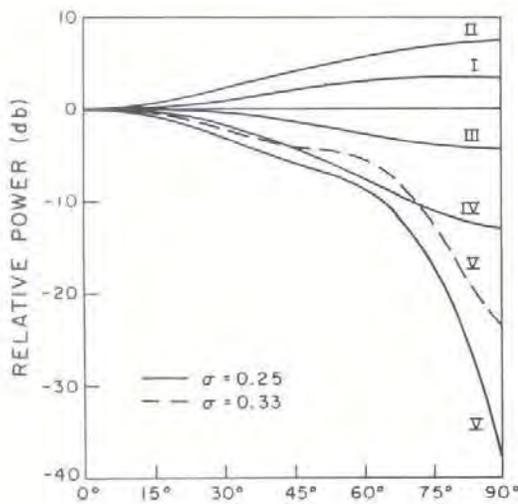


Figure 15. Backscattered power for an incident longitudinal wave.

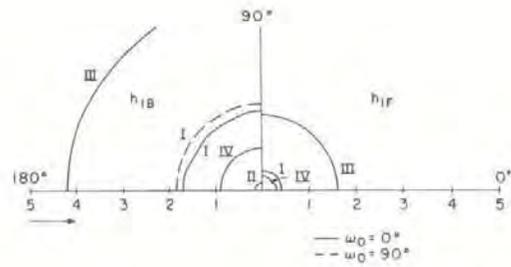


Figure 16. Back and forward scattered amplitudes of scattered shear wave (Incident shear wave).

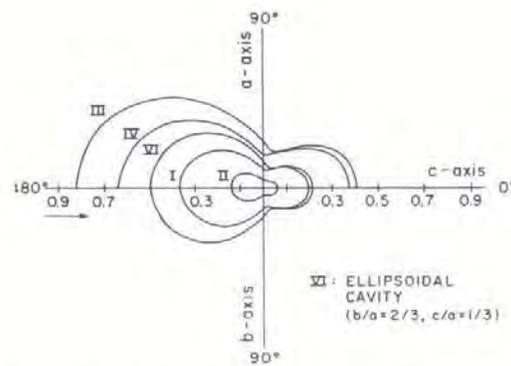


Figure 17. Far field displacement amplitudes for scattered longitudinal wave (Incident longitudinal wave).

In conclusion we would like to point out that MAE can also be used to solve diffraction problems in a half-space. This is discussed in Reference [12] in the context of diffraction by an elliptical inclusion and a circular cylindrical shell buried in a half-space. This extends the analysis for circular cavities presented in Reference [4].

#### References

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