Generalized integral transforms

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Dale Murray Rognlie

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I. INTRODUCTION

A very extensive literature on the Fourier and Laplace transforms exists today. Among the classical treatments are the books by Titchmarsh (1937) and Wiener (1933) on the Fourier transform and those by Doetsch (1950) and Widder (1941) on the Laplace transform. The Stieltjes transform has not been investigated as thoroughly, but Widder (1941) devotes one chapter to it. Brief recent surveys are given by Goldberg (1965) for the Fourier transform and by Hirschman and Widder (1965) for the Laplace and Stieltjes transforms.

Various types of generalizations have been made for these transforms. The best known is the multidimensional Laplace transform of the form

$$ \bar{f}(s_1, \ldots, s_k) = \int_0^\infty \cdots \int_0^\infty e^{i \sum s_i t_i} f(t_1, \ldots, t_k) dt_1 \cdots dt_k, $$

where the transform is a function of several variables. Other types of generalizations of the Laplace transform are given by Meijer (1941) and Varma (1951), in which the kernel $e^{-st}$ is replaced by a Whittaker function $W_{k,m}(st)$, sometimes multiplied by an elementary function of $s$ and $t$. For appropriate values of $k$ and $m$, the new kernel reduces to $e^{-st}$. In the same paper Varma (1951) defines a generalization of the Stieltjes transform which also involves $W_{k,m}$. How-
over, these generalizations lead to transforms which are functions of a single variable.

In a recent paper Carlson (1969) developed a method for generalizing an analytic function $f$ of a single complex variable to an analogous function $F$ of several complex variables $z_1, \ldots, z_k$. Two integral representations are given. One is a multiple-integral representation which is defined as a suitable average of $f$ over the convex hull of $\{z_1, \ldots, z_k\}$. The other is a contour-integral representation which is a generalization of Cauchy's integral formula. The latter representation can be used to continue $F$ analytically in the variables as well as in the parameters which enter through the generalization process. The analogue function $F$ has many useful properties and is relevant to applied mathematics in that it is related to hypergeometric functions, repeated integrals, fractional integrals, mean values, divided differences, and integral transforms.

In this dissertation we shall use the results of Carlson (1969) to generalize the Fourier, Laplace and Stieltjes transforms to functions of several variables. We shall use both the multiple-integral representation and the contour-integral representation for the direct transforms but only the multiple-integral representation for the inverse Fourier and inverse Laplace transforms. We shall not be concerned with the inverse Stieltjes transform since no suitable
integral formula for it exists.

In Chapter II we discuss the multiple-integral representation of $F$ as defined by Carlson (1969) when $F$ is the generalization of an analytic function $f$ of one complex variable. We study also the existence of this representation of $F$, both as a (possibly improper) Riemann integral and as a Lebesgue integral when $f$ is a function of a single real variable. We specify rather minimal conditions on $f$ which guarantee the existence of $F$ for each type of integral.

We also determine some functional properties of $F$ which are related to those of $f$, and find the generalization of Fourier, Laplace and Stieltjes kernels.

In Chapter III we collect some useful lemmas concerning the properties of the ordinary Fourier, Laplace and Stieltjes transforms and also the inverse Fourier and inverse Laplace transforms. We state these propositions as sufficient conditions on a function which guarantee that its ordinary transform exists as a Lebesgue integral, is finite and possesses certain properties (such as continuity or analyticity) on a specified domain.

In Chapter IV we use the results of the previous two chapters to generalize the Fourier, Laplace and Stieltjes transforms and the inverse Fourier and inverse Laplace transforms to functions of several variables. In each case it is shown (by using Fubini's theorem) that replacing the kernel
by the generalized kernel has the effect of replacing the ordinary transform by its generalization in several variables. In this chapter, all generalizations are defined by the multiple-integral representation.

In Chapter V we discuss the contour-integral representation of $F$ as defined by Carlson (1969). We then follow the same procedure as in the previous chapter, this time using the contour-integral representation but limiting the discussion to the three direct transforms. The results of the two chapters are similar in character but the conditions of validity are somewhat different.

Finally, in Chapter VI we show that the generalized integral transforms possess some but not all of the operational properties of the corresponding ordinary transforms. We show also that the generalized transforms are related to each other and to some other integral transforms and that the generalized Laplace transform of $f$ is not equivalent to the multidimensional Laplace transform of $F$, where $F$ is the generalization of $f$. We then use both the generalized Laplace and Stieltjes transforms to find some ordinary Laplace and Stieltjes transforms which are not listed in existing tables of transforms.
II. GENERALIZATION PROCEDURE, A MULTIPLE-INTEGRAL REPRESENTATION

A. Definitions

Let $f$ be a function defined on a domain $D$ in the complex plane $\mathbb{C}^1$. We will show here a method for generalizing $f$ to an analogous function $F$ (denoted by $f \leftrightarrow F$) of $k$ complex variables $z_i$, $(i=1,\ldots,k)$, by means of a multiple integral representation which defines an average of $f$ over the convex hull $K(z)$ of $\{z_1,\ldots,z_k\}$ (Carlson, 1969).

We denote the $k$ complex variables by the ordered $k$-tuple $z=(z_1,\ldots,z_k)$ which may be visualized as either a set of $k$ points in the complex plane $\mathbb{C}^1$ or as a single point in the space $\mathbb{C}^k$ of $k$ complex dimensions. Let $b=(b_1,\ldots,b_k)$ be an ordered $k$-tuple of complex parameters such that $\text{Re}(b_i)>0$, $(i=1,\ldots,k)$, which we abbreviate to $\text{Re}(b)>0$, and define $c = \sum_{i=1}^{k} b_i$. Also let $u=(u_1,\ldots,u_k)$ be an ordered $k$-tuple of real weights such that the following conditions hold:

\begin{align}
\text{a)} & \quad 0 \leq u_i \leq 1, \quad (i = 1,\ldots,k), \\
\text{b)} & \quad \sum_{i=1}^{k-1} u_i \leq 1, \quad u_k = 1-u_1-u_2-\cdots-u_{k-1}. \tag{2.1}
\end{align}

We denote by $E$ the simplex in $\mathbb{R}^{k-1}$ defined by (2.1) and let $u'=(u_1,\ldots,u_{k-1})$ represent a point in $E$. 
Let $P(b,u)$ denote a weight function and choose it to be a complex-valued density function corresponding to the $(k-1)$-variate Dirichlet distribution, which is the generalization of the well-known Beta distribution. We define $P(b,u)$ as follows:

$$P(b,u) = \begin{cases} \frac{1}{B(b)} \frac{b_1^{l-1} b_2^{l-1} \ldots b_{k-1}^{l-1}}{u_2 \ldots u_{k-1} (1-u_1-u_2-\ldots-u_{k-1})^l} & \text{at any interior point of } E, \\ 0 & \text{on the boundary and complement of } E, \end{cases} \quad (2.2)$$

where $B(b)$ is the Beta function of $b$ defined by

$$B(b) = \frac{\Gamma(b_1) \ldots \Gamma(b_k)}{\Gamma(c)}, \quad (2.3)$$

and $\Gamma(a)$ is the Gamma function of $a$ defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, dt, \quad \text{Re}(a)>0. \quad (2.4)$$

The $(k-1)$-variate Dirichlet distribution, with positive integral values of the $b_i$'s, is basic to the probability theory of order statistics (Wilks, 1962, p. 177). In the present context, however, the $b_i$'s need not be integral nor even real and $P(b,u)$ is not a statistical density function in the usual sense.

The integral of $P(b,u)$ over the simplex $E$ is unity, as is verified by making the transformation (Wilks, 1962, p. 178).
Then we have

\[
P(b,u) \, du_1 \ldots du_{k-1} = \frac{1}{B(b)} \frac{b_1^{-1}}{\theta_1} \frac{b_2 + \ldots + b_k}{(1-\theta_1)} \frac{b_3 + \ldots + b_k}{\theta_2} \ldots \frac{b_k}{\theta_{k-1}} \, d\theta_1 \ldots d\theta_{k-1},
\]

where the range of the \( \theta \)'s is the \((k-1)\)-dimensional unit cube:

\[
\{ (\theta_1, \ldots, \theta_{k-1}) : 0 < \theta_i < 1, \ (i=1, \ldots, k-1) \}.
\]

Thus if we define \( du' = du_1 \ldots du_{k-1} \),

\[(2.5) \quad \int_{E} P(b,u) du' \]

\[
= \frac{1}{B(b)} \int_0^1 \ldots \int_0^1 \frac{b_1^{-1}}{\theta_1} \frac{b_2 + \ldots + b_k}{(1-\theta_1)} \frac{b_k}{\theta_{k-1}} \, d\theta_1 \ldots d\theta_{k-1}
\]

\[
= \frac{1}{B(b)} \frac{\Gamma(b_1) \Gamma(b_2 + \ldots + b_k)}{\Gamma(c)} \cdot \frac{\Gamma(b_2)}{\Gamma(b_2 + \ldots + b_k)} \frac{\Gamma(b_3 + \ldots + b_k)}{\Gamma(b_2 + \ldots + b_k)} \frac{\Gamma(b_{k-1}) \Gamma(b_k)}{\Gamma(b_{k-1} + b_k)}
\]

\[
= 1, \quad \text{Re}(b) > 0.
\]
For \( \mu \in E \), \( \mu \cdot z = \sum_{i=1}^{k} \mu_i z_i \) is a weighted average of the points \( z_i \in D \), \( i = 1, \ldots, k \), and \( \mu \cdot z \in K(z) \) for all combinations of the \( \mu_i \)'s such that (2.1) holds.

We first form \( f(\sum_{i=1}^{k} \mu_i z_i) \), which is defined for all \( \mu \in E \) provided that \( K(z) \subseteq D \). We then multiply \( f(\sum_{i=1}^{k} \mu_i z_i) \) by \( P(b,\mu) \) and integrate this product over the simplex \( E \).

Thus the generalization of \( f \) to \( k \) variables is

\[
(2.6) \quad F(b,z) = F(b_1,\ldots,b_k; z_1,\ldots,z_k) = \int_{E} f(\sum_{i=1}^{k} \mu_i z_i) P(b,\mu) d\mu',
\]

where \( \text{Re}(b)>0 \) and \( K(z) \subseteq D \).

If \( z_1 = z_2 = \ldots = z_k = z \), then \( f(\sum_{i=1}^{k} \mu_i z_i) = f(z) \) and

\[
(2.7) \quad F(b,z) = \int_{E} f(z) P(b,\mu) d\mu' = f(z) \int_{E} P(b,\mu) d\mu' = f(z).
\]

We note that for \( \text{Re}(b)>1 \),

\[
(2.8) \quad |P(b,\mu)| = \frac{1}{|B(b)|} |u_1^{b_1-1} \cdots u_k^{b_k-1}| \leq \frac{1}{|B(b)|}.
\]
B. Existence of $F$ in the Riemann Sense

It is not essential that $f$ be defined on a domain in the complex plane. The generalization procedure remains the same if $f$ is a real-valued or a complex-valued function defined on an interval $I$ of the real line. In this case we shall usually write $t=(t_1,\ldots,t_k)$ in place of $z=(z_1,\ldots,z_k)$. Let $t_{\min} = \min_i \{t_1,\ldots,t_k\}$ and $t_{\max} = \max_i \{t_1,\ldots,t_k\}$. The convex hull $K(t)$ of the set of points $\{t_1,\ldots,t_k\}$ is a closed subinterval $[t_{\min},t_{\max}]$ of $I$.

For a fixed set of $z$'s (or $t$'s), the condition that $f$ be continuous on $K(z)$ (or $K(t)$) is sufficient but certainly not necessary for the integral (2.6) to exist. In the complex domain case $f$ will usually be assumed to be analytic and therefore continuous, but in the real case it is useful to find sufficient conditions on $f$, which are less restrictive than continuity and such that $F$ exists as a (possibly improper) Riemann integral.

Suppose that $f$ is defined on an interval $I \subseteq \mathbb{R}$ and let $t=(t_1,\ldots,t_k) \in I^k$ be fixed. The function

$$g(u') = f(\sum_{i=1}^k u_it_i), \quad (u' \in \mathbb{E})$$

is defined on $E \subseteq \mathbb{R}^{k-1}$. If $f$ has a discontinuity at $\tau \in I$, then $\sum_{i=1}^k u_it_i = \tau$ represents a hyperplane $H \subseteq \mathbb{R}^{k-1}$ and $g$ is discontinuous at every point of $H \cap E$. More generally, if the discontinuities of $f$ are denoted by $\{\tau_\alpha : \alpha \in A\}$, where $A$ is some
index set, then there is a corresponding set \( \{ H_\alpha : \alpha \in A \} \) of pairwise disjoint hyperplanes in \( \mathbb{R}^{k-1} \) and \( g \) is discontinuous at every point of \( \bigcup_{\alpha \in A} (H_\alpha \cap E) \).

**Example 2.1.** Consider the case where \( k=3 \) and \( f \) has a discontinuity at \( r=0 \). Let \((t_1, t_2, t_3) = (-1, 1, 2)\). Then

\[ H: \quad u_1 t_1 + u_2 t_2 + (1-u_1-u_2) t_3 = 0 \]

or

\[ 3u_1 + u_2 = 2 \]

is a line in \( \mathbb{R}^2 \), each point of which is a discontinuity of \( g(u_1, u_2) = f(\Sigma u_i t_i) \). Its intersection with \( E \) is the line segment in the \((u_1, u_2)\) plane joining the points \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{2}{3}, 0)\).

Consider any fixed point \( \bar{u}'=(\bar{u}_1, \ldots, \bar{u}_k) \in E \) and

\[ \bar{u}_k = 1 - \Sigma u_i \]

such that \( \Sigma \bar{u}_i t_i = r \) is not a discontinuity of \( f \).

Then on a neighborhood \( N \) of \( \bar{u}' \), \( g(u') = f(\Sigma u_i t_i) \) is a continuous function of a continuous function and is therefore continuous for every \( u' \in N \). Hence \( \bigcup_{\alpha \in A} (H_\alpha \cap E) \) is precisely the set of points of discontinuity of \( g \). We now consider conditions under which this set has Lebesgue measure zero in \( \mathbb{R}^{k-1} \).

For \( u' \in E \) and \( u_k = 1 - \Sigma u_i \), the function

\[ (2.9) \quad T(u') = \Sigma t_i u_i', \quad (u' \in E), \]

represents (for fixed real \( t \)) a linear transformation

\( T: E \to \mathbb{R} \). \( T \) is the restriction to \( E \) of a linear transformation

\( L: \mathbb{R}^{k-1} \to \mathbb{R} \), defined by
\begin{align}
&L(u^i) = \sum_{i=1}^{k} u_i t_i = \sum_{i=1}^{k-1} (t_i - t_{i+1}) u_i + t_k. \\
&L(u^i) = \sum_{i=1}^{k-1} (t_i - t_{i+1}) u_i + t_k.
\end{align}

For any set \( S \subseteq R \),
\begin{equation}
T^{-1}(S) = L^{-1}(S) \cap E.
\end{equation}

Moreover, if \( S \) is the set of discontinuities \( \{ r_\alpha : \alpha \in A \} \) of \( f \), then
\begin{equation}
T^{-1}(S) = \bigcup_{\alpha \in A} (H_\alpha \cap E)
\end{equation}
is the set of discontinuities of \( g \).

**Lemma 2.1.** Let \( L \) be a linear function from \( R^m \) into \( R \) and let \( S \subseteq R \) be any Lebesgue measurable set. Then \( L^{-1}(S) \) is Lebesgue measurable. Moreover, provided that \( L \) is not a constant function, \( L^{-1}(S) \) has measure zero in \( R^m \) if and only if \( S \) has measure zero in \( R \).

**Proof:** Let \( x = (x_1, \ldots, x_m) \in R^m \) and denote \( L(x) \) by
\[ L(x) = \sum_{i=1}^{m} a_i x_i + b, \]
where \( b \) and \( a_i, (i=1, \ldots, m), \) are real. We define
\[ ||a|| = \left( \sum_{i=1}^{m} a_i^2 \right)^{\frac{1}{2}}. \]
We may assume that not all of the \( a_i \)'s are zero, for if \( L(x) = b \) then \( L^{-1}(S) \) is the void set if \( b \notin S \) and \( L^{-1}(S) = R^m \) if \( b \in S \). In either case \( L^{-1}(S) \) is Lebesgue measurable. Thus we assume that \( ||a|| \neq 0. \)
We define \( n_i = a_i / ||a|| \), \( i = 1, \ldots, m \), and let \( h: \mathbb{R}^m \to \mathbb{R} \) and \( \mathcal{L}: \mathbb{R} \to \mathbb{R} \) be defined by

\[
(2.13) \quad h(x) = \sum_{i=1}^{m} n_i x_i = n \cdot x
\]

and

\[
(2.14) \quad \mathcal{L}(w) = ||a|| w + b.
\]

Then for \( x \in \mathbb{R}^m \)

\[
(2.15) \quad L(x) = \sum_{i=1}^{m} a_i x_i + b = ||a|| \sum_{i=1}^{m} n_i x_i + b = \mathcal{L}(h(x)).
\]

Let \( M = \mathcal{L}^{-1}(S) \), where \( S \) is Lebesgue measurable. Since \( \mathcal{L} \) is a linear transformation of the real line onto itself, \( M \) is also a measurable set (Halmos, 1950, Thm. D, p. 64). We observe from (2.15) that

\[
(2.16) \quad L^{-1}(S) = h^{-1}(\mathcal{L}^{-1}(S)) = h^{-1}(M) = \{ x \in \mathbb{R}^m : n \cdot x \in M \}.
\]

The null space of \( h \) is the hyperplane \( H_0 = \{ y \in \mathbb{R}^m : n \cdot y = 0 \} \) and the unit vector \( n \) is perpendicular to \( H_0 \). Any \( x \in \mathbb{R}^m \) has a unique decomposition

\[
x = wn + y = (w, y),
\]

where \( w = n \cdot x \) and \( y \in H_0 \). Thus by (2.16)

\[
(2.17) \quad L^{-1}(S) = \{(w, y) : w \in M, y \in H_0 \} = M \times H_0 = \mathcal{L}^{-1}(S) \times H_0.
\]

Geometrically, \( M \times H_0 \) is the union of disjoint hyperplanes in \( \mathbb{R}^m \), one for each point in \( M \). Since \( M \) and \( H_0 \) are measurable sets and the cartesian product of measurable sets is measurable (Williamson, 1962, Thm. 2.3i, p. 30), then \( L^{-1}(S) \) is measurable. Moreover, since \( H_0 \) has measure \( 0 \) in \( \mathbb{R}^{m-1} \), \( M \times H_0 \) has measure zero in \( \mathbb{R}^m \) if and only if \( M \) has measure zero.
in R. But $M = \mathcal{L}^{-1}(S)$ has measure zero in $R$ if and only if $S$ has measure zero in $R$ (Halmos, 1950, Thm. D, p. 64). Therefore $L^{-1}(S)$ has measure zero in $R^m$ if and only if $S$ has measure zero in $R$. \footnote{Throughout this thesis \# will be used to indicate the end of a proof of a theorem or lemma.}

We now wish to establish somewhat minimal conditions on $f$ which will guarantee the existence of $F$ in the Riemann sense at every point of $I^k$. Let $S = \{t_\alpha \in I: \alpha \in \mathcal{A}\}$ be the set of discontinuities of $f$ and assume that $S$ has measure zero. The set of discontinuities of $g$ on $E$ is

\[(2.18) \bigcup_{\alpha \in \mathcal{A}} (H_\alpha \cap E) = T^{-1}(S) = L^{-1}(S) \cap E.\]

If we exclude the case $t_1 = t_2 = \ldots = t_k$, then $L^{-1}(S)$ has measure zero in $R^{k-1}$ by Lemma 2.1 and hence $T^{-1}(S)$ also has measure zero.

