2014

Sign patterns that require eventual exponential nonnegativity

Craig Jeremy Erickson
Iowa State University

Sign patterns that require eventual exponential nonnegativity

by

Craig Jeremy Erickson

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:

Leslie Hogben, Major Professor

Ryan Martin

Roger Maddux

Justin Peters

Sung-Yell Song

Iowa State University

Ames, Iowa

2014

DEDICATION

This dissertation is dedicated to my wife Bonny, who has supported me wherever my mathematical adventures have taken me.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1. GENERAL INTRODUCTION

1.1 Introduction

A *sign pattern matrix* (or *sign pattern*) is a matrix with entries in $\{+,-,0\}$. For $A=[a_{ij}]\in\mathbb{R}^{n\times n}$, the sign pattern of A is given by $\operatorname{sgn}(A)=[\operatorname{sgn}(a_{ij})]$. The *qualitative class* of the $n\times n$ sign pattern $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all matrices, $A\in\mathbb{R}^{n\times n}$, such that $\operatorname{sgn}(A)=\mathcal{A}$. A matrix in $\mathcal{Q}(\mathcal{A})$ is called a *realization* of $\mathcal{A}$. Sign pattern $\mathcal{A}$ *allows* property $\mathcal{P}$ (or is *potentially* $\mathcal{P}$) if there exists a realization $A\in\mathcal{Q}(\mathcal{A})$ that has property $\mathcal{P}$ and sign pattern $\mathcal{A}$ *requires* property $\mathcal{P}$ if every realization $A\in\mathcal{Q}(\mathcal{A})$ has property $\mathcal{P}$.

The study of sign patterns began in the field of economics, particularly with Samuelson's [48] introduction of qualitative economics. Samuelson raised the question as to whether one could completely determine the signs of the entries in the solution to a linear system from only qualitative knowledge of the linear system. The study of sign patterns has applications to many fields (e.g., economics, biology, chemistry, and sociology) in which the qualitative information about a dynamical system is known, but the quantitative information is unknown or unreliable. For a more thorough historical perspective of the study of sign patterns, see [8, 29].

The *matrix exponential function* e^{tA} is defined by its power series expansion as $e^{tA}=\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}$. An $n\times n$ matrix $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ is *essentially nonnegative* if $a_{ij}\geq 0$ for $i\neq j$ and $i,j\in\{1,\ldots,n\}$. Matrix $A\in\mathbb{R}^{n\times n}$ is called *eventually nonnegative* (*eventually positive*) if there exists $k_0\in\mathbb{Z}^+$ such that for all $k\geq k_0$, $A^k\geq 0$ (respectively, $A^k>0$), where the inequality is entrywise. Matrix A is *exponentially nonnegative* (*exponentially positive*) if for all $t>0$, $e^{tA}\geq 0$ (respectively, $e^{tA}>0$). Matrix A is *eventually exponentially nonnegative* (*eventually exponentially positive*) if there exists some real number $t_0\geq 0$ such that for all $t>t_0$, $e^{tA}\geq 0$ (respectively, $e^{tA}>0$.)

Eventually exponentially nonnegative matrices have important applications. For example, Noutsos and Tsatsomeros [43] studied linear differential systems of the type

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad (A \in \mathbb{R}^{n\times n}, \mathbf{x}_0 = \mathbf{x}(0) \in \mathbb{R}^n, t \geq 0), \tag{1.1}$$

whose solutions are nonnegative (positive) for all $t \geq T$ for some $T \in \mathbb{R}$. These solutions are the $n \times n$ eventually exponentially nonnegative (positive) matrices.

In this dissertation, we study the problem of determining which sign patterns require eventual exponential nonnegativity (Chapter 2) or allow eventual exponential positivity (Chapter 3).

1.1.1 Definitions and notation

If for the sign patterns $\mathcal{A} = [\alpha_{ij}]$ and $\widehat{\mathcal{A}} = [\widehat{\alpha}_{ij}]$, $\alpha_{ij} \neq 0$ implies $\alpha_{ij} = \widehat{\alpha}_{ij}$, then $\mathcal{A}$ is a *subpattern* of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$. The *positive part* and *negative part* of sign pattern $\mathcal{A}$, denoted $\mathcal{A}^+ = [\alpha^+_{ij}]$ and $\mathcal{A}^- = [\alpha^-_{ij}]$, respectively, are defined by

$$\alpha^+_{ij} = \begin{cases} + & \text{if } \alpha_{ij} = + \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \alpha^-_{ij} = \begin{cases} - & \text{if } \alpha_{ij} = - \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\mathcal{A}^+$ and $\mathcal{A}^-$ are subpatterns of $\mathcal{A}$, and that $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$. Given an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, the $n \times n$ sign pattern $\mathcal{A}_{D(+)} = [\widehat{\alpha}_{ij}]$ is defined to be $\widehat{\alpha}_{ij} = \alpha_{ij}$ for $i \neq j$ and $\widehat{\alpha}_{ii} = +$ for $i, j \in \{1, \ldots, n\}$. $\mathcal{A}_{D(-)}$ and $\mathcal{A}_{D(0)}$ are defined analogously, with negative and zero diagonal, respectively. Two $n \times n$ sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if there exists some permutation matrix P such that $\mathcal{B} = P^T \mathcal{A} P$ or $\mathcal{B} = P^T \mathcal{A}^T P$.

An $n \times n$ matrix S is *nonsingular* if it is invertible, that is, if there exists a matrix S^{-1} such that $SS^{-1} = I = S^{-1}S$ (where I is the $n \times n$ identity matrix). Two $n \times n$ matrices A and B are *similar* if there exists a nonsingular matrix S such that $B = S^{-1}AS$.

The *characteristic polynomial* of $A \in \mathbb{R}^{n\times n}$ is $p_A(x) = \det(xI - A)$, where I is the $n \times n$ identity matrix. The *minimal polynomial* of A, denoted $m_A(x)$, is the unique monic polynomial of least degree such that $m_A(A) = 0$. If there exists a scalar λ and nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, then λ is an *eigenvalue* with associated (*right*) *eignevector* $\mathbf{x}$. If the nonzero vector $\mathbf{w}$ is such that $\mathbf{w}^T A = \lambda \mathbf{w}^T$, then we call $\mathbf{w}$ a *left eigenvector* of A associated with the eigenvalue

λ. It is well known that λ is an eigenvalue of A if and only if λ is a root of $p_A(x)$ and the (*algebraic*) *multiplicity* of λ as an eigenvalue of A is the multiplicity of λ as a root of $p_A(x)$. The *spectrum* of A, denoted $\mathrm{spec}(A)$, is the multiset of the eigenvalues of A. The *spectral radius*, $\rho(A)$, is defined to be $\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{spec}(A)\}$. It is well known that similar matrices have the same spectrum. An eigenvalue λ of A is called a *dominant eigenvalue* if $|\lambda| = \rho(A)$. Let $\mathrm{Re}(z)$ denote the real part of $z \in \mathbb{C}$. The *spectral abscissa*, $\alpha(A)$, is defined as $\alpha(A) = \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spec}(A)\}$. Eigenvalue $\gamma \in \mathrm{spec}(A)$ is a *rightmost eigenvalue* of A if $\mathrm{Re}(\gamma) = \alpha(A)$.

Matrix $T = [t_{ij}]$ is *upper triangular* (respectively, *strictly upper triangular*) if $t_{ij} = 0$ for $i > j$ (respectively, $t_{ij} = 0$ for $i \geq j$). An upper triangular (strictly upper triangular) sign pattern is defined similarly. It is well known that the eigenvalues of an upper triangular matrix, $T = [t_{ij}]$, are the diagonal elements t_{ii}.

A $k \times k$ *Jordan block* for $\lambda \in \mathbb{C}$ is a square matrix of the form

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}.$$

A *Jordan matrix* is a block diagonal matrix of the form $J = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_p}(\lambda_m)$. Note that it is possible for J to have more than one block—possibly of different sizes—corresponding to a given eigenvalue, so there is no assumption that the λ_i are distinct. A *Jordan canonical form* of matrix A is a Jordan matrix that is similar to A.

A square matrix A (or sign pattern) is called *reducible* if there exists some permutation matrix P such that $PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are nonempty square matrices (sign patterns) and $\mathbf{0}$ is a (possibly rectangular) block consisting entirely of zero entries. If no such permutation exists, then A is *irreducible.*

A *digraph* $G = (V, E)$ of *order* n is a set of *vertices* $V = \{1, 2, \ldots, n\}$ and *arcs* (directed edges) $E = \{(x, y) : x, y \in V\}$. A *loop* is an arc $e = (x, y) \in E$ such that $y = x$. We do not allow multiple arcs, that is, the arc-set is a set (not a multiset) of ordered pairs. For $x \neq y$, the arcs (x, y) and (y, x) are two distinct arcs and the digraph may contain one, both, or neither of these arcs. A (directed) *walk* (or v_1-v_k *walk*) in digraph G is a sequence of vertices and arcs $v_1, (v_1, v_2), v_2, (v_2, v_3), \ldots, v_{k-1}, (v_{k-1}, v_k), v_k$, provided that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$. If x is a vertex in a walk, we say the walk *passes through* or *visits* vertex x. The number of times a walk visits a vertex x is the number of times x appears in the sequence $v_1, v_2, \ldots, v_k$. Since we do not allow multiple arcs, when writing out a walk, we will list only the vertices through which the walk passes, e.g., $(v_1, v_2, \ldots, v_k)$. The vertices $v_2, \ldots, v_{k-1}$ in the above v_1-v_k walk are referred to as the *interior vertices* of the walk. If $v_1 = v_k$, the walk is called a *closed walk*. A *cycle* is a closed walk in which each interior vertex is visited only once. A *path* is a walk in which no vertex is repeated. The *length* of the walk $(v_1, v_2, \ldots, v_k)$ is the number of arcs traversed in the walk, i.e., $k - 1$. Therefore a path of length k uses $k + 1$ vertices and a cycle of length k uses k vertices. We say that vertex $u \in V$ *has access to* vertex $v \in V$ if there exists a u-v walk in G. For $v \in V$, $\mathrm{In}(v)$ is the set of vertices which have access to v and $\mathrm{Out}(v)$ is the set of vertices to which v has access. Digraph $G = (V, E)$ is *strongly connected* if for any two vertices $u, v \in V$, u has access to v (i.e., there exists a u-v walk in G).

We say that the digraph $G = (V, E)$ is *associated with* the matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ (respectively, $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$), or that A is associated with G, when $a_{ij} \neq 0$ (respectively, $\alpha_{ij} \neq 0$) if and only if $(i, j) \in E$. Therefore a matrix (or sign pattern) has a nonzero diagonal element a_{ii} if and only if its associated digraph G has the loop (i, i) in its arc-set. It is well known that a matrix (or sign pattern) is irreducible if and only if its associated digraph is strongly connected. We denote the *weighted digraph associated with* A by $\Gamma(A)$, where $(i, j) \in E$ if and only if $a_{ij} \neq 0$ and if $(i, j) \in E$, we assign to arc (i, j) the arc-weight a_{ij}. Similarly, we denote the signed digraph (the weighted digraph with arc-weights coming from $\{+, -\}$) associated with the sign pattern $\mathcal{A}$ by $\Gamma(\mathcal{A})$. The *product of a u-v walk* in $\Gamma(A)$ is the product of the weights of the arcs in said u-v walk. The *sign of a u-v walk* in $\Gamma(A)$ (or in $\Gamma(\mathcal{A})$) is the sign of the product of said u-v walk. We call a walk *arc-positive* if each arc used in the walk has positive arc-weight.

A digraph $G = (V, E)$ is *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 so that arc $(i,j) \in E$ implies that either (1) $i \in V_1$ and $j \in V_2$ or (2) $j \in V_1$ and $i \in V_2$. Let $M = [m_{ij}] \in \mathbb{R}^{m\times n}$. The *König digraph* of M, denoted $K(M)$, is a weighted bipartite digraph on $m+n$ vertices, with vertices $V_r = \{r_1, \ldots, r_m\}$ corresponding to the rows of M and vertices $V_c = \{c_1, \ldots, c_n\}$ corresponding to the columns of M. The ordered pair (r_i, c_j) is an arc in $K(M)$ if and only if $m_{i,j} \neq 0$ and the weight of arc (r_i, c_j) is given by $m_{i,j}$. Consider the matrices $X = [x_{ij}] \in \mathbb{R}^{m\times n}$ and $Y = [y_{ij}] \in \mathbb{R}^{n\times p}$ and their König digraphs $K(X)$ and $K(Y)$, with vertices $V_{X,r} \cup V_{X,c}$ and $V_{Y,r} \cup V_{Y,c}$, respectively (where $V_{X,r} = \{r_{X,1}, r_{X,2}, \ldots, r_{X,m}\}$, $V_{X,c} = \{c_{X,1}, c_{X,2}, \ldots, c_{X,n}\}$, $V_{Y,r} = \{r_{Y,1}, r_{Y,2}, \ldots, r_{Y,n}\}$, and $V_{Y,c} = \{c_{Y,1}, c_{Y,2}, \ldots, c_{Y,p}\}$). It is well known that the (i,j)-entry of the product XY can be computed as follows (see, e.g., [6]). First, construct the *composite König digraph*: for $1 \leq k \leq n$, identify vertex $c_{X,k}$ with vertex $r_{Y,k}$ and rename as v_k. Second, for $1 \leq k \leq n$, compute the weight, w_k, of the $(r_{X,i}, v_k, c_{Y,j})$-path as $w_k = x_{i,k} y_{k,j}$. Finally, compute the sum $w_1 + w_2 + \cdots + w_n$. By the definition of matrix multiplication, this sum is the (i,j)-entry of the product XY. This process readily generalizes to products of more than two matrices (see, e.g., Example 2.4.1 in Chapter 2). Note that Brualdi and Cvetković [6] collapse what we call the composite König digraph into the König digraph for the matrix that is the result of computing the product. For example, the arc weight of the arc $(r_{X,i}, c_{Y,j})$ in $K(XY)$ would be the sum $w_1 + w_2 + \cdots + w_n$.

1.1.2 Functions of matrices

We use the interpretation of Higham in [34] for a “function of a matrix.” That is, modifying a scalar function, f, appropriately to have a matrix $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) as an input and an $n \times n$ matrix, $f(A)$, (in either $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$) as an output in such a way as to be a natural generalization of the scalar function. For example, if $p : \mathbb{R} \to \mathbb{R}$ is a polynomial having constant term c, when considering the matrix function $p : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, we must replace the constant term by cI, where I is the $n \times n$ identity matrix.

There are several ways to define a matrix function (see, e.g., [33, 34]) and we use a few of these different (but equivalent) definitions of e^{tA} to prove various results in Chapter 2. If the scalar function $f : \mathbb{C} \to \mathbb{C}$ has a convergent power series, then the matrix function $f : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ can be defined by the power series. However, for many interesting functions (e.g., the exponential function), terminating the power series after a finite number of terms results in an approximation to the function, not the function itself. While this is natural in decimal arithmetic, it presents some challenges for sign patterns, where zero and approximately zero differ significantly.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of $A \in \mathbb{C}^{n\times n}$. Let n_i be the *index* of λ_i, that is, the size of the largest Jordan block for λ_i, for $i = 1, 2, \ldots, s$ (which is equivalent to the multiplicity of λ_i as a root of the minimal polynomial of A). The function f is *defined on the spectrum* of A if $f^{(j)}(\lambda_i)$ exists for $j = 0, 1, \ldots, n_i - 1$; $i = 1, 2, \ldots, s$, where $f^{(j)}$ is the jth derivative (these values are called the *values f takes on the spectrum of A*). The *Jordan canonical form definition of a matrix function* is:

Definition 1.1.1 ([34, p. 3])**.** Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let A have Jordan canonical form $S^{-1}AS = J = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_p}(\lambda_s)$. Then $f(A) := Sf(J)S^{-1} = S\big[f\big(J_{k_1}(\lambda_1)\big) \oplus \cdots \oplus f\big(J_{k_p}(\lambda_s)\big)\big]S^{-1}$, where

$$f\big(J_{k_i}(\lambda_j)\big) := \begin{bmatrix} f(\lambda_j) & f'(\lambda_j) & \cdots & \frac{f^{(k_i-1)}(\lambda_j)}{(k_i-1)!} \\ 0 & f(\lambda_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f'(\lambda_j) \\ 0 & \cdots & 0 & f(\lambda_j) \end{bmatrix}$$

for $i = 1, \ldots, p$.

An advantage of using the Jordan canonical form definition of a matrix function is that a finite process yields the exact function value (rather than an approximation). However, one must compute the Jordan canonical form of A, which can be highly nontrivial, especially if n is large. In fact, if $n \geq 5$ it may not be possible to find the exact values of the eigenvalues.

The definition of a matrix function via polynomial interpolation utilizes the numerical analysis technique of approximating a function f with an interpolating polynomial p. It is important to note that for a polynomial p, $p(A)$ is determined by the values of p on the spectrum of A.

Theorem 1.1.2 ([34, Theorem 1.3])**.** *For polynomials p and q and $A \in \mathbb{C}^{n\times n}$, $p(A) = q(A)$ if and only if p and q take the same values on the spectrum of A.*

The *Hermite interpolation definition of a matrix function* is:

Definition 1.1.3 ([34, p. 5])**.** Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let $m_A(x)$ be the minimal polynomial of A. Then $f(A) := p(A)$, where p is the polynomial of degree less than $\sum_{i=1}^{s} n_i = \deg m_A(x)$ that satisfies the interpolation conditions $p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i)$, for $j = 0, 1, \ldots, n_i - 1$ and $i = 1, \ldots, s$.

It is well known that the polynomial p in Definition 1.1.3 is unique; and this polynomial is called the *Hermite interpolating polynomial.* Moreover, the Hermite interpolating polynomial is given explicitly [34, p. 6] by

$$p(z) = \sum_{i=1}^{s} \left[\left(\sum_{k=0}^{n_i-1} \frac{1}{k!} \phi_i^{(k)}(\lambda_i)(z-\lambda_i)^k \right) \prod_{j\neq i} (z-\lambda_j)^{n_j} \right],$$

where

$$\phi_i(z) = \frac{f(z)}{\prod_{j\neq i}(z-\lambda_j)^{n_j}}.$$

1.1.3 Calculating the matrix exponential function

Calculating e^{tA} can be very difficult for arbitrary $A \in \mathbb{R}^{n\times n}$, especially when n gets large. There are many methods used to calculate e^{tA} for a given matrix (see, e.g., [47, 40, 41, 1, 27]). However, as Moler and van Loan note in [40, 41], in practice, most of these methods only provide an approximation and determining which method is best to use depends greatly on the combinatorial structure (sparse versus dense) and eigenstructure of A and how accurate of an approximation is needed. For this dissertation, we do not necessarily need to know the exact entries of e^{tA}, but we do need to know whether those entries are positive, negative, or zero as $t \to \infty$. Therefore an approximation is not good enough; moreover, we need to be able to analyze the sign of each entry as $t \to \infty$. Therefore we use only the power series, Jordan canonical form, and Hermite interpolation definitions from Section 1.1.2 to calculate e^{tA}.

