RECIPROCITY THEORIES FOR FLAW ANALYSIS

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ABSTRACT

A new method of theoretical analysis based on the use of the reciprocity theorem has been developed. With this technique, one can predict the signal scattered by a flaw from one transducer to another in terms of the signals at the actual electrical terminals. The technique is particularly useful for dealing with focused beams and for taking account of near field and far field excitation and reception. We have applied this technique, using static assumptions, to determine scattering of a Rayleigh wave from a penny-shaped crack; by using the Born approximation we have determined the scattering of Rayleigh waves from a cylindrical hole. We have also used the theory to indicate the type of reflected signals we would expect from a focused beam illuminating a crack, and the peaks in the reflection characteristics as a function of frequency due to resonant modes of the flaws. The technique can be adapted for variational calculations. Basic variational theories have been derived but have not been applied to practical problems yet.

The scattering theories that are presently available generally consider plane wave excitation and reception by a point receiver. Basically the reason is historical. The first type of scattering theory was essentially that of Lord Rayleigh, who considered scattering of light from rain drops with plane wave excitation by the sun, the receiver being the eye, a point receiver.

We have developed a method of scattering analysis based on the use of the reciprocity theorem commonly employed in electromagnetic theory, which considers the transmitter to be a transducer, and the receiver to be the same transducer or another transducer located in a different position. The technique has the merit of great flexibility; it is particularly needed for our work in imaging systems, where both the transmitted and received beams may be focused. The technique is able to reproduce the results obtained by other methods, for the Green's function we employ is essentially that of the transducer, rather than that of a point source. With this approach we have been able to develop variational scattering formulae, elasto-static approximations, and normal mode expansions in terms of the resonant modes of a flaw.

The analysis follows from a generalization of the reciprocity theorem of electromagnetic theory:

$$\int_S (\underline{E}^a \times \underline{H}^b - \underline{E}^b \times \underline{H}^a) \cdot dS$$
$$= \int_V (\underline{J}^a \cdot \underline{E}^b - \underline{J}^b \cdot \underline{E}^a) \, dV \tag{1}$$

which relates two possible field solutions $\underline{E}^a$, $\underline{H}^a$, and $\underline{E}^b$, $\underline{H}^b$ within a volume V to each other. The terms J^a, J^b are current sources within the volume associated with the appropriate fields respectively. By using the piezoelectric constitutive equations, in addition to Maxwell equations, it is possible to generalize this reciprocity theorem to include both electromagnetic fields and acoustic fields.[1] In the problem of interest here it is convenient to carry out the analysis for the region shown in Fig. 1, including a flaw, whose volume is V_F and area S_F and whose outer surface is exterior to the sample and intersects the coaxial lines feeding the two transducers. We now take the field "a" to correspond to that which is present when one transducer is excited and no flaw is present, and the fields "b" to correspond to the situation when the other transducer is excited and there is a flaw present. In this case, we obtain the result

$$\int_{S_1+S_2} (\underline{E}^a \times \underline{H}^b - \underline{E}^b \times \underline{H}^a) \cdot dS$$
$$+ j\omega \int_{S_1+S_2} (\sigma_{ij}^b u_j^a - \sigma_{ij}^a u_j^b) n_i \, dS \tag{2}$$
$$= j\omega \int (u_j^b \sigma_{ij}^a - \sigma_{ij}^b u_j^a) n_i \, dS ,$$

where n_i is the outward normal from the surface of interest into the interior of the medium, and the notation is standard since the surface $S_1 + S_2$ is exterior to the sample and the acoustic fields are zero (assuming a vacuum) or negligibly small (assuming a normal atmosphere). Consequently, only the electromagnetic terms are of interest and can be related to the voltage and current at the terminals of the coaxial line. By this means, we have managed to write the results in terms of scattering matrix formalism using acoustic fields corresponding to unit power excitation at the transducer. In this case, the expression for the reflection coefficient at the transducer, due to the wave scattered from a void, is

$$\Gamma = S_{11} = \frac{j\omega}{4} \int_{S_F} (\sigma_{ij}^i u_j - \sigma_{ij} u_j^i) n_i \, dS , \tag{3}$$

where the superscript i corresponds to the incident unperturbed acoustic wave field due to the transducer when the flaw is not present.
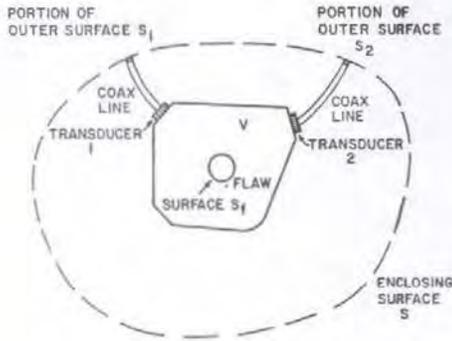
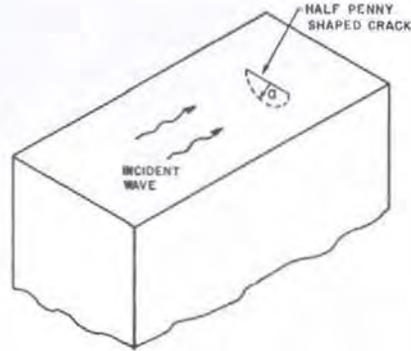Figure 1. A schematic of the region of integration used for the reciprocity theorem.