\textbf{Lemma 2.2.} (Apostol, 1957, Thm. 10-24, p. 267) Let a function $g$ be defined and bounded on a Jordan-measurable set $E$ in $R^n$ and let $G \subseteq E$ be the set of discontinuities of $g$. Then $g$ is Riemann integrable on $E$ if and only if $G$ has Lebesgue measure zero.

\textbf{Lemma 2.3.} (Fulks, 1961, Extension of Problem Bl, p. 416) Let $\int_E h(u)du$ be an absolutely convergent improper Riemann integral and $g$ be a bounded Riemann integrable function on $E$. Then $\int_E h(u)g(u)du$ converges absolutely.
Lemma 2.4. (Apostol, 1957, Generalization of Thm. 14-15, p. 437. For proof see Thm. 14-5, p. 432.) If \( f \) is Riemann integrable on every closed subset of \((E-\partial E) \subset \mathbb{R}^n\), where \( \partial E \) is the boundary of \( E \), and if \( \int_{E} |f|\,du \) converges, then \( \int_{E} f\,du \) converges.

Theorem 2.1. Let \( f \) be a real-valued function defined and continuous almost everywhere on an interval \( I \subset \mathbb{R} \). Assume that \( f \) is bounded on every closed subinterval of \( I \). Then, if \( \text{Re}(b) > 0 \), \( F(b,t) \) exists as a (possibly improper) Riemann integral at every point of \( I^k \).

Proof: Let \( t = (t_1, \ldots, t_k) \in I^k \) be fixed. If \( t_1 = t_2 = \ldots = t_k \), then \( F(b,t) = f(t) \) and we may henceforth exclude this case.

Since \( f \) is continuous almost everywhere on \( I \), the set \( S \subset I \) of discontinuities of \( f \) has measure zero in \( \mathbb{R} \). By Lemma 2.1 and (2.18) the set of discontinuities of
\[
g(u') = f(\sum_{i=1}^{k} u_i t_i),
\]
where \( u' \in E \), has measure zero in \( \mathbb{R}^{k-1} \). Since \( [t_{\min}, t_{\max}] \) is a closed subinterval of \( I \) and
\[
t_{\min} \leq \sum_{i=1}^{k} u_i t_i \leq t_{\max}
\]
for every \( u' \in E \), \( g \) is defined and bounded on \( E \). Thus, by Lemma 2.2, \( g \) is a bounded integrable function on \( E \). Since \( \int_{E} P(b,u)\,du' \) is absolutely convergent if \( \text{Re}(b) > 0 \), by Lemmas 2.3 and 2.4, \( F(b,t) \) exists as a (possibly improper) Riemann integral at every point of \( I^k \).
Example 2.2. Consider the following example (Olmstead, 1956, Ex. 2, p. 142). Let $f$ be defined on $[0,1]$ by

$$f(x) = \begin{cases} 
1 & \text{if } x=0 \text{ or } x=1, \\
0 & \text{if } x \text{ is irrational}, \\
\frac{1}{q} & \text{if } x=p/q, \text{ where } p, q \text{ are relatively prime positive integers.}
\end{cases}$$

Then $f$ is continuous at the irrational points and discontinuous at every rational point of $[0,1]$. Thus $f$ has a countably infinite number of discontinuities in $[0,1]$ but is Riemann integrable ($\int_0^1 f(x)dx=0$). We note that $f$ is neither piecewise continuous nor of bounded variation on $[0,1]$.

With this choice of $f$, consider the case where $k=2$ and $\Re(b)>0$. Then, if $t_1>t_2$, 

$$F(b_1,b_2; t_1,t_2) = \frac{1}{B(b)} \int_0^1 f[ut_1+(1-u)t_2]u^{b_1-1}(1-u)^{b_2-1}du$$

$$= \frac{1}{B(b)} \int_{t_2}^{t_1} f(t) \frac{b_1-1}{(t_1-t)} \frac{b_2-1}{(t_1-t)} dt$$

$$= 0 \text{ since } f(t)=0 \text{ almost everywhere.}$$

Similarly $F(b,t)=0$ if $t_2>t_1$, whereas $F(b_1,b_2; t,t)=f(t)$. Thus $F(b,t)$ exists and is finite at every point of $[0,1]^2$ but is discontinuous in $t_1$ and $t_2$ separately as well as jointly at a countably infinite number of points.
C. Existence of F in the Lebesgue Sense

Restricting our attention to the real domain, we wish to consider conditions on $f$ under which the integral (2.6) exists as a Lebesgue integral and is finite.

**Theorem 2.2.** Let $f$ be a function which is defined and Lebesgue measurable on an interval $I \subset \mathbb{R}$, with values in a topological space $X$. Let $t=(t_1,\ldots,t_k) \in I^k$ be fixed and let $g: E \to X$ be defined by $g(u') = f(T(u')) = f(\sum_{i=1}^{k} u_i t_i)$ for all $u' \in E$. Then $g$ is Lebesgue measurable on $E$.

**Proof:** The function $g$ is measurable on $E$ if $g^{-1}(S)$ is measurable for every open set $S \subset X$. But we have

$$ (2.19) \quad g^{-1}(S) = (fT)^{-1}(S) = T^{-1}(f^{-1}(S)) = L^{-1}(f^{-1}(S)) \cap E, $$

where $L$ is defined by (2.10). Since $f$ is a measurable function, $f^{-1}(S)$ is measurable. Since $L$ is a linear function from $\mathbb{R}^{k-1}$ to $\mathbb{R}$, $L^{-1}(f^{-1}(S))$ is measurable by Lemma 2.1. Since $E$ is a closed set in $\mathbb{R}^{k-1}$ and therefore measurable, $L^{-1}(f^{-1}(S)) \cap E$ is measurable. Thus $g$ is measurable on $E$. 

In the next theorem we shall assume that a function is finite on an interval $I \subset \mathbb{R}$ and is essentially bounded on every closed subinterval of $I$ ($f$ essentially bounded means that there exists a finite constant $M > 0$ such that the set $\{t: f(t) > M\}$ has measure zero). The following examples
illustrate what this assumption means; it is satisfied by the first two but not by the third.

Example 2.3. Let \( f \) be a function from \([0,1]\) into \( \mathbb{R} \), where

\[
f(x) = \begin{cases} 
  n, & \text{if } x = \frac{1}{n}, \ n=1,2,3,\ldots, \\
  0, & \text{otherwise.}
\end{cases}
\]

This function is defined, finite and essentially bounded but not bounded on \([0,1]\).

Example 2.4. Let \( f \) be a function from \((0,1)\) into \( \mathbb{R} \), where

\[
f(x) = \frac{1}{x}.
\]

This function is defined, finite but not essentially bounded on \((0,1)\). It is bounded and therefore essentially bounded on every closed subinterval of \((0,1)\).

Example 2.5. Let \( f \) be a function from \([0,1]\) into \( \mathbb{R} \), where

\[
f(x) = \begin{cases} 
  \frac{1}{x}, & 0 < x < 1, \\
  0, & x=0.
\end{cases}
\]

This function is defined and finite on \([0,1]\), but is not essentially bounded on any closed subinterval \([0,a]\), where \((0 < a < 1)\).
Theorem 2.3. Let $f$ be a complex-valued function which is defined, Lebesgue-measurable and finite on an interval $I \subset \mathbb{R}$. Assume that $f$ is essentially bounded on every closed subinterval of $I$. Then, if $\text{Re}(b) > 0$, $F(b,t)$ exists as a Lebesgue integral and is finite at every point of $I^k$.

Proof: Let $t=(t_1,\ldots,t_k) \in I^k$ be fixed. If $t_1=t_2=\ldots=t_k$, then $F(b,t)=f(t)$ and we may henceforth exclude this case. The set $D=[t_{\min},t_{\max}]$ is a closed subinterval of $I$ on which $f$ is defined, measurable, finite and essentially bounded. Since $t_{\min} \leq \sum_{i=1}^{k} u_{i} t_{i} \leq t_{\max}$ for all $u \in E$,

$$g(u')=f(T(u'))=f(\sum_{i=1}^{k} u_{i} t_{i})$$

is finite on $E$.

Since $f$ is essentially bounded on $D$, there exists a positive, finite constant $K$ such that

$$(2.20) \quad D_K = \{t \in D: |f(t)| > K\}$$

has Lebesgue measure zero. We define

$$(2.21) \quad E_K = \{u' \in E: |g(u')| > K\}.$$ 

Then, by definition we have

$$(2.22) \quad E_K = T^{-1}(D_K) = L^{-1}(D_K) \cap E,$$

where $L$ is defined by (2.10). But by Lemma 2.1, $L^{-1}(D_K)$ has measure zero in $\mathbb{R}^{k-1}$ and therefore $E_K$ has measure zero. Thus $g$ is essentially bounded on $E$ since $|g(u')| \leq K$ almost everywhere on $E$. 

The function $P(b,u)$ is continuous almost everywhere on $E$ and is therefore measurable. By Theorem 2.2 $g$ is measurable on $E$. Therefore, the product $g(u') P(b,u)$ is measurable on $E$, and $F(b,t)$ exists as a Lebesgue integral. Moreover, $F(b,t)$ is finite, for

$$|F(b,t)| \leq \int_{E} |g(u')||P(b,u)|du' \leq K \int_{E} |P(b,u)|du'$$

$$= \frac{B[\text{Re}(b)]}{|B(b)|} < \infty, \text{ Re}(b)>0. \#$$

D. Properties of $F$

We have seen that under the assumptions of Theorems 2.1 and 2.3, $F(b,t)$ exists and is finite. We also observed from Example 2.1 that discontinuities of $f$ may yield discontinuities of $F$. We are then led to the proposition of the next theorem.

**Theorem 2.4.** Let $f$ be continuous on an interval $I \subset \mathbb{R}$. If $\text{Re}(b)>0$, then $F(b,t)$ is a continuous function of $t$ on $I^k$.

**Proof:** Since $f$ is continuous on $I$, then $F$ exists and is finite for $t \in I^k$, by Theorem 2.3. Furthermore, given $\zeta>0$ there exists a $\delta>0$ such that $t', t'' \in I^k$ and $|t'-t''| < \delta$ imply

$$|f(t') - f(t'')| < \zeta |B(b)|/B[\text{Re}(b)].$$

Now let $t'=(t'_{1},...,t'_{k}) \in I^k$ and $t''=(t''_{1},...,t''_{k}) \in I^k$ such that $|t'_{i}-t''_{i}| < \delta, \quad (i=1,...,k)$. 
Then
\[ \left| \sum_{i=1}^{k} u_i t_i' - \sum_{i=1}^{k} u_i t_i'' \right| \leq \sum_{i=1}^{k} u_i |t_i' - t_i''| < \delta, \]
and
\[ |F(b, t') - F(b, t'')| \leq \int_{E} |f(\sum_{i=1}^{k} u_i t_i')| \]
\[ \leq \zeta \cdot \frac{|B(b)|}{B[\text{Re}(b)]} \int_{E} |p(b, u)| du' = \zeta, \]
\[ \text{Re}(b) > 0. \]
Therefore $F(b, t)$ is continuous in $t$ on $I^k$. 

**Lemma 2.5.** (Carlson, 1969, Thm. 1) If $f(z)$ is analytic on a domain $D \subset \mathbb{C}^1$, then $F$ is an analytic function of $b$ and $z$ on the domain $G \subset \mathbb{C}^{2k}$ defined by the conditions $\text{Re}(b) > 0$ and $K(z) \subset D$.

**E. Some Special Cases**

**Generalization of $e^{tz}$ and $(z+t)^{-a}$:**

1. Let $f(z) = e^{tz}$, where $t$ is a finite constant. Since $f(z)$ is entire, $D$ is the finite complex plane. We define
\[ S(b, tz) = S(b_1, \ldots, b_k; tz_1, \ldots, tz_k) \]
\[ = \int_{E} \exp(t \sum_{i=1}^{k} u_i z_i) p(b, u) du', \quad \text{Re}(b) > 0, \]
to be the k-variable analogue of $e^{tz}$ and we refer to (2.23) as the S-function. We note that the S-function is an entire function of $z$ by Lemma 2.5.

2. Let $f(z) = (z+t)^{-a}$, where $t$ is a real constant such that $0 < t < \infty$ and $a$ is an arbitrary complex constant. Let $D$ denote the $z$-plane cut along the nonpositive real axis and assume $K(z) \subseteq D$. We define

$$R(a,b,z+t) = R(a; b_1, \ldots, b_k; z_1 + t, \ldots, z_k + t)$$

$$= \int \left( \sum_{i=1}^{k} u_i z_i + t \right)^{-a} p(b,u) du', \quad \text{Re}(b) > 0,$$

(2.24) to be the k-variable analogue of $(z+t)^{-a}$. We refer to (2.24) as the hypergeometric R-function. Since $(z+t)^{-a}$ is an analytic function of $z$ for $z \in D$, then $R(a,b,z+t)$ is an analytic function of $z$ for $K(z) \subseteq D$ by Lemma 2.5.

It is easily seen in view of (2.7) that if $z_1 = z_2 = \ldots = z_n = z$, then

(2.25) $S(b,tz) = e^{tz}$ and $R(a,b,z+t) = (z+t)^{-a}$. 

III. PROPERTIES OF THE FOURIER, LAPLACE AND STIELTJES TRANSFORMS

A. Introduction

In Chapters IV and V we shall discuss the generalization of certain integral transforms by means of a multiple-integral representation and a contour-integral representation, respectively. Certain properties of the ordinary transforms will be assumed which guarantee the existence of the generalized transforms. Therefore, in this chapter, various lemmas are presented for reference in which certain properties of a function are assumed in order to yield the desired properties of the ordinary transforms.

B. The Fourier Transform

We first consider the exponential Fourier transform as a function of a real variable. If \( h \in L^1 \) (i.e., if \( h \) is a complex-valued function defined and integrable with respect to Lebesgue measure on \( \mathbb{R} \)), then the Fourier transform of \( h \) is defined as

\[
\hat{h}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-ixt} \, dt, \quad -\infty < x < \infty.
\]

Lemma 3.1. (Bremermann, 1965, Thm. 8.2, p. 79) If \( h \in L^1 \), then \( \hat{h} \) is uniformly continuous and bounded on \( (-\infty, \infty) \).

We next consider the exponential Fourier transform as a
function of a complex variable. If $h$ is a function defined on $\mathbb{R}$ and $s = \sigma + i\tau$, then the Fourier transform of $h$ is defined as

$$
(3.2) \quad \overline{h}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-ist} dt,
$$

provided that this integral exists. We wish to impose conditions on $h$ which make $\overline{h}$ an analytic function of $s$ in some region of the complex plane. We begin with a lemma which will be useful in proving several succeeding lemmas and theorems.

**Lemma 3.2.** (Rudin, 1966, Extension of Problem 15, p. 220) Let $\mu$ be a real or complex measure on a measure space $X$, $\Theta$ an open set in the complex plane, and $\phi$ a function defined on $\Theta \times X$. Assume that $\phi(s,t)$ is a measurable function of $t$ for each $s \in \Theta$, $\phi(s,t)$ is analytic on $\Theta$ for each $t \in X$, and $|\phi(s,t)|$ is majorized on $\Theta$ by a function $\psi(t)$, where $\psi(t)$ is Lebesgue integrable with respect to $\mu$ on $X$. For each $s \in \Theta$, define

$$
(3.3) \quad \overline{\phi}(s) = \int_X \phi(s,t) \, d\mu(t).
$$

Then $\overline{\phi}$ is analytic on $\Theta$. If we assume further that $\phi(s,t)$ is defined and continuous on $\Theta \cup \partial \Theta$ for each $t \in X$ and that $\overline{\phi}$ is defined by (3.3) on $\Theta \cup \partial \Theta$, then $\overline{\phi}$ is continuous on $\Theta \cup \partial \Theta$. 
Proof: From the definition of $\tilde{\phi}$ we have

$$
\frac{\tilde{\phi}(s)-\tilde{\phi}(s_o)}{s-s_o} = \int_X \frac{\phi(s,t)-\phi(s_o,t)}{s-s_o} \, d\mu(t), \quad s, s_o \in \Theta, \quad s \neq s_o.
$$

Let $K \subset \Theta$ be a compact set and let $\gamma \subset \Theta$ be a positively oriented rectifiable Jordan curve of length $L$ such that $\gamma$ and its inner region $I(\gamma)$ lie in $\Theta$ and $K \subset I(\gamma)$. Let $\delta$ be the distance between $K$ and $\gamma$. Then, since $\phi$ is analytic on $\gamma \cup I(\gamma)$, we apply Cauchy's integral formula and obtain

$$
\left| \frac{\phi(s,t)-\phi(s_o,t)}{s-s_o} \right| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{\phi(z,t)}{z-s} \, dz - \int_{\gamma} \frac{\phi(z,t)}{z-s_o} \, dz \right|
$$

$$
= \frac{1}{2\pi} \left| \int_{\gamma} \frac{\phi(z,t)}{(z-s)(z-s_o)} \, dz \right|
$$

$$
\leq \frac{1}{2\pi} \int_{\gamma} \frac{|\phi(z,t)|}{|z-s||z-s_o|} \, |dz|
$$

$$
\leq \frac{L}{2\pi \delta^2} \psi(t), \quad s, s_o \in K, \quad s \neq s_o, \quad t \in X.
$$

Define

$$
\chi_s(t) = \frac{\phi(s,t)-\phi(s_o,t)}{s-s_o}, \quad s \neq s_o, \quad t \in X.
$$

Then $\chi_s(t)$ is measurable in $t$ for each $s \in K$, $s \neq s_o$, and does not exceed $\frac{L}{2\pi \delta^2} \psi(t)$ in absolute value. We note that since $\phi(s,t)$ is analytic on $\Theta$, 

(3.5) \( \lim_{s \to s_0} \chi_s(t) = \phi_1(s_0, t), \quad t \in X, \)

where \( \phi_1(s_0, t) = \frac{\partial \phi(s, t)}{\partial s} \bigg|_{s=s_0} \) is analytic at \( s_0 \) for every \( t \in X. \)

In order to apply Lebesgue's dominated convergence theorem, let \( \{s_n\} \) be a sequence of points in \( K \) converging to \( s_0 \in K \) and define the corresponding sequence of functions

\[ g_n(t) = \chi_{s_n}(t), \quad (n=1,2,...), \quad t \in X. \]

Then, if \( s_n \to s_0, \)

\[ \lim_{n \to \infty} g_n(t) = \lim_{n \to \infty} \chi_{s_n}(t) = \phi_1(s_0, t) \]

by (3.5). The Lebesgue dominated convergence theorem applies and we have that \( \phi_1(s_0, t) \) is Lebesgue integrable with respect to \( \mu \) and

\[ \lim_{n \to \infty} \frac{\phi(s_n) - \phi(s_0)}{s_n - s_0} = \int_X \phi_1(s_0, t) d\mu(t) \]

for every sequence \( \{s_n\} \) which converges to \( s_0 \). It follows that

\[ \lim_{s \to s_0} \frac{\phi(s) - \phi(s_0)}{s - s_0} = \int_X \phi_1(s_0, t) d\mu(t), \quad s, s_0 \in K. \]

Provided that \( s_0 \) is an interior point of \( K \), it follows that the complex derivative \( \overline{\phi}' \) exists at \( s_0 \) and is given by

\[ \overline{\phi}'(s_0) = \int_X \phi_1(s_0, t) d\mu(t). \]

For every \( s_0 \in \Theta \) we can find a compact \( K \subset \Theta \) containing \( s_0 \) in its interior and so \( \overline{\phi} \) is analytic on \( \Theta \). Moreover, suppose that \( \phi(s, t) \) is defined and continuous on \( \Theta \cup \overline{\Theta} \) for each \( t \in X. \) Since
|φ(s,t)| is majorized by ψ(t) for s ∈ Θ, it is majorized by ψ(t) for s ∈ Θ U Θ, and hence it follows directly from Lebesgue's dominated convergence theorem that φ(s) is continuous on Θ U Θ. #

(For an alternative proof of Lemma 3.2 one can use the Lebesgue dominated convergence theorem to prove that φ(s) is continuous on Θ. If φ is integrated around the boundary of any triangle contained in Θ, the order of integration may be interchanged by Fubini's theorem, and the integral of φ around the boundary of the triangle vanishes by the Cauchy integral theorem. But since φ is continuous and has a vanishing integral around every triangle in Θ, it is analytic on Θ by Morera's theorem.)