1.2 Literature Review

1.2.1 Generalizations of positive matrices

Nonnegative matrices, positive matrices, and their generalizations have been studied for over one hundred years, with some of the earliest results published by the German mathematicians Oskar Perron and Georg Frobenius. Hawkins provides a thorough history of what is now called the Perron-Frobenius theorem in [31]. Perron's namesake theorem first appeared in [45], with its proof appearing in [46].

Theorem 1.2.1 (Perron's Theorem)**.** *Let $A \in \mathbb{R}^{n\times n}$ be entrywise positive. Then $\rho = \rho(A) > 0$ is a simple eigenvalue of A and there exist positive vectors $\mathbf{x}, \mathbf{w}$ such that $A\mathbf{x} = \rho\mathbf{x}$ and $\mathbf{w}^T A = \rho\mathbf{w}^T$.*

Perron also established what Hawkins calls Perron's Corollary.

Corollary 1.2.2. *Let $A \in \mathbb{R}^{n\times n}$ be entrywise nonnegative. If $A^k > 0$ for some power $k \geq 1$, then $\rho = \rho(A) > 0$ is a simple eigenvalue of A and there exist positive vectors $\mathbf{x}, \mathbf{w}$ such that $A\mathbf{x} = \rho\mathbf{x}$ and $\mathbf{w}^T A = \rho\mathbf{w}^T$.*

Perron's proofs of these results relied on limits and he set forth a challenge to prove these results using purely algebraic techniques (i.e., without limits). Frobenius answered this challenge in his papers on positive and nonnegative matrices [24, 25, 26], in which the ideas of irreducibility and primitivity were introduced. Frobenius did not use graph theory to define these ideas; however, we shall: Digraph G is *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is one. An $n \times n$ nonnegative matrix A (or sign pattern) is primitive if the digraph associated with A is primitive. It is well known (see, e.g., [7]) that a primitive matrix is eventually positive.

Theorem 1.2.3 (Perron-Frobenius Theorem)**.** *Let $A \in \mathbb{R}^{n\times n}$ be irreducible and $A \geq 0$. Then $\rho = \rho(A) > 0$ is a simple eigenvalue of A and there exist positive vectors $\mathbf{x}, \mathbf{w}$ such that $A\mathbf{x} = \rho\mathbf{x}$ and $\mathbf{w}^T A = \rho\mathbf{w}^T$.*

Matrix A has the *strong Perron-Frobenius property* if $\lambda_1 = \rho(A) > 0$ is a simple eigenvalue of A with a positive eigenvector (called a *Perron vector*) and $|\lambda| < \lambda_1$ for all $\lambda \in \text{spec}(A)$ such that $\lambda \neq \lambda_1$. It is well known that $A \in \mathbb{R}^{n\times n}$ is eventually positive if and only if both A and A^T have the strong Perron-Frobenius property (see, e.g., [30, 37, 42]). Both Handelman [30] and Johnson and Tarazaga [37] consider matrices with this property but Noutsos introduced the terminology "strong Perron-Frobenius property" after others (e.g., [50]) had started to consider matrices with similar, but weaker, eigenstructure.

See [4, 32, 49] for surveys of results relating the combinatorial structure of nonnegative matrices and their eigenstructure (the eigenvalues—including multiplicity—and eigenvectors). Since eventually nonnegative matrices were introduced by Friedland [23] in 1978, many people have studied eventually nonnegative matrices (see, e.g., [23, 30, 53, 9, 52, 10, 17, 18]) in an attempt to better understand the combinatorial structure of nonnegative matrices, eventually nonnegative matrices, and other generalizations, and how it relates to the eigenstructure of the matrix. Matrix A has the *Perron-Frobenius property* if $\rho = \rho(A)$ is an eigenvalue of A and there exists $\mathbf{x} \geq 0$ such that $A\mathbf{x} = \rho\mathbf{x}$. The term Perron-Frobenius property was used in [50], where the authors extend the Perron-Frobenius theorem to matrices having some negative entries. The following result appeared in [42, Theorem 2.3] but without the necessary hypothesis that A must not be nilpotent, as was pointed out in [17]; the correct version appears in [43]

Theorem 1.2.4 ([43, Theorem 3.12])**.** *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both A and A^T have the Perron-Frobenius property.*

Unfortunately, Theorem 1.2.4 is not an "if and only if" test. In an attempt to recover eigenstructure closer to the strong Perron-Frobenius property, and an "if and only if" test for eventual nonnegativity, the class of strongly eventually nonnegative matrices were introduced in [11] (an earlier preprint version of [12]) and [35]. Matrix $A \in \mathbb{R}^{n\times n}$ is *strongly eventually nonnegative* if A is eventually nonnegative and there exists some power of A that is both nonnegative and irreducible. Along with strongly eventually nonnegative matrices, [11] also introduces an eigenstructure property that is stronger than the Perron-Frobenius property, but weaker than the strong Perron-frobenius property. Matrix A has the *semi-strong Perron-Frobenius property* if $\rho = \rho(A) > 0$ is a simple eigenvalue of A having a positive eigenvector. Hogben [35] provides an "if and only if" test for strong eventual nonnegativity.

Noutsos and Tsatsomeros provide the following result relating eventually exponentially positive and eventually positive matrices.

Theorem 1.2.5 ([43, Theorem 3.3]). *For a matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:*

(i) There exists $a \geq 0$ such that both matrices $A + aI$ and $A^T + aI$ have the strong Perron-Frobenius property (where I is the $n \times n$ identity matrix).

(ii) $A + aI$ is eventually positive for some $a \geq 0$.

(iii) $A^T + aI$ is eventually positive for some $a \geq 0$.

(iv) A is eventually exponentially positive.

(v) A^T is eventually exponentially positive.

They also showed that certain eventually nonnegative matrices are also eventually exponentially nonnegative. The notation $\mathrm{index}_0(A)$ is used to denote the size of the largest Jordan block for the eigenvalue 0, where $\mathrm{index}_0(A) := 0$ if 0 is not an eigenvalue of A.

Theorem 1.2.6 ([43, Theorem 3.7]). *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix with* $\mathrm{index}_0(A) \leq 1$. *Then A is an eventually exponentially nonnegative matrix.*

Noutsos and Tsatsomeros also studied the eigenstructure of the solutions to Equation (1.1):

Theorem 1.2.7 ([43, Theorem 3.14]). *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix. Then the following hold:*

(i) e^A and $(e^A)^T$ have the Perron-Frobenius property.

(ii) If $\rho(e^A)$ is a simple eigenvalue of e^A and $\rho(e^A) = e^{\rho(A)}$, then there exists $a_0 \geq 0$ such that

$$\lim_{k\to\infty} \frac{1}{\rho(A+aI)^k}(A+aI)^k = \mathbf{x}\mathbf{y}^T$$

for all $a > a_0$, where $\mathbf{x}$ and $\mathbf{y}$ are, respectively, right and left nonnegative eigenvectors of A corresponding to $\rho(A)$, satisfying $\mathbf{x}^T\mathbf{y} = 1$.

Generalizations of eventual positivity are not the only eventual properties to be studied. For a matrix property $\mathcal{P}$, the $n \times n$ matrix A is *eventually* $\mathcal{P}$ if there exists $k_0 \in \mathbb{Z}^+$ such that A^k has property $\mathcal{P}$ for all $k \geq k_0$. Digraph $G = (V, E)$ is *cyclically* r*-partite* if there exists a disjoint partition $V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_r = V$ of r nonempty subsets such that for each arc $(i, j) \in E$, there exists $\ell \in \{1, \ldots, r\}$ with $i \in V_\ell$ and $j \in V_{\ell+1}$ (with $r + 1$ interpreted as 1). For $r \geq 2$, matrix $A \in \mathbb{R}^{n \times n}$ (or sign pattern $\mathcal{A}$) is called r-cyclic if its associated digraph is cyclically r-partite. Hogben [35] introduced the notion of an eventually r-cyclic matrix. Eventually irreducible matrices were introduced in [53], where it was shown that a matrix that is both eventually irreducible and eventually nonnegative is in fact eventually positive. Hogben and Wilson [36] further studied eventually r-cyclic matrices, as well as eventually reducible matrices, and showed that if $A \in \mathbb{C}^{n \times n}$ is eventually reducible, then the reducibility index k_0 is at most n.

Corollary 1.2.8 ([36, Corollary 3.7])**.** *Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

(i) A is eventually reducible.

(ii) A^n and A^{n+1} are reducible.

(iii) For every integer $k \geq n$, A^k is reducible.

1.2.2 Sign patterns that require or allow property $\mathcal{P}$

Whether a sign pattern requires or allows property $\mathcal{P}$ has been studied for many properties related to generalizations of positivity and nonnegativity. In general, determining which sign patterns require property $\mathcal{P}$ is more tractable than the problem of determining those sign patterns that allow $\mathcal{P}$. See [29] for a general survey of results on various sign pattern problems. Sign patterns that require or allow various special eigenstructures have been studied (see, e.g., [38, 16, 5, 44, 14, 11, 12, 15]). The allows problem for the properties eventual nonnegativity and eventual exponential nonnegativity are wide open, whereas those sign patterns that require eventual positivity, eventual nonnegativity, and eventual exponential positivity have been completely characterized [19] and Chapter 2 of this dissertation presents results on sign patterns that require eventual exponential nonnegativity. Some progress has been made on the allows problem for eventual positivity (in [3]) and eventual exponential positivity (in Chapter 3 of this dissertation). These are discussed in more detail below.

A sign pattern that requires (respectively, allows) nilpotence necessarily requires (respectively, allows) eventual nonnegativity. It is clear that sign pattern $\mathcal{A}$ requires nilpotence if and only if $\mathcal{A}$ is permutationally similar to a strictly upper triangular sign pattern, i.e., that the digraph associated with $\mathcal{A}$ contains no loops or cycles of length 2 or more. The problem of characterizing those sign patterns that allow nilpotence (generally called potentially nilpotent sign patterns) is still open, despite being thoroughly studied. In [51], Yeh characterizes various star sign patterns (those patterns that have an associated digraph that is a star) as to whether or not they are potentially nilpotent, as well as characterizing which 7×7 tridiagonal sign patterns, $\mathcal{A} = [\alpha_{ij}]$, such that $\alpha_{ii} = 0$ and $\alpha_{ij} \neq 0$ if and only if $\alpha_{ji} \neq 0$ are potentially nilpotent. Eschenbach and Li [22] provide some necessary conditions for a sign pattern to be potentially nilpotent. The *minimum rank of sign pattern* $\mathcal{A}$ is $\mathrm{mr}(\mathcal{A}) = \min\{\mathrm{rank}(A) : A \in \mathcal{Q}(\mathcal{A})\}$. The *nilpotence index* of a nilpotent matrix is the minimum $k \in \mathbb{Z}^+$ such that $A^k = 0$ and the *nilpotence index of sign pattern* $\mathcal{A}$ is the minimum nilpotence index over all nilpotent realizations $A \in \mathcal{Q}(\mathcal{A})$. Eschenbach and Li characterized the potentially nilpotent sign patterns $\mathcal{A}$ that have nilpotence index 2 and $\mathrm{mr}(\mathcal{A}) = 1$. They also characterized the 3×3 potentially nilpotent sign patterns and determined almost all 4×4 potentially nilpotent sign patterns with nilpotence index 2 and conjectured that the remaining four sign patterns whose nilpotence index they could not determine did not have nilpotence index 2. This conjecture was confirmed in [21]. Much work has also been done on so called spectrally arbitrary sign patterns, i.e., those sign patterns that have a realization achieving every possible characteristic polynomial and therefore every possible spectrum (see, e.g., [16, 39, 5, 44, 14, 15]). It is well known that $A \in \mathbb{R}^{n \times n}$ is nilpotent if and only if 0 is the only eigenvalue of A, so a spectrally arbitrary sign pattern is potentially nilpotent.

Sign patterns that require eventual positivity, eventual nonnegativity, exponential positivity, or eventual exponential positivity have been characterized in [19].

Theorem 1.2.9 ([19, Theorem 2.3]). *The sign pattern $\mathcal{A}$ requires eventual positivity if and only if $\mathcal{A}$ is nonnegative and primitive.*

Theorem 1.2.10 ([19, Theorem 2.6]). *The sign pattern $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity if and only if for every s, t $(s \neq t)$ such that $\alpha_{st} = -$, $\mathcal{A}[\mathrm{In}(s)]$ and $\mathcal{A}[\mathrm{Out}(t)]$ require nilpotence.*

Theorem 1.2.11 ([19, Theorem 2.9]). *Let $\mathcal{A}$ be a square sign pattern. Then the following are equivalent.*

(i) $\mathcal{A}$ requires eventual exponential positivity.

(ii) $\mathcal{A}$ is irreducible and its off-diagonal entries are nonnegative.

(iii) $\mathcal{A}$ requires exponential positivity.

Sign patterns that allow eventual positivity (also called potentially eventually positive sign patterns) were studied in [3]. Several necessary or sufficient conditions are given for a sign pattern to allow eventual positivity.

Theorem 1.2.12 ([3, Theorem 2.1]). *Let $\mathcal{A}$ be a sign pattern such that $\mathcal{A}^+$ is primitive. Then $\mathcal{A}$ is potentially eventually positive.*

The minimum number of positive entries in a potentially eventually positive sign pattern was also established.

Corollary 1.2.13 ([3, Corollary 4.5]). *For $n \geq 2$, the minimum number of $+$ entries in an $n \times n$ potentially eventually positive sign pattern is $n + 1$.*

The 2×2 and 3×3 potentially eventually positive sign patterns are characterized in [3].

Theorem 1.2.14 ([3, Theorem 6.1]). *A 2×2 sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}^+$ is primitive.*

Theorem 1.2.15 ([3, Theorem 6.4]). *A 3×3 sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}^+$ is primitive or $\mathcal{A}$ is equivalent to a sign pattern of the form*

$$\mathcal{B} = \begin{bmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{bmatrix},$$

where $?$ is one of $0, +, -$ and $\ominus$ is one of $0, -$.

Matrix A is *power-positive* if $A^k > 0$ for some $k \in \mathbb{Z}^+$. Clearly eventually positive matrices are power-positive and so a sign pattern that requires (respectively, allows) eventual positivity also requires (respectively, allows) power-positivity. In [13], it is shown that sign pattern $\mathcal{A}$ requires power-positivity if and only if either $\mathcal{A}$ or $-\mathcal{A}$ requires eventual positivity. Furthermore, it is shown in [13] that sign pattern $\mathcal{A}$ allows power-positivity if and only if either $\mathcal{A}$ or $-\mathcal{A}$ allows eventual positivity.

The preprint [11] provides an analysis of the relationships between the classes of sign patterns that allow eventual positivity, nilpotence, eventual nonnegativity, strong eventual nonnegativity, and the semi-strong Perron-Frobenius property. The next theorem and the Euler diagram in Figure 1.1, which summarize these relationships, appear in [11] but were removed before publication as [12] when the focus was later narrowed. They are included here for completeness.

Theorem 1.2.16 ([11, Theorem 4.7])**.** *The Euler diagram in Figure 1.1 gives the relationship among the following eight classes of sign patterns (for patterns of order at least two):*

1. *potentially eventually positive sign patterns (PEP),*
2. *potentially strongly eventually nonnegative sign patterns (PSEN),*
3. *sign patterns that allow the semi-strong Perron-Frobenius property (PSSPF),*
4. *irreducible sign patterns (irreducible),*
5. *potentially eventually nonnegative sign patterns (PEN),*
6. *r-cyclic sign patterns (r-cyclic),*
7. *potentially nilpotent sign patterns (PN),*
8. *nonnegative sign patterns (nonnegative).*

1.3 Dissertation Organization

This dissertation is organized in the format of a dissertation containing journal papers. In the General Introduction the research problem and related background information are

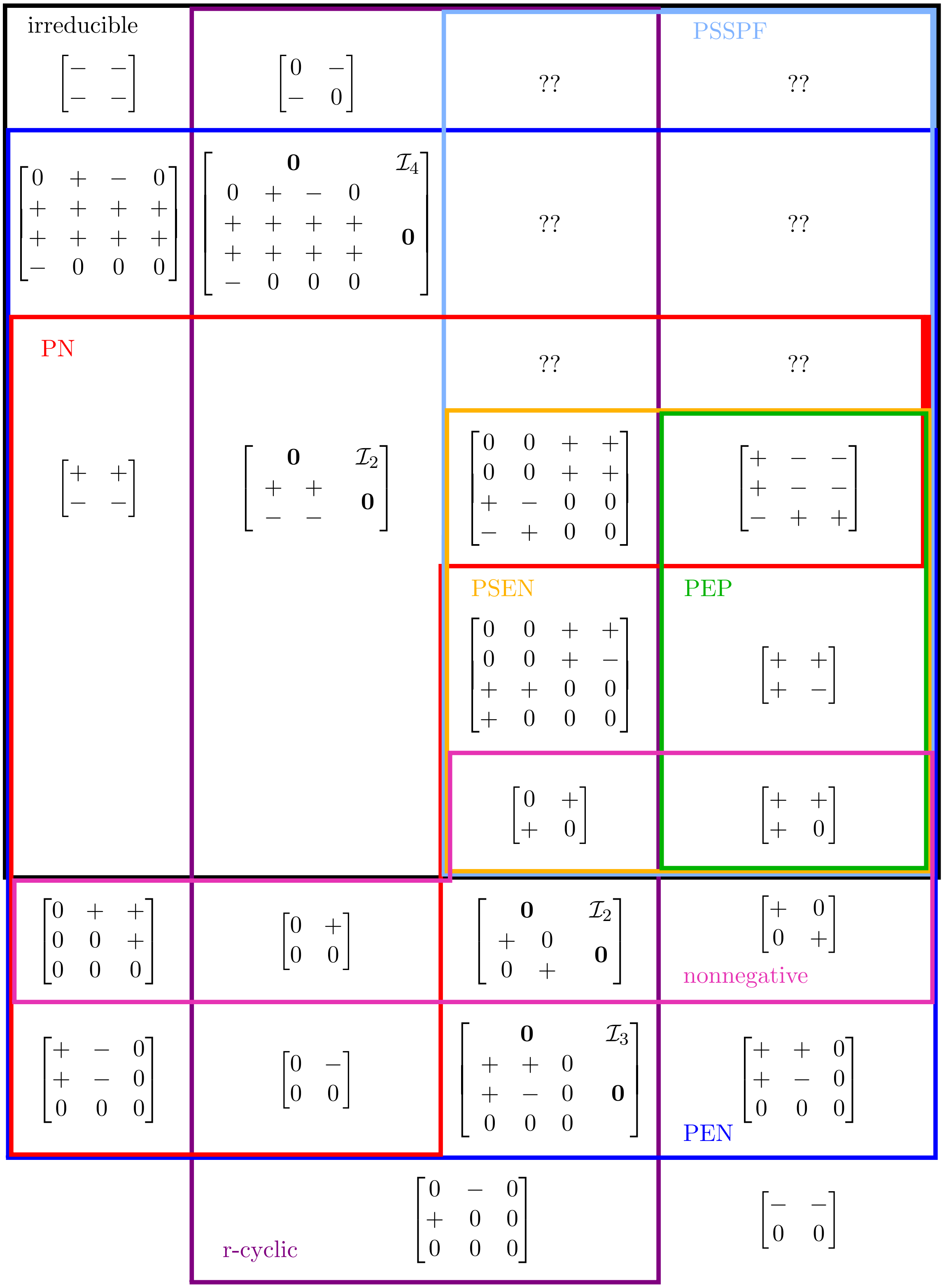

Figure 1.1 [11, Figure 4.1] An Euler diagram of classes of sign patterns related to potentially eventually nonnegative sign patterns, for patterns of order at least 2. A sign pattern in a given region is an example of a sign pattern belonging to that region. We do not have examples of sign patterns belonging to regions containing ??. It is conjectured that those regions are in fact empty (i.e., that $PSSPF = PSEN$).

presented. A review of literature on the subject is also given.