Figure 2. Schematic of half penny shaped crack.

One example which we have tackled is the scattering of a Rayleigh surface wave from a half penny shaped crack, whose radius a is much less than the wavelength λ, as shown in Fig. 2. In this case, only the longitudinal component of stress parallel to the surface is of importance and we can write

$$u_0 = \frac{\sigma_{zz}^{i}(1 - \nu)}{\pi \mu} \sqrt{a^2 - r^2} \qquad (4)$$

where σ_{zz}^{i} is the incident stress, ν is Poisson's ratio and μ the shear elasticity. Using this result in Eq. (3) we find that

$$S_{11}(\text{near field}) \approx \frac{a^3}{6\lambda^2 w} \eta g_z , \qquad (5)$$

$$S_{11}(\text{far field}) = \frac{a^3 w}{6\lambda^3 z} \eta q_z , \qquad (6)$$

$$\frac{S_{11}(\text{far})}{S_{11}(\text{near})} = \frac{w^2}{\lambda z} , \qquad (7)$$

where the transducer width is w; λ is the Rayleigh wavelength; the crack radius is $a \ll \lambda$; $g_z \approx 1$ and is a dimensionless calculable parameter; η is the tranducer efficiency; z the distance of the transducer from the crack. We note that this formula shows that the scattering depends on the radius cubed, i.e., it is as if the crack occupies a volume $V \sim a^3$. It will be observed that it is easy to carry out the two calculations of interest, one for the crack in the far field of the transducer, and one for the crack located in the near field of the transducer, it is assumed that the wave is straight crested in this region.

A similar analysis can be carried out for volume perturbations rather than for surface scattering. If we use the Born approximation for longitudinal waves, i.e., we assume $u_i = u_i^{i}$; $\sigma_{ij} = \sigma_{ij} = \sigma_{ij}^{i}$, it can be shown from Eq. (3) that the back scattered signal is

$$S_{11} = -\frac{j\omega}{4} \int \left(\omega^2(\Delta \rho)u - \epsilon_{ij}(\Delta c_{ijkl})\epsilon_{kl}\right) dV \quad (8)$$

where $\Delta \rho$ and Δc_{ijkl} are the perturbation in the density and elastic constants where the flaw is present. We can write this result for isotropic materials with a cross section much less than a wavelength in the form

$$|S_{11}| \approx \frac{\pi}{\lambda} \left(\frac{\Delta\rho}{\rho} - \frac{\Delta c}{c}\right) V_{\text{flaw}} \times \frac{\text{beam intensity at flaw}}{\text{power input to transducer}} \quad (9)$$

The last term contains information on whether the flaw is in the near field or far field of the transducer, and determines how in large part $|S_{11}|$ varies with frequency.

In the opposite limit when the flaw size is large we can consider the specular reflections from it. In this case Eq. (4) enables us to write

$$S_{11} = \frac{1}{4} \int (\sigma_{ij}^{i}\sigma_j^{s} - \sigma_{ij}^{s}\sigma_j^{i}) n_i dS \qquad (10)$$

where the superscript i stands for an incident wave, and s for the scattered waved. If we suppose that we know the reflection coefficient of a plane wave incident on the flaw surface from the transducer as illustrated in Fig. 3, we can write

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$$S_{11} \approx \frac{\pi}{2} \int_{\text{flaw}} \Gamma(\theta) \, F(\theta) e^{-jkR\theta^2} \, \sigma_{33}{}^i v_z{}^i R^2 \theta d\theta \, , \quad (11)$$

where $\Gamma(\theta)$ is the reflection coefficient at the flaw and $F(\theta)$ is a slowly varying function of θ with $F(\theta) = 1$ at $\theta = 0$. Assuming that the main contributions to the integral are from regions where θ is small we find that

$$|S_{11}| \approx \frac{R}{2} \, F_{\text{flaw}} (\theta = 0)$$

$$\times \quad \frac{\text{Beam intensity at flaw}}{\text{Power input to transducers}} \, , \qquad (12)$$

where R is the radius of curvature of the flaw at the point on the axis of the transducer. Thus the indication is that the return signal is proportional to the radius of curvature of the flaw.
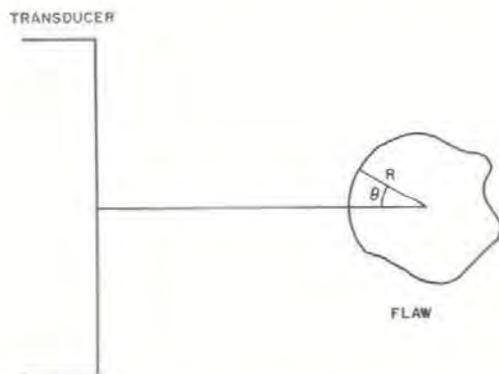
We shall not deal with other cases here, except to say that full variational scattering theorems have been derived using this technique and a start made on a scattering theorem in terms of the resonant modes of the flaw.

We believe that this technique is a simple but powerful method of analysis, and are carrying it further to examine a wide range of NDT problems of interest.

Reference

1. B. A. Auld, "Acoustic Fields and Waves in Solids, Vol. II," pp. 152-154, Wiley-Interscience, New York, 1973.

Figure 3. A schematic illustration of the notation used for the calculation of the specular reflection.