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EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH MULTIVALUED RIGHT-HAND SIDE

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TABLE OF CONTENTS

I. INTRODUCTION

In this study we consider sufficient conditions for the existence of solutions of differential equations with multivalued right-hand sides. In this chapter we give a brief discussion of such equations.

A relation of the form

$$\dot{x}(t) \in F(t,x(t)) \text{ a.e.} \tag{1.1}$$

is called a differential equation with multivalued right-hand side, where $x(t)$ is an unknown, absolutely continuous, n-dimensional vector-valued function, while $F(t,x)$ is a function which associates with every point (t,x) from a certain region of $(n+1)$-dimensional space a subset of n-dimensional space. Thus, the notion of a differential equation with multivalued right-hand side is a generalization of the usual notion of an ordinary differential equation.

The following are ways in which relations of the form (1.1) arise: differential equations of the form $f(t,x,\dot{x}) = 0$, differential inequalities $f(t,x,\dot{x}) \geq 0$, contingent equations (Alimov and Barbashin (1)), and control systems described by equations of the form

$$\dot{x}(t) = f(t,x(t),u(t)) \text{ a.e.,} \quad u(t) \in U \text{ a.e.}$$

where $x(t)$ and $u(t)$ are the unknown function and the control function, respectively, and U is a given set (Boltyanskii, et al. (2)).

The relation (1.1) together with an initial condition

$$\dot{x}(t) \in F(t,x(t)) \text{ a.e.,} \quad x(t_o) = x_o \qquad (1.2)$$

is called an initial value problem.

Some conditions must be imposed on the set-valued function $F(t,x)$ in order to guarantee the local existence of a solution of (1.2). Certainly it is sufficient to require that there exist a continuous single-valued function $f(t,x)$ such that $f(t,x) \in F(t,x)$ for all (t,x). Such a function f is called a continuous selection for F. In this case the Cauchy-Peano existence theorem applies (Coddington and Levinson, (4)). However, it is not necessary that such a function $f(t,x)$ exists in order that a solution of (1.2) exists, even though $F(t,x)$ is compact and continuous in the Hausdorff metric (Hermes (9)). If $F(t,x)$ is upper semicontinuous with respect to set inclusion in (t,x) and $F(t,x)$ is a compact, convex set for each (t,x), then a solution of (1.2) exists

(Alimov and Barbashin (1)). It is apparently unknown whether requiring that F(t,x) be compact and continuous in the Hausdorff metric is sufficient to guarantee the existence of a solution of (1.2). For other existence theorems, see Filippov (6).

In Chapter II we state some definitions and preliminary results. In Chapter III we prove some theorems in which conditions are imposed on F(t,x) so that the local existence of a solution of (1.2) is guaranteed. One of our lemmas is a generalization of Filippov's implicit function lemma (Filippov (7)). We also investigate continuity with respect to initial conditions for (1.2).

II. PRELIMINARIES

A. Notation

R^m denotes m-dimensional Euclidean space, and R^1 is written R. If

$$x,y \in R^m, x = (x^1,x^2,\cdots,x^m), y = (y^1,y^2,\cdots,y^m),$$

then

$$x \cdot y = x^1y^1 + x^2y^2 + \cdots + x^my^m$$

and

$$|x| = ((x^1)^2 + (x^2)^2 + \cdots + (x^m)^2)^{1/2}.$$

B_M denotes a closed ball in R^m of radius M centered at the origin. $\dot{x} = \frac{dx}{dt}$. The measure of a subset A of R is Lebesgue measure and is denoted $m(A)$. Almost everywhere is abbreviated as a.e.. If $\{a_i\}$ is a sequence of real numbers, then $a_i \downarrow a_o$ means that

$$a_i > a_{i+1}, i = 1,2,\cdots$$

and

$$\lim_{i\to\infty} a_i = a_o.$$

The closure of a set A is denoted by $cl A$. $[a,b]$ denotes a compact subinterval of R. $L_1[a,b]$ denotes the space of real-valued Lebesgue integrable functions on $[a,b]$. If we say that a vector-valued function is in $L_1[a,b]$, we shall mean that each of its components is in this space. If $f : [a,b] \to R^m$, then $V_{[a,b]}(f)$ denotes the variation of the function f on the interval $[a,b]$.

The distance between subsets of R^m is denoted by ρ. If F and G are subsets of R^m, the Hausdorff deviation between F and G, $\alpha(F,G)$, is defined by

$$\alpha(F,G) = \max\{\sup_{x\in F} \rho(x,G), \sup_{y\in G} \rho(y,F)\}$$

The Hausdorff deviation is a metric for the collection of nonempty compact subsets of R^m (Dugundji, (5)).

If F is a subset of R^m and ε is a positive real number,

$$F^\varepsilon = \{x \mid \rho(x,F) \leq \varepsilon\}.$$

Let $K(R^m)$ denote the collection of nonempty compact subsets of R^m.

B. Preliminary Results

Let A denote a subset of R^n, and let F be a set-valued function defined on A with values in $K(R^n)$.

Let $x_o \in A$ and let δ be a positive real number.

$$N_\delta(x_o) = \{x \mid x \in A \text{ and } |x - x_o| \leq \delta\}.$$

$$F(x_o, \delta) = \bigcup_{x \in N_\delta(x_o)} F(x).$$

<u>Definition 2.1</u>. F is upper semicontinuous with respect to set inclusion (u.s.c.i. for short) at $x_o \in A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_o| < \delta \text{ and } x \in A \text{ implies } F(x) \subset F^\varepsilon(x_o).$$

<u>Proposition 2.2</u>. If F is continuous in the Hausdorff metric at $x_o \in A$, then F is u.s.c.i. at x_o.

<u>Proof</u>. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that $|x - x_o| < \delta$ and $x \in A$ implies $\alpha(F(x), F(x_o)) < \varepsilon$. But $\alpha(F(x), F(x_o)) < \varepsilon$ implies $F(x) \subset F^\varepsilon(x_o)$. Q.E.D.

Proposition 2.3. Suppose A is closed. If $\bar{F}$ is u.s.c.i. on A, then the graph of F

$$M = \{(x,y) \mid x \in A, y \in F(x)\}$$

is a closed subset of R^{m+n}.

Proof. Let $\{(x_i,y_i)\}$ be a sequence of elements of M such that

$$(x_i,y_i) \to (x_o,y_o) \quad \text{as} \quad i \to \infty.$$

It follows that

$$x_i \to x_o, x_o \in A.$$

Let $\varepsilon > 0$ be given. There exists δ such that

$$|x - x_o| < \delta \quad \text{and} \quad x \in A \quad \text{imply} \quad F(x) \subset F^{\varepsilon}(x_o).$$

There exists k such that

$$i > k \quad \text{implies} \quad |x_i - x_o| < \delta.$$

Thus $y_i \in F^{\varepsilon}(x_o)$ for $i > k$. Hence $y_o \in F^{\varepsilon}(x_o)$. Since ε is arbitrary and $F(x_o)$ is closed, it follows that $y_o \in F(x_o)$. Q.E.D.