Chapter 2 contains the paper "Sign patterns that require eventual exponential nonnegativity" [20], submitted to the *Electronic Journal of Linear Algebra*. In the paper, we develop conditions necessary for a sign pattern to require eventual exponential nonnegativity. We also provide conditions sufficient for a sign pattern that is permutationally similar to an upper triangular sign pattern to require eventual exponential nonnegativity, as well as sufficient conditions for a sign pattern that requires eventual nonnegativity to also require eventual exponential nonnegativity.

Chapter 3 contains the paper "Potentially eventually exponentially positive sign patterns" [2], which was published in *Involve: A journal of mathematics*. In the paper, we introduce the study of potentially eventually exponentially positive sign patterns (sign patterns that allow eventual exponential positivity). It is shown that the set of sign patterns that allow eventual positivity and the set of potentially eventually exponentially positive sign patterns are not equal, i.e., that that the former is properly contained within the latter. We also characterize all 2×2 and 3×3 potentially eventually exponentially positive sign patterns. This paper was produced based on results from a Research Experiences for Undergraduates project held at Iowa State University in the summer of 2010. Drs. Catral and Hogben were the faculty mentors and C. Erickson was the graduate student research assistant on the project. This paper is included in this dissertation because it was C. Erickson's introduction to sign pattern problems involving the matrix exponential function. C. Erickson was the primary writer and one of the main researchers of the paper.

Chapter 4 summarizes results and discusses plans for future research.

Bibliography

[1] W.A. Adkins and M.G. Davidson. Putzer's algorithm for e^{At} via the Laplace transform. *Mathematics Magazine,* 83 (2010), no. 4, 267–275.

[2] M. Archer, M. Catral, C. Erickson, R. Haber, L. Hogben, X. Martinez-Rivera, and

A. Ochoa. Potentially eventually exponentially positive sign patterns. *Involve: A journal of mathematics*, 6 (2013), no. 3, 261–271.

[3] A. Berman, M. Catral, L.M. DeAlba, A. Elhashash, F. Hall, L. Hogben, I.-J. Kim, D.D. Olesky, P. Tarazaga, M.J. Tsatsomeros, P. van den Driessche. Sign patterns that allow eventual positivity. *Electronic Journal of Linear Algebra*, 19 (2010), 108–120.

[4] A. Berman and B. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.

[5] B.D. Bingham, D.D. Olesky, P. van den Driessche. Potentially nilpotent and spectrally arbitrary even cycle sign patterns. *Linear Algebra and its Applications*, 421 (2007), 24–44.

[6] R. Brualdi and D. Cvetković. *A Combinatorial Approach to Matrix Theory and its Applications*. CRC Press, Boca Raton, FL, 2009.

[7] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge, 1991.

[8] R.A. Brualdi and B.L. Shader. *Matrices of Sign-solvable Linear Systems*. Cambridge University Press, Cambridge, 1995.

[9] S. Carnochan Naqvi and J.J. McDonald. The combinatorial structure of eventually nonnegative matrices. *Electronic Journal of Linear Algebra*, 9 (2002), 255–269.

[10] S. Carnochan Naqvi and J.J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices. *Linear Algebra and its Applications*, 281 (2004), 245–258.

[11] M. Catral, C. Erickson, L. Hogben, D.D. Olesky, and P. van den Driessche. Sign patterns that allow strong eventual nonnegativity. Preprint version of [12].

[12] M. Catral, C. Erickson, L. Hogben, D.D. Olesky, and P. van den Driessche. Sign patterns that allow strong eventual nonnegativity. *Electronic Journal of Linear Algebra*, 23 (2012), 1–10.

[13] M. Catral, L. Hogben, D.D. Olesky, and P. van den Driessche. Sign patterns that require or allow power-positivity. *Electronic Journal of Linear Algebra*, 19 (2010), 121–128.

[14] M. Catral, D.D. Olesky, P. van den Driessche. Allow problems concerning spectral properties of sign pattern matrices: A survey. *Linear Algebra and its Applications*, 430 (2009), 3080–3094.

[15] M.S. Cavers and S.M. Fallat. Allow problems concerning spectral properties of patterns. *Electronic Journal of Linear Algebra*, 23 (2012), 731–754.

[16] J.H. Drew, C.R. Johnson, D.D. Olesky, P. van den Driessche. Spectrally arbitrary sign patterns. *Linear Algebra and its Applications*, 308 (2000), 121–137.

[17] A. Elhashash and D. Szyld. On general matrices having the Perron-Frobenius property. *Electronic Journal of Linear Algebra*, 17 (2008), 389–413.

[18] A. Elhashash and D. Szyld. Matrix functions preserving sets of generalized nonnegative matrices. *Electronic Journal of Linear Algebra*, 20 (2010), 673–690.

[19] E.M. Ellison, L. Hogben, and M.J. Tsatsomeros. Sign patterns that require eventual positivity or require eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19 (2010), 98–107.

[20] C. Erickson. Sign patterns that require eventual exponential nonnegativity. Submitted to *Electronic Journal of Linear Algebra*.

[21] C. Erickson and I.-J. Kim. On nilpotence indices of sign patterns. *Communications of the Korean Mathematical Society*, 25 (2010), no. 1, 11-18.

[22] C.A. Eschenbach and Z. Li. Potentially nilpotent sign pattern matrices. *Linear Algebra and its Applications*, 299 (1999), 81–99.

[23] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29 (1978), no. 1, 43–60.

[24] G. Frobenius. Über Matrizen aus positiven Elementen. *Sitzungsberichte der Akademie der Wissenschaften zu Berlin*, (1908), 471–476. Reprinted in *Gesammelte Abhandlungen*, vol. 3, pp. 404–409, J.-P. Serre, (editor). Springer-Verlag, Berlin-New York, 1968.

[25] G. Frobenius. Über Matrizen aus positiven Elementen II. *Sitzungsberichte der Akademie der Wissenschaften zu Berlin*, (1909), 514–518. Reprinted in *Gesammelte Abhandlungen*, vol. 3, pp. 410–414, J.-P. Serre, (editor). Springer-Verlag, Berlin-New York, 1968.

[26] G. Frobenius. Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Akademie der Wissenschaften zu Berlin*, (1912), 456–477. Reprinted in *Gesammelte Abhandlungen*, vol. 3, pp. 546–567, J.-P. Serre, (editor). Springer-Verlag, Berlin-New York, 1968.

[27] P.-L. Giscard, S.J. Thwaite, and D. Jaksch. Evaluating matrix functions by resummations on graphs: The method of path-sums. *SIAM Journal on Matrix Analysis and Applications*, 34 (2013), 445–469.

[28] G.H. Golub and C.F. Van Loan. *Matrix Computations* (third edition). Johns Hopkins University Press, Baltimore, 1996.

[29] F.J. Hall and Z. Li. Sign Pattern Matrices. In *Handbook of Linear Algebra* (second edition), L. Hogben (editor). Chapman & Hall/CRC Press, Boca Raton, 2014.

[30] D. Handelman. Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains. *Journal of Operator Theory*, 6 (1981), 55–74.

[31] T. Hawkins. Continued fractions and the origins of the Perron-Frobenius theorem. *Archive for History of Exact Sciences*, 62 (2008), no. 6, 655–717.

[32] D. Hershkowitz. The combinatorial structure of generalized eigenspaces–from nonnegative matrices to general matrices. *Linear Algebra and its Applications*, 302/303 (1999), 173–191.

[33] N. Higham. Functions of Matrices. In *Handbook of Linear Algebra* (second edition), L. Hogben (editor). Chapman & Hall/CRC Press, Boca Raton, 2014.

[34] N. Higham. *Functions of Matrices: Theory and Computation*. SIAM, Philadelphia, 2008.

[35] L. Hogben. Eventually cyclic matrices and a test for strong eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19 (2010), 129–140.

[36] L. Hogben and U. Wilson. Eventual properties of matrices. *Electronic Journal of Linear Algebra*, 23 (2012), 953–965.

[37] C.R. Johnson and P. Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8 (2004), 327–338.

[38] S.J. Kirkland, J.J. McDonald, and M.J. Tsatsomeros. Sign-patterns which require a positive eigenvalue. *Linear and Multilinear Algebra*, 41 (1996), no. 3, 199–210.

[39] J.J. McDonald, D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche. On the spectra of striped sign patterns. *Linear and Multilinear Algebra*, 51 (2003), no. 1, 39–48.

[40] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix. *SIAM Review*, 20 (1978), no. 4, 801–836.

[41] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review*, 45 (2003), no. 1, 3–49.

[42] D. Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Linear Algebra and its Applications*, 412 (2006), 132–152.

[43] D. Noutsos and M.J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM Journal on Matrix Analysis and Applications*, 30 (2008), 700–712.

[44] R. Pereira. Nilpotent matrices and spectrally arbitrary sign patterns. *Electronic Journal of Linear Algebra*, 16 (2007), 232–236.

[45] O. Perron. Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus. *Mathematische Annalen*, 64 (1907), no. 1, 1–76.

[46] O. Perron. Zur Theorie der Matrices. *Mathematische Annalen*, 64 (1907), no. 2, 248–263.

[47] E.J. Putzer. Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients. *The American Mathematical Monthly*, 73 (1966), 2–7.

[48] P.A. Samuelson. *Foundations of Economic Analysis*. Harvard University Press, Cambridge, 1947.

[49] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey. *Linear Algebra and its Applications*, 84 (1986), 161–189.

[50] P. Tarazaga, M. Raydan, A. Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Linear Algebra and its Applications*, 328 (2001), 57–68.

[51] L. Yeh. Sign pattern matrices that allow a nilpotent matrix. *Bulletin of the Australian Mathematical Society*, 53 (1996), no. 2, 189–196.

[52] B.G. Zaslavsky and J.J. McDonald. A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices with the properties of nonnegative matrices. *Linear Algebra and its Applications*, 372 (2003), 253–285.

[53] B.G. Zaslavsky and B.-S. Tam. On the Jordan form of an irreducible matrix with eventually non-negative powers. *Linear Algebra and its Applications*, 302-303 (1999), 303–330.

CHAPTER 2. SIGN PATTERNS THAT REQUIRE EVENTUAL EXPONENTIAL NONNEGATIVITY

A paper submitted to the *Electronic Journal of Linear Algebra*

Craig Erickson

Abstract

Sign patterns that require exponential nonnegativity are characterized. A set of conditions necessary for a sign pattern to require eventual exponential nonnegativity are established. It is shown that these conditions are also sufficient for an upper triangular sign pattern to require eventual exponential nonnegativity and it is conjectured that these conditions are both necessary and sufficient for any sign pattern to require eventual exponential nonnegativity. It is also shown that the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is $\frac{(n-1)(n-2)}{2} + 2$.

2.1 Introduction

A real square matrix is eventually exponentially nonnegative (positive) if there exists some $\tau_0 \geq 0$ such that $e^{\tau A}$ is an entrywise nonnegative (positive) matrix for all $\tau > \tau_0$. If $e^{\tau A}$ is entrywise nonnegative (positive) for all $\tau > 0$, then A is called exponentially nonnegative (positive). Ellison, Hogben, and Tsatsomeros [4] showed that a sign pattern $\mathcal{A}$ requires exponential positivity if and only if $\mathcal{A}$ requires eventual exponential positivity and characterized such sign patterns.

In Section 2.2, we establish some eigenstructure of eventually exponentially nonnegative matrices, which is analogous to the eigenstructure of eventually exponentially positive matrices established in [1]. In Section 2.3, we develop a set of necessary conditions for a sign pattern to require eventual exponential nonnegativity and conjecture that these conditions are also sufficient for a sign pattern to require eventual exponential nonnegativity. We utilize the Hermite interpolation method for evaluating $e^{\tau A}$ in confirming this conjecture in the case that the sign pattern is permutationally similar to an upper triangular sign pattern. The remainder of the current section contains definitions, notation, and results cited throughout this paper.

2.1.1 Definitions and notation

A *sign pattern* $\mathcal{A}$ is a matrix with entries in $\{+, -, 0\}$. The class of all real matrices for which $\text{sgn}(A) = \mathcal{A}$ is called the *qualitative class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, and a *realization* of $\mathcal{A}$ is a real matrix $A \in \mathcal{Q}(\mathcal{A})$. Sign pattern $\mathcal{A}$ *allows* property $\mathcal{P}$ if there exists a realization $A \in \mathcal{Q}(\mathcal{A})$ which has property $\mathcal{P}$. The sign pattern $\mathcal{A}$ *requires* property $\mathcal{P}$ if every realization $A \in \mathcal{Q}(\mathcal{A})$ has property $\mathcal{P}$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix, we denote the (i, j)-entry of A^ℓ by $a_{ij}^{(\ell)}$ and use similar notation for the (i, j)-entry of the power of a sign pattern.

Definition 2.1.1. An $n \times n$ matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is

- *eventually nonnegative (positive)* if there exists $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k \geq 0$ ($A^k > 0$), where the inequality is entrywise.
- *exponentially nonnegative (positive)* if for all $\tau > 0$, $e^{\tau A} = \displaystyle\sum_{k=0}^{\infty} \frac{\tau^k A^k}{k!} \geq 0$ ($e^{\tau A} > 0$).
- *eventually exponentially nonnegative (positive)* if there exists $\tau_0 \geq 0$ such that for all $\tau > \tau_0$, $e^{\tau A} = \displaystyle\sum_{k=0}^{\infty} \frac{\tau^k A^k}{k!} \geq 0$ ($e^{\tau A} > 0$).
- *essentially nonnegative* if $a_{ij} \geq 0$ for all $i \neq j$.

Another (equivalent) definition of an eventually exponentially nonnegative matrix is: for all $i, j \in \{1, 2, \ldots, n\}$, if $(e^{\tau A})_{ij} \neq 0$ then $\lim_{\tau\to\infty}(e^{\tau A})_{ij} > 0$. The *dominant term* (or *dominating term*) of $(e^{\tau A})_{ij}$ is the term which determines $\lim_{\tau\to\infty}(e^{\tau A})_{ij}$. So if A is eventually exponentially nonnegative, either $(e^{\tau A})_{ij} = 0$ or the dominating term for the (i, j)-entry is positive for all $i, j \in \{1, \ldots, n\}$.

Definition 2.1.2. Given an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, we denote by $\mathcal{A}_{D(+)} = [\widehat{\alpha}_{ij}]$ the $n \times n$ sign pattern such that $\widehat{\alpha}_{ij} = \alpha_{ij}$ for $i \neq j$ and $\widehat{\alpha}_{ii} = +$ for $i, j \in \{1, \ldots, n\}$. $\mathcal{A}_{D(0)}$ and $\mathcal{A}_{D(-)}$ are defined analogously, with zero and negative diagonal, respectively.

A square matrix (or sign pattern) A is called *reducible* if there exists some permutation matrix P such that PAP^T is upper block triangular with square diagonal blocks. If no such permutation exists, then A is *irreducible.*

We say that the digraph $G = (V, E)$ is *associated with* the matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, or that A is associated with G, when $a_{ij} \neq 0$ if and only if $(i, j) \in E$. Therefore A has a nonzero diagonal element a_{ii} if and only if G contains the loop (i, i). We denote the weighted digraph associated with A by $\Gamma(A)$, that is, (i, j) is an arc in $\Gamma(A)$ with weight a_{ij} if and only if $a_{ij} \neq 0$. Likewise, $\Gamma(\mathcal{A})$ is the signed digraph associated with $\mathcal{A}$.

Let $G = (V, E)$ be a digraph and $u, v \in V$. A vertex $u \in V$ *has access to* vertex $v \in V$ if there exists a u-v walk in G or $u = v$. For $v \in V$ we define $\mathrm{In}(v)$ to be the set of vertices which have access to v and define $\mathrm{Out}(v)$ to be the set of vertices to which v has access. The *product of a u-v walk* in $\Gamma(A)$ is the product of the weights of the arcs in the walk. The *sign of a u-v walk* in $\Gamma(A)$ (or $\Gamma(\mathcal{A})$), is the sign of the product of the u-v walk. An *arc-positive walk (path)* is a walk (path) that uses only positive arcs. An arc-positive walk W may pass through a vertex that has a negative loop, so long as the negative loop is not included in W.

The *spectral abscissa* of matrix A is defined as $\alpha(A) := \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spec}(A)\}$. Eigenvalue $\gamma \in \mathrm{spec}(A)$ is a *rightmost eigenvalue of* A if $\mathrm{Re}(\gamma) = \alpha(A)$. In [1] an eigenvalue is called a rightmost eigenvalue if it is real and equal to the spectral abscissa, we allow for a rightmost eigenvalue to be complex (in which case it would not be the unique rightmost eigenvalue). Note that if $\alpha(A)$ is an eigenvalue of A, then it is a rightmost eigenvalue of A. Furthermore, if the spectral radius $\rho(A)$ is an eigenvalue of A, then $\rho(A) = \alpha(A)$ is a rightmost eigenvalue of A.

It is well known that e^{λ} is an eigenvalue of e^A if and only if λ is an eigenvalue of A. Suppose that A is eventually exponentially nonnegative. Then $(e^A)^k = e^{kA} \geq 0$ for large enough integers k, therefore e^A is eventually nonnegative and either e^A is nilpotent (which is not possible) or both e^A and $(e^A)^T = e^{A^T}$ have the Perron-Frobenius property (i.e., $\rho(e^A) \in \mathrm{spec}(e^A)$, and $\rho(e^A)$ has corresponding nonnegative left and right eigenvectors).