The following lemmas illustrate a variety of assumptions on a function h which make the Fourier transform of h analytic on various regions of the complex plane.

Lemma 3.3. Let h ∈ L_1 vanish on (-∞,0). Then its Fourier transform h̄ is analytic on the lower half plane and continuous on the union of the lower half plane and the real axis.

Proof: We have

h̄(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-ist} dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} h(t) e^{-ist} dt.

Let X = (0,∞), Θ = {s: \text{Im}(s)<0}, dμ(t)=dt and φ(s,t) = h(t) e^{-ist}. We observe that if s = σ + iτ, where σ and τ are real, then
\[ |\phi(s,t)| = |h(t)e^{-ist}| = |h(t)|e^{it} \leq |h(t)|, \text{ for } t>0 \text{ and } \tau<0.\]

Thus \( |\phi(s,t)| \) is majorized on \( \Theta \) by an integrable function, and \( \phi(s,t) \) is measurable in \( t \) for each \( s \in \Theta \) and analytic on \( \Theta \) for each \( t \in \mathbb{R} \). By Lemma 3.2, \( \overline{h}(s) \) is analytic on the lower half plane. Moreover, \( \phi(s,t) \) is defined and continuous on the union of the lower half plane and the real axis for every \( t \in \mathbb{R} \), and hence \( \overline{h} \) also is continuous there. #

**Lemma 3.4.** Let \( h \in L_1 \) vanish outside a finite interval \((a,b)\). Then \( \overline{h} \) is entire.

**Proof:** We have
\[
\overline{h}(s) = \frac{1}{\sqrt{2\pi}} \int_a^b h(t)e^{-ist}dt.
\]
Given \( c>0 \), let \( \Theta \) be the strip \( \{s: |\text{Im}(s)|<c\} \), \( X=(a,b) \), \( d\mu(t)=dt \) and \( \phi(s,t)=h(t)e^{-ist} \). Then \( \phi(s,t) \) is measurable in \( t \) for each \( s \in \Theta \) and analytic on \( \Theta \) for each \( t \in \mathbb{R} \). Let \( a=\max(|a|,|b|) \). Then
\[
|h(t)e^{-ist}| = |h(t)|e^{it} \leq |h(t)|e^{ac}.
\]
Therefore \( |\phi(s,t)| \) is majorized by an integrable function of \( t \), and by Lemma 3.2 \( \overline{h}(s) \) is analytic on \( \Theta \). Given any \( s \) in the plane, we can choose \( c \) large enough so that \( s \in \Theta \). Hence \( \overline{h} \) is entire. #

**Lemma 3.5.** Let \( K_1 \) and \( K_2 \) be positive constants. Assume that \( h \) is Lebesgue measurable on \( \mathbb{R} \) and that
\[ |h(t)| \leq K_1 e^{-at}, \quad -\infty < t < 0 \text{ and } |h(t)| \leq K_2 e^{-bt}, \quad 0 \leq t < \infty, \quad a < b. \]

Then \( \overline{h} \) is analytic on the open strip \( a < \text{Im}(s) < b. \)

**Proof:** We write

\[ \overline{h}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-ist} dt = \overline{h}_1(s) + \overline{h}_2(s) \]

where

\[ (3.6) \overline{h}_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} h(t) e^{-ist} dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} h(-t) e^{ist} dt \]

and

\[ (3.7) \overline{h}_2(s) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} h(t) e^{-ist} dt. \]

Suppose that \( a < \rho_1 < \rho_2 < b. \) Let \( d\mu(t) = dt, \quad X = \{0, \infty\}, \)
\( \Theta_1 = \{s: \text{Im}(s) > \rho_1\}, \quad \Theta_2 = \{s: \text{Im}(s) < \rho_2\}, \quad \phi(s, -t) = h(-t) e^{ist} \) and \( \phi(s, t) = h(t) e^{-ist}, \quad t \in X. \) Then \( \phi(s, -t) \) and \( \phi(s, t) \) are analytic in \( s \) for each \( t \in X \) and measurable in \( t \) for each finite \( s. \)

Moreover, for \( t \in X, \)

\[ |\phi(s, -t)| = |h(-t) e^{ist}| = |h(-t)| e^{-\tau t} \leq K_1 e^{at} e^{-\rho_1 t} \]

\[ = K_1 e^{-(\rho_1 - a)t}, \quad s \in \Theta_1, \]

and

\[ |\phi(s, t)| = |h(t) e^{-ist}| = |h(t)| e^{\tau t} \leq K_2 e^{-bt} \]

\[ = K_2 e^{-(b - \rho_2)t}, \quad s \in \Theta_2. \]

Thus \( |\phi(s, -t)| \) and \( |\phi(s, t)| \) are majorized by integrable functions of \( t, \) and by Lemma 3.2, \( \overline{h}_1(s) \) is analytic on \( \Theta_1 \) and \( \overline{h}_2(s) \) is analytic on \( \Theta_2. \) Therefore \( \overline{h}(s) = \overline{h}_1(s) + \overline{h}_2(s) \) is analytic on \( \Theta_1 \cap \Theta_2 = \{s: \rho_1 < \text{Im}(s) < \rho_2\}. \) Given any \( s \) such that
a<\text{Im}(s)<b$, we can choose $\rho_1$ and $\rho_2$ so that $a<\rho_1<\text{Im}(s)<\rho_2<b$. Hence $\bar{h}$ is analytic on the open strip $a<\text{Im}(s)<b$. 

We note that $\bar{h}$ need not be continuous nor even defined on the closed strip $a<\text{Im}(s)<b$. For example, if $h(t)=1$ for $t<0$ and $h(t)=e^{-t}$ for $t\geq 0$, we can take $a=0$ and $b=1$. Then $\bar{h}(t)e^{-ist}$ is defined and continuous on $a<\text{Im}(s)<b$. However, $\bar{h}(s)$ does not exist as a Lebesgue integral for $\text{Im}(s)=0$ or 1.

C. The Inverse Fourier Transform

If $h(t)$ and its Fourier transform $\widehat{h}(x)$, defined by (3.1), are both Lebesgue integrable on the real line, then the Fourier inversion theorem holds and the inverse Fourier transform is the same as the direct Fourier transform except for a change of sign in the exponent of the kernel. However, if the Fourier transform $\overline{h}(s)$, defined by (3.2), exists on a region of the complex $s$-plane which does not include the real axis, it may still be possible to recover $h$ from $\overline{h}$ by an integration along a suitable path in the complex plane. For instance, if $\overline{h}(s)$ is defined on a strip $0<a<\text{Im}(s)<b$ (as in Lemma 3.5), then the Fourier transform of $e^{\gamma t}h(t)$ is defined on the strip $a-\gamma<\text{Im}(s)<b-\gamma$, and we can choose $\gamma$ so that this strip contains the real axis. If we assume that the ordinary Fourier inversion theorem applies to $e^{\gamma t}h(t)$, then
where the integration in the s-plane is along a straight line \( \text{Im}(s) = \gamma \). This is the complex inversion integral for the Fourier transform which we call the inverse Fourier transform. (In particular, this inverse Fourier transform includes the ordinary transform as the special case \( \gamma = 0 \).)

The preceding development serves only as motivation for determining the nature of the inverse Fourier transform, and we shall not be concerned with the conditions of validity for the Fourier inversion theorem. However, we shall be interested in imposing conditions on \( \overline{h}(s) \) which guarantee that the function defined by

\[
(3.8) \quad h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} e^{ist} \overline{h}(s) ds
\]

is continuous on \((-\infty, \infty)\) and independent of \( \gamma \).
Lemma 3.6. Let \( \tilde{h} \) be analytic on a strip \( a<\text{Im}(s)<b \), where \( a \) may be \(-\infty\) and \( b \) may be \(+\infty\), and on this strip let \( \tilde{h}(\sigma+i\tau)\to 0 \) as \( |\sigma|\to\infty \). Assume that \( \tilde{h}(s) \) is Lebesgue integrable on some line \( \text{Im}(s)=\gamma \), \( a<\gamma<b \). Then the integral in (3.8) defines a continuous function of \( t \), \(-\infty<t<\infty\), which is independent of \( \gamma \) for \( a<\gamma<b \).

Proof: (The proof given here is similar in part to that for the inverse Laplace transform given by Churchill, 1958, pp. 178-179.) Let \( \gamma, a<\gamma<b, \) be fixed. Then

\[
h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \tilde{h}(s)e^{ist} ds
\]

\[
= e^{-\gamma t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{h}(\sigma+i\gamma)e^{i\sigma t} d\sigma, \ t\in\mathbb{R}.
\]

Since \( \tilde{h} \) is Lebesgue integrable along the line \( \text{Im}(s)=\gamma \), \( a<\gamma<b \), then this last integral exists and defines a continuous function of \( t \), \(-\infty<t<\infty\), (Lemma 3.1 with \( x \) replaced by \(-\sigma\)). Hence \( h(t) \) is continuous \( \circ \ (\infty, \infty) \).

To show that (3.8) is independent of \( \gamma \) we use any second path \( \text{Im}(s)=\gamma', a<\gamma<\gamma'<b \). The integrand is bounded on the closed str. \( \gamma \leq \text{Im}(s) \leq \gamma' \) because \( \tilde{h}(\sigma+i\tau)e^{i(\sigma+i\tau)t} \) is continuous and tends to zero as \( |\sigma|\to\infty \). Since \( \tilde{h}(s)e^{ist} \) is analytic when \( a<\text{Im}(s)<b \), the integral of this function around
the boundary of the rectangle ABCD is zero, by Cauchy's integral theorem. On the side AB, \( s = a + i\tau \). The integral along this side satisfies

\[
\lim_{\alpha \to \infty} \int_{a + i\gamma}^{a + i\gamma'} h(s) e^{i\alpha\tau} e^{i\tau\tau} d\tau = 0,
\]

by Lebesgue's dominated convergence theorem (because the integrand is bounded). The same argument applies to the integral along the side CD.

Since the sum of the integrals over the four sides is zero, then

\[
\lim_{\alpha \to \infty} \int_{-a + iy}^{a + iy'} h(s) e^{ist} ds = \lim_{\alpha \to \infty} \int_{-a + iy}^{a + iy} h(s) e^{ist} ds.
\]

We find that \( h(t) \) has the same value for the path \( \text{Im}(s) = \gamma' \) as for \( \text{Im}(s) = \gamma \). #
D. The Laplace Transform

If the complex s-plane is rotated through 90°, that is, if we let \( s' = is \), then the Fourier transform (3.2) becomes

\[
\mathcal{F}(-is') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-st}dt.
\]

Moreover, if we let \( \mathcal{F}(s') = \sqrt{2\pi} \mathcal{F}(-is') \), then

\[
\mathcal{F}(s') = \int_{-\infty}^{\infty} h(t)e^{-st}dt.
\]

This last expression is called the two-sided Laplace transform of \( h \). We shall henceforth be interested in the one-sided Laplace transform or simply the Laplace transform of a function \( f \) defined on \((0, \infty)\),

(3.9) \( \mathcal{F}(s) = \int_{0}^{\infty} e^{-st}f(t)dt \),

provided the integral exists as a Lebesgue integral and is finite.

In the next two lemmas we assume conditions on \( f \) which make \( \mathcal{F} \) analytic on some region of the complex plane. This property of \( \mathcal{F} \) will be sufficient to guarantee the existence of both the multiple-integral representation and the contour-integral representation of the generalized Laplace transform. We will use the notation \( R_+ \) for \((0, \infty)\).
Lemma 3.7. Let $f \in L^1(0,\infty)$. Then, on the half plane $\text{Re}(s)>0$, $\hat{f}$ is analytic. Moreover, $\hat{f}$ is continuous for $\text{Re}(s)>0$.

Proof: Let $X = \mathbb{R}^+$, $\Theta = \{s: \text{Re}(s)>0\}$, $d\mu(t) = dt$ and $\phi(s,t) = f(t)e^{-st}$. Then $\phi(s,t)$ is a measurable function of $t$ for each $s \in \Theta$, and $\phi(s,t)$ is analytic on $\Theta$ for each $t \in X$. Since

$$|\phi(s,t)| = |f(t)|e^{-\sigma t} \leq |f(t)|, \quad \sigma = \text{Re}(s)>0, \ t>0,$$

then $|\phi(s,t)|$ is majorized by a function of $t$ which is integrable on $X$. By Lemma 3.2, $\hat{f}$ is analytic on the half plane $\text{Re}(s)>0$. Moreover, since $\phi(s,t)$ is defined and continuous on $\{s: \text{Re}(s)>0\}$, then $\hat{f}(s)$ is continuous on this set. 

The following lemma assumes more familiar conditions on $f$ and the proof is similar to that of Lemma 3.5.

Lemma 3.8. Let $f$ be Lebesgue measurable on the positive real axis $\mathbb{R}^+$. For some real constants $M>0$ and $\sigma_a$, assume that $|f(t)| \leq Me^{\sigma_a t}$, $0<t<\infty$. Then $\hat{f}$ is analytic on the half plane $\text{Re}(s)>\sigma_a$.

E. The Inverse Laplace Transform

Churchill (1958, p. 176) suggests a heuristic argument leading to the inversion of the Laplace transform in terms of a Bromwich contour. Cauchy's integral formula can be extended
to the case in which a closed contour is replaced by a straight line parallel to the imaginary axis, provided it is applied to a function which is analytic to the right of the line and satisfies certain other conditions. If $\mathcal{F}$ is such a function, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\mathcal{F}(s)}{z-s} ds, \quad \text{Re}(z) > \gamma. \quad (3.10)$$

Suppose that $\mathcal{F}(z)$ is the Laplace transform of $f(t)$, and note that $(z-s)^{-1}$ is the Laplace transform of $e^{st}$. This suggests that

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \mathcal{F}(s) ds, \quad -\infty < t < \infty. \quad (3.11)$$

This is the complex inversion integral for the Laplace transform which we call the inverse Laplace transform. We could have also obtained this expression from the inverse Fourier transform by means of a rotation of the plane.

As with the inverse Fourier transform, we are not concerned with the conditions of validity for the inversion, but only with the conditions on $\mathcal{F}(s)$ which guarantee that the function defined by (3.11) is continuous on $(-\infty, \infty)$ and independent of $\gamma$. 
Lemma 3.9. Let \( f \) be analytic on a strip \( a < \text{Re}(s) < b \), where \( a \) may be \(-\infty\) and \( b \) may be \(+\infty\), and on this strip let \( \bar{f}(s+it) \to 0 \) as \( |t| \to \infty \). Assume that \( f \) is Lebesgue integrable on some line \( \text{Re}(s)=\gamma \), \( a < \gamma < b \). Then (3.11) defines a function of \( t \) which is continuous on \((-\infty,\infty)\) and independent of \( \gamma \) for \( a < \gamma < b \).

Proof: The proof is similar to that of Lemma 3.6. 

F. The Stieltjes Transform

The Stieltjes transform arises naturally as an iteration of the Laplace transform; that is, if

\[
\bar{g}(s) = \int_0^\infty e^{-su} \phi(u) du
\]

where

\[
\phi(u) = \int_0^\infty e^{-ut} g(t) dt,
\]

then formally,

\[
\bar{g}(s) = \int_0^\infty e^{-su} \int_0^\infty e^{-ut} g(t) dt du
\]

\[
= \int_0^\infty g(t) \int_0^\infty e^{-(s+t)u} du dt
\]

\[
= \int_0^\infty g(t) (s+t)^{-1} dt.
\]

We shall henceforth be concerned with the general Stieltjes transform which, for some arbitrary complex constant \( a \), is
defined as

\[
(3.12) \quad \overline{g}_a(s) = \int_0^\infty \frac{g(t)}{(s+t)^a} \, dt,
\]

whenever this integral exists.

In the following lemma we assume conditions on \( g \) which make \( \overline{g}_a \) analytic on some region of the \( s \)-plane. This property of \( \overline{g}_a \) will be sufficient to guarantee the existence of both the multiple-integral representation and the contour-integral representation of the generalized Stieltjes transform.

**Lemma 3.10.** Let \( g \) be Lebesgue measurable on \( \mathbb{R}^+ \) and let \( M \) be a positive constant. Fix the complex constant \( a \) and assume that \( |g(t)| \leq Mt^{\zeta-1}(1+t)^{\Re(a)-2\zeta} \) for all \( t > 0 \) and some \( \zeta > 0 \). Then \( \overline{g}_a \) is analytic on the \( s \)-plane cut along the nonpositive real axis.

**Proof:** In Lemma 3.2 let \( X=(0,\infty) \) and let \( \Theta \) be an open disc whose closure is contained in the cut \( s \)-plane. Take \( d\mu(t)=dt \) and \( \phi(s,t)=g(t)/(s+t)^a \). Then \( \phi(s,t) \) is measurable in \( t \) for each \( s \in \Theta \) and analytic on \( \Theta \) for each \( t \in X \).