<u>Proposition 2.4</u>. If F is u.s.c.i. at $x_o \in A$, then

$$F(x_o) = \bigcap_{\delta>0} \text{cl}\, F(x_o, \delta).$$

<u>Proof</u>. Clearly

$$F(x_o) \subset \bigcap_{\delta>0} F(x_o, \delta).$$

Let $\{\varepsilon_i\}$ be given such that ε_i 0. There exists $\{\delta_i\}$, $\delta_i \downarrow 0$, such that

$$F(x_o, \delta_i) \subset F^{\varepsilon_i}(x_o), i = 1, 2, \cdots$$

by Definition 2.1. Hence

$$\text{cl}\, F(x_o, \delta_i) \subset F^{\varepsilon_i}(x_o), i = 1, 2, \cdots .$$

Thus

$$\bigcap_{i \geq 1} \text{cl}\, F(x_o, \delta_i) \subset \bigcap_{i \geq 1} F^{\varepsilon_i}(x_o).$$

But

$$\bigcap_{\delta>0} \mathrm{cl}\, F(x_o,\delta) = \bigcap_{i\geq 1} \mathrm{cl}\, F(x_o,\delta_i)$$

and

$$\bigcap_{i\geq 1} F^{\varepsilon_i}(x_o) = F(x_o). \qquad \text{Q.E.D.}$$

Proposition 2.5. Suppose A is compact. F is u.s.c.i. on A if and only if M is compact.

Proof. Assume F is u.s.c.i. on A. By Proposition 2.3, M is closed. Suppose M is unbounded. Then there exists a sequence $\{(x_i,y_i)\}$, such that $|y_i| \to \infty$ as $i \to \infty$, and $(x_i,y_i) \in M, i = 1,2,\cdots$. Since A is compact, there exists a subsequence of $\{x_i\}$, say $\{x_{i_j}\}$, such that

$$x_{i_j} \to x_o \quad \text{as} \quad j \to \infty.$$

Let ε be a positive number. Since F is u.s.c.i. at x_o, there exists k such that $j > k$ implies

$$y_{i_j} \in F^{\varepsilon}(x_o).$$

This is a contradiction since $F^{\varepsilon}(x_o)$ is bounded.

Assume M is compact. Suppose that there exists $x_o \in A$ such that F is not u.s.c.i. at x_o. Then there exists $\bar{\varepsilon} > 0$, $\{\delta_i\}$ such that $\delta_i \downarrow 0$, and $(x_i, y_i) \in M$ such that

$$|x_i - x_o| < \delta_i \text{ and } \rho(y_i, F(x_o)) > \bar{\varepsilon}.$$

Since M is compact, there exists a subsequence

$$\{(x_{i_j}, y_{i_j})\}$$

such that

$$(x_{i_j}, y_{i_j}) \to (x_o, y_o) \text{ as } j \to \infty.$$

It follows that $y_o \in F(x_o)$. This contradicts

$$\rho(y_{i_j}, F(x_o)) > \bar{\varepsilon}, j = 1, 2, \cdots.$$

Q.E.D.

<u>Proposition 2.6</u>. For $\bar{x} \in R^n$ and $F, G \in K(R^n)$,

$$|\rho(\bar{x}, F) - \rho(\bar{x}, G)| \leq \alpha(F, G)$$

Proof. We first show that

$$\rho(\bar{x},F) - \rho(\bar{x},G) \leq \alpha(F,G)$$

Let $E = \{x \mid \rho(x,F) \leq \alpha(F,G)\}$. If $\bar{x} \notin E$, then $\rho(\bar{x},E) = |\bar{x} - y|$ for some y on the boundary of E, and $\rho(y,F) = \alpha(F,G)$. $|\bar{x} - x| \leq |\bar{x} - y| + |y - x|$ for any $x \in F$. Taking the infimum of both sides in the latter inequality for $x \in F$, we obtain $\rho(\bar{x},F) \leq \rho(\bar{x},E) + \alpha(F,G)$. If $\bar{x} \notin E$, then $\rho(\bar{x},E) \leq \rho(\bar{x},G)$, since $G \subset E$. Therefore, for $\bar{x} \notin E, \rho(\bar{x},F) \leq \rho(\bar{x},G) + \alpha(F,G)$. If $\bar{x} \in E$, $\rho(\bar{x},F) \leq \alpha(F,G) \leq \alpha(F,G) + \rho(\bar{x},G)$. By reversing the roles of F and G, $\rho(\bar{x},G) - \rho(\bar{x},F) \leq \alpha(F,G)$. Q.E.D.

Proposition 2.7. For $x,y \in R^n$ and $F \in K(R^n)$,

$$|\rho(x,F) - \rho(y,F)| \leq |x - y|$$

Proof. Let $x_o,y_o \in F$ such that $|x - x_o| = \rho(x,F)$ and $|y - y_o| = \rho(y,F)$. Suppose that

$$|x - y| < |x - x_o| - |y - y_o|.$$

We have $|x - x_o| > |x - y| + |y - y_o|$. Hence

$|x - x_o| > |x - y_o|$. This is a contradiction. By a similar argument, $|y - y_o| - |x - x_o| \leq |x - y|$. Q.E.D.

We next state two theorems which will be needed in Chapter III.

Theorem 2.8. (Helley's Theorem): Let $f_m : [a,b] \to R^n$, $m = 1,2,\cdots$. Suppose that there exists a constant K such that

$$|f_m(t) \leq K \quad \text{for all} \quad t \in [a,b], m = 1,2,\cdots$$

and

$$V_{[a,b]}(f_m) \leq K, m = 1,2,\cdots.$$

Then there exists a subsequence of $\{f_m\}$ which converges pointwise on $[a,b]$ to a limit function f which is of bounded variation on $[a,b]$.

Proof. See Taylor (11).

Theorem 2.9. (Scorza-Dragoni's Theorem): Suppose $f : E \times G \to R^n$, where G is a closed subset of R^r and E is a closed subset of $[a,b]$. Suppose that $f(t,u)$ is continuous in u for fixed t and measurable in t for fixed u. Then for any $\varepsilon > 0$, there corresponds a perfect set $S \subset E$, such that $m(S) > m(E) - \varepsilon$, and f is jointly continuous in (t,u) on $S \times G$.

Proof. See Goodman (8).