2.1.2 Results cited

Theorem 2.1.3. [5, p. 323] *Let* $A, B \in \mathbb{R}^{n\times n}$*. If* λ *is a simple eigenvalue of* A *and* $A(\varepsilon) = A + \varepsilon B$*, then in a neighborhood of the origin there exist differentiable (and hence continuous) functions* $\lambda(\varepsilon)$ *and* $\mathbf{x}(\varepsilon)$*, with* $\lambda(0) = \lambda$ *and* $\mathbf{x}(0) = \mathbf{x}$*, such that* $A(\varepsilon)\mathbf{x}(\varepsilon) = \lambda(\varepsilon)\mathbf{x}(\varepsilon)$*.*

Lemma 2.1.4. [7, Lemma 2.2] *Let* $A \in \mathbb{R}^{n\times n}$*. The following are equivalent:*

(i) A *is eventually exponentially nonnegative.*

(ii) There exists $a \in \mathbb{R}$ *such that* $A + aI$ *is eventually exponentially nonnegative.*

(iii) For all $a \in \mathbb{R}$*,* $A + aI$ *is eventually exponentially nonnegative.*

Theorem 2.1.5. [4, Theorem 2.9] *Let* $\mathcal{A}$ *be a square sign pattern. Then the following are equivalent.*

(i) $\mathcal{A}$ *requires eventual exponential positivity.*

(ii) $\mathcal{A}$ *is irreducible and its off-diagonal entries are nonnegative.*

(iii) $\mathcal{A}$ *requires exponential positivity.*

Note that for $\beta \geq 0$, since A and βI commute,

$$e^{\tau A} = e^{-\tau\beta I}e^{\tau(A+\beta I)} = e^{-\tau\beta}e^{\tau(A+\beta I)},$$

so A may have negative diagonal entries and be (eventually) exponentially nonnegative. In fact, the sign pattern $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$ requires (eventual) exponential positivity by Theorem 2.1.5.

Theorem 2.1.6. [4, Theorem 2.6] *The sign pattern* $\mathcal{A} = [\alpha_{ij}]$ *requires eventual nonnegativity if and only if for every* s, t *such that* $\alpha_{st} = -$*,* $\mathcal{A}[\mathrm{In}(s)]$ *and* $\mathcal{A}[\mathrm{Out}(t)]$ *require nilpotence.*

Theorem 2.1.6 can be rephrased in graph theory language as follows:

Theorem 2.1.7. *The sign pattern* $\mathcal{A} = [\alpha_{ij}]$ *requires eventual nonnegativity if and only if for every* s, t *such that* $\alpha_{st} = -$*, every directed walk in* $\Gamma(\mathcal{A})$ *that contains the arc* (s, t) *is a path.*

2.2 Eventually exponentially nonnegative matrices

This section introduces some results on eventually exponentially nonnegative matrices that are interesting in themselves. We will use them primarily as tools in proving sign pattern results in Section 2.3.

Observation 2.2.1. *Let A be a block triangular matrix with square diagonal blocks $A_1, A_2, \ldots, A_m$. Then $e^{\tau A}$ is block triangular and the diagonal blocks of $e^{\tau A}$ are $e^{\tau A_1}, e^{\tau A_2}, \ldots, e^{\tau A_m}$.*

The following lemma appears as an aside in [7] and is a result of the application of [6, Theorem 1.36] to the matrix exponential function.

Lemma 2.2.2. *Let $A \in \mathbb{R}^{n \times n}$. Then $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of e^A if and only if $\mathbf{x}$ is an eigenvector of A.*

In [1], it was shown that matrix A is eventually exponentially positive if and only if the spectral abscissa of A is a simple (real) eigenvalue of A with corresponding positive left and right eigenvectors. Eventually exponentially nonnegative matrices also have a special eigenstructure.

Theorem 2.2.3. *Let $A \in \mathbb{R}^{n \times n}$ be eventually exponentially nonnegative. Then the spectral abscissa of A is an eigenvalue with corresponding nonnegative left and right eigenvectors. Equivalently, A has a real rightmost eigenvalue with corresponding nonnegative left and right eigenvectors.*

Proof. Since A is eventually exponentially nonnegative, there exists $\tau_0 \geq 0$ such that for all $\tau > \tau_0$, $e^{\tau A} \geq 0$. Let $k_0 = \lceil \tau_0 \rceil$, then for all $k \in \mathbb{Z}^+$, $k \geq k_0$, $\left(e^A\right)^k = e^{kA} \geq 0$ and hence e^A is eventually nonnegative. Therefore $\rho\left(e^A\right)$ is a (nonzero) eigenvalue of e^A with corresponding nonnegative left and right eigenvectors. By Lemma 2.2.2, there exists $\mu \in \operatorname{spec}(A)$ such that $e^\mu = \rho\left(e^A\right)$ and μ has corresponding nonnegative left and right eigenvectors. Moreover, since

$$\left|e^{a+ib}\right| = e^a \left|\cos(b) + i\sin(b)\right| = e^a \left(\cos^2(b) + \sin^2(b)\right) = e^a,$$

$\rho(e^A) = e^{\mathrm{Re}(\mu)} \geq e^{\mathrm{Re}(\lambda)}$ for all $\lambda \in \operatorname{spec}(A)$, and since $e^z : \mathbb{R} \to \mathbb{R}$ is nonnegative and strictly monotonically increasing, this implies that $\mathrm{Re}(\mu) \geq \mathrm{Re}(\lambda)$ for all $\lambda \in \operatorname{spec}(A)$. Therefore $\mathrm{Re}(\mu)$ is the spectral abscissa of A, and μ has nonnegative left and right eigenvectors.

Moreover, since A is real and μ has corresponding nonnegative (and therefore real) left and right eigenvectors, μ must be real. □

However, as the following example shows, a matrix can have a simple real rightmost eigenvalue with corresponding nonnegative left and right eigenvectors without being eventually exponentially nonnegative.

Example 2.2.4. Let $A = \begin{bmatrix} 0 & 4 & 1 & 0 & 0 \\ 9 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$. Then $\operatorname{spec}(A) = \{-6, -1, 0, 1, 6\}$ and $\begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 6 & 4 & 1 & 1 & 0 \end{bmatrix}^T$ are right and left eigenvectors, respectively, corresponding to the eigenvalue 6. However, the $(1,5)$-entry of $e^{\tau A}$ is

$$-\frac{1}{35}e^{-6\tau} + \frac{e^{-\tau}}{10} - \frac{e^{\tau}}{14},$$

which is negative for $\tau > 0$.

2.3 Sign patterns that require exponential nonnegativity or eventual exponential nonnegativity

In [4] Ellison, Hogben, and Tsatsomeros showed that a sign pattern requires eventual exponential positivity if and only if it requires exponential positivity. This is not the case for nonnegativity. If a matrix A is exponentially nonnegative, then A is also eventually exponentially nonnegative; therefore the class of sign patterns that require exponential nonnegativity is contained in the class of sign patterns that require eventual exponential nonnegativity. However, as Example 2.3.3 below shows, a sign pattern can require eventual exponential nonnegativity without requiring exponential nonnegativity, so the two classes of sign patterns are not equivalent. It is well known that matrix A is exponentially nonnegative if and only if A has no negative off-diagonal entries (i.e., is essentially nonnegative, see, e.g., [2, Chapter 6, Theorem (3.12)], [7]). This immediately leads to a classification of those sign patterns that require exponential nonnegativity.

Proposition 2.3.1. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. $\mathcal{A}$ requires exponential nonnegativity if and only if $\alpha_{ij} \neq -$ for $i \neq j$.*

So if $\mathcal{A}$ is irreducible and requires exponential nonnegativity, then $\mathcal{A}$ requires (eventual) exponential positivity by Theorem 2.1.5.

It is clear from the power series definition of $e^{\tau A}$ that for a sign pattern to require eventual exponential nonnegativity, the following necessary condition must hold.

Observation 2.3.2. *Let $\mathcal{A}$ be a sign pattern that requires eventual exponential nonnegativity. If $\Gamma(\mathcal{A})$ has a negative i-j walk of length k, then there must exist a positive i-j walk of length greater than k.*

While it is true that if $\mathcal{A}$ requires eventual exponential nonnegativity, then $\mathcal{A}_{D(+)}$ allows eventual exponential nonnegativity; as the following example shows, it is not necessarily the case that $\mathcal{A}_{D(+)}$ requires eventual exponential nonnegativity.

Example 2.3.3. Consider the matrix

$$A = \begin{bmatrix} 0 & a_{12} & -a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

where $a_{12}, a_{13}, a_{23} > 0$. Clearly

$$e^{\tau A} = \begin{bmatrix} 1 & \tau a_{12} & -\tau a_{13} + \tau^2 a_{12} a_{23} \\ 0 & 1 & \tau a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

and for $\tau > \dfrac{a_{13}}{a_{12}a_{23}}$, we have $e^{\tau A} \geq 0$, therefore $\mathcal{A} = \operatorname{sgn}(A)$ requires eventual exponential nonnegativity. However, the matrix

$$A = \begin{bmatrix} 1 & 1 & -10 \\ 0 & \frac{1}{10} & 1 \\ 0 & 0 & \frac{1}{10} \end{bmatrix},$$

which is in $\mathcal{Q}(\mathcal{A}_{D(+)})$, is not eventually exponentially nonnegative. The $(1,3)$-entry of $e^{\tau A}$ is

$$\frac{10e^{\tau/10}\left(80 - 80e^{9\tau/10} - 9\tau\right)}{81}$$

which is negative for $\tau > 0$. Therefore $\mathcal{A}_{D(+)}$ does not require eventual exponential nonnegativity.

In [4] it was shown that a sign pattern requires exponential positivity if and only if it requires eventual exponential positivity. The preceding example also illustrates that a sign pattern may require eventual exponential nonnegativity without requiring exponential nonnegativity.

The following proposition gives a condition which is sufficient for the sign pattern $\mathcal{A}$ to require eventual exponential nonnegativity.

Proposition 2.3.4. *Let $\mathcal{A}$ be an $n \times n$ sign pattern such that*

1. *$\mathcal{A}$ requires eventual nonnegativity, and*
2. *if there is a negative (directed) s-t walk of length k in $\Gamma(\mathcal{A})$, then there exists an $\ell > k$ such that every s-t walk of length ℓ is positive.*

Then $\mathcal{A}$ requires eventual exponential nonnegativity.

Proof. Let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Then, denoting the entries of A^m by $a_{ij}^{(m)}$, we have

$$\left(e^{\tau A}\right)_{ij} = \begin{cases} 1 + \tau a_{ii} + \dfrac{\tau^2}{2} a_{ii}^{(2)} + \dfrac{\tau^3}{3!} a_{ii}^{(3)} + \cdots & \text{if } j = i, \\ \tau a_{ij} + \dfrac{\tau^2}{2} a_{ij}^{(2)} + \dfrac{\tau^3}{3!} a_{ij}^{(3)} + \cdots & \text{if } j \neq i. \end{cases}$$

Suppose that $a_{st}^{(m)} < 0$ for some $m \in \mathbb{Z}^+$ and let m_0 be the greatest such integer (which exists since $\mathcal{A}$ requires eventual nonnegativity). Then by hypothesis, there exists some $\ell > m_0$ such that $a_{st}^{(\ell)} > 0$. Denote the degree ℓ Maclaurin polynomial for $\left(e^{\tau A}\right)_{st}$ by $p_\ell(\tau)$. This polynomial has a finite number of roots and a positive leading coefficient, therefore there exists $\tau_0(s,t) \geq 0$ such that $p_\ell(\tau) > 0$ for all $\tau > \tau_0(s,t)$ (namely, $\tau_0(s,t) = 0$ if all the roots of $p_\ell(\tau)$ are nonpositive, otherwise $\tau_0(s,t)$ is the greatest real positive root of $p_\ell(\tau)$). Then $\left(e^{\tau A}\right)_{st} = p_\ell(\tau) + r(\tau)$, where $r(\tau) \geq 0$ for all $\tau > 0$, and hence $\left(e^{\tau A}\right)_{st} > 0$ for $\tau > \tau_0(s,t)$.

If $a_{st}^{(m)} \geq 0$ for all $m \in \mathbb{Z}^+$, define $\tau_0(s,t) = 0$. Then $e^{\tau A} \geq 0$ for $\tau \geq \max\limits_{1 \leq s,t \leq n} \{\tau_0(s,t)\}$ and $\mathcal{A}$ requires eventual exponential nonnegativity. □

For matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ (or an $n \times n$ sign pattern) with associated digraph $\Gamma(A) = (V, E)$ let $\widehat{V}(s,t) := \{v \in V : v \in \operatorname{Out}(s) \cap \operatorname{In}(t)\}$. Then the *embedding* $\widetilde{A[\widehat{V}]} = [\widetilde{a}_{ij}]$ of A is the

$n \times n$ matrix defined by $\widetilde{a}_{ij} = a_{ij}$ if $i, j \in \widehat{V} = \widehat{V}(s,t)$ and $\widetilde{a}_{ij} = 0$ otherwise. Note that the (i,j)-entry of $e^{\tau A}$ is only affected by the nonzero entries in $\widetilde{A[\widehat{V}]}$ (where $\widehat{V} = \widehat{V}(i,j)$), that is,

$$\left(e^{\tau A}\right)_{ij} = \left(e^{\tau \widetilde{A[\widehat{V}]}}\right)_{ij}$$

for $i, j \in \{1, \ldots, n\}$.

Another method to calculate the matrix exponential is by use of an interpolating polynomial. We make use of the Hermite interpolation formula from [6, Chapter 1], applied to the matrix exponential function, $f(A) = e^{\tau A}$, which is reproduced below for the reader's convenience. Let $A \in \mathbb{R}^{n \times n}$ have m distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. The Hermite interpolation conditions are: $p^{(k)}(\lambda_i) = f^{(k)}(\lambda_i)$ for $0 \le k \le n_i - 1$ and $1 \le i \le m$ and the Hermite interpolating polynomial $p(z)$ is given by

$$p(z) = \sum_{i=1}^{m} \left[\left(\sum_{k=0}^{n_i-1} \frac{1}{k!} \phi_i^{(k)}(\lambda_i)(z-\lambda_i)^k \right) \prod_{j \ne i} (z-\lambda_j)^{n_j} \right], \tag{2.1}$$

where

$$\phi_i(z) = \frac{e^{\tau z}}{\prod_{j \ne i}(z-\lambda_j)^{n_j}}$$

and n_i is the multiplicity of λ_i as a root of the minimal polynomial of A, i.e., the size of the largest Jordan block associated with λ_i in the Jordan canonical form of A. Then $e^{\tau A} = p(A)$. However, as noted by Higham in [6, Remark 1.5], it is often convenient to use a higher degree interpolating polynomial $q(z)$, for example, replacing n_i in (2.1) with the multiplicity of λ_i as a root of the characteristic polynomial rather than the minimal polynomial. This is allowed since as long as $p(z)$ divides $q(z)$, $q(z)$ also satisfies the Hermite interpolation conditions.

It is clear that the (s,t)-entry of a power of A is affected only by the entries associated with $\widehat{V} = \widehat{V}(s,t)$, so the dominating term of the (s,t)-entry of A is the same as the dominating term of the (s,t)-entry of $\widetilde{A} = \widetilde{A[\widehat{V}]}$. Let $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots$ be the distinct eigenvalues of the principal submatrix $A[\widehat{V}]$ with $\widehat{n}_1, \widehat{n}_2, \ldots$ the multiplicities of $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots$ as roots of the minimal polynomial of $A[\widehat{V}]$, respectively. Note that in the Hermite interpolation formula, τ appears only in $\phi_i^{(k)}(\widehat{\lambda}_i)$, for $0 \le k \le \widehat{n}_i - 1$ and $1 \le i \le m$. Furthermore, for $\tau > 0$, $e^{\tau\mu_1} \ge e^{\tau\mu_2}$ if and only if $\mathrm{Re}(\mu_1) \ge \mathrm{Re}(\mu_2)$. Thus, if the principal submatrix $A[\widehat{V}(s,t)]$ has a unique rightmost eigenvalue, which we denote by $\widehat{\gamma} = \widehat{\lambda}_\nu$, then the dominating term of the (s,t)-entry of $e^{\tau A}$ is precisely the (s,t)-entry of

$$\left(\sum_{k=0}^{\widehat{n}_\nu - 1} \frac{1}{k!} \phi_\nu^{(k)}(\widehat{\gamma})(\widetilde{A} - \widehat{\gamma} I)^k\right) \prod_{j \neq \nu} (\widetilde{A} - \widehat{\lambda}_j I)^{\widehat{n}_j}. \tag{2.2}$$

Moreover, $\widehat{\gamma}$ is real, and hence $\phi_\nu(\widehat{\gamma}) = \dfrac{e^{\tau\widehat{\gamma}}}{\prod_{j\neq\nu}(\widehat{\gamma} - \widehat{\lambda}_j)^{\widehat{n}_j}}$ is real and positive since $\widehat{\gamma} \geq \mathrm{Re}(\widehat{\lambda}_j)$ for $\widehat{\lambda}_j \in \mathrm{spec}(A[\widehat{V}(s,t)])$ and any complex eigenvalues of $A[\widehat{V}(s,t)]$ come in conjugate pairs. Furthermore,

$$\phi_\nu'(\widehat{\gamma}) = \tau \cdot \phi_\nu(\widehat{\gamma}) - \left(\frac{\phi_\nu(\widehat{\gamma})}{\prod_{j\neq\nu}(\widehat{\gamma} - \widehat{\lambda}_j)^{\widehat{n}_j}}\right) \frac{d}{dz}\left[\prod_{j\neq\nu}(z - \widehat{\lambda}_j)^{\widehat{n}_j}\right]\Big|_{z=\widehat{\gamma}} = \phi_\nu(\widehat{\gamma})\,(\tau - K),$$

where K is a constant. Therefore $\phi_\nu'(\widehat{\gamma}) \to \infty$ as $\tau \to \infty$. It is apparent that $\phi_\nu^{(k)}(\widehat{\gamma})$ is a k-th degree polynomial in τ with leading coefficient $\phi_\nu(\widehat{\gamma}) > 0$ and hence $\phi_\nu^{(k)}(\widehat{\gamma}) \to \infty$ as $\tau \to \infty$. Since $\phi_\nu^{(k)}(\widehat{\gamma})$ is a k-th degree polynomial (in τ) with leading coefficient $\phi_\nu(\widehat{\gamma})$, the dominating term of the (s,t)-entry of (2.2) is the (s,t)-entry of

$$\frac{e^{\tau\widehat{\gamma}}\tau^{\widehat{n}_\nu - 1}}{(n_\nu - 1)!\prod_{j\neq\nu}(\widehat{\gamma} - \widehat{\lambda}_j)^{\widehat{n}_j}}(\widetilde{A} - \widehat{\gamma} I)^{\widehat{n}_\nu - 1}\prod_{j\neq\nu}(\widetilde{A} - \widehat{\lambda}_j I)^{\widehat{n}_j}.$$

Combining the fact that an eventually exponentially nonnegative matrix has a (real) rightmost eigenvalue (Theorem 2.2.3) with the above Hermite interpolation analysis, we have the following observation.