Let \( \delta \) be the distance between \( \Theta \) and the nonpositive real axis and let \( \rho=2 \sup\{|s|: s \in \Theta\} \). We note that \( \delta > 0 \) and that \( 2 \Re(s) \geq -\rho \). Hence we may write the following inequalities for every \( s \in \Theta \):
\[ |s+t| > \rho, \quad 0 < t < \rho; \]
\[ |s+t| = \left[ t^2 + 2 \operatorname{Re}(s)t + |s|^2 \right]^\frac{1}{2} \geq \left[ t^2 - pt + \delta^2 \right]^\frac{1}{2}, \quad t > \rho; \]
\[ |s+t| \leq t + |s| \leq t + \rho/2, \quad t > 0. \]

For \( \operatorname{Re}(a) > 0 \), define
\[ f_1(t) = \begin{cases} 
\delta - \operatorname{Re}(a), & 0 < t < \rho \\
\frac{- \frac{\operatorname{Re}(a)}{2}}{\left[ t^2 - pt + \delta^2 \right]^\frac{1}{2}}, & t > \rho,
\end{cases} \]
and for \( \operatorname{Re}(a) < 0 \), define
\[ f_2(t) = \left[ t + \rho/2 \right]^{\operatorname{Re}(a)}, \quad t > 0. \]

Then both \( f_1(t) \) and \( f_2(t) \) are continuous on \( X \) and
\[ |s+t|^{-\operatorname{Re}(a)} \leq f_1(t), \quad \operatorname{Re}(a) > 0, \]
\[ |s+t|^{-\operatorname{Re}(a)} \leq f_2(t), \quad \operatorname{Re}(a) < 0. \]

Let \( s+t = |s+t|e^{i\theta} \) where \( |\theta| < \pi \). Then
\[ |(s+t)^{-a}| \leq |s+t|^{-\operatorname{Re}(a)} e^{\pi |\operatorname{Im}(a)|}. \]
Hence, for \( \operatorname{Re}(s) > 0 \),
\[ |\phi(s,t)| = |g(t)||s+t|^{-a} \]
\[ \leq M e^{\pi|\operatorname{Im}(a)|}\int_{\zeta=1}^{\infty}(t+1)^{\operatorname{Re}(a)-2\zeta}|s+t|^{-\operatorname{Re}(a)} \]
\[ \leq M e^{\pi|\operatorname{Im}(a)|}\int_{\zeta=1}^{\infty}(t+1)^{\operatorname{Re}(a)-2\zeta}f_1(t), \quad s \in \Theta, \quad t \in X \]
and for Re(a) < 0,

\[ |\phi(s,t)| \leq Me^{\pi |\text{Im}(a)| t} \frac{\zeta-1}{(1+t)^{\zeta-1}} \frac{\text{Re}(a)-2\zeta}{2^\zeta f_2(t)}, \ s \in \Theta, \ t \in \mathbb{X}. \]

In either case, \(|\phi(s,t)|\) is bounded by a continuous function of \(t\), which is of order \(t^{\zeta-1}\) as \(t \to 0\) and \(t^{-\zeta-1}\) as \(t \to \infty\), and which is therefore integrable on \(X\). By Lemma 3.2 \(\overline{g}_a\) is analytic on \(\Theta\). For any \(s\) in the cut plane we can find a \(\Theta\) such that \(s \in \Theta\). Therefore \(\overline{g}_a\) is analytic on the cut plane. 

\#
IV. GENERALIZED TRANSFORMS BY THE METHOD OF MULTIPLE INTEGRALS

A. Introduction

In this chapter we shall generalize the Fourier, Laplace and Stieltjes transforms to functions of several variables. We shall use the multiple-integral representation outlined in Chapter II.

In the process of generalizing the transforms by this method we will obtain expressions of the form

\[(4.1) \int \int \psi(x) \phi(\sum_{i=1}^{k} u_i y_i, x) P(b, u) dx \, du',\]

where \( \lambda \) is a path of integration in \( C^1 \), \( \phi \) is the kernel of the transform, and the \( y_i \)'s and \( x \) may be real or complex. We will want to interchange the order of integration in (4.1) to obtain

\[\int \int \psi(x) \phi(\sum_{i=1}^{k} u_i y_i, x) P(b, u) du' \, dx \]

\[= \int \psi(x) \phi(b, y, x) dx, \quad \text{Re}(b) > 0,\]

where \( \phi \) is the \( k \)-variable analogue of the kernel \( \phi \) given by (2.6). The following lemma provides conditions on \( \psi \) and \( \phi \) which are sufficient for the validity of the interchange of order of integration in (4.1).
Lemma 4.1. Let \( y_1, \ldots, y_k \) be fixed complex numbers and let \( \lambda \) be a connected subset of a straight line in \( \mathbb{C}^1 \). Assume that the complex-valued function \( \psi(x) \) is Lebesgue measurable on \( \lambda \), \( \phi(\sum u_i y_i, x) \) is measurable on \( \text{Ex} \lambda \), and \( |\phi(\sum u_i y_i, x)| \leq f(x) \) on \( \text{Ex} \lambda \), where \( |\psi|f \) is Lebesgue integrable on \( \lambda \). Then for \( \text{Re}(b) > 0 \),

\[
\int \int_{E \times \lambda} \psi(x) \phi(\sum u_i y_i, x) P(b,u) dx \, du' = \int \int_{E \times \lambda} \psi(x) \int \phi(\sum u_i y_i, x) P(b,u) du' \, dx
\]

\[
=\int \psi(x) \phi(b,y,x) dx,
\]

where \( \phi \) is the \( k \)-variable analogue of \( \phi \) given by (2.6).

Proof: The function \( P(b,u) \) is measurable on \( E \) and therefore, by the assumptions, the integrand of the inner integral of (4.1) is measurable on \( \text{Ex} \lambda \). We also have

\[
\int \int_{E \times \lambda} |\psi(x)\phi(\sum u_i y_i, x)| P(b,u) dx \, du' = \int_{E} |P(b,u)| \int_{\lambda} |\psi(x)| |\phi(\sum u_i y_i, x)| dx \, du'
\]

\[
\leq \int_{E} |P(b,u)| du' \int_{\lambda} |\psi(x)| f(x) dx
\]
By Fubini's theorem (Rudin, 1966, Thm. 7.8, pp. 140-141) we may interchange the order of integration in (4.1) to obtain

\[ \int_E \int_{\lambda} \psi(x) \phi(\sum_{i=1}^{k} u_i y_i, x) P(b, u) du dx \]

\[ = \int_{\lambda} \psi(x) \int_{E} \phi(\sum_{i=1}^{k} u_i y_i, x) P(b, u) du dx \]

\[ = \int_{\lambda} \psi(x) \phi(b, y, x) dx, \quad \text{Re}(b) > 0. \]

B. The Fourier Transform in the Case of Real Variables

If a function \( h \in L_1 \), then the ordinary Fourier transform of \( h \), defined by

\[ \hat{h}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \exp(-ixt) dt, \quad x \in \mathbb{R}, \]

is continuous on \( \mathbb{R} \) (Lemma 3.1). Hence by Theorem 2.1 or 2.3, \( \hat{h} \) may be generalized to a function \( \hat{H} \) of \( k \) real variables by the method of multiple integrals which was outlined in Chapter II. Then we have
(4.2) \( \overline{\mathcal{H}}(b,x) = \int \int_{E} h_{(\sum_{i=1}^{k} u_{i}x_{i})} P(b,u)du' \), \( \text{Re}(b) > 0 \),

where \( x=(x_{1},\ldots,x_{k}) \in \mathbb{R}^{k} \) (the k-th Cartesian power of \( \mathbb{R} \))

and \( E \) is the simplex defined by (2.1).

**Theorem 4.1.** Let \( h \in L_{1} \). If \( \text{Re}(b) > 0 \) and \( x \in \mathbb{R}^{k} \), then

\[
(4.3) \quad \overline{\mathcal{H}}(b,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S(b,-ix_{i})dt,
\]

where \( S(b,-ix_{i}) = S(b_{1},\ldots,b_{k}; -ix_{1},\ldots,-ix_{k}) \) is obtained

from (2.23) by replacing \( t \) by \( -it \) and \( z \) by \( x \).

**Proof.** In (3.1) replace \( x \) by \( \sum_{i=1}^{k} u_{i}x_{i} \) and substitute in

(4.2). Then

\[
(4.4) \quad \overline{\mathcal{H}}(b,x) = \frac{1}{\sqrt{2\pi}} \int \int_{E} h(t) \exp(-it \sum_{i=1}^{k} u_{i}x_{i}) P(b,u)du' dt.
\]

Since \( h \in L_{1} \), it is measurable on \( \mathbb{R} \). The function

\( \exp(-it \sum_{i=1}^{k} u_{i}x_{i}) \) is continuous on \( \mathbb{R} \times \mathbb{R} \). Since

\[
\left| \exp(-it \sum_{i=1}^{k} u_{i}x_{i}) \right| = 1,
\]

then

\[
\int_{-\infty}^{\infty} |h(t) \exp(-it \sum_{i=1}^{k} u_{i}x_{i})| dt = \int_{-\infty}^{\infty} |h(t)| dt < \infty.
\]

In Lemma 4.1 we put \( \lambda = \mathbb{R} \) and \( \psi = h \) and observe that the

exponential function \( \phi \) is majorized by \( f(t) = 1 \). Then we have

\[
\overline{\mathcal{H}}(b,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \int_{E} \exp(-it \sum_{i=1}^{k} u_{i}x_{i}) P(b,u)du' dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S(b,-ix_{i})dt, \quad \text{Re}(b) > 0. \]

\#
We note that if \( x_1 = x_2 = \ldots = x_n = \xi \), then \( \overline{H}(b, x) = \overline{h}(\xi) \).

Also, since \( h(x) \) is continuous on \( R \), then by Theorem 2.4, \( \overline{H}(b, x) \) is continuous in \( x \) on \( R^k \).

C. The Fourier Transform in the Case of Complex Variables

If the assumptions of either Lemma 3.3, 3.4 or 3.5 are imposed upon a function \( h \), then the ordinary Fourier transform of \( h \), defined by

\[
(3.2) \quad \overline{h}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \exp(-ist) dt,
\]

is analytic on either the lower half-plane, the entire plane or an open strip, respectively. Let \( s_1, \ldots, s_k \) be elements in the domain \( D \) of analyticity of \( \overline{h} \). Since \( D \) is convex in each of the three cases, then the convex hull \( K(s) \) is contained in \( D \) and the analogue function

\[
(4.5) \quad \overline{H}(b, s) = \int_{E} \overline{h}(\sum_{i=1}^{k} u_i s_i) P(b, u) du', \quad \text{Re}(b) > 0,
\]

is analytic on \( D^k \) by Lemma 2.5.

Theorem 4.2. Let \( h \) satisfy the assumptions of either Lemma 3.3, 3.4 or 3.5 and let \( D \) be the corresponding domain of analyticity of \( \overline{h} \). If \( \text{Re}(b) > 0 \) and \( s \in D^k \), then

\[
(4.6) \quad \overline{H}(b, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \overline{S}(b, -ist) dt, \quad \text{Re}(b) > 0,
\]
where $S$ is defined by (2.23). In the case of Lemma 3.3, the assumption $s \in D^k$ may be replaced by $s \in D^k_1$ where $D_1$ is the union of the lower half-plane and the real axis.

Proof: In (3.2) replace $s$ by $\sum_{i=1}^{k} u_i s_i$ and substitute in (4.5). Then

\[(4.7) \quad H(b,s) = \frac{1}{\sqrt{2\pi}} \int_{E} \int_{-\infty}^{\infty} h(t) \exp(-it \sum_{i=1}^{k} u_i s_i) P(b,u) dt \, du,\]

\[= \frac{1}{\sqrt{2\pi}} \int_{E} \int_{0}^{\infty} h(-t) \exp(it \sum_{i=1}^{k} u_i s_i) P(b,u) dt \, du' + \frac{1}{\sqrt{2\pi}} \int_{E} \int_{0}^{\infty} h(t) \exp(-it \sum_{i=1}^{k} u_i s_i) P(b,u) dt \, du'.\]

As in the proof of Theorem 4.1, and under the assumptions of either Lemma 3.3, 3.4 or 3.5, the integrand of the inner integral in each of the above terms is measurable on $E \times R^+$.

If $s_i = \sigma_i + it_i$, let $m = \min \{\tau_i\}$ and $M = \max \{\tau_i\}$. Then

\[\int_{0}^{\infty} |h(-t)\exp(it \sum_{i=1}^{k} u_i s_i)| dt = \int_{0}^{\infty} |h(-t)|\exp(-t \sum_{i=1}^{k} u_i \tau_i) dt \leq \int_{0}^{\infty} |h(-t)| \exp(-tm) dt,\]

and

\[\int_{0}^{\infty} |h(t)\exp(-it \sum_{i=1}^{k} u_i s_i)| dt = \int_{0}^{\infty} |h(t)|\exp(t \sum_{i=1}^{k} u_i \tau_i) dt \leq \int_{0}^{\infty} |h(t)| \exp(tM) dt.\]
Under the assumptions of Lemma 3.3, $h(-t) = 0$, $0 < t < \infty$, and $M < 0$ (or $M \leq 0$ if $s \in D_1^k$), so each of these integrals is finite. Under the assumptions of Lemma 3.4, $h$ vanishes outside a finite interval and so both integrals are finite. Under the assumptions of Lemma 3.5, the integrands are both bounded by integrable functions. We apply Lemma 4.1 to each of these integrals with $\lambda = \mathbb{R}_+$, $\psi(t) = h(-t)$ and $f(t) = \exp(-tm)$ for the first integral and $\lambda = \mathbb{R}_+$, $\psi(t) = h(t)$ and $f(t) = \exp(tM)$ for the second. Then

$$H(b,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \int_{\mathbb{R}} \exp(-it \sum_{i=1}^k u_i s_i) P(b,u) du \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S(b,-ist) dt, \quad \text{Re}(b) > 0. \#$$

We note that if $s_1 = s_2 = \ldots = s_k = z$, then $H(b,s) = \overline{h}(z)$. Also since $\overline{h}$ is analytic on a domain $D \subset \mathbb{C}^1$, then by Lemma 2.5, $H(b,s)$ is analytic in $b$ and $s$ on a domain in $\mathbb{C}^{2k}$ defined by $\text{Re}(b) > 0$ and $s \in D^k$.

D. The Inverse Fourier Transform

If $\overline{h}$ satisfies the assumptions of Lemma 3.6, then the ordinary inverse Fourier transform of $\overline{h}$, defined by

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{h}(s) \exp(its) dt, \quad t \in \mathbb{R}, \quad (3.5)$$
is continuous on \( \mathbb{R} \), and according to Theorem 2.1 or 2.3 it may be generalized to a function \( H \) of \( k \) real variables by the method of Chapter II. Then, if \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \),

\[
(4.8) \quad H(b, t) = \int_{E}^{\infty} h(\sum_{i=1}^{k} u_i t_i) P(b, u) du', \quad \text{Re}(b) > 0.
\]

Theorem 4.3. Let \( \bar{h}(s) \) satisfy the conditions of Lemma 3.6. If \( \text{Re}(b) > 0 \) and \( t \in \mathbb{R}^k \), then

\[
(4.9) \quad H(b, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}(s) S(b, its) ds,
\]

where \( S \) is defined by (2.23).

**Proof:** In (3.5) replace \( t \) by \( \sum_{i=1}^{k} u_i t_i \) and substitute in (4.8). Then

\[
(4.10) \quad H(b, t) = \frac{1}{\sqrt{2\pi}} \int_{E}^{\infty} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}(s) \exp(is \sum_{i=1}^{k} u_i t_i) P(b, u) ds du',
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{E}^{\infty} \int_{-\infty}^{\infty} \bar{h}(\sigma+i\gamma) \exp[i(\sigma+i\gamma) \sum_{i=1}^{k} u_i t_i] P(b, u) d\sigma du'.
\]

Since \( \bar{h}(s) \) was assumed to be integrable on the line \( \text{Im}(s) = \gamma \), then \( \bar{h}(\sigma+i\gamma) \), considered as a function of the real variable \( \sigma \), is integrable on \( \mathbb{R} \). Then the integrand of the inner integral of (4.10) is measurable on \( E \times \mathbb{R} \). Let \( \bar{\epsilon} = \max\{|t_i|\} \). Then
$$\left[ \int_{-\infty}^{\infty} |\bar{h}(\sigma+i\gamma)| \exp \left[ i(\sigma+i\gamma) \sum_{i=1}^{k} u_i t_i \right] d\sigma \right]^k$$

$$= \int_{-\infty}^{\infty} |\bar{h}(\sigma+i\gamma)| \exp \left[ -\gamma \sum_{i=1}^{k} u_i t_i \right] d\sigma$$

$$\leq e^{\gamma|\bar{E}|} \int_{-\infty}^{\infty} |\bar{h}(\sigma+i\gamma)| d\sigma$$

$$< \infty.$$  

By Lemma 4.1 with \( f(t) = e^{\gamma|E|} \) we have

$$H(b,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}(s) \exp(i s \sum_{i=1}^{k} u_i t_i) P(b,u) du \ ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}(s) S(b, ist) ds, \ Re(b)>0. \$$

If \( t_1 = t_2 = \ldots = t_k = \xi \), then \( H(b,t) = h(\xi) \). Also, since \( h(t) \) is continuous on \( R \), then by Theorem 2.4, \( H(b,t) \) is continuous in \( t \) on \( R^k \).

**E. The Laplace Transform**

If the conditions of either Lemma 3.7 or 3.8 are imposed upon a function \( f \), then the ordinary Laplace transform of \( f \), defined by

$$\mathcal{L}\{f(t)\} = \mathcal{F}(s) = \int_{0}^{\infty} \exp(-st) f(t) \ dt,$$
is analytic on the half plane \( \text{Re}(s)>0 \), respectively. Moreover, \( \overline{f} \) is continuous for \( \text{Re}(s)>0 \) in the case of Lemma 3.7. Let \( s_1, \ldots, s_k \) be elements in the domain \( D \) of analyticity of \( \overline{h} \). Since \( D \) is convex, \( K(s) \subseteq D \), and the analogue function

\[
(4.11) \quad \overline{F}(b,s) = \int_{E} \overline{f}(\sum_{i=1}^{k} u_i s_i) P(b,u)du', \quad \text{Re}(b)>0,
\]

is analytic in \( s \) on \( D_k \) by Lemma 2.5.

**Theorem 4.4.** Let \( f \) satisfy the assumptions of either Lemma 3.7 or 3.8, and let \( D \) be the corresponding domain of analyticity of \( \overline{f} \). If \( \text{Re}(b)>0 \) and \( s \in D_k \), then

\[
(4.12) \quad \overline{F}(b,s) = \int_{0}^{\infty} f(t) S(b,-ts)dt, \quad \text{Re}(b)>0,
\]

where \( S \) is defined by (2.23). In the case of Lemma 3.7, the assumption \( s \in D_k \) may be replaced by \( s \in D_1 \), where \( D_1 \) is the union of the right half-plane and the imaginary axis.