III. EXISTENCE OF SOLUTIONS

In this chapter we prove theorems on local existence for (1.2), theorems relating to continuity with respect to initial conditions, and a generalization of Filippov's implicit function lemma.

Lemma 3.1. Let U be a set-valued function from $[a,b]$ to $K(R^r)$. Let U be u.s.c.i. on $[a,b]$. Let G be a closed subset of R^r such that

$$\bigcup_{t\in[a,b]} U(t) \subset G.$$

Let $f:[a,b] \times G \to R^n$ be such that $f(t,u)$ is continuous in u for fixed t and measurable in t for fixed u. Let B be a closed subset of R^n. If there exists a function $\bar{u}:[a,b] \to G$ such that $\bar{u}(t) \in U(t)$ a.e., and $f(t,\bar{u}(t)) \in B$ a.e., then there exists a measurable function $u:[a,b] \to G$ such that $u(t) \in U(t)$ a.e., and $f(t,u(t)) \in B$ a.e.

Proof. By assumption $\bar{u}(t) \in U(t)$ and $f(t,\bar{u}(t)) \in B$ for all t in a measurable set E, $m(E) = b - a$. Choose a closed set $E_i \subset E$ for each positive integer i, such that

$$m(E_i) > \max(0,(b - a) - \tfrac{1}{i}).$$

We define $P(t) = \{u \mid u \in U(t), f(t,u) \in B\}$ for each $t \in E$. By assumption, $P(t)$ is not empty for all $t \in E$. Since E_i is closed, we may apply Theorem 2.9 and conclude that for each positive integer j, there exists $C_{ij} \subset E_i$ with C_{ij} perfect, and

$$m(C_{ij}) > \max\,(0, m(E_i) - \tfrac{1}{j})$$

such that f is jointly continuous in (t,u) on $C_{ij} \times G$.

We show next that $P(t)$ is compact for each $t \in E$. Since $U(t)$ is compact, $P(t)$ is bounded. Suppose $u_m \to \bar{u}$ as $m \to \infty$ and $u_m \in P(t)$, $m = 1,2,\cdots$. Since f is continuous in u for fixed t, $f(t,\bar{u}) \in B$. $\bar{u} \in U(t)$ since $U(t)$ is closed. Hence $P(t)$ is closed.

For each $t \in E$, we define $u(t)$ as follows: Let P_1 be the subset of $P(t)$ with u^1 minimum, let P_2 be the subset of P_1 with u^2 minimum, $\cdots$, let P_r be the subset of P_{r-1} with u^r minimum. P_r is a single point in $U(t)$, $u = u(t)$.

We now show by induction that $u^1(t), \cdots, u^r(t)$ are measurable. Let us assume that $u^1(t), \cdots, u^{s-1}$ are measurable. (If $s = 1$, nothing need be assumed.)

For each positive integer k there exists a closed subset D_{ijk} of C_{ij} such that

$$m(D_{ijk}) > \max(0, m(C_{ij}) - \frac{1}{k})$$

and $u^1(t), \cdots, u^{s-1}(t)$ are continuous on D_{ijk}. We next show that $u^s(t)$ is measurable on D_{ijk}. We do this by showing that, for every real c, the set of all $t \in D_{ijk}$ with $u^s(t) \leq c$ is closed. Suppose that this is not the case. Then there exists a sequence $\{t_n\}$ with $t_m \in D_{ijk}$, $u^s(t_m) \leq c$, and $t_m \to \bar{t} \in D_{ijk}$, $u^s(\bar{t}) > c$. Then $u^\alpha(t_m) \to u^\alpha(\bar{t})$, $\alpha = 1, 2, \cdots, s-1$. Since $U(t)$ is u.s.c.i. on $[a,b]$ it follows by Proposition 2.5 that there is a constant L such that $|u^\beta(t_m)| \leq L$ for $\beta = s, s+1, \cdots, r$. We select a subsequence $\{t_{m_n}\}$ such that

$$u^\beta(t_{m_n}) \to \tilde{u}^\beta$$

as $n \to \infty$, for $\beta = s, s+1, \cdots, r$ for some real numbers $\tilde{u}^\beta$. As

$$t_{m_n} \to \bar{t}, u(t_{m_n}) \to \tilde{u}$$

where $\tilde{u} = (u^1(\bar{t}), \cdots, u^{s-1}(\bar{t}), \tilde{u}^s, \cdots, \tilde{u}^r)$. Given any number $\delta > 0$, we have

$$u(t_{m_n}) \in U(t_{m_n})$$

and

$$U(t_{m_n}) \subset c\ell\, U(\bar{t},\delta)$$

provided n is sufficiently large. Thus

$$\tilde{u} \in \bigcap_{\delta>0} c\ell\, U(\bar{t},\delta)$$

and

$$\bigcap_{\delta>0} c\ell\, U(\bar{t},\delta) = U(\bar{t})$$

by Proposition 2.4. Since

$$f(t_{m_n},u(t_{m_n})) \in B$$

and B is closed, $f(\bar{t},\tilde{u}) \in B$. Since

$$u^S(t_{m_n}) \leq c,$$

$\tilde{u}^s \leq c$. Also $\bar{t} \in D_{ijk}$ and $D_{ijk} \subset E$, $f(\bar{t},u(\bar{t})) \in B$ and $u^s(\bar{t}) > c$. This is a contradiction to the way $u^s(\bar{t})$ was chosen. Thus $u^s(t)$ is measurable on D_{ijk}. For any $\varepsilon > 0$, there exist integers i_1, j_1, and k_1 such that

$$m(D_{i_1,j_1,k_1}) > (b - a) - \varepsilon.$$

Hence $u^s(t)$ is measurable on $[a,b]$. Thus, by induction, $u(t) = (u^1(t),\cdots,u^r(t))$ is measurable on $[a,b]$. Q.E.D.

Lemma 3.2. If $U : [a,b] \to K(R^r)$ and $\tilde{U} : [a,b] \to K(R^r)$, $u : [a,b] \to R^r$, $u(t) \in U(t)$ a.e., and $\alpha(\tilde{U}(t),U(t)) \leq k(t)$ a.e. where $k(t)$ is measurable, then there exists $\tilde{u}(t)$ measurable on $[a,b]$, $\tilde{u}(t) \in \tilde{U}(t)$ a.e. such that $|\tilde{u}(t) - u(t)| \leq k(t)$ a.e. $[a,b]$.