Observation 2.3.5. *$A \in \mathbb{R}^{n\times n}$ is eventually exponentially nonnegative if and only if for $1 \leq s,t \leq n$ (i) the (s,t)-entry of $e^{\tau A}$ is 0 or (ii) the (s,t)-entry of*

$$(\widetilde{A} - \widehat{\gamma} I)^{\widehat{n}_\nu - 1}\prod_{j\neq\nu}(\widetilde{A} - \widehat{\lambda}_j I)^{\widehat{n}_j}$$

is positive.

2.3.1 Necessary conditions

In this section, we discuss the following questions: What properties does every sign pattern that requires eventual exponential nonnegativity have? What type of structure prohibits a sign pattern from requiring eventual exponential nonnegativity? We establish conditions that are necessary for a sign pattern to require eventual exponential nonnegativity.

A sign pattern is called *acyclic* if there are no (directed) cycles or loops in $\Gamma(\mathcal{A})$. An acyclic sign pattern requires nilpotence, and therefore requires eventual nonnegativity. Proposition 2.3.4 leads to the following result.

Corollary 2.3.6. *Let $\mathcal{A}$ be an acyclic sign pattern. $\mathcal{A}$ requires eventual exponential nonnegativity if and only if for any negative u-v path of length k, there exists a positive u-v path of length greater than k.*

Furthermore, an acyclic sign pattern allows eventual exponential nonnegativity if and only if it requires eventual exponential nonnegativity.

Lemma 2.3.7. *Let $C = [c_{ij}]$ be an $n \times n$ matrix with entries in $\{-1, 0, 1\}$ such that $\Gamma(C)$ is a (directed) n-cycle. Then the characteristic polynomial of C is $x^n + (-1)^{m+1}$, where m is the number of -1 entries in C.*

Proof. Without loss of generality, suppose that $c_{ij} \neq 0$ implies $j \equiv i + 1 \pmod{n}$. Then by cofactor expansion along the first column, $\det(C) = (-1)^{n+1}(-1)^m$. Since each $k \times k$ principal minor of C is 0 for $k < n$, the characteristic polynomial is $p_C(x) = x^n + (-1)^n \det(C) = x^n + (-1)^n(-1)^{n+1}(-1)^m = x^n + (-1)^{m+1}$. □

Proposition 2.3.8. *Let $\mathcal{C}$ be an $n \times n$ sign pattern such that $\Gamma(\mathcal{C})$ is a (directed) n-cycle, with $n \geq 2$. $\mathcal{C}$ requires eventual exponential nonnegativity if and only if $\mathcal{C}$ is nonnegative.*

Proof. If $\mathcal{C}$ is nonnegative, then $\mathcal{C}$ requires (eventual) exponential positivity by Theorem 2.1.5 and therefore $\mathcal{C}$ requires eventual exponential nonnegativity.

Suppose that $\mathcal{C}$ has at least one negative entry and let $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ be the characteristic matrix of $\mathcal{C}$, that is, $C \in Q(\mathcal{C})$ and $c_{ij} \in \{-1, 0, 1\}$. If there are an odd number of negative entries in C, then the characteristic polynomial of C is $x^n + 1$ and the eigenvalues of C are the n roots of -1; therefore the spectral abscissa of C is not an eigenvalue and C is not eventually exponentially nonnegative by Theorem 2.2.3.

If there are an even number of negative entries in C, then the characteristic polynomial of C is $x^n - 1$ and the eigenvalues of C are the n roots of unity. Therefore 1 is the spectral abscissa of C. Suppose that $C\mathbf{x} = \mathbf{x}$, where $\mathbf{x} \geq 0$. Then

$$\begin{aligned} c_{12}x_1 &= x_2 \\ c_{23}x_2 &= x_3 \\ &\vdots \\ c_{n1}x_n &= x_1 \end{aligned}$$

and since $c_{i,i+1} = -1$ for some $i \in \{1, 2, \ldots, n\}$, this implies $x_i = 0$ for $i = 1, 2, \ldots, n$ and hence C does not have a nonnegative eigenvector corresponding to a real rightmost eigenvalue. Therefore C is not eventually exponentially nonnegative by Theorem 2.2.3. □

Proposition 2.3.9. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If $\Gamma(\mathcal{A})$ has a cycle of length at least 2 that contains a negative arc, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let $\mathcal{A}$ be as prescribed and let $W = (i_1, i_2, \ldots, i_k, i_1)$ denote a cycle of length at least 2 where $\alpha_{st} = -$ for some $s = i_j$, $t = i_{j+1}$ (with $i_{k+1} = i_1$). Consider the matrix C obtained from the characteristic matrix of $\mathcal{A}$ by setting all of the entries not associated with W equal to zero. For $\varepsilon > 0$, consider $A(\varepsilon) = C + \varepsilon B \in Q(\mathcal{A})$. As in the proof of Proposition 2.3.8, C does not have nonnegative left and right eigenvectors for a real rightmost eigenvalue. Therefore by Theorem 2.1.3, for small enough ε, $A(\varepsilon)$ does not have nonnegative left and right eigenvectors for a real rightmost eigenvalue and hence is not eventually exponentially nonnegative by Theorem 2.2.3. □

Hence if an $n \times n$ sign pattern $\mathcal{A}$ is irreducible (with $n \geq 2$) one of two things is true: either $\mathcal{A}$ is essentially nonnegative and therefore requires exponential positivity by Theorem 2.1.5, or $\mathcal{A}$ has an off-diagonal negative entry and therefore does not require eventual exponential nonnegativity by Proposition 2.3.9.

The proof of the following result closely follows that of [4, Theorem 2.5].

Proposition 2.3.10. *Let the $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ require eventual exponential nonnegativity. Then there exists no walk in $\Gamma(\mathcal{A})$ which includes both a negative arc (s,t), $s \neq t$, and either a positive loop or arc-positive cycle.*

Proof. We proceed by way of contradiction. Let $\mathcal{A}$ be as prescribed. Without loss of generality, suppose there exists a walk W

$$(s = 1, t = 2, 3, \ldots, u, \ldots, v, \ldots, \ell = u)$$

in which the cycle $(u, \ldots, v, \ldots, u)$ contains no negative arc (note that it is possible that $u = 2$ and/or $v = \ell = u$). Consider the matrix C obtained from the characteristic matrix of $\mathcal{A}$ by setting all entries to zero except those associated with the walk W. If we let $U = \{1, \ldots, u-1\}$ and $V = \{u, \ldots, v, \ldots, \ell - 1\}$, then C has the block form

$$C = \begin{bmatrix} C[U] & C[U,V] & 0 \\ 0 & C[V] & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $C[U]$ is nilpotent $\rho(C[V]) = \rho(C)$, denote this by ρ. Since $C[V]$ is nonnegative and irreducible, by the Perron-Frobenius theorem, $\rho > 0$. Moreover, there exists $\mathbf{v} \in \mathbb{R}^n$ such that $C\mathbf{v} = \rho\mathbf{v}$ and $\mathbf{v}[V] > 0$. Working backwards from u, we have $v_i \neq 0$ for $i = 1, \ldots, u-1$, and $v_k < 0$, where k is the greatest index in $\{1, \ldots, u-1\}$ such that $c_{k,k+1} = -1$. Note that ρ is a simple eigenvalue of C, and therefore is a (real) rightmost eigenvalue of C. So for sufficiently small $\varepsilon > 0$, $A(\varepsilon) = C + \varepsilon B \in Q(\mathcal{A})$ does not have a nonnegative (right) eigenvector corresponding to its spectral abscissa and hence is not eventually exponentially nonnegative. This contradicts the assumption that $\mathcal{A}$ requires eventual exponential nonnegativity; therefore W cannot contain a positive loop or arc-positive cycle of length 2 or more. The case of a positive loop or arc-positive cycle coming before the negative arc (s,t) in a walk is similar and involves considering C^T rather than C. □

Propositions 2.3.9 and 2.3.10 lead to the following result.

Corollary 2.3.11. *Let $\mathcal{A}$ be an $n \times n$ sign pattern that requires eventual exponential nonnegativity. No walk containing a negative non-loop arc also contains either a positive loop or any cycle of length 2 or more. Thus $\mathcal{A}_{D(0)}$ requires eventual nonnegativity.*

Proposition 2.3.12. *Let the $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ require eventual exponential nonnegativity. If $\alpha_{st} = -$, $s \neq t$, then there exists an arc-positive s-t path in $\Gamma(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ require eventual exponential nonnegativity and suppose that $\alpha_{st} = -$ for some $s, t \in \{1, \ldots, n\}$, $s \neq t$. Let W be the longest s-t walk in $\Gamma(\mathcal{A}_{D(0)})$. If W is an arc-positive walk, then it contains an arc-positive s-t path.

Suppose that W has a negative arc, (w_i, w_j). By Observation 2.3.2 there exists a positive w_i-w_j walk P_{w_i,w_j} in $\Gamma(\mathcal{A}_{D(0)})$ (which Corollary 2.3.11 implies is a path). Note that the only vertices that W and P_{w_i,w_j} share are w_i and w_j, otherwise $\Gamma(\mathcal{A}_{D(0)})$ would have a walk containing both a negative non-loop arc and a cycle, contradicting the assumption that $\mathcal{A}$ requires eventual exponential nonnegativity. Replacing the arc (w_i, w_j) in the path W with the path P_{w_i,w_j} creates a longer s-t path, contradicting the maximality of W. Therefore the longest s-t walk does not have any negative non-loop arcs and there exists an arc-positive s-t path. □

Lemma 2.3.13. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If there exists $s \neq t$ such that $\alpha_{st} = -$, at least one of α_{ss}, α_{tt} is $-$, and each interior vertex in every arc-positive s-t walk in $\Gamma(\mathcal{A})$ has a negative loop, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let the sign pattern $\mathcal{A}$, with associated graph $\Gamma(\mathcal{A}) = (V, E)$, be as hypothesized. If for some $s \neq t$, $\alpha_{st} = -$ and $\alpha_{vv} = -$ for all $v \in \widehat{V} = \widehat{V}(s,t)$ (recall that $s, t \in \widehat{V}(s,t)$), then $\mathcal{A}$ does not require eventual exponential nonnegativity since this is equivalent to $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}[\widehat{V}]}$ with $\alpha_{st} = -$, $s \neq t$, and $v \in \widehat{V}$ implies $\alpha_{vv} = +$ requiring eventual exponential nonnegativity, which contradicts Proposition 2.3.10.

Suppose that (i) for every $\alpha_{st} = -$, and $s \neq t$, there exists an arc-positive s-t path; (ii) no cycle of length 2 or more contains a negative arc; and (iii) no walk containing a negative arc also contains a positive loop or positive cycle of length 2 or more (otherwise $\mathcal{A}$ does not require eventual exponential nonnegativity by one of Propositions 2.3.12, 2.3.9, or 2.3.10). Choose s, t so that $|\widehat{V}|$ is minimized over all $\widehat{V}$ such that $i \neq j$, $\alpha_{ij} = -$, $\alpha_{ii} = 0$, $\alpha_{jj} = -$, and each interior vertex in every arc-positive i-j walk in $\Gamma(\mathcal{A})$ has a negative loop. Then for $i, j \in \widehat{V}$, $i \neq j$, $(i, j) \neq (s, t)$ implies $\alpha_{ij} \geq 0$ (due to the minimality of $|\widehat{V}|$, assumptions (i)-(iii), and the finiteness of $\Gamma(\mathcal{A})$).

Without loss of generality, let $s = 1$, $t = |\widehat{V}|$, and $v \in \widehat{V}$ imply $v \le t$. Construct $\widetilde{A} = [a_{ij}] \in \mathcal{Q}(\widetilde{\mathcal{A}})$ by setting (1) $a_{ii} = -(t-1)$ for $i = 2, \ldots, t-1$, (2) $a_{tt} = -1$, (3) $a_{1t} = -t$, (4) $a_{it} = 1$ if $\alpha_{it} \neq 0$ for $i = 2, \ldots, t-1$, and (5) choose the remaining nonzero a_{ij} so that column j has column sum zero for $j = 2, \ldots, t-1$ (note that column 1 is a zero column). Then 0 is both a simple eigenvalue and the spectral abscissa of $\widetilde{A}$. Let $\mathbf{x} = [x_i] \in \mathbb{R}^n$ with $x_i = 1$ for $i = 1, \ldots, t-1$, $x_t = -(t-m)$ where $m = \sum_{i=2}^{t-1} a_{it}$ (so $m < t$), and $x_i = 0$ for $i = t+1, \ldots, n$. Then $\mathbf{x}$ is a left eigenvector of $\widetilde{A}$ corresponding to the simple eigenvalue 0. Since $\mathbf{x}$ has both positive and negative entries, $\widetilde{A}$ does not have a nonnegative left eigenvector corresponding to its spectral abscissa.

Let B be the characteristic matrix of $\mathcal{A}$ and $A(\varepsilon) = \widetilde{A} + \varepsilon B \in \mathcal{Q}(\mathcal{A})$. For small $\varepsilon > 0$, $A(\varepsilon) = \widetilde{A} + \varepsilon B$ does not have a nonnegative left eigenvector corresponding to its spectral abscissa ε, hence $A(\varepsilon)$ is not eventually exponentially nonnegative. Therefore $\mathcal{A}$ does not require eventual exponential nonnegativity.

The case for $\alpha_{ss} = -, \alpha_{tt} = 0$ is similar, considering the right eigenvector rather than the left. □

Lemma 2.3.14. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If there exists $s \neq t$ such that $\alpha_{st} = -$, $\alpha_{ss} = \alpha_{tt} = 0$, and each interior vertex in every arc-positive s-t walk in $\Gamma(\mathcal{A})$ has a negative loop, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let the sign pattern $\mathcal{A}$, with associated graph $\Gamma(\mathcal{A}) = (V, E)$, be as hypothesized and let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Suppose that (i) for every $\alpha_{st} = -$ such that $s \neq t$, there exists an arc-positive s-t path; (ii) every cycle of length 2 or more is arc-positive; and (iii) if a walk contains a negative non-loop arc it does not contain any positive loop or positive cycle of length 2 or more (otherwise $\mathcal{A}$ does not require eventual exponential nonnegativity by one of Propositions 2.3.12, 2.3.9, or 2.3.10).

Suppose that there exists s, t, $s \neq t$, such that $\alpha_{st} = -$, $\alpha_{ss} = \alpha_{tt} = 0$, and $\alpha_{vv} = -$ for $v \in \widehat{V}(s,t) \setminus \{s, t\}$. Let $\widehat{V} = \widehat{V}(s,t)$, $\widetilde{A} = \widetilde{A[\widehat{V}]}$, and $m = |\widehat{V}|$. It follows from assumptions (ii) and (iii) that $\mathrm{In}(s) \cap \widehat{V} = s$ and $\mathrm{Out}(t) \cap \widehat{V} = t$, and hence $\widetilde{A}$ has at most $m-2$ nonzero eigenvalues, each of which is independent of a_{st}. By choosing $|a_{vv}|$ large and spread out from the other nonzero diagonal elements of $\widetilde{A}$, we can ensure that the nonzero eigenvalues of $\widetilde{A}$ are real, distinct, and negative and that for the principal submatrix $A[\widehat{V}]$ the geometric multiplicity of 0 as an eigenvalue is one. Therefore the rightmost eigenvalue of $\widetilde{A}$ is 0. Let $\widehat{\lambda}_1 = 0, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m-1}$ be the distinct eigenvalues of $\widetilde{A}$. Note that there are $n-m+1$ zero columns in $\widetilde{A}$, so the geometric multiplicity of 0 as an eigenvalue of $\widetilde{A}$ is $n-m+1$. Hence $\widetilde{A}$ has $n-m+1+m-2 = n-1$ linearly independent eigenvectors and the size of the largest Jordan block corresponding to eigenvalue 0 in the Jordan canonical form of $\widetilde{A}$ is 2.

Since 0 is the rightmost eigenvalue of $\widetilde{A}$, and it is a double root of the minimal polynomial of $\widetilde{A}$, by Observation 2.3.5 it is clear that $\widetilde{A}$ is eventually exponentially nonnegative only if $\widetilde{A}\prod_{k=2}^{m-1}(\widetilde{A} - \widehat{\lambda}_k I) \geq 0$. Note that

$$\begin{aligned} \widetilde{A}\prod_{k=2}^{m-1}(\widetilde{A} - \widehat{\lambda}_k I) &= \widetilde{A}^{m-1} - \operatorname{tr}(\widetilde{A})\widetilde{A}^{m-2} + \cdots + (-1)^{m-2}(\lambda_2 \cdots \lambda_{m-1})\widetilde{A} \\ &= \widetilde{A}^{m-1} - \operatorname{tr}(\widetilde{A})\widetilde{A}^{m-2} + \cdots + |\lambda_2 \cdots \lambda_{m-1}|\widetilde{A}, \end{aligned}$$

since $\lambda_k < 0$ for $k = 2, \ldots, m-1$. By assumption no s-t walk of length 2 or more includes the arc (s,t), so $\widetilde{a}_{st} = a_{st}$ is not in the (s,t)-entry of $\widetilde{A}^k$ for $k = 2, \ldots, m-1$. Recall that $\widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m-1}$ are independent of a_{st}, and that $a_{st} < 0$, therefore we may choose a_{st} so that the (s,t)-entry of $\widetilde{A}\prod_{k=2}^{m-1}(\widetilde{A} - \widehat{\lambda}_k I)$ is negative and hence $\widetilde{A}$ is not eventually exponentially nonnegative.

Therefore A is not eventually exponentially nonnegative and hence $\mathcal{A}$ does not require eventual exponential nonnegativity. □

The previous results lead to the following necessary conditions for the sign pattern $\mathcal{A}$ to require eventual exponential nonnegativity.

Theorem 2.3.15. *If the $n \times n$ sign pattern $\mathcal{A}$ requires eventual exponential nonnegativity, then*

(i) every cycle in $\Gamma(\mathcal{A})$ of length 2 or more is arc-positive,

(ii) if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop or (arc-positive) cycle of length 2 or more, and

(iii) for every negative arc (s,t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with an interior vertex that does not have a negative loop.

It is interesting to note that for a sign pattern $\mathcal{A} = [\alpha_{ij}]$ that requires eventual exponential nonnegativity with $\alpha_{st} = -$, and $s \neq t$, the arc-positive s-t path with an interior negative-loop-free vertex need not be the longest arc-positive s-t path.