**Proof:** In (3.9) replace \( s \) by \( \sum_{i=1}^{k} u_i s_i \) and substitute in (4.11). Then

\[
(4.13) \quad \overline{F}(b,s) = \int_{E} \int_{0}^{\infty} \exp(-t \sum_{i=1}^{k} u_i s_i) f(t) P(b,u)dt du'.
\]

Under either set of assumptions, \( f \) is measurable on \( R_+ \), and so the integrand of the inner integral of (4.13) is measurable on \( E \times R_+ \). If \( s_i = \sigma_i + it_i \), let \( \overline{\sigma} = \min \{ \sigma_i \} \). Then
\[
\int_0^\infty |f(t)| \exp(-t\sigma) dt \leq \int_0^\infty |f(t)| dt < \infty.
\]

Under the assumptions of Lemma 3.8, \( \sigma > \sigma_a \) and so

\[
\int_0^\infty |f(t)| \exp(-t\sigma) dt \leq M \int_0^\infty \exp[t(\sigma_a - \sigma)] dt = M/(\sigma_a - \sigma).
\]

By Lemma 4.1, with \( \lambda = \mathbb{R}_+ \) and \( f(t) = \exp(-t\sigma) \), we have

\[
\bar{F}(b,s) = \int_0^\infty f(t) \int_E \exp(-t \sum_{i=1}^k u_i s_i) P(b,u) du dt
\]

\[
= \int_0^\infty f(t) \cdot S(b, -ts) dt, \quad \text{Re}(b) > 0.
\]

If \( s_1 = s_2 = \ldots = s_k = 1 \), then \( \bar{F}(b,s) = \bar{F}(z) \). Also, since \( \bar{F}(s) \) is analytic on a half plane \( D \), then by Lemma 2.5, \( \bar{F}(b,s) \) is analytic in \( b \) and \( s \) on a domain in \( \mathbb{C}^{2k} \) defined by \( \text{Re}(b) > 0 \) and \( s \in D^k \).
F. The Inverse Laplace Transform

If a function \( \overline{f} \) satisfies the conditions of Lemma 3.9, then the ordinary inverse Laplace transform of \( \overline{f} \), defined by

\[
(3.11) \quad f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp(st)\overline{f}(s)\,ds, \quad t \in \mathbb{R},
\]

is continuous on \( \mathbb{R} \), and by Theorem 2.1 or 2.3 we may generalize \( f \) to a function \( F \) of \( k \) real variables by the method of multiple integrals. Then, if \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \),

\[
(4.14) \quad F(b, t) = \int_{E} \left( \sum_{i=1}^{k} u_i t_i \right) P(b, u)\,du', \quad \text{Re}(b) > 0.
\]

**Theorem 4.5.** Let \( \overline{f} \) satisfy the assumptions of Lemma 3.9. If \( \text{Re}(b) > 0 \) and \( t \in \mathbb{R}^k \), then

\[
(4.15) \quad F(b, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \overline{f}(s) S(b, st)\,ds, \quad \text{Re}(b) > 0,
\]

where \( S \) is defined by (2.23).

**Proof:** The proof is similar to that of Theorem 4.3. #

If \( t = (t_1, \ldots, t_k) = \xi \), then \( F(b, t) = f(\xi) \). Also, since \( f(t) \) is continuous on \( \mathbb{R} \), then \( F(b, t) \) is continuous in \( t \) on \( \mathbb{R}^k \) by Theorem 2.4.

G. The Stieltjes Transform

If the conditions of Lemma 3.10 are imposed on a function \( g \), then the ordinary Stieltjes transform of \( g \), defined by

\[
(3.12) \quad \overline{g}_a(s) = \int_{0}^{\infty} \frac{g(t)}{(s+t)^a} \,dt,
\]
where a is some fixed complex constant, is analytic on the s-plane cut along the nonpositive real axis. If Re(b) > 0 and s₁,...,sₖ are points in the cut plane D and if K(s)⊂D, then the analogue function is

\[ (4.16) \quad \overline{g}_a(b,s) = \int_{E} \overline{g}_a(\sum_{i=1}^{k} u_i s_i) P(b,u)du', \quad \text{Re}(b) > 0. \]

Moreover, \( \overline{g}_a \) is analytic on a domain in \( \mathbb{C}^{2k} \) defined by \( \text{Re}(b) > 0 \) and \( K(s)\subset D \), by Lemma 2.5.

**Theorem 4.6.** Let \( g \) be Lebesgue measurable on \( \mathbb{R}^+ \) and let \( M \) be a positive constant. Fix the complex constant \( a \) and assume that \( |g(t)| \leq Mt^{\zeta-1}(1+t)^{\text{Re}(a)-2\zeta} \) for all \( t > 0 \) and some \( \zeta > 0 \). If \( \text{Re}(b) > 0 \) and \( K(s) \) is contained in the cut plane \( D \), then

\[ (4.17) \quad \overline{g}_a(b,s) = \int_{0}^{\infty} g(t) R(a,b,s+t)dt, \]

where \( R(a,b,s+t) = R(a; b_1,\ldots,b_k; s_1+t,\ldots,s_k+t) \) is the hypergeometric R-function defined by (2.24).

**Proof:** In (3.12) replace \( s \) by \( \sum_{i=1}^{k} u_i s_i \) and substitute in (4.16). Then

\[ (4.18) \quad \overline{g}_a(b,s) = \int_{E} \int_{0}^{\infty} g(t) \left( \sum_{i=1}^{k} u_i s_i + t \right)^{-a} P(b,u)dt \, du'. \]

Since the point \( \sum_{i=1}^{k} u_i s_i \in K(s) \) for all \( u' \in E \) and \( K(s) \) does not intersect the nonpositive real axis, then \( \left( \sum_{i=1}^{k} u_i s_i + t \right)^{-a} \) is jointly continuous in the \( u_i \)'s and \( t \). Therefore the integrand of the inner integral of (4.18) is measurable on \( E \times \mathbb{R}^+ \). Choose an open disc \( \Theta \subset D \) such that \( K(s) \subset \Theta \). For every \( u' \in E \),
\[ \sum_{i=1}^{k} u_i s_i + t \in \mathbb{C}. \] As in the proof of Lemma 3.10,
\[ |\sum_{i=1}^{k} u_i s_i + t|^{-\text{Re}(a)} \leq f_i(t), \]
where \( i = 1 \) if \( \text{Re}(a) > 0 \) and \( i = 2 \) if \( \text{Re}(a) < 0 \) and both \( f_1 \) and \( f_2 \) are continuous on \( \mathbb{R}_+ \).

Let \( \sum_{i=1}^{k} u_i s_i + t = \sum_{i=1}^{k} |u_i s_i + t| e^{i\theta}, |\theta| < \pi. \) We note that
\[ \sum_{i=1}^{k} |u_i s_i + t|^{-\text{Re}(a)} | \text{Im}(a)| \theta \]
\[ \leq \sum_{i=1}^{k} |u_i s_i + t|^{-\text{Re}(a)} | \text{Im}(a)|. \]

Then
\[ \int_{0}^{\infty} |g(t) (\sum_{i=1}^{k} u_i s_i + t)^{-a}| dt \]
\[ \leq M \int_{0}^{\infty} t^{-1} (1+t) \text{Re}(a)^{-2s} |(\sum_{i=1}^{k} u_i s_i + t)^{-a}| dt \]
\[ \leq Me^{\pi |\text{Im}(a)|} \int_{0}^{\infty} t^{-1} (1+t) \text{Re}(a)^{-2s} \sum_{i=1}^{k} |u_i s_i + t|^{-\text{Re}(a)} dt \]
\[ \leq Me^{\pi |\text{Im}(a)|} \int_{0}^{\infty} t^{-1} (1+t) \text{Re}(a)^{-2s} f_i(t) dt. \]

In Lemma 4.1 let \( \lambda = \mathbb{R}_+, \psi(t) = 1, \phi(\sum_{i=1}^{k} u_i s_i + t) = g(t) (\sum_{i=1}^{k} u_i s_i + t)^{-a} \)
and \( f(t) = Me^{\pi |\text{Im}(a)|} t^{-1} (1+t) \text{Re}(a)^{-2s} f_i(t) \). The integrand of
the last integral is continuous on \( \mathbb{R}_+ \) and is of order \( t^{-1} \) as \( t \to 0 \) and \( t^{-1} \) as \( t \to \infty \), and so the integral is finite. Then
\[ \tilde{G}_a(b,s) = \int_0^\infty g(t) \int_E \left( \sum_{i=1}^k u_i s_i + t \right)^{-a} p(b,u) du \, dt \]

\[ = \int_0^\infty g(t) R(a,b,s+t) dt, \quad \text{Re}(b)>0. \]

If \( s_1 = s_2 = \ldots = s_k = z \), then \( \tilde{G}_a(b,s) = \tilde{g}_a(z) \). Since \( \tilde{g}_a \) is analytic on the cut plane \( D \), then by Lemma 2.5, \( \tilde{G}_a(b,s) \) is analytic in \( b \) and \( s \) on a domain in \( \mathbb{C}^{2k} \) defined by \( \text{Re}(b)>0 \) and \( K(s) \subset D \).
V. GENERALIZED TRANSFORMS BY THE METHOD
OF CONTOUR INTEGRATION

A. Introduction

Cauchy's integral formula has the same structure as
the defining equations for the ordinary Fourier, Laplace and
Stieltjes transforms, namely,

\( \overline{f}(s) = \int f(t) \phi(s, t) \, dt, \)

where the integration is extended over some path or contour in
\( C^1. \) Therefore it is not surprising that Cauchy's formula
has a generalization similar to the generalized transforms
discussed earlier. Since \( \overline{f} = f \) in the Cauchy case, the
generalized Cauchy formula provides for the analogue function
\( F \) a contour-integral representation which can be used to
advantage in place of the multiple-integral representation
used in Chapter IV.

Let \( \gamma \) denote a positively oriented rectifiable Jordan
curve in \( C^1. \) We shall denote its interior region by \( I(\gamma), \)
its exterior region by \( X(\gamma), \) and the closures of these
regions by \( \overline{I}(\gamma) = I(\gamma) \cup \gamma \) and \( \overline{X}(\gamma) = X(\gamma) \cup \gamma. \) If \( f \) is analytic
on \( I(\gamma) \) and continuous on \( \overline{I}(\gamma), \) then by Cauchy's integral
formulas for \( f \) and its derivatives,

\[
(5.2) \quad f^{(n)}(s) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-s)^{n+1}} \, dz, \quad (n=0, 1, \ldots),
\]

for every \( s \in I(\gamma). \) We would expect that (5.2) would have the
generalization
\[(5.3)\quad F^{(n)}(b, s) = \frac{n!}{2\pi i} \int_{\gamma} f(z) R(n+1, b, z-s)dz, \quad (n=0,1, \ldots),\]

where \(s_1, \ldots, s_N\) are assumed to lie in \(I(\gamma)\) and \(F^{(n)}\) is the analogue in several variables of \(f^{(n)} = \frac{d^n f}{ds^n}\). The \(R\) function in the integrand is the analogue of \((z-s)^{-n-1}\) and is the analytic continuation of the function defined by \((2.24)\). Carlson (1969, Thm. 3) has shown that \((5.3)\) is indeed valid and can be used to continue \(F\) analytically in the variables as well as in the parameters. By contrast with the method of multiple integrals used in Chapter IV, the method of contour integration will then make it possible to eliminate assumptions about the convex hull of the transformed variables and also to relax the assumption that \(\text{Re}(b) > 0\). This will prove especially useful for the generalization of the Stieltjes transform, where we may wish to locate the variables on both sides of the nonpositive real axis. The transform to be generalized must be either an analytic function of a complex variable or the restriction to the real axis of such a function in order that the Cauchy integral formula \((5.2)\) can be applied. This is no hindrance in the cases of the Fourier, Laplace and Stieltjes transforms (see Lemmas 3.3, 3.4, 3.5, 3.7, 3.8 and 3.10). However, the inverse Fourier transform \(h(t)\) and the inverse Laplace transform \(f(t)\), defined by \((3.8)\) and \((3.11)\), do not exist for values of \(t\) both above and below the real axis unless \(\overline{h}(s)\)
and $f(s)$ are required to decay exponentially at large distances in both directions on the path of integration. This very restrictive asymptotic condition seems a high price to pay for relaxing the condition on $\text{Re}(b)$, and so we shall not generalize the inverse Fourier and inverse Laplace transforms by the method of contour integration.

B. Analytic Continuation of $R$

In (5.3), $R(n+1, b, z-s)$ can be defined by the multiple-integral representation

$$R(n+1, b, z-s) = \int \left[ z - \sum_{i=1}^{k} u_i s_i \right]^{-n-1} P(b,u)du', \quad (n=0,1,2,...),$$

so long as $z \in \gamma$, $K(s) \subseteq \Omega(\gamma)$ and $\text{Re}(b) > 0$. It is important that $n$ be an integer because $R$ is then single-valued on $\gamma$. Carlson (1969, Thm. 4) showed that this principal branch of $R(n+1, b, z-s)$ can be continued analytically in the $b$'s and $s$'s so long as $z \in \gamma$, $\{s_1, ..., s_k\} \subseteq \Omega(\gamma)$ and $c \neq 0, -1, -2, ...$

(where $c = \sum_{i=1}^{k} b_i$).

C. Analytic Continuation of $F^{(n)}$

Let $D \subseteq \mathbb{C}^1$ be a simply connected domain and assume that $f$ is analytic on $D$. The analogue $F^{(n)}$ of $f^{(n)}$ can be defined by the multiple-integral representation

$$F^{(n)}(b,s) = \int \left[ f^{(n)} \left( \sum_{i=1}^{k} u_i s_i \right) P(b,u)du', \right.$$
so long as Re(b) > 0 and K(s) ⊂ D. Carlson (1969, Thm. 8) showed that F^{(n)} (n=0,1,2,...), can be continued analytically in the b's and s's by using the generalized Cauchy formula (5.3), so long as s ∈ D^k and c≠0,-1,-2,...

D. Some Special Cases

We are now in a position to give certain formulas relating the S-function and the R-function, which will be required in the generalization procedure (see Carlson, 1969).

The contour integral representation of e^{-ist}, where t is an arbitrary constant, is

(5.4) e^{-ist} = \frac{1}{2\pi i} \int_{\gamma} e^{-itz} (z-s)^{-1} dz, \quad s \in \Gamma (\gamma).

Equation (5.4) is a special case of (5.2) with n=0. The generalization of (5.4), which is a special case of (5.3) with n=0, is

(5.5) S(b, -ist) = \frac{1}{2\pi i} \int_{\gamma} e^{-itz} R(l,b,z-s) dz, \quad (c≠0,-1,-2,...),

where γ encircles s_1,...,s_k in the positive direction.

Similarly, the generalization of

(5.6) e^{-st} = \frac{1}{2\pi i} \int_{\gamma} e^{-zt} (z-s)^{-1} dz

is

(5.7) S(b, -st) = \frac{1}{2\pi i} \int_{\gamma} e^{-zt} R(l,b,z-s) dz, \quad (c≠0,-1,-2,...).

If t>0 and a is a fixed complex constant, then (t+s)^{-a} is analytic in s on the s-plane cut along the nonpositive
real axis. If \( \gamma \) is contained in the cut plane, then

\[
(t+s)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma} (t+z)^{-\alpha}(z-s)^{-1}dt, \quad \text{sec}I(\gamma).
\]

According to (5.3) with \( n=0 \), the generalization of (5.8) is

\[
R(a,b,t+s) = \frac{1}{2\pi i} \int_{\gamma} (t+z)^{-\alpha} R(1,b,z-s)dz,
\]

\((c\neq 0, -1, -2, \ldots)\), where \( s_1, \ldots, s_k \) are points in \( I(\gamma) \).

In case \( s_1=s_2=\ldots=s_k=s \), then \( R(1,b,z-s)=(z-s)^{-1} \) and (5.5), (5.7) and (5.9) reduce to (5.4), (5.6) and (5.8), respectively.

E. Sufficient Conditions for the Existence of the Generalized Transform by Contour Integration

As in Chapter IV, the generalization of the ordinary transforms by the method of contour integration will result in expressions of the form

\[
\int_{\gamma} \int_{\lambda} \psi(t) \phi(z,t)R(1,b,z-s)dt\,dz,
\]

where \( \gamma \) is a rectifiable Jordan curve in \( C^1 \), \( \lambda \) is a path of integration in \( C^1 \), and \( \phi \) is the kernel of the transform. We will want to interchange the order of integration in (5.10) to obtain

\[
\int_{\lambda} \psi(t) \int_{\gamma} \phi(z,t)R(1,b,z-s)dz\,dt = 2\pi i \int_{\lambda} \psi(t) \phi(b,s,t)dt,
\]

where \( \phi \) is the analogue of \( \phi \) as given by (5.3) with \( n=0 \).

The following lemma provides conditions on \( \psi \) and \( \phi \) which are
sufficient for the validity of the interchange of order of integration in (5.10).

**Lemma 5.1.** Let $\gamma$ be a rectifiable Jordan curve in $C^1$ and let $s_1, \ldots, s_k$ be fixed points in $I(\gamma)$. Let $\lambda$ be a connected subset of a straight line in $C^1$ and assume that $\psi$ is Lebesgue measurable on $\lambda$ and that $\phi(z,t)$ is continuous on $\gamma \times \lambda$. If \[ \int_\gamma |\psi(t) \phi(z,t)| \, dt \] is bounded on $\gamma$ and if $c \neq 0, -1, -2, \ldots$, then

\[
\int_\gamma \int_\lambda \psi(t) \phi(z,t) R(l,b,z-s) \, dt \, dz = \int_\lambda \psi(t) \int_\gamma \phi(z,t) R(l,b,z-s) \, dz \, dt = 2\pi \int_\lambda \psi(t) \phi(b,s,t) \, dt,
\]

where $\phi$ is the analogue of $\phi$ given by (5.3) with $n=0$.

**Proof:** The function $R(l,b,z-s)$ has been shown by Carlson (1969, Theorem 4) to be continuous in $z$ on $\gamma$. Therefore by the assumptions of the theorem, the integrand of the inner integral of (5.10) is measurable on $\gamma \times \lambda$. Moreover,

\[
\int_\gamma \int_\lambda |\psi(t) \phi(z,t) R(l,b,z-s)| \, dt \, dz = \int_\gamma |R(l,b,z-s)| \int_\lambda |\psi(t)||\phi(z,t)| \, dt \, dz.
\]

The inner integral on the right is bounded on $\gamma$ and the same is true of $|R(l,b,z-s)|$ since $R(l,b,z-s)$ is continuous on $\gamma$ and $\gamma$ is compact. Since $\gamma$ is rectifiable, the iterated
integral on the right is finite. By Fubini's theorem (Rudin, 1966, Thm. 7.8, p. 140) the order of integration of (5.10) can be interchanged and we have the conclusion of the theorem.

We proceed next to generalize the Fourier, Laplace and Stieltjes transforms by use of (5.3).

F. The Fourier Transform

We have already seen that if a function $h$ satisfies the assumptions of either Lemma 3.3, 3.4 or 3.5, then its Fourier transform $\tilde{h}$ is analytic on the lower half-plane, the entire plane or a strip, respectively. Denote by $D$ the domain of analyticity of $\tilde{h}$ and let $s_1, \ldots, s_k$ be points in $D$. Let $\gamma$ be a positively oriented rectifiable Jordan curve in $D$ which encircles $s_1, \ldots, s_k$. Then by Cauchy's integral formula

$$\tilde{h}(s) = \frac{1}{2\pi i} \int_{\gamma} \tilde{h}(z) \frac{1}{z-s} dz, \quad s \in \gamma.$$

The generalization of this expression as a special case of (5.3) with $n=0$ is

$$(5.11) \quad \widetilde{H}(b,s) = \frac{1}{2\pi i} \int_{\gamma} \tilde{h}(z) R(1,b,z-s) dz, \quad (c \neq 0,-1,-2, \ldots),$$

where $s \in I^k(\gamma)$. In the case of Lemma 3.3, $\tilde{h}$ is continuous on the union $D_1$ of the lower half-plane and the real axis. However, by contrast with the method of multiple integrals, we can no longer allow any one of $s_1, \ldots, s_k$ to be real.
Theorem 5.1. Let \( h \) satisfy the assumptions of either Lemma 3.3, 3.4 or 3.5, and let \( D \) be the corresponding domain of analyticity of \( \bar{h} \). If \( c \neq 0, -1, -2, \ldots \), and \( s \in D^k \), then

\[
(5.12) \quad \bar{h}(b, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S(b, -ist) dt,
\]

where \( S \) is defined by (5.5).

