TWO APPROACHES TO SOLVING THE INVERSION PROBLEM
FOR EDDY CURRENT NDE

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ABSTRACT

The eddy current NDE inversion problem is to determine flaw parameters from eddy current sensor impedance changes. Two approaches to solving this problem are discussed for geometries with two components of eddy current. The first is to use the Finite Element Method of numerical analysis to compute the sensor impedance change for each flaw parameter value. The second approach is to combine the Finite Element Method with an analytical scattering technique. These two approaches are applied to the problem of an infinitely long coil surrounding an infinitely long conducting bar with an infinitely long surface crack. The calculated impedance changes show good agreement with known analytical and experimental results.

INTRODUCTION

The eddy current NDE inversion problem is to determine flaw parameters from the measured changes in the eddy current sensor impedance. This is equivalent to determining the transformation between the flaw parameters and the impedance changes caused by the flaw. A method of obtaining this transformation is to find the electromagnetic fields induced in the material by the sensor, with and without the flaw, and use these fields to calculate the change in sensor impedance. The principal difficulty is solving Maxwell's equations in the complex geometries involved. Two approaches to overcoming this difficulty are shown for problems with two component eddy current fields, and both are applied to an infinitely long coil surrounding an infinitely long conducting bar with an infinitely long radial surface crack. The results are compared to previous analytical and experimental work.

The first approach is to use the Finite Element Method (FEM) to compute the sensor impedance with and without the flaw. The impedance change can then be found by subtracting the two. An example of the use of this method has been previously reported for a problem with a one component eddy current field. [1]. The FEM approach has the advantage that it can be applied to almost any geometry and is capable of very high accuracy. The disadvantage is that a separate computation must be made for each flaw parameter, which can be expensive. In addition, the lack of an analytical transformation equation hinders "understanding" of the relationship between the flaw parameters and the impedance change. The disadvantage is that the formula can only be applied to flaws which fit the scattering model flaw geometry.

TWO COMPONENT FINITE ELEMENT ANALYSIS

This section outlines the analysis required for computing two component eddy current fields by the finite element method. The two component eddy current problem can be formulated directly in terms of a single component diffusion equation in the magnetic field intensity H. Thus the formulation follows closely that previously presented by the authors [1], requiring only some notational changes.

Assumptions Underlying the Analysis

The following assumptions are made in modeling the eddy current problem and obtaining the field solution.
1. Displacement currents are neglected and the problem is treated as quasi-stationary.
2. The source current is assumed to be free of eddy current and proximity effects.
3. The resistivity of the conducting parts is constant and single valued.
4. The problem is assumed to be two-dimensional and all field quantities are considered to be harmonic functions of time.
5. The current density is assumed to have components along the x and y directions, while the magnetizing field H has only one component along the z direction.

Linear Diffusion Equation

For the two-component linear eddy current field problem, subject to the assumptions stated above, the magnetic field intensity vector $\mathbf{H}$ is a single component vector given by the solution to the linear diffusion equation,

$$\frac{1}{\sigma} \nabla^2 \mathbf{H} = j \omega \mathbf{H}$$

(1)
where: \( \sigma = \) conductivity
\( \mu = \) permeability
\( \omega = \) radian frequency

**Finite Element Representation**

Equation (1) can be reformulated in variational terms by energy expressions called functionals. The finite element method consists of discretizing the field region into sub-regions or elements and projecting approximations to the solutions \( H \) which minimize the corresponding functionals. This process results in a matrix equation, which when solved yields the solution to the eddy current problem. The accuracy of the solution depends largely on the discretization of the field region and the prescription of a good solution approximation. For the sake of completeness, the salient steps of the finite element method are presented below, using a second order approximation to the field solution.

Representing the diffusion equation formulation (1) in terms of a single differential equation

\[
D \psi = f
\]

where \( D = \) the differential operator
\( \psi = \) the potential function
\( f = \) the source or forcing function,

the expression for the energy functional is obtained as

\[
F = \langle \psi | D \psi \rangle - 2 \langle \psi | f \rangle
\]

where the inner product \( < > \), represents volume integration of dot product of the variables.

![Fig. 1 Second Order Triangular Element](image)

We now sub-divide the field region into triangular elements as shown in Fig. 1 and prescribe the potential function \( \psi \) in each element in terms of interpolation polynomials called shape functions, weighted by function values at the nodes.