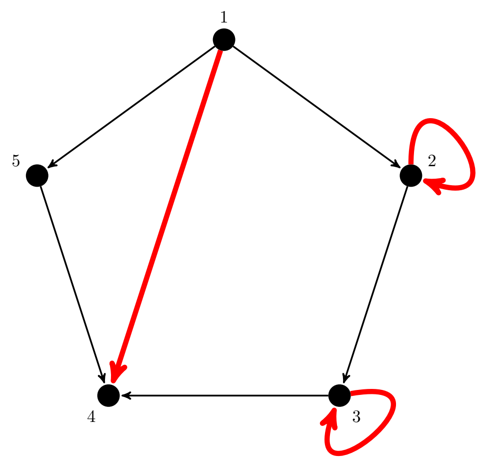

Figure 2.1 The digraph $\Gamma(\mathcal{A})$ for sign pattern $\mathcal{A}$ in Example 2.3.16. The negative arcs are represented by **thick** lines.

Example 2.3.16. Consider the matrix

$$A = \begin{bmatrix} 0 & a_{12} & 0 & -a_{14} & a_{15} \\ 0 & -a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & -a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix},$$

with $a_{12}, a_{14}, a_{15}, a_{22}, a_{23}, a_{33}, a_{34}, a_{54} > 0$. See Figure 2.1 for the digraph $\Gamma(\mathcal{A})$ for $\mathcal{A} = \text{sgn}(A)$. Note that the principal submatrices $A[\{1, 2, 3\}]$ and $A[\{2, 3, 4, 5\}]$ are each essentially nonnegative and therefore are each eventually exponentially nonnegative. Clearly $(e^{\tau A})_{41} = (e^{\tau A})_{51} = 0$, so the only questionable entries are the $(1, 4)$- and $(1, 5)$-entries.

Note that $(A^k)_{15} = 0$ for $k \geq 2$ so by the power series for $e^{\tau A}$,

$$(e^{\tau A})_{15} = \tau a_{15},$$

which is positive for $\tau > 0$.

The distinct eigenvalues of A are $0, -a_{22}$, and $-a_{33}$. The minimal polynomial of A is the characteristic polynomial, so the size of the largest Jordan block for 0 (the rightmost eigenvalue) is three. The $(1,4)$-entry of $A^2(A+a_{22}I)(A+a_{33}I)$ is $a_{15}a_{54}a_{22}a_{33}$, which is positive. Hence by the previous analysis of the Hermite interpolating polynomial, the dominating term of $(e^{\tau A})_{14}$ is positive.

Therefore the sign pattern $\mathcal{A} = \text{sgn}(A)$ requires eventual exponential nonnegativity.

2.3.2 Sufficient conditions for upper triangular sign patterns

It is conjectured that the converse of Theorem 2.3.15 is also true, that is:

Conjecture 2.3.17. *The $n \times n$ sign pattern $\mathcal{A}$ requires eventual exponential nonnegativity if and only if*

(i) every cycle in $\Gamma(\mathcal{A})$ of length 2 or more is arc-positive,

(ii) if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop or (arc-positive) cycle of length 2 or more, and

(iii) for every negative arc (s,t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with an interior vertex that does not have a negative loop.

Note that if a sign pattern has no cycles of length 2 or more, it can be simultaneously permuted into an upper triangular pattern. We will use Hermite interpolation to prove Conjecture 2.3.17 in the case of upper triangular sign patterns, but first we develop tools to analyze the sign of the dominating term of the (i,j)-entry of $e^{\tau A}$ for every realization $A \in \mathcal{Q}(\mathcal{A})$ when $\mathcal{A}$ is an upper triangular sign pattern that satisfies the hypotheses of Conjecture 2.3.17.

The technique of using König digraphs to compute the product of several matrices motivates the following terminology (see, e.g., [3] for more on König digraphs and Section 2.4 for a discussion of how our use of König digraphs differs slightly from that of Brualdi and Cvetković). Let $M^{(1)} = [m_{ij}^{(1)}], M^{(2)} = [m_{ij}^{(2)}], \ldots, M^{(\ell)} = [m_{ij}^{(\ell)}]$ be real $n \times n$ upper triangular matrices. The product

$$m_{i,k_1}^{(1)} m_{k_1,k_2}^{(2)} m_{k_2,k_3}^{(3)} \cdots m_{k_{\ell-1},j}^{(\ell)} \tag{2.3}$$

is called a *loop-path product* of length ℓ. Note that the loop-path product (2.3) is the weight of the walk $(i, k_1, k_2, \ldots, k_{\ell-1}, j)$ in the composite König digraph used to compute the product

$M^{(1)}M^{(2)}\cdots M^{(\ell)}$. Although König digraphs have no loops, we take the convention of calling arc (r_i, c_i) in the König digraph $K(M)$ a loop, since this corresponds to a nonzero diagonal entry m_{ii}. The *underlying path* of a loop-path product is obtained by ignoring the loops in the walk $(i, k_1, k_2, \ldots, k_{\ell-1}, j)$. For example, the loop-path product $m^{(1)}_{1,2}m^{(2)}_{2,4}m^{(3)}_{4,4}m^{(4)}_{4,5}$ has the underlying path $(1, 2, 4, 5)$.

Consider the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}.$$

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A and let n_j be the multiplicity of λ_j as a root of the characteristic polynomial of A for $j = 1, \ldots, m$. Choose $\mu \in \operatorname{spec}(A)$ and let $S = \{k : a_{kk} = \mu\}$. Define

$$B^{(k)} := \begin{cases} (A - a_{kk}I) & \text{for } k \notin S \\ I & \text{for } k \in S \end{cases}.$$

Then since any two scalar shifts of A commute with each other,

$$\prod_{\lambda_j \neq \mu} (A - \lambda_j I)^{n_j} = \prod_{k \notin S} (A - a_{kk}I) = \prod_{k \notin S} B^{(k)} = \prod_{1 \leq k \leq n} B^{(k)}.$$

Note that this last equality is true since $B^{(k)} = I$ for $k \in S$. By definition of matrix multiplication, the $(1, n)$-entry of $\prod_{1 \leq k \leq n} B^{(k)}$ is the sum of loop-path products of the form

$$B^{(1)}_{1,k_1} B^{(2)}_{k_1,k_2} B^{(3)}_{k_2,k_3} \cdots B^{(n)}_{k_{n-1},n} \tag{2.4}$$

where

$$B^{(k)}_{i,j} = \begin{cases} a_{ii} - a_{kk} & \text{for } j = i, \text{ and } k \notin S \\ a_{ij} & \text{for } j \neq i, \text{ and } k \notin S \end{cases} \quad \text{and} \quad B^{(k)}_{i,j} = \begin{cases} 1 & \text{for } j = i, \text{ and } k \in S \\ 0 & \text{for } j \neq i, \text{ and } k \in S \end{cases}.$$

Note that in the following result, the case $S = \{1\}$ is the reason for the inclusion of the phrase "at least one of $s, t \in S$" in (iii).

Lemma 2.3.18. *Suppose $\mathcal{W}$ is a nonzero loop-path product of the form* (2.4). *Then (i) $\mathcal{W}$ includes a factor $B^{(k)}_{kk}$ with $k \in S$, (ii) if ℓ is the least integer such that $\mathcal{W}$ includes the factor $B^{(\ell)}_{k_{\ell-1},k_\ell}$, with $k_{\ell-1} = k_\ell$, then $\ell \le k_{\ell-1}$, and (iii) $\mathcal{W}$ includes a factor a_{st} with at least one of $s, t \in S$ and $s < t$.*

Proof. If S contains either 1 or n then (*iii*) is clear. Suppose $1, n \notin S$ (and therefore $S \neq \{1\}$). We show that $\mathcal{W}$ includes a factor a_{st} for some $t \in S$. Observe that $B^{(k)}$ is upper triangular for $k = 1, \ldots, n$ and $B^{(k)}_{k,k} = 0$ if $k \notin S$. Therefore $\mathcal{W}$ is nonzero only if $k_i \le k_{i+1}$ for $1 \le i \le n-2$; furthermore $k_i = k_{i+1}$ implies that $k_i = k_{i+1} \in S$. By the pigeonhole principle a factor $B^{(k)}_{k,k}$ appears in $\mathcal{W}$ and hence $k \in S$. Let ℓ be the least integer $(2 \le \ell \le n-1)$ such that $B^{(\ell)}_{k_{\ell-1},k_\ell}$ is in $\mathcal{W}$ with $k_{\ell-1} = k_\ell$. Due to the triangular structure, $\ell \le k_{\ell-1}$ and if $k_{\ell-1} \notin S$, then $\ell < k_{\ell-1}$. Since $1, n \notin S$, for the loop-path product (2.4) to be nonzero, it includes a factor $B^{(\ell)}_{st} \neq 0$, with $s < t$ and $t \in S$ (and hence $\ell \notin S$, since $B^{(\ell)} = I$ for $\ell \in S$). Therefore, recalling the definition of $B^{(k)}$ for $k \notin S$, each nonzero loop-path product of the form (2.4) has a factor a_{st} appear for some $t \in S$, with $s < t$. □

For an upper triangular sign pattern that satisfies the conditions of Conjecture 2.3.17, if the loop-path products (2.4) are positive when μ is the rightmost eigenvalue, then the sign pattern requires eventual exponential nonnegativity. However, as Example 2.3.19 below shows, Lemma 2.3.18 does not preclude a loop-path product of the form (2.4) from including a loop at a vertex not in S (i.e., a factor $B^{(\ell)}_{kk}$ with $k > \ell$), which would have a loop weight of unknown sign. Example 2.3.19 also shows how one can rewrite the sum of two loop-path products as a single loop-path product with the loop occurring at a vertex in S; and when μ is a rightmost eigenvalue all loops at vertices in S have nonnegative loop weights.

Example 2.3.19. Consider the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

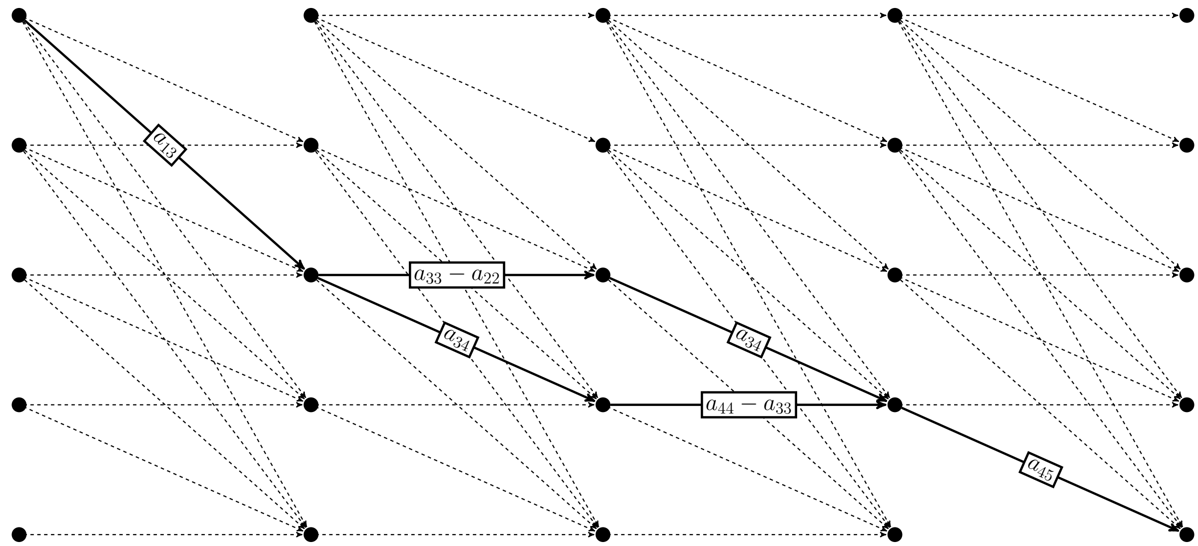

Figure 2.2 The composite König digraph for the matrix product $B^{(1)}B^{(2)}B^{(3)}B^{(5)}$ in Example 2.3.19. The loop-paths $B^{(1)}_{13}B^{(2)}_{33}B^{(3)}_{34}B^{(5)}_{45}$ and $B^{(1)}_{13}B^{(2)}_{34}B^{(3)}_{44}B^{(5)}_{45}$ are in solid lines with arc weights displayed.

with $\mu = a_{44}$. Then $B^{(4)} = I$ so $B^{(1)}B^{(2)}B^{(3)}B^{(4)}B^{(5)} = B^{(1)}B^{(2)}B^{(3)}B^{(5)}$ and the product $B^{(1)}_{13}B^{(2)}_{33}B^{(3)}_{34}B^{(5)}_{45} = a_{13}(a_{33} - a_{22})a_{34}a_{45}$ is not combinatorially zero (see Figure 2.2). However, the product $B^{(1)}_{13}B^{(2)}_{34}B^{(3)}_{44}B^{(5)}_{45} = a_{13}a_{34}(a_{44} - a_{33})a_{45}$ is also not combinatorially zero and summing these together we get $a_{13}a_{34}(a_{44} - a_{22})a_{45}$, which is a loop-path product utilizing a loop at vertex $\nu = 4$ with loop weight $(a_{44} - a_{22})$. Note that the underlying path of the loop-path product $a_{13}a_{34}(a_{44} - a_{22})a_{45}$ does not pass through vertex 2.

Lemma 2.3.20. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be an upper triangular matrix. If the loop weight $(a_{ii} - a_{\ell\ell})$, with $i \notin S$, appears in a nonzero loop-path product of the form* (2.4)*, with underlying path P, then additional loop-path products exist—also with the underlying path P—such that simplification of the sum of these loop-path products results in a single loop-path product, all of whose loops are at a vertex in S.*

Proof. Suppose $\ell \in \{1, \ldots, n\}$ is the least number such that a nonzero loop-path product in the $(1, n)$-entry of $\prod_{k\notin S}(A - a_{kk}I)$ can be written as $(a_{ii} - a_{\ell\ell})\mathcal{W}$ for some loop-path product $\mathcal{W}$. By Lemma 2.3.18 the loop-path product $\mathcal{W}$ passes through a vertex $\nu \in S$. We first consider the case $\ell < \nu$. The product $(a_{ii} - a_{\ell\ell})\mathcal{W}$ can be viewed as a loop-path product that uses a loop with weight $a_{ii} - a_{\ell\ell}$ at vertex i in conjunction with the loop-path product $\mathcal{W}$. If $i = \nu$, we are done.

Note that due to the upper triangularity of A, $i \leq \nu$ and $\ell < i$ (otherwise the loop-path product would be zero). Suppose that $i < \nu$. Let j be the next vertex (different from i) through which $\mathcal{W}$ passes. Then $(a_{jj} - a_{ii})\mathcal{W}$ is a loop-path product that differs from $(a_{ii} - a_{\ell\ell})\mathcal{W}$ by shifting the use of a loop from vertex i to vertex j. Moreover, $(a_{ii} - a_{\ell\ell})\mathcal{W} + (a_{jj} - a_{ii})\mathcal{W} = [(a_{ii} - a_{\ell\ell}) + (a_{jj} - a_{ii})]\,\mathcal{W} = (a_{jj} - a_{\ell\ell})\mathcal{W}$. If $j = \nu$, we are done. If not, then taking k to be the next vertex, after j, visited by $\mathcal{W}$ and simplifying the sum $(a_{jj} - a_{\ell\ell})\mathcal{W} + (a_{kk} - a_{jj})\mathcal{W}$ we get $(a_{kk} - a_{\ell\ell})\mathcal{W}$. Repeating this process, we can shift use of the loop with weight $a_{ii} - a_{\ell\ell}$ at vertex i in conjunction with $\mathcal{W}$ to the use of the loop with weight $a_{\nu\nu} - a_{\ell\ell}$ at vertex ν in conjunction with $\mathcal{W}$.

The case for $\ell > \nu$ is similar. □

Repeated application of this process allows for the sum of all nonzero loop-path products for the $(1, n)$-entry of $\prod_{k \neq \nu}(A - a_{kk}I)$, where A is upper triangular, to be written as loop-path products in which all loops occur at vertex ν. Note that this could require using multiple loops of different weights at vertex ν.

Theorem 2.3.21 follows from Lemmas 2.3.18 and 2.3.20.

Theorem 2.3.21. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be upper triangular with m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then for $\mu \in \{\lambda_1, \ldots, \lambda_m\}$, $S = \{k : a_{kk} = \mu\}$, and $M = n - |S| - 1$, the $(1, n)$-entry of the product $\prod_{k \notin S}(A - a_{kk}I)$ can be expressed as*

$$\sum_{\nu \in S}\Bigg[\left(\prod_{k \notin (S \cup \{1,n\})}(a_{\nu\nu} - a_{kk})\right) a_{1\nu}a_{\nu n} + \sum_{\substack{1<\ell_1<\ell_2<n \\ \nu \in \{\ell_1,\ell_2\}}}\left(\prod_{k \notin (S \cup \{1,\ell_1,\ell_2,n\})}(a_{\nu\nu} - a_{kk})\right) a_{1,\ell_1}a_{\ell_1,\ell_2}a_{\ell_2,n}$$
$$+ \cdots + \sum_{\substack{1<\ell_1<\cdots<\ell_M<n \\ \nu \in \{\ell_1,\ldots,\ell_M\}}} a_{1,\ell_1}a_{\ell_1,\ell_2}\cdots a_{\ell_M,n}\Bigg].$$

Now we are ready to prove Conjecture 2.3.17 in the case that the sign pattern $\mathcal{A}$ is permutationally similar to an upper triangular sign pattern. Recall that if $\mathcal{A}$ is upper triangular, then $\Gamma(\mathcal{A})$ has no cycles of length 2 or more.

Theorem 2.3.22. *Let $\mathcal{A}$ be an $n \times n$ upper triangular sign pattern such that (i) if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop and that (ii) for every negative arc (s,t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with a negative-loop-free interior vertex. Then $\mathcal{A}$ requires eventual exponential nonnegativity.*

Proof. Let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Since A is upper triangular, $\operatorname{spec}(A) = \{a_{ii} : 1 \leq i \leq n\}$ and each eigenvalue of A is real. Suppose that $\left(e^{\tau A}\right)_{ij} \neq 0$. We show that the dominating term of $\left(e^{\tau A}\right)_{ij}$ is positive. For any given $i, j \in \{1, \ldots, n\}$,

$$\left(e^{\tau A}\right)_{ij} = \left(e^{\tau \widetilde{A}}\right)_{ij},$$

where $\widetilde{A} = \widetilde{A[\widehat{V}]}$ and $\widehat{V} = \widehat{V}(i,j) = \operatorname{Out}(i) \cap \operatorname{In}(j)$, so we need only show that the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is positive. From the power series for $e^{\tau A}$, we have that the strictly lower triangular part of $e^{\tau A}$ is all zeros and $\left(e^{\tau A}\right)_{ii} = e^{\tau a_{ii}} > 0$ for $i = 1, \ldots, n$. So we consider only $i < j$. Since $\mathcal{A}$ is upper triangular, $\widehat{V}(i,j) \subseteq \{i, i+1, \ldots, j-1, j\}$.