Proof. Since $u(t) \in U(t)$, and $\rho(u(t),\tilde{U}(t)) \leq \alpha(U(t),\tilde{U}(t))$ it follows that there exists $\bar{u}(t) \in \tilde{U}(t)$ such that

$$|\bar{u}(t) - u(t)| = \rho(u(t),\tilde{U}(t))$$

and

$$|\bar{u}(t) - u(t)| \leq k(t) \text{ a.e.}$$

In Lemma 3.1, let $f(t,u) = |u - u(t)| - k(t)$ and let $B = \{x \mid x \leq 0\}$. The existence of $\tilde{u}(t)$ now follows from Lemma 3.1. Q.E.D.

We next state some hypotheses which will be needed later on. Let $t \in R$ and $x \in R^n$.

(a) $F(t,x)$ is a nonempty compact set contained in R^n

(b) F is u.s.c.i. in (t,x)

(c) There exists an integrable function $k(t)$ such that for any t,x,x'

$$\alpha(F(t,x'),F(t,x)) \leq k(t)|x' - x|$$

(d) F is continuous in (t,x), that is,

$$\alpha(F(t',x'),F(t,x)) \to 0 \text{ as } t' \to t, x' \to x.$$

<u>Theorem 3.3</u>. Let $F(t,x)$ satisfy conditions (a), (b), and (c) in the region $A = \{(t,x) \mid a \leq t \leq b, |x - y(t)| \leq c\}$, where $y(t)$ is absolutely continuous on $[a,b]$. Let M be a real number such that $|\dot{y}(t)| \leq M$ a.e. $[a,b]$. Let $a < t_o < b$ and $a < \tilde{t}_o < b$. Let $\rho(t)$ be integrable and suppose there exists $u(t) \in F(t,y(t))$ such that $u(t)$ is measurable and $|\dot{y}(t) - u(t)| \leq \rho(t)$ a.e. Let

$$|y(t_o) - \tilde{x}_o| \le \frac{\sigma}{2}$$

and let

$$|t_o - \tilde{t}_o| \le \frac{\sigma}{2M}$$

where $\sigma < c$. Then a solution $x(t)$ to the problem

$$\dot{x} \in F(t,x), x(\tilde{t}_o) = \tilde{x}_o \qquad (3.1)$$

exists such that

$$|x(t) - y(t)| \le \xi(t) \qquad (3.2)$$

$|\dot{x}(t) - \dot{y}(t)| \le k(t)\xi(t) + \rho(t)$ a.e., where

$$\xi(t) = \sigma e^{m(t)} + \left|\int_{\tilde{t}_o}^{t} e^{m(t)-m(s)}\rho(s)ds\right|$$

$$m(t) = \left|\int_{\tilde{t}_o}^{t} k(r)dr\right|.$$

(The solution exists for all $t \in E$, where

$$E = \{t \mid \xi(t) \le c,\ a \le t \le b\}.)$$

Proof. Let $x_o(t) = y(t)$ and $u_o(t) = u(t)$ for $t \in [a,b]$. Let

$$x_1(t) = \tilde{x}_o + \int_{\tilde{t}_o}^{t} u_o(s)ds.$$

It follows that

$$|\dot{x}_1(t) - \dot{x}_o(t)| \le \rho(t) \tag{3.3}$$

and

$$\begin{aligned} |x_1(t) - x_o(t)| &\le |y(t_o) - \tilde{x}_o| + \left|\int_{t_o}^{t} \dot{y}(s)ds - \int_{\tilde{t}_o}^{t} u_o(s)ds\right| \\ &\le |y(t_o) - x_o| + \left|\int_{t_o}^{\tilde{t}} |\dot{y}(s)|ds\right| + \left|\int_{\tilde{t}_o}^{t} |\dot{y}(s) - u_o(s)|ds\right| \\ &\le |y(t_o) - x_o| + M|t_o - \tilde{t}_o| + \left|\int_{\tilde{t}_o}^{t} \rho(s)ds\right| \\ &\le \sigma + \left|\int_{\tilde{t}_o}^{t} \rho(s)ds\right| \end{aligned} \tag{3.4}$$

$x_2(t)$ is defined on E as follows: Let $u_1(t)$ be defined and measurable on E such that $u_1(t) \in F(t,x_1(t))$ and

$$|u_1(t) - u_o(t)| \leq k(t)|x_1(t) - x_o(t)| \text{ a.e.}$$

(u_1 exists by Lemma 3.2).

$$x_2(t) = \tilde{x}_o + \int_{\tilde{t}_o}^{t} u_1(s)ds.$$

It follows that

$$|\dot{x}_2(t) - \dot{x}_1(t)| \leq k(t)\{\sigma + |\int_{\tilde{t}_o}^{t} \rho(s)ds|\}$$

and

$$|x_2(t) - x_1(t)| \leq \sigma m(t) + |\int_{\tilde{t}_o}^{t} [m(t) - m(s)]\rho(s)ds|.$$

The proof of the latter inequality is based on the following observations:

For $t \geq \tilde{t}_o$,

$$\frac{d}{dt}[\sigma m(t) + m(t) \int_{\tilde{t}_o}^{t} \rho(s)ds - \int_{\tilde{t}_o}^{t} m(s)\rho(s)ds]$$

$$= \sigma k(t) + k(t) \int_{\tilde{t}_o}^{t} \rho(s)ds + m(t)\rho(t) - m(t)\rho(t), \text{ a.e.}$$

For $t \leq t_o$,

$$\frac{d}{dt}[\sigma m(t) + m(t) \int_{t}^{\tilde{t}_o} \rho(s)ds - \int_{t}^{\tilde{t}_o} m(s)\rho(s)ds]$$

$$= \sigma(-k(t)) + (-k(t)) \int_{t}^{\tilde{t}_o} \rho(s)ds - m(t)\rho(t) + m(t)\rho(t), \text{ a.e.}$$

We proceed to define a sequence $\{x_i\}$ of functions on E by induction.

Assume that $x_1, x_2, \cdots, x_i, x_{i+1}$ are defined on E, and that

$$|\dot{x}_{i+1}(t) - \dot{x}_i(t)| \leq k(t)\{\sigma \frac{[m(t)]^{i-1}}{(i-1)!} + |\int_{\tilde{t}_o}^{t} \frac{[m(t) - m(s)]^{i-1}}{(i-1)!} \rho(s)ds|\} \tag{3.5}$$

and

$$|x_{i+1}(t) - x_i(t)| \leq \sigma \frac{[m(t)]^i}{i!} + |\int_{\tilde{t}_o}^{t} \frac{[m(t) - m(s)]^i}{i!} \rho(s)ds| \tag{3.6}$$

Then x_{i+2} is defined as follows:

From (3.6) it follows that $|x_{i+1}(t) - y(t)| \leq \xi(t)$. Hence for $t \in E$, there exists $u_{i+1}(t)$ measurable and such that $u_{i+1}(t) \in F(t, x_{i+1}(t))$ a.e. and

$$|u_{i+1}(t) - u_i(t)| \leq k(t)|x_{i+1}(t) - x_i(t)| \tag{3.7}$$

Let $x_{i+2}(t) = \tilde{x}_o + \int_{\tilde{t}_o}^{t} u_{i+1}(s)ds$. (3.6) and (3.7) imply

$$|\dot{x}_{i+2}(t) - \dot{x}_{i+1}(t)| \leq k(t)\{\sigma \frac{[m(t)]^i}{i!} + |\int_{\tilde{t}_o}^{t} \frac{[m(t) - m(s)]^i}{i!} \rho(s)ds|\} \tag{3.5'}$$

which is (3.5) with i replaced by $i + 1$. (3.5') implies

$$|x_{i+2}(t) - x_{i+1}(t)| \leq \sigma \frac{[m(t)]^{i+1}}{(i + 1)!} + |\int_{\tilde{t}_o}^{t} \frac{[m(t) - m(s)]^{i+1}}{(i + 1)!} \rho(s)ds| \tag{3.6'}$$

which is (3.6) with i replaced by $i + 1$. The proof that (3.5') implies (3.6') is based on an observation similar to the one made in the case $i = 0$. By induction, we conclude that $\{x_i\}$ is defined on E and (3.5) and (3.6) hold. (3.6) implies that $|x_i(t) - y(t)| \leq \xi(t)$ on E since

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^i}{i!} \leq e^z \quad \text{for} \quad z \geq 0.$$

From (3.5) it follows that $u_i(t) \to v(t)$ as $i \to \infty$ for some v, a.e. on E. From (3.6) it follows that $x_i(t) \to x(t)$ uniformly for some $x(t)$ on E. Since $F(t,x)$ is u.s.c.i. and $u_i(t) \in F(t,x_i(t))$ it follows that $v(t) \in F(t,x(t))$ a.e. From (3.3) and (3.5)

$$|\dot{x}(t) - \dot{y}(t)| \leq k(t)\xi(t) + \rho(t) \text{ a.e.}$$

Since the u_i are uniformly bounded on E, they are integrable, and

$$x(t) = \lim_{i\to\infty} x_i(t) = \lim_{i\to\infty}(\tilde{x}_o + \int_{\tilde{t}_o}^{t} u_{i-1}(s)ds)$$

$$= \tilde{x}_o + \int_{\tilde{t}_o}^{t} v(s)ds$$

by Lebesgue's dominated convergence theorem. Q.E.D.

Corollary 3.4. If $F(t,x)$ satisfies conditions (a), (b), and (c) in the region $A = \{(t,x) \mid a \leq t \leq b, |x - \tilde{x}_o| \leq c\}$, then a solution of (1.2) exists on an open interval containing t_o. (Here, $x_o = \tilde{x}_o$ and $t_o = \tilde{t}_o$.)

Proof. Let $y(t) = \tilde{x}_o$, $a \leq t \leq b$. Choose $u(t)$ measurable, $u(t) \in F(t,\tilde{x}_o)$ by Lemma 3.1. Since $F(t,x)$ is u.s.c.i., $u(t)$ is bounded by some constant L. Let $\rho(t) = L$. We now apply Theorem 3.3. Q.E.D.

Definition 3.5. The attainable set $A(t,t_o,x_o)$ for the problem (1.2) is $A(t,t_o,x_o) = \{x(t) \mid x$ is a solution of (1.2) on $[t_o,t]\}$ for $t \geq t_o$, and $A(t,t_o,x_o) = \{x(t) \mid x$ is a solution of (1.2) on $[t,t_o]\}$ for $t < t_o$.

Theorem 3.6. Let F satisfy conditions (a), (b), and (c) in the region $[a,b] \times R^n$. Let B be a subset of R^n, and let $B^* = [a,b] \times B$. Suppose that there exists L such that for all solutions x satisfying (1.2), where (t_o,x_o) is any point of B^*, it is true that $|x(t)| \leq L$ for all t for which the solution x is defined.

Then $A(t,t_o,x_o)$ is uniformly continuous on $[a,b]^2 \times B$.

Proof. Since all solutions x with initial data in B^* are assumed to satisfy $|x(t)| \leq L$, and since $F(t,x)$ is compact and u.s.c.i., it follows that there exists a constant M such that the absolute values of the derivatives of such solutions are bounded by M. Let $m(t)$ be as in Theorem 3.3, that is,

$$m(t) = \left|\int_{\tilde{t}_o}^{t} k(r)dr\right| \leq \int_a^b k(r)dr.$$

Let

$$\int_a^b k(r)dr = k_1.$$

Let $\varepsilon > 0$ be given. Let $(t,t_o,x_o) \in [a,b]^2 \times B$ and let $(\tilde{t},\tilde{t}_o,\tilde{x}_o)$ be any point in $[a,b]^2 \times B$ such that

$$|t_o - \tilde{t}_o| < \frac{\varepsilon}{3e^{k_1}},\ |x - \tilde{x}_o| < \frac{\varepsilon}{3Me^{k_1}},\ |t - \tilde{t}| < \frac{\varepsilon}{3M}.$$

By Corollary 3.4, and the assumption of uniform boundedness of solutions, we conclude that there is at least one solution having initial data (t_o,x_o) for each $(t_o,x_o) \in B^*$, and any solution may be continued to all of $[a,b]$.

Let $y(t)$ be any solution of (1.2) on $[a,b]$ with initial data (t_o,x_o). Let $u(t) = \dot{y}(t)$. By Theorem 3.3, there exists a solution $x(t)$ of

$$\dot{x} \in F(t,x), x(t_o) = \tilde{x}_o$$

such that

$$|x(t) - y(t)| \leq \frac{2\varepsilon}{3e^{k_1}} e^{m(t)} \leq \frac{2\varepsilon}{3}$$

and

$$|x(\tilde{t}) - x(t)| \leq M|t - \tilde{t}| \leq \frac{\varepsilon}{3}.$$

Hence,

$$|x(\tilde{t}) - y(t)| \leq |x(\tilde{t}) - x(t)| + |x(t) - y(t)| \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}.$$

Thus $A(t,t_o,x_o) \subset A^{\varepsilon}(\tilde{t},\tilde{t}_o,\tilde{x}_o)$.