Thus we may write

\[
\psi = \sum_{k=1}^{n} \zeta_k \phi_k
\]

For the second order approximation, the shape functions are given as

\[
\zeta_k = \zeta_k (2z_k - 1) \quad \text{for} \ k = 1, 3 \ or \ 5
\]

\[
\zeta_k = 4z_k \phi_q \quad \text{for} \ k = 2, 4 \ or \ 6
\]

with \( p,q \) respectively (1,3), (3,5), (5,1), and

\[
\zeta_k = (a_k + b_k x + c_k y) / 2\Delta
\]

where \( \Delta \) is the element area, and \( a_k, b_k \) and \( c_k \) are defined in progressive modulo 3 as:

\[
a_k = \frac{x_k y_m - y_k x_m}{y_k y_m} ; \quad b_k = (y_k - y_m) ; \quad c_k = (x_k - x_m)
\]

Thus the final set of complex linear equations is expressed in matrix form in terms of the coefficient matrix \([S]\) and the related numerical matrix \([T]\), as

\[
K_1 [S] [\psi] + jk_2 [T] [\psi] = [T] [f]
\]

where \( k_1 = \frac{1}{\sigma} ; \quad k_2 = \mu \omega ; \quad \psi = \bar{H}\)

Equation (9) is readily recognized as identical to equation (16) of reference [1].

**Boundary Conditions and Forcing Function**

For the two-component eddy current field problem described above, with a single component magnetic field \( H \), the field external to the infinitely long coil is zero. Also, the magnetic field distribution in the coil is unaffected by the circulating currents in the conducting bar.

The value of the forcing function \( H \) on the inner surface of the coil is related to the coil current density \( J \) which is uniform such that

\[
H_{T1} - H_{T2} = J
\]

**Winding Resistance and Inductance**

The total energy stored in the system is obtained by integrating the product of free space permeability and the square of the magnetic field \( H \) over the volume. Thus the energy per unit length is

\[
W_s / \chi = 1 / 2 \mu_0 |H|^2 \Delta
\]

where \( \chi \) is the length of the bar or solenoid.

Equating the above expression to the well known stored energy in the terminal inductance, and dividing by the square of the bar coil current, \( I \), we have

\[
L / \chi = \frac{\mu_0 |H|^2 \Delta}{I^2}
\]
The power dissipated per unit length in the bar resistance is obtained by integrating the ohmic losses over the volume, giving

\[ P_{d/\xi} = \frac{1}{2\xi} \int J^2_e \, d\xi \]  \hspace{1cm} (13)

where the sum is over all triangles, and

\[ J_e = \text{eddy current density} \]
\[ \rho = \text{resistivity of the bar} \]

Note that \( J_e \) can be calculated from \( H \) by Maxwell's equation

\[ \nabla \times H = J_e \] \hspace{1cm} (14)

Equating the above power loss to the \( I^2R \) product one obtains

\[ \frac{R}{\xi} = \frac{\frac{1}{2} \int J^2_e \, d\xi}{\frac{1}{2} \int J^2_e \, d\xi} = \frac{1}{\frac{1}{2} \int J^2_e \, d\xi} \] \hspace{1cm} (15)

The results of applying the FEM analysis to the problem of the infinitely long conducting bar surrounded by an infinitely long coil are shown in Figs. (2) through (7). The eddy current density profiles in the bar cross-section without and with a crack are shown in Fig. (2) through (5). The corresponding impedance plane diagrams are illustrated in Figs. (6) and (7). The experimental results in Fig. 7 are those of Forster [4].
CONTOUR DIVISIONS = 0.8632E-01
NO. OF CONTOURS = 24

M U = 0.1257E-05
R H O = 0.5000E-06
FREQ = 0.1470E 03

Fig. 5 Imaginary Part Eddy Current Profile in Cross-Section of Round Bar With Crack

Fig. 6 Normalized Impedance Plane Diagram for Circular Cross-Section

Fig. 7 Comparison of Finite Element Results with Experiment for a Crack in a Bar with Circular Cross-Section