If the principal submatrix $A[\widehat{V}(i,j)]$ has no off-diagonal negative entry, then $A[\widehat{V}(i,j)]$ is eventually exponentially nonnegative by Proposition 2.3.1. Note that hypothesis (ii) implies that the first super-diagonal is nonnegative, therefore $A[\{i, i+1\}]$ is essentially nonnegative and hence eventually exponentially nonnegative for $i = 1, \ldots, n-1$. Let $j > i+1$ and suppose $A[\widehat{V}(i,j)]$ has an off-diagonal negative entry. From the interpolation method for calculating the matrix exponential, the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is determined by the rightmost eigenvalue of the principal submatrix $A[\widehat{V}(i,j)]$.

Since by assumption there exist s, t, with $i \leq s < t \leq j$, such that $\alpha_{st} = -$, hypothesis (ii) guarantees the existence of $\kappa \in \{s+1, \ldots, t-1\}$ such that $\alpha_{\kappa\kappa} \geq 0$. Hence the spectral abscissa $\alpha(A[\widehat{V}(i,j)])$ is nonnegative, moreover, since A is a real triangular matrix, $\alpha(A[\widehat{V}(i,j)])$ is an eigenvalue of $A[\widehat{V}(i,j)]$. Let $\widehat{\gamma} = \alpha(A[\widehat{V}(i,j)]), \widehat{\lambda}_1, \ldots, \widehat{\lambda}_m$ be the distinct eigenvalues of $A[\widehat{V}(i,j)]$. Let $\widehat{n}_\nu$ be the size of the largest Jordan block for $\widehat{\gamma}$ in the Jordan canonical form of $A[\widehat{V}(i,j)]$ and $\widehat{n}_k$ be the algebraic multiplicity of $\widehat{\lambda}_k$ (for the submatrix). Let $S = \{k : i \leq k \leq j, a_{kk} = \widehat{\gamma}\}$.

By Observation 2.3.5, the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is positive if and only if the (i,j)-

entry of

$$(\widetilde{A}-\widehat{\gamma}I)^{\widehat{n}_\nu-1}\prod_{i=1}^{m}(\widetilde{A}-\widehat{\lambda}_iI)^{\widehat{n}_i}=(\widetilde{A}-\widehat{\gamma}I)^{\widehat{n}_\nu-1}\prod_{\substack{i\le k\le j\\ k\notin S}}(\widetilde{A}-a_{kk}I)$$

is positive. Let $M := j-(i-1)-|S|-1 = j-i-|S|$. By Theorem 2.3.21, and the fact that $a_{kk}-\widehat{\gamma}=0$ for $k\in S$, the (i,j)-entry of this product can be expressed as

$$\sum_{\nu\in S}\Bigg[\Bigg(\prod_{\substack{i<k<j\\ k\notin\{i,\nu,j\}}}(a_{\nu\nu}-a_{kk})\Bigg)a_{i\nu}a_{\nu j}+\sum_{\substack{i<\ell_1<\ell_2<j\\ \nu\in\{\ell_1,\ell_2\}}}\Bigg(\prod_{\substack{i<k<j\\ k\notin(\{i,\ell_1,\ell_2,j\})}}(a_{\nu\nu}-a_{kk})\Bigg)a_{i,\ell_1}a_{\ell_1,\ell_2}a_{\ell_2,j}$$
$$+\cdots+\sum_{\substack{i<\ell_1<\cdots<\ell_M<j\\ \nu\in\{\ell_1,\ldots,\ell_M\}}}a_{i,\ell_1}a_{\ell_1,\ell_2}\cdots a_{\ell_M,j}\Bigg].$$

Since $a_{\nu\nu}=\widehat{\gamma}=\alpha(A[\widehat{V}(i,j)])$, $a_{\nu\nu}-\widehat{\lambda}_k>0$ for $k=1,\ldots,m$. By hypothesis (i), if $a_{\nu\nu}=\alpha(A[\widehat{V}(i,j)])>0$, every i-ν-j path is arc-positive and hence each product in the sum is positive. If $a_{\nu\nu}=\alpha(A[\widehat{V}(i,j)])=0$, then there could be an i-ν-j path that uses a negative arc (x,y); however, in using arc (x,y), the path avoids an x-y arc-positive path with a negative-loop-free vertex z. So any path product which includes a_{xy} would then be multiplied by $(a_{\nu\nu}-a_{zz})=0$ (since $0\le a_{zz}\le\alpha(A[\widehat{V}(i,j)])=a_{\nu\nu}=0$). Therefore the nonzero products in the sum are positive.

Hence either $\left(e^{\tau A}\right)_{ij}=0$ or the dominating term of $\left(e^{\tau A}\right)_{ij}$ is positive for any $i,j\in\{1,\ldots,n\}$ and A is eventually exponentially nonnegative. Therefore $\mathcal{A}$ requires eventual exponential nonnegativity. □

2.3.3 Maximum number of negative entries

When studying generalizations of positive or nonnegative matrices, one often considers the minimum number of positive entries or the maximum number of negative entries (the former is not relevant for generalizations of nonnegativity). In this section we determine the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity.

It is clear that for $n=1$, the answer is one.

Example 2.3.23. For $n \geq 2$, define

$$\mathcal{T}_n := \begin{bmatrix} - & + & - & \cdots & \cdots & - \\ 0 & 0 & + & - & \cdots & - \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & + & - \\ 0 & \cdots & \cdots & \cdots & 0 & + \\ 0 & \cdots & \cdots & \cdots & 0 & - \end{bmatrix}.$$

By Theorem 2.3.22, $\mathcal{T}_n$ requires eventual exponential nonnegativity. Therefore the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is at least $\frac{(n-1)(n-2)}{2} + 2$ for $n \geq 2$. Note that if any of the 2nd through $(n-1)$st entries of the main diagonal of $\mathcal{T}_n$ are changed to $-$, the resulting sign pattern does not require eventual exponential nonnegativity since it would violate condition (iii) of Theorem 2.3.15.

Theorem 2.3.24. *If the $n \times n$ sign pattern $\mathcal{A}$ $(n \geq 2)$ requires eventual exponential nonnegativity, then $\mathcal{A}$ has at most $\frac{(n-1)(n-2)}{2} + 2$ negative entries.*

Proof. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern $(n \geq 2)$ that requires eventual exponential nonnegativity. We claim that by Theorem 2.3.15, (parts (i) and (iii)) there exists a permutation pattern P such that for $\mathcal{A}' = P\mathcal{A}P^T = [\alpha'_{ij}]$, $\alpha'_{ij} \geq 0$ if $i > j$ or $j = i + 1$.

We construct one such permutation by relabeling each vertex in $\Gamma(\mathcal{A})$ twice, first we relabel each vertex exactly once with a label in $\mathbb{Z}$, then we shift that labeling so that each vertex is relabeled with a positive integer. (Note that the goal of this relabeling is so that $\alpha'_{st} = -$ with $s \neq t$ implies α'_{st} is above the first super-diagonal, the location of positive entries is of no concern to us.) Let $E_N \subset E$ be the set of all non-loop negative arcs in $\Gamma(\mathcal{A})$ and for $(i, j) \in E_N$ let $L(i, j)$ denote the length of the longest arc-positive i-j path in $\Gamma(\mathcal{A})$. The rule for the intermediate relabeling of the vertices is given by: Let $V_0 = \{\emptyset\}$ and $V_k \subseteq V$ be the set of vertices that have been relabeled after iteration k. Let N_k be the set of integers used in the intermediate relabeling of V_k. Choose $s_1, t_1 \in V$ such that $(s_1, t_1) \in E_N$ and $L(s_1, t_1) = \max\{L(s, t) : (s, t) \in E_N\}$. Choose an arc-positive path of length $L(s_1, t_1)$, denoted $P(s_1, t_1) = (s_1, p_2, \ldots, p_{L(s_1,t_1)}, t_1)$. Relabel the vertices in $P(s_1, t_1)$ as $s_1 \mapsto 0', p_2 \mapsto 1', \ldots, t_1 \mapsto (L(s_1, t_1))'$. While there exists

$(s,t) \in E_N$ such that $\{s,t\} \not\subseteq V_{k-1}$: Choose $s_k, t_k \in V \setminus V_{k-1}$ such that $(s_k,t_k) \in E_N$ with $L(s_k,t_k) = \max\{L(s,t) : \{s,t\} \not\subseteq V_{k-1}$ and $(s,t) \in E_N\}$ (note that $L(s_k,t_k) \geq 2$). Choose an arc-positive s_k-t_k path of length $L(s_k,t_k)$, denoted $P(s_k,t_k) = (s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k)$. The rule for relabeling the vertices in $P(s_k,t_k)$ for $k \geq 2$ depends on which (if any) vertices in $P(s_k,t_k)$ have already been relabeled: If $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap V_{k-1} = \emptyset$, then relabel the vertices in $P(s_k,t_k)$ as $s_k \mapsto (\ell_M+1)', p_2 \mapsto (\ell_M+2)', \ldots, t_k \mapsto (\ell_M + L(s_k,t_k)+1)'$ where $\ell_M = \max N_{k-1}$. If $s_k \in V_{k-1}$, relabel the vertices in $\{p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}$ sequentially along the path $P(s_k,t_k)$ as $(\ell_M+1)', (\ell_M+2)', \ldots, (\ell_M+c)'$, where $\ell_M = \max N_{k-1}$ and $c = \left|\{p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}\right|$. If $s_k \notin V_{k-1}$ and $p_i \in V_{k-1}$ for some $i \in \{2, \ldots, L(s_k,t_k)\}$, then let $j = \min\{i : p_i \in V_{k-1}\}$ and relabel the vertices $\{s_k, p_2, \ldots, p_{j-1}\}$ as $s_k \mapsto (\ell_m - j + 1)', \ldots, p_{j-1} \mapsto (\ell_m - 1)'$, where $\ell_m = \min N_{k-1}$, and relabel the vertices in $\{p_{j+1}, \ldots, p_{L(s_k,t_k)}\} \cap \overline{V}_{k-1}$ sequentially along the path $P(s_k,t_k)$ as $(\ell_M+1)', \ldots, (\ell_M+c)'$, $\ell_M = \max N_{k-1}$, and $c = \left|\{p_{j+1}, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}\right|$. If $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap V_{k-1} = \{t_k\}$, relabel the vertices $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}\}$ as $s_k \mapsto (\ell_m - L(s_k,t_k))', \ldots, p_{j-1} \mapsto (\ell_m - 1)'$, where $\ell_m = \min N_{k-1}$. After the process has been completed (i.e., there exists no $(s,t) \in E_N$ such that $\{s,t\} \not\subseteq V_{k-1}$), the remaining vertices are mapped into $\{\ell_M+1, \ldots, \ell_M + |V \setminus V_{k-1}|\}$, where $\ell_M = \max N_{k-1}$. Note that the vertices have now been relabeled into a set of consecutive integers, at least one of which is nonpositive. Finally, we relabel each vertex again by adding $|\ell_m|+1$ to each intermediate label so that the final label of each vertex comes from the integers $\{1, 2, \ldots, |V|\}$.

Since $j \geq i+2$ if $\alpha'_{ij} = -$, $\mathcal{A}'$ has at most $\frac{(n-1)(n-2)}{2} + n$ negative entries. By part 3 of Theorem 2.3.15, if $\alpha'_{st} = -$ (where $s \neq t$), then there exists $k \in \widehat{V}(s,t)$ such that $\alpha'_{kk} \geq 0$. Specifically, if $\alpha'_{k-1,k+1} = -$, then $\alpha'_{kk} \geq 0$ and conversely, if $\alpha'_{kk} = -$, then $\alpha'_{k-1,k+1} \geq 0$. So for $k = 2, \ldots, n-1$, at most one of α'_{kk} and $\alpha'_{k-1,k+1}$ is negative. Therefore $\mathcal{A}$ has at most $\frac{(n-1)(n-2)}{2} + n - (n-2) = \frac{(n-1)(n-2)}{2} + 2$ negative entries. □

It is interesting to note that $\mathcal{T}_n$ is not the unique (even when taking into account graph isomorphism) way for a sign pattern to have $\frac{(n-1)(n-2)}{2} + 2$ negative entries and to require eventual exponential nonnegativity. For example,

$$\begin{bmatrix} - & + & - & - & - \\ 0 & 0 & + & 0 & - \\ 0 & 0 & - & + & - \\ 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - \end{bmatrix}$$

has $\frac{(5-1)(5-2)}{2} + 2 = 8$ negative entries and requires eventual exponential nonnegativity by Theorem 2.3.22.

Corollary 2.3.25. *For $n \geq 2$ the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is $\frac{(n-1)(n-2)}{2} + 2$.*

2.4 Appendix: König digraphs

The following is known (see, e.g., [3]) but is included here to aid the reader. Brualdi and Cvetković [3] use a slightly different (but mathematically equivalent) method in using König digraphs to determine a matrix product. The specific difference is noted after we define our method. Let $M = [m_{ij}]$ be a (real) $m \times n$ matrix. The *König digraph* of M, denoted $K(M)$, is a weighted bipartite digraph on $m + n$ vertices, with vertices $V_r = \{r_1, \ldots, r_m\}$ corresponding to the rows of M and vertices $V_c = \{c_1, \ldots, c_n\}$ corresponding to the columns of M. The ordered pair (r_i, c_j) is an arc in $K(M)$ if and only if $m_{i,j} \neq 0$ and the weight of arc (r_i, c_j) is given by $m_{i,j}$. Consider the matrices $X = [x_{ij}] \in \mathbb{R}^{m \times n}$ and $Y = [y_{ij}] \in \mathbb{R}^{n \times p}$ and their König digraphs $K(X)$ and $K(Y)$, with vertices $V_{X,r} \cup V_{X,c}$ and $V_{Y,r} \cup V_{Y,c}$, respectively (where $V_{X,r} = \{r_{X,1}, r_{X,2}, \ldots, r_{X,m}\}$, $V_{X,c} = \{c_{X,1}, c_{X,2}, \ldots, c_{X,n}\}$, $V_{Y,r} = \{r_{Y,1}, r_{Y,2}, \ldots, r_{Y,n}\}$, and $V_{Y,c} = \{c_{Y,1}, c_{Y,2}, \ldots, c_{Y,p}\}$). The (i, j)-entry of the product XY can be computed as follows. First, construct the *composite König digraph*: for $1 \leq k \leq n$, identify vertex $c_{X,k}$ with vertex $r_{Y,k}$ and rename as v_k. Second, for $1 \leq k \leq n$, compute the weight, w_k, of the $(r_{X,i}, v_k, c_{Y,j})$-path as $w_k = x_{i,k}y_{k,j}$. Finally, compute the sum $w_1 + w_2 + \cdots + w_n$. By the definition of matrix multiplication, this sum is the (i, j)-entry of the product XY. This process generalizes to products of more than two matrices, as shown in Example 2.4.1. Note that Brualdi and

Cvetković collapse what we call the composite König digraph into the König digraph for the matrix that is the result of computing the product. For example, the arc weight of the arc $(r_{X,i}, c_{Y,j})$ in $K(XY)$ would be the sum $w_1 + w_2 + \cdots + w_n$.

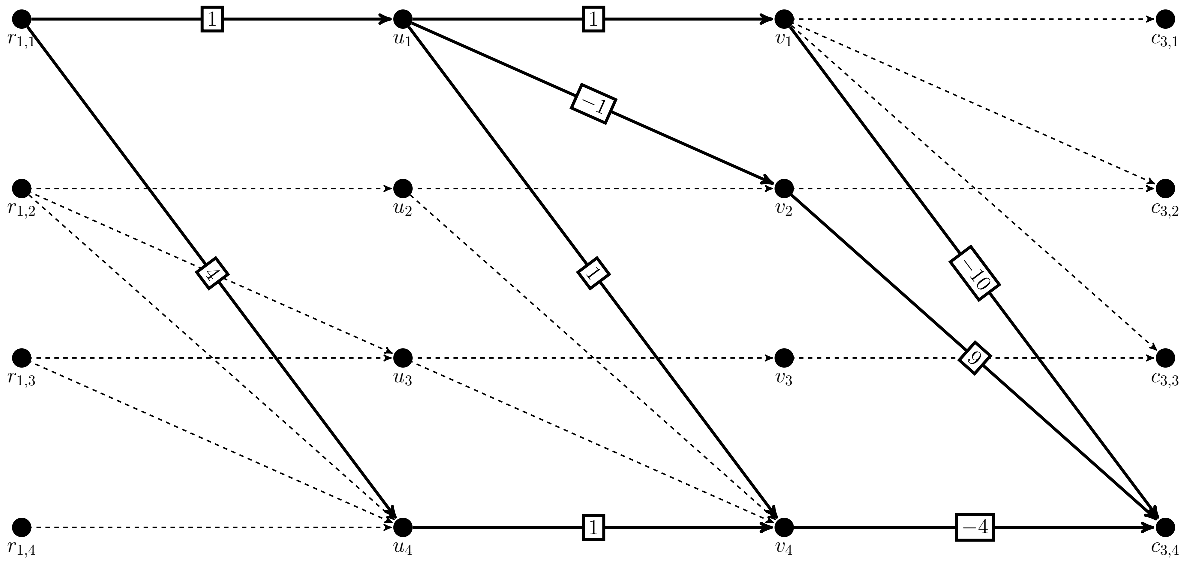

Figure 2.3 The composite König digraph for the matrix product in Example 2.4.1. The arcs not belonging to any $r_{X,1}$-$c_{Z,4}$-path are shown as dashed lines, and their arc weights are omitted, as they are not used in the calculation of the $(1,4)$-entry of the product XYZ.

Example 2.4.1. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 5 & -6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 5 & 8 & -10 \\ 0 & 2 & 0 & 9 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

We can use the König digraphs, $K(X)$, $K(Y)$, and $K(Z)$ (see Figure 2.3) to determine the $(1,4)$-entry of the product XYZ. There are four $r_{X,1}$-$c_{Z,4}$-paths in the composite König digraph in Figure 2.3, adding the weights of these paths, we get $(1)(1)(-10) + (1)(-1)(9) + (1)(1)(-4) + (4)(1)(-4) = -39$, which is the $(1,4)$-entry of the product XYZ.

Bibliography

[1] M. Archer, M. Catral, C. Erickson, R. Haber, L. Hogben, X. Martinez-Rivera, and A. Ochoa. Potentially eventually exponentially positive sign patterns. *Involve: A journal of mathematics*, 6 (2013), no. 3, 261–271.