Similarly, let $z(t)$ be any solution of $\dot{x} \in F(t,x)$ with initial data $(\tilde{t}_o,\tilde{x}_o)$. We may use the same argument as above to obtain a solution $w(t)$ of $\dot{x} \in F(t,x)$ with initial data (t_o,x_o) such that

$$|z(\tilde{t}) - w(t)| \leq \varepsilon.$$

Thus $A(\tilde{t},\tilde{t}_o,\tilde{x}_o) \subset A^{\varepsilon}(t,t_o,x_o)$, and therefore

$\alpha(A(t,t_o,x_o),A(\tilde{t},\tilde{t}_o,\tilde{x}_o)) \leq \varepsilon$ provided $|t_o - \tilde{t}_o| < \delta$, $|x_o - \tilde{x}_o| < \delta$, $|t - \tilde{t}| < \delta$ where

$$\delta = \min\left(\frac{\varepsilon}{3e^{k_1}}, \frac{\varepsilon}{3Me^{k_1}}, \frac{\varepsilon}{3M}\right). \qquad \text{Q.E.D.}$$

In the preceding theorem, if B is compact and there is a constant K such that

$$|u \cdot x| \leq K[1 + |x|^2] \tag{3.8}$$

for all $u \in F(t,x)$, $(t,x) \in [a,b] \times R^n$, then all solutions of (3.8) with initial data in B^* are uniformly bounded on $[a,b]$.

We remark that (3.8) is a special case of a condition involving a Liapunov function which implies uniform boundedness of solutions.

From now on, unless explicitly stated otherwise, we assume that F satisfies conditions (a) and (b) on $[a,b] \times B_r$. Let x_o be such that $|x_o| < r$ and let t_o be such that $a < t_o < b$.

Since $F(t,x)$ is u.s.c.i. and compact on $[a,b] \times B_r$, it follows that there exists $M > 0$ such that

$$\bigcup_{(t,x)\in[a,b]\times B_r} F(t,x) \subset B_M.$$

We choose α sufficiently small that the cone C,

$$C = \{(t,x) \mid |x - x_o| \leq M(t - t_o), t_o \leq t \leq t_o + \alpha\}$$

is contained in $[t_o,b] \times B_r$.

Definition 3.7. The collection, E, of Euler polygons associated with $F(t,x)$ at (t_o,x_o), is defined as follows: $x \in E$ if and only if there exists a partition $t_o < t_1 < \cdots < t_m = t_o + \alpha$ of $[t_o,t_o + \alpha]$ and $v_o \in F(t_o,x_o)$ such that $x(t_o) = x_o$ and $x(t) = x(t_i) + (t - t_i)v_i$, $t_i \leq t \leq t_{i+1}$, $i = 0,1,2,\cdots,m-1$, where v_i is a point of the set $F(t_i,x(t_i))$ closest to the point v_{i-1}. (If more than one such point exists, select one.)

Definition 3.8. The collection, D, of derivatives of Euler polygons associated with $F(t,x)$ at (t_o,x_o) is defined as follows: $\varphi \in D$ if and only if there exists $x \in E$ such that $\varphi(t_i) = v_i, i = 0,1,\cdots,m$ and φ is constant on the intervals $[t_{i-1},t_i)$, $i = 1,2,\cdots,m$.

Lemma 3.9. Suppose that $\{\varepsilon_j\}$ is a sequence of real numbers such that $\varepsilon_j \downarrow 0$. Suppose that $\{x_j\}$ is a sequence of functions, $x_j : [a,b] \to B_r$, and $x_j(t) \to x(t)$ as $j \to \infty$, $a \le t \le b$. Suppose that $\{\varphi_j\}$ is a sequence of functions, $\varphi_j : [a,b] \to R^n$, such that $\varphi_j(t) \to \varphi(t)$ as $j \to \infty$ a.e. on $[a,b]$ and such that

$$\varphi_j(t) \in F^{\varepsilon_j}(t,x_j(t)) \text{ a.e. on } [a,b],\ j = 1,2,\cdots.$$

Then $\varphi(t) \in F(t,x(t))$ a.e. $[a,b]$.

Proof. Let $\varepsilon > 0$ be given. Choose I_1 sufficiently large that $j > I_1$ implies $\varepsilon_j < \varepsilon/2$. Choose $I_2 = I_2(\varepsilon,t)$ such that $j > I_2$ implies

$$F(t,x_j(t)) \subset F^{\varepsilon/2}(t,x(t)).$$

Thus $j > \max(I_1,I_2)$ implies

$$F^{\varepsilon_j}(t,x_j(t)) \subset [F^{\varepsilon/2}(t,x(t))]^{\varepsilon/2}.$$

Suppose that

$$\varphi_j(\bar{t}) \in F^{\varepsilon_j}(\bar{t},x_j(\bar{t})), j = 1,2,\cdots .$$

We have that $j > \max[I_1, I_2]$ implies $\varphi_j(\bar{t}) \in F^{\varepsilon}(\bar{t}, x(\bar{t}))$. Hence $\varphi(t) \in F^{\varepsilon}(t, x(t))$ a.e. $[a,b]$. Since $F(t,x)$ is closed and ε is arbitrary, it follows that $\varphi(t) \in F(t,x(t))$ a.e. $[a,b]$. Q.E.D.

Lemma 3.10. Suppose that x_j is a solution on $[a,b]$ of

$$\dot{x} \in F^{\varepsilon_j}(t,x), x(t_o) = x_o, \; j = 1,2,\cdots$$

where $a < t_o < b$ and $\varepsilon_j \downarrow 0$. Suppose also that $\dot{x}_j(t) \to \varphi(t)$ a.e. $[a,b]$ as $j \to \infty$. Then

$$x(t) = x_o + \int_{t_o}^{t} \varphi(s)ds$$

is a solution of (1.2) on $[a,b]$.

Proof. Let $\varphi_j = \dot{x}_j$ in Lemma 3.9. Since $F(t,x)$ is u.s.c.i. and compact, there exists M such that $|\dot{x}_j(t)| \leq M$ a.e., $j = 1,2,\cdots$. By Lebesgue's dominated convergence theorem, $x_j(t) \to x(t)$ as $j \to \infty$, $a \leq t \leq b$.

Q.E.D.

Lemma 3.11. Let F satisfy (d) in addition to previous hypotheses.

Let $\{\varepsilon_k\}$ be a sequence of real numbers such that

$\varepsilon_k \downarrow 0$. Then the collection E contains a sequence of elements $\{\tilde{x}_k\}$ such that $\tilde{x}_k$ is a solution of

$$\dot{x} \in F^{\varepsilon_k}(t,x), x(t_o) = x_o$$

on $[t_o, t_o + \alpha]$ for $k = 1,2,3,\cdots$.

<u>Proof</u>. Let x_j be constructed as follows: Choose $v_o \in F(t_o, x_o)$. Let $x_j(t_o) = x_o$. Let $h = h_j = 2^{-j}\alpha$ and $t_i = t_o + ih$ for $i = 1,2,\cdots,2^j$. Now finish the construction of x_j as in Definition 3.7.