TWO COMPONENT SCATTERING THEORY

The Scattering Model

In the scattering theory approach, the change in sensor impedance is found from the incident and scattered fields of the flaw by using the reciprocity theorem, as explained by Auld [5]. For a void flaw in a homogeneous, isotropic, conducting medium with permittivity $\varepsilon_0$ and permeability $\mu_0$, the change $\Delta Z$ in sensor impedance is given by

$$\Delta Z = \frac{1}{2} \int_{V_f} \sigma \left( \mathbf{E} \cdot \mathbf{E}' \right) \, dv$$  \hspace{1cm} (16)

where: $I$ = the sensor terminal current

$\mathbf{E}$ = the electric field without the flaw

$\mathbf{E}'$ = the electric field with the flaw

$V_f$ = the volume of the flaw.

Therefore, to compute the sensor impedance change, it is necessary to compute the electric fields within the boundaries of the flaw both when the flaw is present and when it is not.

The strategy for computing these electric fields is to approximate the incident field (i.e. the field without the flaw) by a constant plus a linearly varying component, as shown in Fig. 8. The respective scattered fields can then be computed for an elliptic cylinder flaw by assuming dipole and quadrupole scattered fields,
and matching boundary conditions at the flaw. The resultant field (i.e. the field with the flaw) is then the sum of the incident and scattered fields. (Figs. 9 and 10)

\[ \mathbf{J}_{ci} = \mathbf{T}_y J_{ci} \]  

(17)

The associated scattered current field is a dipole field. The resultant sum field satisfies the static form of Maxwell's equations and matches the current boundary condition for a void i.e. zero current normal to the flaw boundary \( \mathbf{p} = \mathbf{p}_0 \).

\[ \mathbf{E}_c = \frac{\mathbf{T}_y J_{ci}}{\sigma} \]  

(18)

\[ \mathbf{E}_c' = \frac{\mathbf{T}_y J_{ci} a + b}{\sigma b} \]  

(19)

where \( a \) and \( b \) are the major and minor axes of the ellipse. This field satisfies the static form of Maxwell's equations and matches the boundary condition for the electric field i.e. no change in the tangential electric field across the flaw boundary.

Similarly, the linearly varying component of the incident eddy current field is given by

\[ \mathbf{J}_{v1} = \mathbf{T}_y M x \]  

(20)

The associated scattered current field is a quadrupole field. When there is no flaw, the electric field interior to the flaw boundary is the linearly varying incident current field (20) divided by the conductivity.

\[ \mathbf{E}_v = \frac{\mathbf{T}_y M x}{\sigma} \]  

(21)

When there is a flaw, the electric field inside the flaw is the sum of a linearly varying and a "saddle" electric field.

Fig. 8 Incident Eddy Current Fields

In cartesian coordinates, the constant component of the incident eddy current field is given by

\[ \mathbf{J}_{ci} = \mathbf{T}_y J_{ci} \]  

(17)

The associated scattered current field is a dipole field. The resultant sum field satisfies the static form of Maxwell's equations and matches the current boundary condition for a void i.e. zero current normal to the flaw boundary \( \mathbf{p} = \mathbf{p}_0 \).

\[ \mathbf{E}_c = \frac{\mathbf{T}_y J_{ci}}{\sigma} \]  

(18)

\[ \mathbf{E}_c' = \frac{\mathbf{T}_y J_{ci} a + b}{\sigma b} \]  

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\[ \mathbf{E}_v = \frac{\mathbf{T}_y M x}{\sigma} \]  

(21)

When there is a flaw, the electric field inside the flaw is the sum of a linearly varying and a "saddle" electric field.
In elliptic cylinder coordinates (see Fig. 11), this field is given by
\[ E_y = \frac{M}{\rho} \sin \theta \cos \theta + \frac{N}{\rho} \cosh \theta \sin \theta (1 + \cos \theta) \]
\[ + \frac{P}{\rho^2} e^{-2\rho_0} \cosh \rho_0 \sin \theta \]
\[ + \frac{Q}{\rho^2} e^{-2\rho_0} \cosh \rho_0 \sin \theta \cos \theta \]
where \( h \) is the metric coefficient for elliptic cylinder coordinates. Note that the first two terms are the linearly varying field \( \frac{J}{m} \) in elliptic cylinder coordinates.