Proof: Let \( \gamma \) be a positively oriented rectifiable Jordan curve in \( D \) which encircles \( s_1, \ldots, s_k \). In (3.2) replace \( s \) by \( z \) and substitute in (5.11). Then

\[
(5.13) \quad \bar{h}(b, s) = \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \int_{\gamma} h(t)e^{-itz} R(l, b, z-s) dt \, dz
\]

\[
= \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \int_{\gamma} h(-t)e^{itz} R(l, b, z-s) dt \, dz
\]

\[
+ \frac{1}{2\pi i} \frac{1}{\sqrt{2\pi}} \int_{\gamma} h(t)e^{-itz} R(l, b, z-s) dt \, dz.
\]

Under the assumptions of either Lemma 3.3, 3.4 or 3.5, \( h \) is measurable on \( R \). Therefore the integrand of the inner integral in each term of (5.13) is measurable on \( \gamma \times R_+ \).

Let \( m = \inf_{z \in \gamma} \text{Im}(z) \) and \( M = \sup_{z \in \gamma} \text{Im}(z) \). Then

\[
\int_{0}^{\infty} |h(-t)e^{itz}| dt = \int_{0}^{\infty} |h(-t)| e^{-tM} dt \leq \int_{0}^{\infty} |h(-t)| e^{-tm} dt,
\]

and

\[
\int_{0}^{\infty} |h(t)e^{-itz}| dt = \int_{0}^{\infty} |h(t)| e^{tM} dt \leq \int_{0}^{\infty} |h(t)| e^{tm} dt.
\]
Under any one of the assumptions, the integrals on the right are finite and those on the left are therefore bounded on $\gamma$. By Lemma 5.1 and (5.5),

$$H(b,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) \int_{\gamma} \frac{1}{2\pi i} e^{-it\zeta} R(1,b,z-s)dz \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S(b,-ist)dt, \quad (c \neq 0,-1,-2,...).$$

G. The Laplace Transform

If a function $f$ satisfies the conditions of either Lemma 3.7 or 3.8, then its Laplace transform $\mathcal{F}$ is analytic on the half plane $\text{Re}(s) > 0$ or $\text{Re}(s) > \sigma$, respectively. Let $D$ be the domain of analyticity of $\mathcal{F}$ and again let $\gamma \subset D$ be a positively oriented rectifiable Jordan curve. Then by Cauchy's integral formula (5.2) with $n=0$,

$$(5.14) \quad \mathcal{F}(s) = \frac{1}{2\pi i} \int_{\gamma} \mathcal{F}(z)(z-s)^{-1}dz, \quad s \in I(\gamma).$$

The generalization of (5.14) by use of (5.3) with $n=0$ is

$$(5.15) \quad \mathcal{F}(b,s) = \frac{1}{2\pi i} \int_{\gamma} \mathcal{F}(z) R(1,b,z-s)dz, \quad (c \neq 0,-1,-2,...),$$

where $s \in I^k(\gamma)$.

Theorem 5.2. Let $f$ satisfy the assumptions of either Lemma 3.7 or 3.8 and let $D$ denote the corresponding domain of analyticity of $\mathcal{F}$. If $s \in D^k$ and $c \neq 0,-1,-2,...$, then

$$(5.16) \quad \mathcal{F}(b,s) = \int_{0}^{\infty} f(t) S(b,-st)dt,$$

where $S$ is defined by (5.7).
Proof: Let \( \gamma \) be a positively oriented rectifiable Jordan curve in \( D \) which encircles \( s_1, \ldots, s_k \). In (3.9) replace \( s \) by \( z \) and substitute in (5.15). Then

\[
(5.17) \quad \overline{F}(b,s) = \frac{1}{2\pi i} \int_{\gamma} \int_{0}^{\infty} f(t)e^{-zt} R(l,b,z-s)dt \, dz.
\]

Under either assumption the integrand of the inner integral of (5.17) is measurable on \( \gamma \times \mathbb{R}^+ \). Let \( m = \inf \Re(z) \). Then

\[
\int_{0}^{\infty} |f(t)e^{-zt}| dt = \int_{0}^{\infty} |f(t)| e^{-t \Re(z)} dt \leq \int_{0}^{\infty} |f(t)| e^{-mt} dt.
\]

Under either assumption the integral on the right is finite and the integral on the left is therefore bounded on \( \gamma \). By Lemma 5.1 and (5.7) we have

\[
\overline{F}(b,s) = \int_{0}^{\infty} f(t) \frac{1}{2\pi i} \int_{\gamma} e^{-zt} R(l,b,z-s)dz \, dt
\]

\[
= \int_{0}^{\infty} f(t) S(b,-st) dt, \quad (c \neq 0, -1, -2, \ldots). \quad \#
\]

H. The Stieltjes Transform

We have seen previously that if a function \( g \) satisfies the conditions of Lemma 3.10, then its Stieltjes transform \( \overline{g}_a \) is analytic on the complex plane cut along the non-positive real axis. As before, let \( D \) denote the cut plane and let \( \gamma \subset D \) be a positively oriented rectifiable Jordan curve. By Cauchy's formula

\[
(5.18) \quad \overline{g}_a(s) = \frac{1}{2\pi i} \int_{\gamma} \overline{g}_a(z) (z-s)^{-1} dz, \quad s \in \mathbb{I}(\gamma).
\]
The generalization of (5.18) by use of (5.3) with $n=0$ is

$$G^{(b,s)} = \frac{1}{2\pi i} \int_{\gamma} \overline{g}(z) \, R(l,b,z-s) \, dz, \quad (c \neq 0, -1, -2, \ldots),$$

where $s_1, \ldots, s_k$ are points in $I(\gamma)$. These points may be on both sides of the negative real axis whereas in the case of multiple integrals, the convex hull $K(s)$ must be contained in $D$.

**Theorem 5.3.** Let $g$ satisfy the conditions of Lemma 3.10 and let $D$ be the $s$-plane cut along the nonpositive real axis. If $s_1, \ldots, s_k$ are points in $D$ and $c \neq 0, -1, -2, \ldots$, then

$$G(b,s) = \int_0^\infty g(t) \, R(a,b,s+t) \, dt,$$

where $R$ is given by (5.9).

**Proof:** Let $\gamma \subset D$ be a positively oriented rectifiable Jordan curve which encircles $s_1, \ldots, s_k$. In (3.12) replace $s$ by $z$ and substitute in (5.19). Then

$$G_a(b,s) = \frac{1}{2\pi i} \int_{\gamma} \int_0^\infty g(t) (z+t)^{-a} \, R(l,b,s+t) \, dt \, dz.$$

Under the assumptions of Lemma 3.10 $g$ is measurable on $R_+$. For $t > 0$ and $z \in \gamma$, $(z+t)^{-a}$ is continuous on $\gamma \times R_+$. Therefore the integrand of the inner integral of (5.21) is measurable on $\gamma \times R_+$. Let $\delta$ be the distance from $\gamma$ to the nonpositive real axis and let $\rho = 2 \sup \{|z|: z \in \gamma\}$. Then, as in the proof of Lemma 3.10,
where $i=1$ if $\text{Re}(a)>0$ and $i=2$ if $\text{Re}(a)<0$. In either case this last integral exists and is finite. By Lemma 5.1 and (5.9)

$$
\bar{\mathcal{G}}_a(b,s) = \int_0^\infty g(t) \frac{1}{2\pi i} \int_{\gamma} (z+t)^{-a} R(1,b,z-s) \, dz \, dt
$$

$$
= \int_0^\infty g(t) R(a,b,s+t) \, dt, \quad (c \neq 0,-1,-2,\ldots). \# \]

We observe once again that if the ordinary transform is analytic on a simply connected domain $D \subseteq \mathbb{C}^1$, then the generalized transform is analytic in $b$ and $s$ for $s \in D^k$ and $c \neq 0,-1,-2,\ldots$. 
VI. PROPERTIES AND APPLICATIONS OF
THE GENERALIZED TRANSFORMS

A. Introduction

In this chapter we shall show that the generalized Fourier, Laplace and Stieltjes transforms possess some but not all of the operational properties of the corresponding ordinary transforms. We shall also show that the generalized transforms are related to each other and to other integral transforms and, by means of a counterexample, the generalized Laplace transform of $f$ is not equal to the multidimensional Laplace transform of the generalization of $f$. We shall apply the generalized Laplace transform to determine the ordinary Laplace transform of a function which can be expressed (with some restrictions) as an $S$-function or a product of $S$-functions. We shall also use the generalized Stieltjes transform to find the ordinary Stieltjes transform of a function (again with some restrictions) which can be expressed as an $R$-function.

We begin by listing several properties of the $R$-function and the $S$-function (Carlson, 1968 and Carlson, 1969), which shall be used later in the chapter. These functions are analogues in several variables of $z^{-a}$ and $e^{z}$, respectively, where $a$ may be complex. We define $D_i = \frac{\partial}{\partial z_i}$, $c = \sum_{i=1}^{k} b_i$ and assume that $c \neq 0, -1, -2, \ldots$. 
Symmetry in the $b$-parameters and variables:

(6.1) $R(a; b, b'; x, y) = R(a; b', b; y, x)$.

(6.2) $S(b, b'; x, y) = S(b', b; y, x)$.

Functional relations:

(6.3) $R(a, b, \lambda z) = \lambda^{-a} R(a, b, z)$.

(6.4) $S(b, \lambda + z) = e^{\lambda} S(b, z)$.

Relation of $R$ to Gauss' hypergeometric $\binom{2}{1}$ function:

(6.5) $R(a; b, b'; x, y) = y^{-a} \binom{2}{1}(a, b; b+b'; 1-\frac{x}{y})$.

(6.6) $\binom{2}{1}(a, b; c; x) = R(a; b, c-b; 1-x, 1)$.

Relation of $S$ to the confluent hypergeometric $\binom{1}{1}$ function:

(6.7) $S(b, b'; x, y) = e^{y} \binom{1}{1}(b; b+b'; x-y)$.

(6.8) $\binom{1}{1}(b; c; x) = S(b, c-b; x, 0)$.

Special cases of the $S$-function:

(6.9) $\frac{x}{\sqrt{\pi}} \binom{1}{1}(1; 1; -x^2, 0) = \text{Erf}(x)$, (error function).

(6.10) $\frac{(x/2)^{\nu}}{\Gamma(1+\nu)} \binom{1}{1}(\frac{1}{2}+\nu; \frac{1}{2}+\nu; ix, -ix) = J_\nu(x)$, (Bessel function of order $\nu$).

(6.11) $\frac{(x/2)^{\nu}}{\Gamma(1+\nu)} \binom{1}{1}(\frac{1}{2}+\nu; \frac{1}{2}+\nu; x, -x) = I_\nu(x)$, (modified Bessel function of order $\nu$).
Differential relations:

\[(6.13) \quad (\sum_{i=1}^{k} D_i) S = S, \]

\[(6.14) \quad (\sum_{i=1}^{k} D_i) R(a, b, z) = -aR(a+1, b, z). \]

Special cases of the R-function:

\[(6.15) \quad R(c, b, z) = \prod_{i=1}^{k} \frac{b_i}{z_i}. \]

\[(6.16) \quad R(\alpha; 0, b, b'; x, y, z) = R(\alpha; b, b'; y, z), \]

\[(6.17) \quad R\left(\frac{1}{2}, \frac{1}{2}, 1; x, y\right) = (y-x)^{-\frac{1}{2}} \sin^{-1}\left(\frac{y-x}{x}\right)^{\frac{1}{2}} \]

\[= (y-x)^{-\frac{1}{2}} \cos^{-1}\left(\frac{x}{y}\right)^{\frac{1}{2}}, \]

\[(6.18) \quad R(\alpha; b, b'; x, y) = y^{b-a} R(b; a, c-a; x, y), \]

(exchange transformation),

\[(6.19) \quad R(1; 1, 1; x, y) = (\log x - \log y)/(x-y), \]

\[(6.20) \quad R(b-\frac{1}{2}; b, b; x, y) = \left[\frac{x^2 + y^2}{2}\right]^{1-2b}, \]

\[(6.21) \quad R(-\alpha; \nu, \nu; x, y) = \frac{\Gamma(2\nu)\Gamma(\alpha+1)}{\Gamma(2\nu+\alpha)} \frac{\alpha^\nu}{\Gamma(a+1)} C_{\alpha}^\nu \left(\frac{x+y}{2\sqrt{xy}}\right), \]

where \(C_{\alpha}^\nu\) is the Gegenbauer function,
(6.22) \( R(-1; 1,1; x, y) = \frac{x+y}{2} \).

Relation of \( R \) to \( S \):

(6.23) \( \int_0^\infty t^{a-1} S(b,-tz)dt = \Gamma(a) R(a,b,z), \text{Re}(a)>0, \text{Re}(z)>0. \)

B. Operational Properties of the Generalized Transforms

We shall adopt the following notation.

Ordinary transform of \( \phi \): \( T\{\phi(t)\} = \hat{\phi}(s) \)

Generalized Fourier transform of \( h \): \( F\{h(t)\} = \overline{H}(b,s) \)

Generalized Laplace transform of \( f \): \( L\{f(t)\} = \overline{F}(b,s) \)

Generalized Stieltjes transform of \( g \): \( S_a\{g(t)\} = \overline{G}_a(b,s) \)

In the properties to be examined it will be assumed that \( h, f \) and \( g \) possess corresponding ordinary Fourier, Laplace and Stieltjes transforms which are analytic on a strip \( a<\text{Im}(s)<b \), a half-plane \( \text{Re}(s)>\alpha \), and a domain \( D \) consisting of the \( s \)-plane cut along the nonpositive real axis, respectively. We also assume that \( c = \sum \beta_i k_i \neq 0, -1, -2, \ldots, \alpha \) and \( \beta \) are complex constants, and \( \gamma \) is a nonzero real constant.

1. Linearity - It follows immediately from the definitions of the generalized transforms that

\[
F\{\alpha h_1(t) + \beta h_2(t)\} = \alpha \overline{H}_1(b,s) + \beta \overline{H}_2(b,s), \quad a<\text{Im}(s)<b,
\]
L{af\_1(t) + \beta f\_2(t)} = \alpha \mathcal{F}\_1(b,s) + \beta \mathcal{F}\_2(b,s), \Re(s) > \sigma_a,

\mathcal{S}\_a\{ag\_1(t) + \beta g\_2(t)\} = \alpha \mathcal{G}\_a\_1(b,s) + \beta \mathcal{G}\_a\_2(b,s), \ s \in \mathbb{D}^k.

2. Change of scale - The ordinary Fourier and Laplace transforms satisfy \( T\{\phi(\gamma t)\} = \frac{1}{\gamma} \overline{\phi}(\frac{S}{\gamma}) \). Likewise, we have

\[ F\{h(\gamma t)\} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} h(\gamma t) \ S(b,-ist) \, dt \]

\[ = \frac{1}{\gamma \sqrt{2\pi}} \int\limits_{-\infty}^{\infty} h(x) \ S(b,-\frac{iS}{\gamma}x) \, dx \]

\[ = \frac{1}{\gamma} \overline{H}(b,\frac{S}{\gamma}), \quad a < \Im(\frac{S}{\gamma}) < b, \]

and similarly

\[ L\{f(\gamma t)\} = \frac{1}{\gamma} \mathcal{F}(b,\frac{S}{\gamma}), \quad \Re(\frac{S}{\gamma}) > \sigma_a. \]

The ordinary Stieltjes transform satisfies \( T_a\{\phi(\gamma t)\} = \gamma^{a-1} \phi_a(\gamma S) \), and similarly

\[ \mathcal{S}\_a\{g(\gamma t)\} = \int\limits_0^\infty g(\gamma t) \ \mathcal{R}(a,b,s+t) \, dt \]

\[ = \frac{1}{\gamma} \int\limits_0^\infty g(x) \ \mathcal{R}(a,b,s+\frac{x}{\gamma}) \, dx \]

\[ = \gamma^{a-1} \mathcal{G}\_a(b,\gamma S), \ \text{by (6.3)}, \quad \gamma \in \mathbb{D}^k. \]

Thus each generalized transform retains the change of scale property of the ordinary transform.
3. Shift of origin in the s-plane - The ordinary Fourier and Laplace transforms satisfy $T\{e^{at}\phi(t)\} = \tilde{\phi}(s-a)$.

Similarly we find

$$F\{e^{iat}h(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iat}h(t) S(b,-ist)dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) S[b,-i(s-a)t]dt, \text{ by (6.4)},$$

$$= \tilde{H}(b,s-a), \ a<\text{Im}(s-a)<b,$$

and

$$L\{e^{at}f(t)\} = \tilde{F}(b,s-a), \ \text{Re}(s-a)>\sigma_a.$$ 

There is no shift of origin property for the ordinary Stieltjes transform.

4. Shift of origin in the t-plane - The ordinary Fourier transform satisfies $T\{\phi(t-t_0)\} = e^{-st_0}\tilde{\phi}(s)$. However, the generalized Fourier transform does not have an analogous property, for

$$F\{h(t-t_0)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t-t_0) S(b,-ist)dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) S[b,-is(x+t_0)]dx$$

$$\neq S(b,-ist_0) \tilde{H}(b,s).$$

The ordinary Laplace transform satisfies $T\{\phi(t-t_0) U(t-t_0)\} = e^{-st_0} \tilde{\phi}(s)$, where $U(t)$ is the Heaviside unit function, but
The ordinary Stieltjes transform satisfies \( T^{(s)}(t-t_0) = O(t-t_0) \), and
\[
S^{(s)}(t-t_0) = S^{(s)}(t-t_0) R(a, b, s) + x) dx + S^{(s)}(t-t_0) R(a, b, s+t_0) dx
\]
Therefore only the generalized Stieltjes transform retains a similar property under a shift of origin in the \( t \)-plane.

5. Derivatives of the generalized transforms - Each generalized transform is analytic in \( s \) on the specified domain and we may take derivatives under the integral sign (see Lemma 3.2). The ordinary Fourier and Laplace transforms satisfy \( T^{(n)}(\phi) = \frac{d^n}{ds^n} T(\phi) = \alpha^n T(t^n \phi(t)) \), where \( \alpha = -i \) for the Fourier transform and \( \alpha = -1 \) for
the Laplace transform. Thus we have

\[ F^{(n)}\{h(t)\} = (\sum_{i=1}^{k} D_i)^n F\{h(t)\} \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) (\sum_{i=1}^{k} D_i)^n S(b, -ist) dt \]

\[ = \frac{1}{\sqrt{2\pi}} (-i)^n \int_{-\infty}^{\infty} t^n h(t) S(b, -ist) dt, \text{ by } (6.13), \]

\[ = (-i)^n F\{t^n h(t)\}, \]

and similarly

\[ (6.24) \quad L^{(n)}\{f(t)\} \equiv (\sum_{i=1}^{k} D_i)^n L\{f(t)\} = (-1)^n L\{t^n f(t)\}. \]

The ordinary Stieltjes transform satisfies

\[ T_{a}^{(n)}\{\phi\} = (-1)^n (a,n) T_{a+n}\{\phi\}. \]

We have

\[ S_{a}^{(n)}\{f(t)\} \equiv (\sum_{i=1}^{k} D_i)^n S_{a}\{f(t)\} \]

\[ = \int_{0}^{\infty} f(t) (\sum_{i=1}^{k} D_i)^n R(a, b, s+t) dt \]

\[ = (-1)^n (a,n) \int_{0}^{\infty} f(t) R(a+n, b, s+t) dt, \text{ by } (6.14), \]

\[ = (-1)^n (a,n) S_{a+n}\{f(t)\}. \]

Therefore, each generalized transform retains the derivative
property of the corresponding ordinary transform.