Corresponding to ε_k, choose J_k such that $J_k > J_{k-1}$ (for $k = 1$, the latter may be disregarded) and such that $j \geq J_k$ implies $Mh_j < \delta$ and $h_j < \delta$, where δ is chosen such that $|t - t'| < \delta$, $|x - x'| < \delta$, and $(t,x),(t',x') \in [t_o, t_o + \alpha] \times B_r$ implies $\alpha(F(t,x),F(t',x')) < \varepsilon_k$. Hence,

$$\alpha(F(t_{i-1},x_j(t_{i-1})),F(t,x_j(t)) < \varepsilon_k$$

for $t_{i-1} \leq t \leq t_i$, provided $j \geq J_k$. Hence, $\rho(\varphi_j(t),F(t,x_j(t)) \leq \varepsilon_k$ for $t_{i-1} \leq t \leq t_i$, $j \geq J_k$. Hence

$$\varphi_j(t) \in F^{\varepsilon_k}(t,x_j(t)),$$

$t_o \leq t \leq t_o + \alpha$, for $j \geq J_k$. Define

$$\tilde{x}_k = x_{J_k} . \qquad \text{Q.E.D.}$$

The following example shows that Lemma 3.11 is not true if F is required to be u.s.c.i. and $F(t,x)$ is required to be compact and convex for each (t,x).

Example.

$$F(x) = 2 + \sin \frac{1}{x}, \quad x \neq 0$$

$$= [1,3], \quad x = 0.$$

Clearly $F(x)$ is u.s.c.i. in x, but not continuous in the Hausdorff metric. Let $t_o = 0$ and $x_o = 0$. Let x be any Euler polygon. Since x has positive constant derivative on $[0,t_1)$, there exists an interval $I \subset [0,t_1)$ such that $m(I) > 0$ and

$$\dot{x}(t) \notin F^{1/4}(x(t)) \quad \text{for} \quad t \in I.$$

Definition 3.12. D is said to be of uniformly bounded variation with constant K in case there exists $K < \infty$

such that

$$|\varphi(t)| \leq K, \; t_o \leq t \leq t_o + \alpha \quad \text{and} \quad V_{[t_o, t_o+\alpha]}(\varphi) \leq K$$

for all $\varphi \in D$.

Theorem 3.13. Let F satisfy (d) in addition to previous hypotheses. Suppose that D is of uniformly bounded variation with constant K. Then there exists a solution x of (1.2) on $[t_o, t_o+\alpha]$ such that $\dot{x}(t) = \dot{\varphi}(t)$ a.e. $[t_o, t_o+\alpha]$ where $V_{[t_o, t_o+\alpha]}(\varphi) \leq K$.

Proof. Let $\{\varepsilon_j\}$ be a sequence of real numbers such that $\varepsilon_j \downarrow 0$. Let $\{x_j\}$ be a sequence of Euler polygons associated with $\{\varepsilon_j\}$ as given by Lemma 3.11. Let $\{\varphi_j\}$ be the sequence of derivatives of $\{x_j\}$ (see Definition 3.8). By Helly's theorem, it follows that there exists a function of bounded variation φ such that

$$\varphi_{j_i}(t) \to \varphi(t), t_o \leq t \leq t_o + \alpha \quad \text{as} \quad i \to \infty$$

for some subsequence $\{\varphi_{j_i}\}$ of $\{\varphi_j\}$. By Lemma 3.10, it follows that

$$x(t) = x_o + \int_{t_o}^{t} \varphi(s)ds$$

is the required solution. Furthermore,

$$V_{[t_o,t_o+\alpha]}(\varphi_{j_i}) \leq K, \quad i = 1,2,\cdots$$

implies $V_{[t_o,t_o+\alpha]}(\varphi) \leq K$. Q.E.D.

Theorem 3.14. If there exists a function $f : [t_o, t_o + \alpha] \to R$, f of bounded variation, and a constant L such that

$$\alpha(F(t,x),F(t',x')) \leq |f(t) - f(t')| + L|x - x'|$$

for all $(t,x),(t',x') \in C$, then D is of uniformly bounded variation with constant

$$K = \max[M, V_{[t_o,t_o+\alpha]}(f) + LM\alpha]$$

Proof. Let $x \in E$ and let $\varphi \in D$ be the derivative of x.

$$V_{[t_o,t_o+\alpha]}(\varphi) = \sum_{i=1}^{m} |\varphi(t_{i-1}) - \varphi(t_i)|$$

$$\leq \sum_{i=1}^{m} \rho(\varphi(t_{i-1}),F(t_i,x(t_i))$$

$$\leq \sum_{i=1}^{m} \alpha(F(t_{i-1},x(t_{i-1})),F(t_i,x(t_i)))$$

$$\leq \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|$$

$$+ L \sum_{i=1}^{m} |x(t_i) - x(t_{i-1})|$$

$$\leq V_{[t_o,t_o+\alpha]}(f) + L[M\alpha] \qquad \text{Q.E.D.}$$

Theorem 3.15. If there exist $f:[t_o,t_o+\alpha] \to R$ and $g:[|x_o|,|x_o| + M\alpha] \to R$, f and g of bounded variation and

$$\alpha(F(t,x),F(t',x')) \leq |f(t) - f(t')|$$

$$+ |g(|x|) - g(|x'|)|$$

for all (t,x), $(t',x') \in C$ with $|x_o| \leq |x|$, $|x_o| \leq |x'|$

and if $y \in F(t,x)$ implies $y^i \geq 0$, $i = 1,2,\cdots,n$ for all $(t,x) \in C$, then D is of uniformly bounded variation with constant

$$K = \max[M, V_{[t_o, t_o+\alpha]}(f) + V_{[|x_o|, |x_o|+M\alpha]}(g)].$$

Proof. Let $x \in E$ and let $\varphi \in D$ be the derivative of x. We note that $x^i(t)$ is nondecreasing on $[t_o, t_o+\alpha]$, $i = 1,\cdots,n$ since $\varphi^i(t) \geq 0$, $i = 1,\cdots,n$.

$$V_{[t_o, t_o+\alpha]}(\varphi) = \sum_{i=1}^{m} |\varphi(t_{i-1}) - \varphi(t_i)|$$

$$= \sum_{i=1}^{m} \rho(\varphi(t_{i-1}), F(t_i, x(t_i)))$$

$$\leq \sum_{i=1}^{m} \alpha(F(t_{i-1}, x(t_{i-1})), F(t_i, x(t_i)))$$

$$\leq \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|$$

$$+ \sum_{i=1}^{m} |g(|x(t_i)|) - g(|x(t_{i-1})|)|$$

$$\leq V_{[t_o,t_o+\alpha]}(f) + V_{[|x_o|,|x_o|+M\alpha]}(g).$$

Q.E.D.