Computation of the Change in Sensor Impedance

The change in sensor impedance due to the flaw can be calculated from (16). For the sinusoidal steady state, the electric fields are complex quantities.
\[ E = E_x + j E_y \]
\[ E' = E'_x + j E'_y \]
then the dot product in (16) is given by
\[ E \cdot E' = (E_x \cdot E'_x - E_y \cdot E'_y) + (E_x \cdot E'_y + E_y \cdot E'_x) \]
\[ \rho = \rho_0 \text{ (FLAW BOUNDARY)} \]
\[ \rho = 0 \text{ (FLAW BOUNDARY)} \]
\[ e = \pi \]
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\[
E' = \frac{J(0)}{\sigma} \left[ \frac{1}{r} \frac{a + b}{y} + \frac{A}{y} \right] \frac{x}{r} \frac{2}{r}
\]

(27)

\[
E'_q = \frac{J(0)}{\sigma} \left[ \frac{1}{r} \frac{B}{y} \frac{x}{r} \right]
\]

(28)

The change in impedance \(\Delta Z\) can now be found by substituting (25) through (28) into (24), and performing the integration (16) over the semi-elliptical cylinder volume, and letting the crack width \(b\) go to zero.

It has been implicitly assumed up to this point that the scattering model is valid for the semi-ellipse. This is exactly true for the dipole field, since there is no current normal to the \(y\)-axis. The quadrupole field, however, does have a component normal to the \(y\)-axis as shown in Fig. 10. This component is very small compared to the total current density in the region, and is simply ignored. The placing of a boundary along the \(y\)-axis should not significantly change the distribution of the scattered field.

Carrying out the integration in (16), and letting \(b\) go to zero, gives for the change in impedance per unit length

\[
\Delta Z = -\frac{J(0)^2}{\sigma} a^2 \left[ \left( \frac{x}{r} + \frac{2}{3} A \frac{x}{r} \right) + \frac{1}{3} B \frac{x}{r} \right]
\]

(29)

It is customary in the literature to normalize this change in impedance to the magnitude of the impedance of the coil with no conducting bar inserted. The magnitude of the impedance per unit length is given by the well known expression

\[
|Z_0| = \omega \mu_0 N^2 r^2
\]

(30)

where \(N\) is the number of turns per unit length, and \(\omega\) is the radian frequency of the excitation.

The expression for \(J(0)\) is given by Hochschild [6],

\[
J(0) = \frac{M_0 B_r G e^{i\gamma}}{\nu_0}
\]

(31)

where:

\[
y = [e_1(\gamma) - a_0(\gamma) - \frac{3\pi}{4}]
\]

(32)

\[
G = [M_1(\gamma) / M_0(\gamma)]
\]

(33)

In these equations, the functions \(M_0, M_1, a_0, e_1\) are respectively the moduli and phases of the Kelvin functions of orders 0 and 1. The constant \(B_r\) is the magnetic flux density at the surface of the conducting cylinder, and is given by the well known expression

\[
B_r = \nu_0 N I
\]

(34)

Combining (29) through (32) gives change in normalized impedance of the coil as a function of the crack depth \(a\).

\[
\Delta \frac{Z}{|Z_0|} = -\frac{1}{2} (Ge^{i\gamma})^2 \left[ (1 + \frac{4}{3} A \frac{x}{r}) + \frac{1}{3} B \frac{x}{r} \right]
\]

(35)

Since \(Z\) is a single valued function of the crack depth \(a\), this equation is also a solution to the inversion problem for the given conditions.
Comparison With Other Theory and Experiment

A comparison of the normalized impedance with other theoretical and an experimental result is shown in Figure 14 for \( g_a = \sqrt{5} \). The theoretical results are those Burrows [2] and Spal and Kahn [3]. The experimental result is due to Forster [4]. Burrows dipole model shows good agreement with experiment only for crack depths small compared to the skin depth, as he predicted.

The dipole plus quadrupole model derived here compares favorably with experiment for crack depths up to one half skin depth. The departure of the model from experiment for greater depths is due to the large error in the linear approximation to the imaginary part of the incident field beyond one half skin depth. The exact theoretical model of Spal and Kahn follows the phase of the experimental result more closely than the others, but has about the same difference in magnitude. This suggests that the experimental conditions deviated from the model assumptions.

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