6. Generalized transforms of derivatives - Assuming appropriate asymptotic conditions on \( \phi \) and \( \phi' \), integration by parts is used to show that the ordinary Fourier transform of \( \phi' \) has the property \( T\{\phi'\} = iST\{\phi\} \). The generalized Fourier transform does not have this property because in integrating by parts, we must find \( \frac{d}{dt} S(b,-ist) \), which does not yield a product of \( is \) and \( S \). The same is true for the generalized Laplace transform.

The ordinary Stieltjes transform of \( \phi' \) has the property \( T_{\lambda}\{\phi'\} = a \int \phi(t) R(a,b,s+t)dt \) for provided \( \text{Re}(a) > 0 \) and \( \phi \) is bounded on \( (0,\infty) \) (or that \( \phi \) decays exponentially at large distances). The generalized Stieltjes transform has a similar property, for

\[
S_{\lambda}\{g'(t)\} = \int_{0}^{\infty} g'(t) R(a,b,s+t)dt
= a \int_{0}^{\infty} g(t) R(a+1,b,s+t)dt + R(a,b,s+t)g(t)|_{0}^{\infty}
= a S_{\lambda+1}\{g(t)\} - R(a,b,s)g(0).
\]

7. Generalized transforms of integrals - The ordinary Fourier and Laplace transforms have the property

\[
T\{\int_{0}^{t} \phi(x)dx\} = \frac{1}{as} T\{\phi\}, \text{ where } \alpha = i \text{ for the Fourier trans-}
\]

form and $a = 1$ for the Laplace transform. However, the
generalized Fourier and Laplace transforms do not have
this property for, assuming we can interchange the
order of integration we have, for the generalized
Fourier transform,

$$ F\{\int_{0}^{t} h(x) \, dx \} = \int_{0}^{\infty} \left( \int_{0}^{t} h(x) \, dx \right) S(b, -its) \, dt $$

$$ = \int_{0}^{\infty} h(x) \, dx \int_{x}^{\infty} S(b, -ists) \, dt $$

$$ \neq R(1, b, is) \, F\{h(t)\}, $$

and the same conclusion holds for the generalized Laplace
transform.

The ordinary Stieltjes transform has the property

$$ T_{a} \{ \int_{0}^{t} \phi(x) \, dx \} = \frac{1}{a-1} \, T_{a-1} \{ \phi \}, \text{ for } \text{Re}(a) > 1. $$

Provided we can reverse the order of the repeated integral,

$$ S_{a} \{ \int_{0}^{t} g(t) \, dt \} = \int_{0}^{\infty} [\int_{0}^{t} g(x) \, dx] \, R(a, b, s+t) \, dt $$

$$ = \int_{0}^{\infty} g(x) \, dx \int_{x}^{\infty} R(a, b, s+t) \, dt $$

$$ = \frac{1}{a-1} \int_{0}^{\infty} g(x) \, R(a-1, b, s+x) \, dx $$

$$ = \frac{1}{a-1} \, S_{a-1} \{ g(t) \}, \, \text{Re}(a) > 1. $$
Thus, only the generalized Stieltjes transform retains a similar property for transforms of integrals.

8. Generalized transforms of convolutions - One of the most important properties of the ordinary Fourier and Laplace transforms is that the transform of a convolution of two functions is the product of the transforms. We have not found any corresponding property for the generalized transforms. This is not surprising, since the generalization to several variables of a product of two functions is not in general the product of their generalizations.

C. Relation of the Generalized Transforms to Other Transforms

The generalized Laplace transform may be obtained from the generalized Fourier transform by a suitable change of variables. In the generalized Fourier transform

\[
\mathcal{H}(b,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) s(b,ist) dt,
\]

let

\[
z_i = is_i, \ (i=1,\ldots,k),
\]

and

\[
h(t) = \begin{cases}
0 & , \ t<0, \\
\sqrt{2\pi} f(t) & , \ t\geq0.
\end{cases}
\]
Then
\[ \bar{H}(b,-iz) = \int_0^\infty f(t) S(b,-zt) dt = \bar{F}(b,z), \]
where \( \bar{F}(b,z) \) is the generalized Laplace transform. This change of variables is equivalent to rotating the complex plane through 90°.

We observed in Chapter III that the ordinary Stieltjes transform is obtained as an iteration of the Laplace transform. In a somewhat analogous manner we may obtain a generalized Stieltjes transform by taking the generalized Laplace transform of an ordinary Laplace transform. Let

\[ f(u) = \int_0^\infty g(t)e^{-ut} dt, \quad \text{Re}(u) > 0. \]

Then formally,
\[ \bar{F}(b,s) = \int_0^\infty f(u)S(b,-su) du \]
\[ = \int_0^\infty \left[ \int_0^\infty g(t)e^{-ut} dt \right] S(b,-su) du \]
\[ = \int_0^\infty g(t) \int_0^\infty S[b,-u(s+t)] du dt, \text{ by (6.4)}, \]
\[ = \int_0^\infty g(t) R(l,b,s+t) dt, \text{ by (6.23) with } a=1, \]
\[ = \bar{c}_1(b,s), \]
where $\bar{g}(b,s)$ is the generalized Stieltjes transform corresponding to the ordinary Stieltjes transform

$$\bar{g}(s) = \int_0^\infty \frac{g(t)}{s+t} \, dt.$$  

If we choose a special form for $f$, namely

$$f_a(u) = \frac{u^{a-1}}{\Gamma(a)} \int_0^\infty g(t)e^{-ut} \, dt, \quad \text{Re}(u)>0, \quad \text{Re}(a)>0,$$

then

$$\bar{F}_a(b,s) = \bar{g}(b,s).$$

The generalized Laplace transform is related to the Hankel transform $H_\nu$, where

$$H_\nu(x) = \int_0^\infty h(t) J_\nu(xt)(xt)^\frac{1}{2} \, dt.$$  

By (6.10) we have

$$H_\nu(x) = \int_0^\infty h(t) J_\nu(xt)(xt)^\frac{1}{2} \, dt$$

$$= \int_0^\infty h(t) \frac{(xt)^\nu}{\Gamma(1+\nu)} S(\frac{1}{2}+\nu, \frac{1}{2}+\nu; \text{i}xt,-\text{i}xt)(xt)^\frac{1}{2} \, dt$$

$$= \frac{x^{\nu+\frac{1}{2}}}{2^{\nu} \Gamma(1+\nu)} \int_0^\infty t^{\nu+\frac{1}{2}} h(t) S(\frac{1}{2}+\nu, \frac{1}{2}+\nu; \text{i}xt,-\text{i}xt) \, dt$$

$$= \frac{x^{\nu+\frac{1}{2}}}{2^{\nu} \Gamma(1+\nu)} \bar{F}(\frac{1}{2}+\nu, \frac{1}{2}+\nu; -\text{i}x, \text{i}x),$$

where $\bar{F}$ is the generalized Laplace transform of $f(t)=t^{\nu+\frac{1}{2}} h(t)$, with $b=b'=\frac{1}{2}+\nu$, $s_1=\text{i}x$ and $s_2=-\text{i}x$. When
\[ v = \frac{1}{2} \text{ or } v = -\frac{1}{2}, \] the Hankel transform reduces to the Fourier sine transform or Fourier cosine transform, respectively. Therefore, all three transforms are special cases of a generalized Laplace transform.

Varma (1951) defines a generalization of the Laplace transform to be

\[ \tilde{f}(s; k, m) = \int_0^\infty e^{-\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,m}(st) f(t) dt, \]

where \( W_{k,m} \) is the Whittaker function. This generalized transform reduces to the ordinary Laplace transform if \( k + m = \frac{1}{2} \). The Whittaker function may be expressed in terms of \( S \)-functions as follows:

\[
W_{k,m}(st) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k)} (st)^{\frac{1}{2} + m} S(\frac{1}{2} + m - k, \frac{1}{2} + m + k; \frac{st}{2}, -\frac{st}{2}) \]

\[ + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m - k)} (st)^{\frac{1}{2} - m} S(\frac{1}{2} - m - k, \frac{1}{2} - m + k; \frac{st}{2}, -\frac{st}{2}). \]

Therefore Varma's generalization becomes

\[
\tilde{f}(s; k, m) = \int_0^\infty e^{-\frac{1}{2}st} (st)^{m-\frac{1}{2}} \left[ \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k)} (st)^{\frac{1}{2} + m} S(\frac{1}{2} + m - k, \frac{1}{2} + m + k; \frac{st}{2}, -\frac{st}{2}) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m - k)} (st)^{\frac{1}{2} - m} S(\frac{1}{2} - m - k, \frac{1}{2} - m + k; \frac{st}{2}, -\frac{st}{2}) \right] f(t) dt
\]

\[
= \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k)} \int_0^\infty t^{2m} f(t) S(\frac{1}{2} + m - k, \frac{1}{2} + m + k; 0, -st) dt
\]
\[ + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)} \int_{0}^{\infty} f(t) S(\frac{1}{2}-m-k, \frac{1}{2}+m+k; 0, -st) dt \]

\[ = \frac{\Gamma(-2m)s^{2m}}{\Gamma\left(\frac{1}{2}-m-k\right)} \tilde{F}(2m) \left(\frac{1}{2}+m-k, \frac{1}{2}+m+k; 0, s\right) \]

\[ + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)} \tilde{F}\left(\frac{1}{2}-m-k, \frac{1}{2}+m+k; 0, s\right), \]

where \( \tilde{F} \) is the generalized Laplace transform of \( f \), provided that \( f \) is of order \( e^{\sigma t} \), \( \sigma < 0 \), and \( m \neq 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \ldots \).

A natural question is whether the generalization of the Laplace transform \( \tilde{F} \) to \( k \) variables is equal to the \( k \)-dimensional Laplace transform of the generalization of \( f \), that is, is

\[ \tilde{F}(b,s) = \int_{0}^{\infty} e^{-s_{1}t_{1}} dt_{1} \ldots \int_{0}^{\infty} e^{-s_{k}t_{k}} F(b,t) dt_{k} \]

The conjecture is false, as can be seen by the following example. Let \( k=2, b=b'=1 \) and \( f(t)=t \). Then for \( \text{Re}(s)>0 \)

\[ \tilde{F}(s) = \int_{0}^{\infty} e^{-st} t dt = \frac{1}{s^{2}} \]

\[ \tilde{F}(1, 1; x, y) = R(2; 1, 1; x, y) = \frac{1}{xy}, \quad \text{Re}(x)>0, \quad \text{Re}(y)>0, \]

by (6.15). We have also

\[ F(1, 1; t_{1}, t_{2}) = R(-1; 1, 1; t_{1}, t_{2}) = \frac{t_{1}+t_{2}}{2}, \quad \text{by (6.22)}. \]

But then
D. Applications of the Generalized Laplace Transform

In this section we shall use the generalized Laplace transform to determine the ordinary Laplace transform of functions which can be expressed in terms of an S-function or a product of S-functions. We shall first prove some relations that will be useful in the applications which follow.

The ordinary Laplace transform of $t^{a-1}$ is given by

$$\int_0^\infty e^{-st}t^{a-1}dt = \Gamma(a)s^{-a}, \ Re(s)>0, \ Re(a)>0. \quad (6.25)$$

By Theorem 5.1, the generalized Laplace transform of $t^{a-1}$ is

$$\int_0^\infty t^{a-1}S(b,-st)dt = \Gamma(a)R(a,b,s), \quad (6.26)$$
where \( \text{Re}(s)>0, \text{Re}(a)>0 \) and \( c \neq 0, -1, -2, \ldots \), where \( c = \sum_{i=1}^{k} b_i \).

This integral can also be regarded as the Mellin transform of an \( S \)-function. Let \( s_i = p + z_i \), where \( p \) and \( z_i \), \( (i=1, \ldots, k) \), are complex. Then by (6.4)

\[
(6.27) \quad \int_{0}^{\infty} t^{a-1} S[b, -t(p+z)] dt = \int_{0}^{\infty} t^{a-1} e^{-pt} S(b, -tz) dt
\]

\[
= \Gamma(a) R(a, b, p+z),
\]

where \( \text{Re}(p+z_i)>0, (i=1, \ldots, k), \text{Re}(a)>0 \) and \( c \neq 0, -1, -2, \ldots \).

We next let \( k=2 \) and require \( \text{Re}(c)>0 \) and \( a=c \). Then by (6.15)

\[
(6.28) \quad \int_{0}^{\infty} t^{c-1} e^{-pt} S(b,b'; -tz, -tw) dt = \Gamma(c) (p+z)^{-b} (p+w)^{-b'}.
\]

We regard this equation as the Laplace transform of \( t^{c-1} S(b,b'; -tz, -tw) \), and the right side is analytic in \( p \) for \( \text{Re}(p+z)>0 \) and \( \text{Re}(p+w)>0 \). We apply Theorem 4.4 to (6.28) to obtain

\[
(6.29) \quad \int_{0}^{\infty} t^{c-1} S(\beta, \beta'; -tq, -tr) S(b,b'; -tz, -tw) dt
\]

\[
= \Gamma(c) \int_{0}^{1} [uq+(1-u)r+z]^{-b}[uq+(1-u)r+w]^{-b'} P(\beta, u) du
\]

\[
= \Gamma(c) (r+z)^{-b} (r+w)^{-b'} R(\beta; \gamma-c,b,b'; 1, q+z, q+w),
\]

by (Carlson, 1968, Sec. 6.3, (25)), where \( \gamma = \beta + \beta' \), \( \text{Re}(\beta)>0 \), \( \text{Re}(\beta')>0 \), \( \text{Re}(c)=\text{Re}(b+b')>0 \), \( \text{Re}(r+z)>0 \), \( \text{Re}(r+w)>0 \), \( \text{Re}(q+z)>0 \) and \( \text{Re}(q+w)>0 \). Since \( R \) can be continued analytically in the
parameters, we can replace the conditions \( \Re(\beta) > 0, \Re(\beta') > 0 \) by \( \gamma \neq 0, -1, -2, \ldots \).

As a special case of (6.29) we shall investigate the Laplace transform of a function involving the product of two \( {}_1 F_1 \) (confluent hypergeometric) functions. Consider the integral

\[
\int_0^\infty e^{-st} t^{\nu-1} {}_1 F_1(\lambda; \mu; zt) {}_1 F_1(\delta; \nu; wt) \, dt,
\]

where \( s, \nu, \lambda, \mu, \delta, z \) and \( \beta \) are all complex.

**Example 6.1.**

(6.30) \[
\int_0^\infty e^{-st} t^{\nu-1} {}_1 F_1(\lambda; \mu; zt) {}_1 F_1(\delta; \nu; wt) \, dt
\]

\[
= \int_0^\infty e^{-st} t^{\nu-1} S(\lambda, \mu-\lambda; zt,0) S(\delta, \nu-\delta; wt,0) \, dt,
\]

by (6.8),

\[
= \int_0^\infty t^{\nu-1} S(\lambda, \mu-\lambda; -t(s-z), -ts) S(\delta, \nu-\delta; -t(-w),0) \, dt
\]

\[
= \Gamma(\nu)(s-w)^{\delta-\nu} R(\lambda; \mu-\nu, \delta, \nu; 1, \frac{s-z-w}{s-w}, -\frac{s-z}{s}),
\]

where \( \Re(\nu) > 0, \mu \neq 0, -1, -2, \ldots \), and \( \Re(s) > \Re(z) + \Re(w) \).

As another example of (6.29), we consider the Laplace transform of the product of two error functions, where

\[
\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \, dy.
\]
Example 6.2.

\[
\int_0^\infty e^{-st} t^{-\frac{1}{2}} \operatorname{Erf} (\sqrt{zt}) \operatorname{Erf} (\sqrt{wt}) dt
\]

\[
= \frac{4}{\pi} \int_0^\infty e^{-st} t^{-\frac{1}{2}} \sqrt{zt} S(\frac{1}{2}, 1; -zt, 0) \sqrt{wt} S(\frac{1}{2}, 1; -wt, 0) dt,
\]

by (6.9),

\[
= \frac{4\sqrt{zw}}{\pi} \int_0^\infty t^2 S(\frac{1}{2}, 1; -t(s+z), -ts) S(\frac{1}{2}, 1; -tw, 0) dt
\]

\[
= \frac{4\sqrt{zw}}{\pi} \Gamma\left(\frac{3}{2}\right) (s+w)^{-\frac{1}{2}} s^{-1} R\left(\frac{1}{2}; 0, \frac{1}{2}, 1; \frac{s+z+w}{s+w}, \frac{s+z}{s}\right)
\]

\[
= 2\left(\frac{zw}{\pi s}\right)^{\frac{1}{2}} R\left[\frac{1}{2}; \frac{1}{2}, 1; s(s+z+w), (s+z)(s+w)\right],
\]

by (6.16) and (6.3),

\[
= \frac{2}{\sqrt{\pi s}} \sin^{-1} \left[ \frac{zw}{(s+z)(s+w)} \right]^\frac{1}{2}.
\]

by (6.17), where \(\Re(s) > |\Re(z)| + |\Re(w)|\).

As another example of (6.29), we consider the Laplace transform of the product of an error function and a Bessel function of order zero.
Example 6.3.
\[
\int_0^\infty e^{-st} \frac{1}{\sqrt{t}} \text{Erf}(\sqrt{zt}) J_0(\omega t) dt
\]
\[
= 2 \sqrt{\pi} \int_0^\infty e^{-st} \frac{1}{\sqrt{zt}} S\left(\frac{1}{2}, l; -tz, 0\right) S\left(\frac{1}{2}, \frac{1}{2}; i\omega t, -i\omega t\right) dt,
\]
by (6.10),
\[
= 2\left(\frac{z}{\pi}\right)^{\frac{1}{2}} \int_0^\infty S\left[\frac{1}{2}, l; -t(s+z), -ts\right] S\left[\frac{1}{2}, \frac{1}{2}; -t(i\omega), -t(-i\omega)\right] dt
\]
\[
= 2\left(\frac{z}{\pi}\right)^{\frac{1}{2}} (s+i\omega) \frac{1}{2}(s-i\omega) -\frac{1}{2} R\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; l, \frac{s+z+i\omega}{s+i\omega}, \frac{s+z-i\omega}{s-i\omega}\right]
\]
\[
= 2\left(\frac{z}{\pi}\right)^{\frac{1}{2}} R\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; s^2 + w^2, (s+z+i\omega)(s-i\omega), (s+z-i\omega)(s+i\omega)\right],
\]
by (6.3),

where \(\text{Re}(s) > |\text{Re}(z)| + |\text{Im}(w)|\). This last \(R\)-function is the same as the standard incomplete elliptic integral of the first kind. If we let \(w = iy\), where \(y\) is real, then, since
\[
I_0(yt) = J_0(iyt),
\]
\[
\int_0^\infty e^{-st} \frac{1}{\sqrt{t}} \text{Erf}(\sqrt{zt}) I_0(yt) dt
\]
\[
= 2\left(\frac{z}{\pi}\right)^{\frac{1}{2}} R\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; s^2 - y^2, (s+z-y)(s+y), (s+z+y)(s-y)\right].
\]
Examples 6.2 and 6.3 have not been found in tables of Laplace transforms.