In the following example, F has no continuous selection and F does not satisfy a Lipschitz condition, however, the derivatives of the Euler polygons are of uniformly bounded variation.

Example. F depends only on $x \in R^3$, and $F(x) \subset R^3$. $F(x) = \{y \mid y \cdot x = 0, |y| = \sqrt{|x|}\}$ for each $x \in R^3$. $x_o = 0$, $t_o = 0$. Suppose a continuous selection for F exists, say $f(x)$, in some neighborhood N of 0. Let S be a sphere centered at 0, $S \subset N$. f restricted to S defines a continuous nonzero tangent vector field on S, which is impossible. Let $x = (x^1,0,0)$.

$$\alpha(F(x),F(0)) = \sqrt{|x^1|},$$

and $\sqrt{|x^1|}$ is not less than $K|x^1|$ for all x^1, no matter how K is chosen. The set E contains only the function which is identically zero.

This example is similar to one due to Hermes (9) where, however, F is Lipschitzian.

We next consider a different approach to the existence problem. From now on, assume that F satisfies (a) and (d) on $[a,b] \times R^n$. Let x_o and t_o be given, $a < t_o < b$. Consider the functional H on $L_1[a,b]$ defined by

$$H(\varphi) = \int_a^b \rho(\varphi(t), F(t,x(t)))dt$$

where

$$x(t) = x_o + \int_{t_o}^t \varphi(s)ds,$$

for $\varphi \in L_1[a,b]$. (It follows from a lemma of Filippov (6), that $\rho(\varphi(t), F(t,x(t)))$ is integrable on $[a,b]$.)

<u>Theorem 3.16</u>. $H(\varphi) = 0$ if and only if there exists a solution of (1.2) on $[a,b]$.

<u>Proof</u>. If

$$x(t) = x_o + \int_{t_o}^t \dot{x}(s)ds$$

is a solution of (1.2), then $\rho(\dot{x}(t), F(t,x(t))) = 0$ a.e. $[a,b]$. Hence, $H(\dot{x}) = 0$. Conversely, if $H(\varphi) = 0$, then $\rho(\dot{x}(t), F(t,x(t))) = 0$ a.e. $[a,b]$ where

$$x(t) = x_o + \int_{t_o}^{t} \varphi(s)ds.$$

Thus x is a solution of (1.2) Q.E.D.

<u>Theorem 3.17</u>. If $\varphi, \psi \in L_1[a,b], x(t) = x_o + \int_{t_o}^{t} \varphi(s)ds,$

$y(t) = x_o + \int_{t_o}^{t} \psi(s)ds,$ then

$$|H(\varphi) - H(\psi)| \leq \int_a^b \alpha(F(t,x(t)),F(t,y(t)))dt$$

$$+ \int_a^b |\varphi(t) - \psi(t)|dt.$$

<u>Proof</u>.

$$|H(\varphi) - H(\psi)| \leq \int_a^b |\rho(\varphi(t),F(t,x(t)))$$

$$- \rho(\psi(t),F(t,y(t)))|dt$$

$$\leq \int_a^b |\rho(\varphi(t),F(t,x(t)))$$

$$- \rho(\varphi(t),F(t,y(t)))|dt$$

$$+ \int_a^b |\rho(\varphi(t),F(t,y(t)))$$

$$- \rho(\psi(t),F(t,y(t)))|dt$$

$$\leq \int_a^b \alpha(F(t,x(t)),F(t,y(t)))dt$$

$$+ \int_a^b |\varphi(t) - \psi(t)|dt$$

Where the last step follows from Propositions 2.6 and 2.7.

Q.E.D.

Theorem 3.18. H is continuous on $L_1[a,b]$.

Proof. Let $\{\varphi_i\}$ be a sequence of elements of $L_1[a,b]$ such that

$$\int_a^b |\varphi_i^j(s) - \varphi^j(s)|ds \to 0 \quad \text{as} \quad i \to \infty$$

for $j = 1,2,\cdots,n$, for some $\varphi \in L_1[a,b]$. Since

$$\int_a^b |\varphi_i(s) - \varphi(s)|ds \leq \sum_{j=1}^{n} \int_a^b |\varphi_i^j(s) - \varphi^j(s)|ds,$$

it follows that

$$\int_a^b |\varphi_i(s) - \varphi(s)|ds \to 0 \quad \text{as} \quad i \to \infty.$$

Let

$$y(t) = x_o + \int_{t_o}^{t} \varphi(s)ds$$

for $t \in [a,b]$, and

$$x_i(t) = x_o + \int_{t_o}^{t} \varphi_i(s)ds$$

for $t \in [a,b]$.

$$|x_i(t) - y(t)| \leq |\int_{t_o}^{t} |\varphi_i(s) - \varphi(s)|ds$$

$$\leq \int_a^b |\varphi_i(s) - \varphi(s)|ds.$$

Hence, $x_i \to y$ uniformly on $[a,b]$ as $i \to \infty$. Thus

$$\int_a^b \alpha(F(s,x_i(s)),F(s,y(s)))ds \to 0 \quad \text{as} \quad i \to \infty.$$

Applying Theorem 3.17, we obtain $|H(\varphi_i) - H(\varphi)| \to 0$ as

$i \to \infty$. Q.E.D.

If we restrict H to a compact subset S of $L_1[a,b]$, then there exists $\bar{\varphi} \in S$ such that

$$H(\bar{\varphi}) = \inf_{\varphi \in S} H(\varphi).$$

However, for existence of a solution of (1.2) it must be shown that

$$\inf_{\varphi \in S} H(\varphi) = 0.$$

For example, if we replace $[a,b]$ in the preceding discussion by $[t_o, t_o + \alpha]$ (see the discussion before Definition 3.7) and let

$$\tilde{S} = \{\varphi \mid V_{[t_o, t_o+\alpha]}(\varphi) \leq K, |\varphi(\tau)| \leq K\},$$

where K is a constant, then $\tilde{S}$ is a compact subset of $L_1[t_o, t_o + \alpha]$. The assumption that D is of uniformly bounded variation with constant K and Lemma 3.11 show that

$$\inf_{\varphi \in \tilde{S}} H(\varphi) = 0$$

Here,

$$H(\varphi) = \int_{t_o}^{t_o+\alpha} \rho(\varphi(s), F(s, x(s)))ds.$$

Since $\tilde{S}$ is compact, it follows that there exists $\tilde{\varphi} \in \tilde{S}$ such that $H(\tilde{\varphi}) = 0$. Thus, we have given another proof of Theorem 3.13.

It is possible to consider other compact subsets S of $L_1[a,b]$. See Chang (3) and Riesz (10).

IV. LITERATURE CITED

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