We next use the generalized Laplace transform to find the ordinary transform of a product of two Whittaker functions.
Example 6.4.

\[
\int_0^\infty e^{-st} t^\nu u^{-\frac{1}{2}} M_{\alpha, \mu}(zt) M_{\beta, \nu}(wt) dt
\]

\[
= z^{\frac{1}{2}+\mu} w^{\frac{1}{2}+\nu} \int_0^\infty e^{-st} t^2 v S\left(\frac{1}{2}+\mu-\alpha, \frac{1}{2}+\mu+\alpha; \frac{z}{2}, -\frac{z}{2}\right) S\left(\frac{1}{2}+\nu-\beta, \frac{1}{2}+\nu+\beta; \frac{w}{2}, -\frac{w}{2}\right) dt,
\]

by (6.12),

\[
= z^{\frac{1}{2}+\mu} w^{\frac{1}{2}+\nu} \int_0^\infty t^2 v S\left[\frac{1}{2}+\mu-\alpha, \frac{1}{2}+\mu+\alpha; -t(s-\frac{z}{2}), -t(s+\frac{z}{2})\right] \times
\]

\[
S\left[\frac{1}{2}+\nu-\beta, \frac{1}{2}+\nu+\beta; -t(-\frac{w}{2}), -t(\frac{w}{2})\right] dt
\]

\[
= z^{\frac{1}{2}+\mu} w^{\frac{1}{2}+\nu} \Gamma(2\nu+1) (s+z-w) - (s+z+w) \times
\]

\[
x R\left(\frac{1}{2}+\mu-\alpha, 2\mu-2\nu, \frac{1}{2}+\nu-\beta, \frac{1}{2}+\nu+\beta; 1, \frac{s-z-w}{s+z-w, s+z+w/2}, \frac{s-z-w}{s+z-w/2}\right),
\]

where \( R(\nu) > -\frac{1}{2}, \mu \neq -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \), and \( 2Re(s) > |Re(z)| + |Re(w)| \).

For a list of functions which can be expressed in terms of this Whittaker function (and therefore as \( S \)-functions) see (Erdélyi et al., 1954, Vol. 2, pp. 431-432). Equation (6.29) thus provides for many combinations of functions and includes (6.27) as a special case. We now use (6.27) to determine the Laplace transform of a function involving a Bessel
Consider the integral
\[ \int_0^\infty e^{-st} t^\lambda J_\nu (zt) dt, \]
where \( \lambda, \nu, s \) and \( z \) are complex.

Example 6.5.

\[ \int_0^\infty e^{-st} t^\lambda J_\nu (zt) dt \]

\[ = \int_0^\infty e^{-st} \frac{t^\lambda}{\Gamma(1+\nu)} (zt)^\nu S\left(\frac{1}{2}+\nu, \frac{1}{2}+\nu; izt, -izt\right) dt, \]
by (6.10),

\[ = \frac{(z/2)^\nu \Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)} R(\lambda+\nu+1; \frac{1}{2}+\nu, \frac{1}{2}+\nu; s+iz, s-iz), \]

where \( \text{Re}(\lambda+\nu) > -1, \nu \neq -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \) and \( \text{Re}(s) > |\text{Im}(z)|. \) If \( \lambda = \nu = 0, \) then

\[ \int_0^\infty e^{-st} J_0 (zt) dt = R(1; \frac{1}{2}, \frac{1}{2}; s+iz, s-iz) \]

\[ = (s^2+z^2)^{-\frac{1}{2}}, \text{ Re}(s) > |\text{Im}(z)|. \]

As a final example of (6.29) consider the integral

\[ \int_0^\infty e^{-st} t^{\nu-\mu} I_\nu (zt) I_\mu (wt) dt, \]
where $s, v, \mu, z$ and $w$ are complex.

**Example 6.6.**

\[
\int_0^\infty e^{-st} t^{v-\mu} I_\nu z(t) I_\mu w(t) dt
\]

\[
= \int_0^\infty e^{-st} t^{v-\mu} (zt/2)^\nu \frac{S(1+v, 1+\nu; zt, -zt)}{\Gamma(1+\nu)} \frac{S(1+\mu, 1+\mu; wt, -wt)}{\Gamma(1+\mu)} dt,
\]

by (6.11),

\[
= \frac{(z/2)^\nu (w/2)^\mu}{\Gamma(1+\nu)\Gamma(1+\mu)} \int_0^\infty t^{2\nu} S\left[\frac{1}{2}+\nu, \frac{1}{2}+\nu; -t(s-z), -t(s+z)\right] \times S\left[\frac{1}{2}+\mu, \frac{1}{2}+\mu; -t(w), -tw\right] dt
\]

\[
= \frac{(z/2)^\nu (w/2)^\mu}{\Gamma(1+\nu)\Gamma(1+\mu)} \left[\frac{1}{2}+\nu, \frac{1}{2}+\nu; 1, \frac{s-z-w}{s+z-w}, \frac{s-z+w}{s+z+w}\right].
\]

where $\text{Re}(\nu) > -\frac{1}{2}$, and $\text{Re}(s) > |\text{Re}(z)| + |\text{Re}(w)|$. If $\mu = v = 0$ and $w = z$,

\[
\int_0^\infty e^{-st} J_0(zt)^2 dt = R\left[\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; s^2 + 4z^2, s^2\right],
\]

where $\text{Re}(s) > 2|\text{Im}(z)|$. This $R$-function is a complete elliptic integral of the first kind.

The use of generalized Laplace transforms allows one not only to find the ordinary Laplace transform of a function
expressible as an S-function or a product of S-functions, but also to determine the integral of a product of two S-functions and an appropriate power of the variable of integration. In the case of finding the ordinary Laplace transform of a function expressible as a single S-function, the method may not be of great advantage, since the S-function is entire and the transform is quite easily determined by other methods. It does provide a new view of the problem, however.

In the case of finding the ordinary transform of a function expressible as a product of two S-functions, the method is advantageous because (6.29) provides for many combinations of functions and has produced some apparently new results.

E. Applications of the Generalized Stieltjes Transform

In the preceding section we observed that the Laplace transform (6.27) of an S-function can be determined by use of the generalized Laplace transform. Although this is a different method of finding the transform, the results are well-known and easily determined because the S-function is an entire function. However, this is not the case with the Stieltjes transform. This transform has not been studied as extensively as the Laplace or Fourier transform, perhaps
because the kernel \((z+t)^{-a}\) is not, in general, an entire function of \(z\).

In this section we shall use the generalized Stieltjes transform to determine the integral of the product of an \(R\)-function in several variables and a power of the variable of integration (analogous to (6.26)). We shall also use it to find the ordinary Stieltjes transform of functions which can be expressed as \(R\)-functions (analogous to (6.27)). Both of these are significant and useful applications of the generalized transform because the integrals of the first case are not well-known and in the second case relatively few (general) Stieltjes transforms have been tabulated (Erdélyi et al., 1954, Vol. 2). We have not been able to evaluate the Stieltjes transform of a product of \(R\)-functions, which would be analogous to (6.29).

The ordinary Stieltjes transform of \(t^{v-1}\) is

\[
\int_0^\infty (s+t)^{-a}dt = B(v,a-v)s^{-(a-v)}, \quad \text{Re}(a) > \text{Re}(v) > 0.
\]

Let \(D\) denote the complex plane cut along the nonpositive real axis and let \(s_1,\ldots,s_k\) be points in \(D\). Then, by Theorem 5.2, the generalized Stieltjes transform of \(t^{v-1}\) is

\[
\int_0^\infty (s+t)^{-a}dt = B(v,a-v) R(a-v,\beta,s),
\]

where \(\text{Re}(a) > \text{Re}(v) > 0\) and \(\gamma \neq 0, -1, -2, \ldots\), with \(\gamma = \sum \beta_i\). This result has been developed elsewhere by a different method.
Integrals of this type are not found directly in mathematical tables, even for the case $k=2$.

We let $k=2$ in (6.32) and require that $a=Y$. Then by (6.15)

$$
\int_0^\infty t^{\nu-1}(z+t)^{-\beta}(w+t)^{-\beta'}dt = B(\nu, \gamma-v)R(\gamma-v; \beta, \beta'; z, w)
$$

$$
= B(\nu, \gamma-v)w^{\gamma-(\nu-v)}R(\beta; \gamma-v, v; z, w),
$$

by the exchange transformation (6.18). We regard this equation as the Stieltjes transform of $t^{\nu-1}(w+t)^{-\beta'}$, and we apply Theorem 4.6 to obtain the generalized Stieltjes transform of $t^{\nu-1}(w+t)^{-\beta'}$. We then have

$$
\int_0^\infty t^{\nu-1} R(\beta; b_1, \ldots, b_k; z_1+t, \ldots, z_k+t) (w+t)^{-\beta'}dt
$$

$$
= B(\nu, \gamma-v)w^{\nu-\beta'} \int_{E} R(\beta; \gamma-v, v; \sum_{i=1}^k u_i z_i, w) \frac{P(b, u)}{u'} du'.
$$

If $\gamma-v=c$, where $c=\sum b_i$, then by (Carlson, 1969, (4.21)),

$$
(6.33) \quad \int_0^\infty t^{\nu-1} R(\beta, b, z+t) (w+t)^{-\beta'}dt
$$

$$
= B(\nu, c)w^{\nu-\beta'} R(\beta; b_1, \ldots, b_k, v; z_1, \ldots, z_k, w)
$$

$$
= B(\nu, c)w^{\nu-\beta'} R(\beta; b, v; z, w),
$$

where $v+c=\beta+\beta'$, $c=\sum b_i$, $\operatorname{Re}(c)>0$, $\operatorname{Re}(v)>0$, and $z_1, \ldots, z_k, w$ are points in $D$. We may consider (6.33) either as the generalized Stieltjes transform of $t^{\nu-1}(w+t)^{-\beta'}$ or as the
ordinary Stieltjes transform of $t^{\nu-1} R(\beta, b, z+t)$.

The $R$-function in two variables can be expressed as Gauss' hypergeometric $2F_1$ function, and so (6.32) is then the integral of a $2F_1$ function, expressed in terms of another $2F_1$ function. (This is in contrast with (6.26) in which the integral of a $1F_1$ function (S-function) yields a $2F_1$ function.)

Example 6.7.

$$
\int_0^\infty t^{\nu-1} (t+y)^{-\alpha} 2F_1(\alpha, \beta; \gamma; \frac{x}{y+t}) \, dt
$$

$$
= \int_0^\infty t^{\nu-1} R(\alpha; \beta, \gamma-\beta; t+y-x, t+y) \, dt, \text{ by (6.6)},
$$

$$
= B(\nu, \alpha-\nu) R(\alpha-\nu; \beta, \gamma-\beta; y-x, y)
$$

$$
= B(\nu, \alpha-\nu) y^{\nu-\alpha} 2F_1(\alpha-\nu, \beta; \gamma; \frac{x}{y}), \text{ by (6.5)},
$$

where $\text{Re}(\alpha) > \text{Re}(\nu) > 0$, $\gamma \neq 0, -1, -2, \ldots$, and $y-x$ and $y$ are points in $D$.

Equation (6.33) proves to be very useful in that it provides a method for determining the Stieltjes transform of a wide variety of functions which can be represented by $R$-functions. We shall illustrate this with several examples.
Example 6.8.

\[ \int_0^\infty t^{a-2} \log \left( \frac{x+t}{y+t} \right) \left( z+t \right)^{-a} \, dt \]

\[ = (x-y) \int_0^\infty t^{a-2} \frac{\log(x+t) - \log(y+t)}{(x+t) - (y+t)} \left( z+t \right)^{-a} \, dt \]

\[ = (x-y) \int_0^\infty t^{a-2} R(1; 1,1; x+t,y+t) \left( z+t \right)^{-a} \, dt, \text{ by (6.19)}, \]

\[ = (x-y) B(a-1,2) z^{-1} R(1; 1,1,a-1; x,y,z) \]

where Re(a) > 1 and x, y and z are points in D.

Example 6.9.

\[ \int_0^\infty t^{a-2} \cos^{-1} \left( \frac{x+t}{y+t} \right)^2 \left( z+t \right)^{-a} \, dt \]

\[ = (y-x) \int_0^\infty t^{a-2} \frac{1}{\left( \frac{1}{2} \right)^{a-2}} R \left( \frac{1}{2}; \frac{1}{2},1,a-1; x,y,z \right) \]

where Re(a) > 1 and x, y and z are points in D.

For certain values of a in Example 6.9, the R-function on the right represents an elliptic integral. We shall list the R-function notation for the standard symmetric elliptic integrals (Carlson, 1968, Sec. 8).

**Standard incomplete elliptic integral**

\[ R \left( \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x,y,z \right) = R_p(x,y,z), \text{ first kind}, \]
Standard complete elliptic integral

\[ R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x, y\right) = R_K(x, y), \text{ first kind,} \]

\[ R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x, y\right) = R_E(x, y), \text{ second kind,} \]

\[ R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, 1; x, y, z\right) = R_L(x, y, z), \text{ third kind.} \]

Alternative incomplete elliptic integral of the third kind (Jacobi's integral)

\[ R\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; x, y, z, w\right) = R_J(x, y, z, w). \]

Corresponding complete elliptic integral of the third kind

\[ R\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}, 1; x, y, z\right) = R_M(x, y, z). \]

In Example 6.9 let \( a = \frac{3}{2} \), then

\[ \int_0^{\infty} t^{-\frac{1}{2}} \cos^{-1} \left( \frac{x+t}{y+t} \right)^2 (z+t) \frac{1}{2} \frac{3}{2} \text{dt} = \frac{n}{2} (y-x)^{\frac{1}{2}} z^{-1} R_L(x, z, y). \]

We next consider Stieltjes transforms of elliptic integrals.
Example 6.10.
\[
\int_{0}^{\infty} t^{s-\frac{3}{2}} R_{K}(x+t, y+t)(z+t)^{-s} \, dt
\]
\[
= \int_{0}^{\infty} t^{s-\frac{3}{2}} R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x+t, y+t\right)(z+t)^{-s} \, dt
\]
\[
= B\left(s-\frac{1}{2}, 1\right) \frac{1}{2} R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, s-\frac{1}{2}; x, y, z\right), \text{ by (6.33)},
\]
where \( \text{Re}(s) > \frac{1}{2} \) and \( x, y \) and \( z \) are points in \( D \).

If \( s=1 \), then
\[
\int_{0}^{\infty} t^{\frac{1}{2}} R_{K}(x+t, y+t)(z+t)^{-1} \, dt = 2z \frac{1}{2} R_{F}(x, y, z).
\]

If \( s=\frac{3}{2} \), then
\[
\int_{0}^{\infty} R_{K}(x+t, y+t)(z+t)^{-\frac{3}{2}} \, dt = z \frac{1}{2} R_{L}(x, y, z).
\]

Example 6.11.
\[
\int_{0}^{\infty} t^{s-\frac{5}{2}} R_{E}(x+t, y+t)(z+t)^{-s} \, dt
\]
\[
= \int_{0}^{\infty} t^{s-\frac{5}{2}} R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; x+t, y+t\right)(z+t)^{-s} \, dt
\]
\[
= B\left(s-\frac{3}{2}, 1\right) \frac{3}{2} R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, s-\frac{3}{2}; x, y, z\right),
\]
where \( \text{Re}(s) > \frac{3}{2} \) and \( x, y \) and \( z \) are points in \( D \).

If \( s=2 \), then
\[
\int_{0}^{\infty} t^{\frac{1}{2}} R_{E}(x+t, y+t)(z+t)^{-2} \, dt = 2z \frac{3}{2} R_{G}(x, y, z).
\]
Example 6.12.

\[ \int_0^\infty t^{s-2} R_p(x+t,y+t,z+t)(w+t)^{-s} dt \]

\[ = \int_0^\infty t^{s-2} R(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x+t,y+t,z+t)(w+t)^{-s} dt \]

\[ = B(s-l, \frac{3}{2}) w^{-1} R(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; s-l; x,y,z,w), \]

where Re(s)>1 and x, y, z and w are points in D.

If \( s=\frac{3}{2} \), then

\[ \int_0^\infty t^{\frac{1}{2}} R_p(x+t,y+t,z+t)(w+t)^{-\frac{3}{2}} dt \]

\[ = \frac{\pi}{2^1} w^{-1} R(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x,y,z,w), \]

which is a hyperelliptic integral.

If \( s=2 \), then

\[ \int_0^\infty R_p(x+t,y+t,z+t)(w+t)^{-2} dt = \frac{2}{3} w^{-1} R_H(x,y,z,w). \]

We next consider several miscellaneous examples.

Example 6.13.

\[ \int_0^\infty a^{a-b-\frac{3}{2}} \left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \right) (x+t)^{\frac{1}{2}} (y+t)^{\frac{1}{2}} (z+t)^{-a} dt \]

\[ = \int_0^\infty a^{a-b-\frac{3}{2}} R(b-\frac{1}{2}; b,b; x+t,y+t)(z+t)^{-a} dt, \text{ by (6.21)}, \]

\[ = B(a-b-\frac{1}{2}, 2b) x^{-b-\frac{1}{2}} R(b-\frac{1}{2}; b,b,a-b-\frac{1}{2}; x,y,z), \]
where Re(a-b) > \frac{1}{2} and x, y and z are points in D.


\[
\int_0^\infty t^{a-\alpha-2\nu-1} [(x+t)(y+t)]^{\alpha/2} C_\alpha^\nu \frac{x+y+2t}{2\sqrt{(x+t)(y+t)}} (z+t)^{-\alpha} dt
\]

\[
= \frac{\Gamma(2\nu+\alpha)}{\Gamma(2\nu)\Gamma(\alpha+1)} \int_0^\infty t^{a-\alpha-2\nu-1} R(-\alpha; \nu, \nu; x+t, y+t) (z+t)^{-\alpha},
\]

by (6.21),

\[
= \frac{B(\alpha-\alpha-2\nu, 2\nu)}{\alpha B(\alpha, 2\nu)} z^{-\alpha-2\nu} R(-\alpha; \nu, \nu, \alpha-2\nu; x, y, z),
\]

where Re(a-\alpha) > Re(2\nu) > 0 and x, y and z are points in D. The function \( C_\alpha^\nu \) is the Gegenbauer function. If \( \alpha \) is a nonnegative integer \( n \), then \( C_\nu \) is a Gegenbauer polynomial which reduces to a Legendre polynomial if \( \nu = \frac{1}{2} \).

Example 6.15.

\[
\int_0^\infty t^{\alpha+\gamma-1} (t+y)^{-\alpha} {}_2F_1(\alpha, \beta; \gamma; \frac{x}{y+t}) (z+t)^{-\alpha} dt
\]

\[
= \int_0^\infty t^{\alpha+\gamma-1} R(\alpha; \beta, \gamma-\beta; t+y-x, t+y) (z+t)^{-\alpha} dt, \text{ by (6.6)},
\]

\[
= B(\alpha+\gamma, \gamma) z^{\alpha-\gamma} R(\alpha; \beta, \gamma-\beta, \alpha+\gamma; y-x, y, z),
\]

where Re(\alpha+\lambda) > Re(\gamma) > 0 and y-x, y and z are points in D.

None of Examples 6.8-6.15 have been found in tables of Stieltjes transforms.
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