[2] A. Berman and B. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.

[3] R. Brualdi and D. Cvetković. *A Combinatorial Approach to Matrix Theory and its Applications*. CRC Press, Boca Raton, FL, 2009.

[4] E.M. Ellison, L. Hogben, and M.J. Tsatsomeros. Sign patterns that require eventual positivity or require eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19 (2010), 98–107.

[5] G.H. Golub and C.F. Van Loan. *Matrix Computations* (third edition). Johns Hopkins University Press, Baltimore, 1996.

[6] N. Higham. *Functions of Matrices: Theory and Computation*. SIAM, Philadelphia, 2008.

[7] D. Noutsos and M.J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM Journal on Matrix Analysis and Applications*, 30 (2008), 700–712.

CHAPTER 3. POTENTIALLY EVENTUALLY EXPONENTIALLY POSITIVE SIGN PATTERNS

A paper first published in *Involve: A journal of mathematics* in vol. 6, no. 3, published by Mathematical Sciences Publishers.

Marie Archer, Minerva Catral, Craig Erickson, Rana Haber, Leslie Hogben, Xavier Martinez-Rivera, Antonio Ochoa

Abstract

We introduce the study of potentially eventually exponentially positive (PEEP) sign patterns and establish several results using the connections between these sign patterns and the potentially eventually positive (PEP) sign patterns. It is shown that the problem of characterizing PEEP sign patterns is not equivalent to that of characterizing PEP sign patterns. A characterization of all 2×2 and 3×3 PEEP sign patterns is given.

3.1 Introduction

A matrix $A \in \mathbb{R}^{n \times n}$ is *eventually positive* if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$ (where the inequality is interpreted entrywise). A matrix A is *eventually exponentially positive* if there exists some $t_0 \geq 0$ such that for all $t \geq t_0$,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} > 0.$$

Eventually exponentially positive matrices have applications to dynamical systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter [5]. Noutsos and Tsatsomeros provide the following characterization of eventual exponential positivity in terms of eventual positivity.

Theorem 3.1.1. [5, Theorem 3.3] *The matrix* $A \in \mathbb{R}^{n\times n}$ *is eventually exponentially positive if and only if there exists* $a \geq 0$ *such that* $A + aI$ *is eventually positive (where* I *is the* $n \times n$ *identity matrix).*

A *sign pattern* is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\mathrm{sgn}(A)$ is the sign pattern having entries that correspond to the signs of the entries in A. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *qualitative class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\mathrm{sgn}(A) = \mathcal{A}$; such a matrix A is called a *realization* of $\mathcal{A}$. A sign pattern $\mathcal{A}$ is *potentially eventually positive* (*PEP*) if there exists some realization $A \in \mathcal{Q}(\mathcal{A})$ that is eventually positive. PEP sign patterns were studied in [1], and we adapt several techniques from that paper to study potentially eventually exponentially positive sign patterns.

Definition 3.1.2. A sign pattern $\mathcal{A}$ is *potentially eventually exponentially positive* (*PEEP*) if there exists some realization $A \in \mathcal{Q}(\mathcal{A})$ that is eventually exponentially positive.

Since an eventually positive matrix is eventually exponentially positive, a PEP sign pattern is PEEP. Theorem 3.1.1 leads naturally to consideration of a sign pattern with positive diagonal entries.

Definition 3.1.3. Given an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, we denote by $\mathcal{A}_{D(+)} = [\hat{\alpha}_{ij}]$ the $n \times n$ sign pattern such that $\hat{\alpha}_{ij} = \alpha_{ij}$ for $i \neq j$ and $\hat{\alpha}_{ii} = +$ for $i, j \in \{1, \ldots, n\}$. $\mathcal{A}_{D(0)}$ and $\mathcal{A}_{D(-)}$ are defined analogously, with zero and negative diagonal, respectively.

In [1] it is noted that if $\mathcal{A}$ is PEP then $\mathcal{A}_{D(+)}$ is also PEP. This observation together with Theorem 3.1.1 leads to the following observation.

Observation 3.1.4. *If* $\mathcal{A}$ *is a PEEP sign pattern, then* $\mathcal{A}_{D(+)}$ *is a PEP sign pattern (and hence* $\mathcal{A}_{D(+)}$ *is also PEEP).*

Given a PEEP sign pattern, we can generate a PEP sign pattern by changing every diagonal element to $+$. However, taking a PEP sign pattern and changing $+$ diagonal entries to 0 or $-$

does not always yield a PEEP sign pattern. For example

$$\mathcal{B}_{D(+)} = \begin{bmatrix} + & - & 0 \\ + & + & - \\ - & + & + \end{bmatrix} \tag{3.1}$$

is PEP [1], but in Example 3.2.3 below it is shown that the sign pattern

$$\mathcal{B}_{D(0)} = \begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ - & + & 0 \end{bmatrix} \tag{3.2}$$

is not PEEP. Thus the problem of determining which sign patterns are PEEP is not equivalent to the problem of determining which sign patterns are PEP.

Section 3.2 presents general results on PEEP sign patterns, including those obtained by perturbation analysis and connections with known results on PEP sign patterns. At the end of Section 3.2 the open question of the minimum number of positive entries in an $n \times n$ PEEP sign pattern is discussed. In Section 3.3 small order PEEP sign patterns are characterized. The remainder of this section contains information on eventually exponentially positive matrices and terminology on digraphs and sign patterns.

The *spectrum* of A, denoted $\sigma(A)$, is the multiset of the eigenvalues of A. The *spectral radius* of A is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ and an eigenvalue $\lambda \in \text{spec}(A)$ is a *dominant eigenvalue* if $|\lambda| = \rho(A)$. A nonzero vector $\mathbf{w}$ is called a *left eigenvector* of A if $\mathbf{w}^T A = \lambda \mathbf{w}^T$ for some $\lambda \in \sigma(A)$ (or equivalently, $\mathbf{w}$ is a (right) eigenvector of A^T). The matrix A is eventually positive if and only if A has a unique dominant eigenvalue that is positive and simple, and A has positive right and left eigenvectors for $\rho(A)$ [4] (this is called the *strong Perron-Frobenius test* for eventual positivity).

Definition 3.1.5. A real eigenvalue $\gamma \in \text{spec}(A)$ is called the *rightmost eigenvalue* if it is simple and for all $\lambda \in \text{spec}(A)$, $\lambda \neq \gamma$ implies $\text{Re}(\lambda) < \gamma$, where $\text{Re}(\alpha)$ denotes the real part of a complex number α.

Not every matrix has a rightmost eigenvalue. This definition was motivated by the following test for eventual exponential positivity, which is implicit in the proof of Theorem 3.3 in [5] (and also follows immediately from that theorem, which is Theorem 3.1.1 above).

Proposition 3.1.6. *Let $A \in \mathbb{R}^{n\times n}$. Then A is eventually exponentially positive if and only if A has a rightmost eigenvalue having positive left and right eigenvectors.*

An eventually positive matrix must have a positive entry in each row and column. This need not be the case for an eventually exponentially positive matrix (for example, an eventually exponentially positive matrix that realizes $\mathcal{B}_{D(-)}$ in (3.3) will not have a positive entry in each row and column). However, certain conditions on the eigenvalues require an eventually exponentially positive matrix to have a positive entry in each row and column.

Proposition 3.1.7. *Let A be an eventually exponentially positive matrix.*

1. *If A has an eigenvalue with nonnegative real part, then each row and column of A has a positive entry.*

2. *If A does not have an eigenvalue with positive real part, then each row and column of A has a negative entry.*

Proof. If A has an eigenvalue with nonnegative real part, then the rightmost eigenvalue γ of A is nonnegative. By Proposition 3.1.6, A has positive right and left eigenvectors corresponding to γ. Suppose that row k of A has no positive entry. Since A is an eventually exponentially positive matrix, A is irreducible, so row k has a negative entry. But then if $\mathbf{x} > 0$, $(A\mathbf{x})_k < 0$ and $(\gamma\mathbf{x})_k \geq 0$, so $\mathbf{x}$ is not a (right) eigenvector. Thus every row of A has a positive entry. The result for column k of A is established with the left eigenvector. Similarly, if A has no eigenvalue with positive real part, then $\gamma \leq 0$ and every row and every column of A has a negative entry. □

A square sign pattern $\mathcal{A}$ (or matrix) is *reducible* if there exists a permutation matrix P such that

$$P\mathcal{A}P^T = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are nonempty square sign patterns (or matrices) and 0 is a (possibly rectangular) block consisting entirely of zero entries. If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is called *irreducible* (note any 1×1 matrix is irreducible). Since an eventually exponentially positive matrix must be irreducible, a PEEP sign pattern must be irreducible.

For an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *digraph of* $\mathcal{A}$, denoted $\Gamma(\mathcal{A})$, has vertex set $\{1, \ldots, n\}$ and arc set $\{(i,j) : \alpha_{ij} \neq 0\}$. A nonnegative sign pattern $\mathcal{A}$ is *primitive* if $\mathcal{A}$ is irreducible and the greatest common divisor of the lengths of the cycles of $\Gamma(\mathcal{A})$ is one; for a nonnegative matrix the definition of primitive is analogous. It is well known that a primitive (necessarily nonnegative) matrix is eventually positive.

Let $\mathcal{A} = [\alpha_{ij}], \hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ be sign patterns. If $\alpha_{ij} \neq 0$ implies $\alpha_{ij} = \hat{\alpha}_{ij}$, then $\mathcal{A}$ is a *subpattern* of $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$. Define the *positive part* of $\mathcal{A}$ to be $\mathcal{A}^+ = \left[\alpha_{ij}^+\right]$, where

$$\alpha_{ij}^+ = \begin{cases} + & \text{if} \quad \alpha_{ij} = +, \\ 0 & \text{if} \quad \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -. \end{cases}$$

Note $\mathcal{A}^+$ is a subpattern of $\mathcal{A}$.

3.2 PEEP sign patterns

In this section we establish general properties of PEEP sign patterns. Some of these results will be used in Section 3.3 to determine which sign patterns of order at most 3 are PEEP.

Remark 3.2.1. *If $\mathcal{A}_{D(+)}$ is a PEP sign pattern, then $\mathcal{A}_{D(-)}$ is a PEEP sign pattern, because if $A \in \mathcal{Q}(\mathcal{A}_{D(+)})$ is eventually positive, there exists $t > 0$ such that $A - tI \in \mathcal{Q}(\mathcal{A}_{D(-)})$.*

A PEP sign pattern must have a positive entry in each row and column. This need not be the case for an eventually exponentially positive matrix. The sign pattern

$$\mathcal{B}_{D(-)} = \begin{bmatrix} - & - & 0 \\ + & - & - \\ - & + & - \end{bmatrix} \tag{3.3}$$

is PEEP because the sign pattern $\mathcal{B}_{D(+)}$ in (3.1) is PEP. But $\mathcal{B}_{D(-)}$ does not have a + entry in row 1 nor in column 3. If $A \in \mathbb{R}^{n \times n}$ is an eventually exponentially positive matrix with nonnegative trace, then A has an eigenvalue with nonnegative real part. As a consequence of Proposition 3.1.7, we have the following observation.

Observation 3.2.2. *If $\mathcal{A}$ is a PEEP sign pattern with no $-$ on the diagonal, then $\mathcal{A}$ has a $+$ in each row and column.*

The next example shows that the problem of determining which sign patterns are PEEP is not equivalent to the problem of determining which sign patterns are PEP, because the fact that $\mathcal{A}_{D(+)}$ is PEP does not guarantee that $\mathcal{A}$ is PEEP.

Example 3.2.3. The sign pattern

$$\mathcal{B}_{D(0)} = \begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ - & + & 0 \end{bmatrix}$$

is not PEEP by Observation 3.2.2, because $\mathcal{B}_{D(0)}$ has no $-$ on the diagonal and no $+$ in row 1. Note that $\left(\mathcal{B}_{D(0)}\right)_{D(+)} = \mathcal{B}_{D(+)}$ from (3.1) is PEP.

Related sign patterns are discussed in Corollary 3.3.4 and Theorem 3.3.5 below.

Matrix perturbations are used extensively in the study of potential eventual positivity. It is well known that for any matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of A are continuous functions of the entries of A. For a simple eigenvalue, the same is true of the eigenvector [3, p. 323]. Because a matrix is eventually positive if and only if it passes the strong Perron-Frobenius test, eventual positivity is inherited by matrices that are small perturbations of eventually positive matrices. That is, if $A \in \mathbb{R}^{n\times n}$ is eventually positive and $C \in \mathbb{R}^{n\times n}$ is any matrix, then for ε sufficiently small, $A(\varepsilon) = A + \varepsilon C$ is eventually positive (see, for example, [2] for applications of this technique). The analogous result for eventually exponentially positive matrices follows from Proposition 3.1.6 and perturbation theory.

Theorem 3.2.4. *If $A \in \mathbb{R}^{n\times n}$ is eventually exponentially positive and $C \in \mathbb{R}^{n\times n}$ is any matrix, then for ε sufficiently small, $A(\varepsilon) = A + \varepsilon C$ is eventually exponentially positive.*

If $\hat{\mathcal{A}}$ is a superpattern of a PEEP sign pattern $\mathcal{A}$, and $A \in \mathcal{Q}(\mathcal{A})$ is eventually exponentially positive, then a matrix $\hat{A}$ realizing $\hat{\mathcal{A}}$ can be obtained by a small perturbation of A.

Corollary 3.2.5. *If $\mathcal{A}$ is a PEEP sign pattern, then every superpattern of $\mathcal{A}$ is PEEP. If $\hat{\mathcal{A}}$ is a sign pattern that is not PEEP, then no subpattern of $\hat{\mathcal{A}}$ is a PEEP sign pattern.*

If a sign pattern $\mathcal{A}$ has a primitive positive part, it is PEP. There is an analogous result for PEEP sign patterns.

Theorem 3.2.6. *Let $\mathcal{A}$ be a sign pattern such that $\mathcal{A}^+$ is irreducible. Then $\mathcal{A}$ is PEEP.*

Proof. Let B be the matrix obtained from $\mathcal{A}^+$ by replacing $+$ by 1. Since $B+I \geq 0$, has positive entries on its diagonal, and is irreducible, $B+I$ is primitive and thus eventually positive. So B is eventually exponentially positive and $\mathcal{A}^+$ is PEEP. Since $\mathcal{A}$ is a superpattern of $\mathcal{A}^+$, $\mathcal{A}$ is PEEP. □

The converse of Theorem 3.2.6 is false because the sign pattern $\mathcal{B}_{D(+)}$ (3.1) is a PEP sign pattern with reducible positive part.

Several necessary or sufficient conditions for PEP sign patterns were established in [1]. The sign patterns

$$\mathcal{B}_1 = \begin{bmatrix} - & - & + \\ + & - & - \\ - & + & - \end{bmatrix}, \qquad \mathcal{B}_2 = \begin{bmatrix} - & - & - \\ + & - & - \\ - & + & - \end{bmatrix}$$

are PEEP and demonstrate that the following statements about PEP sign patterns do not necessarily hold for PEEP sign patterns:

1. For $n \geq 2$, an $n \times n$ sign pattern that has exactly one positive entry in each row and each column is not PEP.

2. For $n \geq 2$, the minimum number of $+$ entries in an $n \times n$ PEP sign pattern is $n+1$.

3. If A is PEP, then $\Gamma(\mathcal{A})$ has a cycle (of length one or more) consisting entirely of $+$ entries.

Certain conditions that prevent a sign pattern from being PEP also prevent a sign pattern from being PEEP.

Theorem 3.2.7. [1] *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern with $n \geq 2$ such that for every $k = 1, \ldots, n$,* *1.* $\alpha_{kk} = +$, *and*

2. *(a) no off-diagonal entry in row* k *is* $+$, *or*

(b) no off-diagonal entry in column k *is* $+$.

Then $\mathcal{A}$ *is not PEP.*

Corollary 3.2.8. *Let* $\mathcal{A} = [\alpha_{ij}]$ *be an* $n \times n$ *sign pattern with* $n \geq 2$ *such that for every* $k = 1, \ldots, n$,

(a) no off-diagonal entry in row k *is* $+$, *or*

(b) no off-diagonal entry in column k *is* $+$.

Then $\mathcal{A}$ *is not PEEP.*

Proof. By Theorem 3.2.7, $\mathcal{A}_{D(+)}$ is not PEP, so $\mathcal{A}$ is not PEEP. □

Corollary 3.2.9. *If* $\mathcal{A}$ *is a PEEP sign pattern, then there exists* k *such that both row and column* k *have an off-diagonal* $+$. *Hence, a PEEP sign pattern must have at least* 2 *positive off-diagonal entries.*

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is a *Z sign pattern* if $\alpha_{ij} \neq +$ for all $i \neq j$.

Corollary 3.2.10. *If* $\mathcal{A}$ *is an* $n \times n$ *Z sign pattern with* $n \geq 2$, *then* $\mathcal{A}$ *is not PEEP.*

Proposition 3.2.11. [1] *Let*

$$\mathcal{K} = \begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be a square checkerboard block sign pattern where the block $[+]$ *(respectively,* $[-]$*) consists of entirely positive (respectively, entirely negative) entries, and the diagonal blocks are square. Then* $-\mathcal{K}$ *is not PEP, and if* $\mathcal{K}$ *has a negative entry, then* $\mathcal{K}$ *is not PEP.*

Corollary 3.2.12. *No subpattern of a checkerboard pattern* $\mathcal{K}$ *that contains a negative entry is PEEP.*

Remark 3.2.13. *Provided the sign pattern $\mathcal{K}$ contains a negative entry,*

$$-\mathcal{K} = \begin{bmatrix} [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is PEEP because the positive part of $(-\mathcal{K})_{D(+)}$ is primitive.

For a PEP sign pattern $\mathcal{A}$, Lemma 4.3 in [1] establishes the existence of a standard form of a matrix $C \in \mathcal{Q}(\mathcal{A})$ with $\rho(C) = 1$ and $C\mathbb{1} = \mathbb{1}$. We have a related result for PEEP sign patterns.

Proposition 3.2.14. *Let $\mathcal{A}$ be a PEEP sign pattern. There is an eventually exponentially positive matrix $C \in \mathcal{Q}(\mathcal{A})$ such that the rightmost eigenvalue $\gamma(C) \in \{-1, 0, 1\}$ and $C\mathbb{1} = \gamma(C)\mathbb{1}$.*

Proof. There exists $A \in \mathcal{Q}(\mathcal{A})$ that is eventually exponentially positive. Let $\gamma(A)$ be the rightmost eigenvalue of A and $\mathbf{v} = [v_1, \ldots, v_n]^T$ be the corresponding positive eigenvector. If $\gamma(A) \neq 0$, let $B = \frac{1}{|\gamma(A)|}A$; otherwise, $B = A$. Then $B \in \mathcal{Q}(\mathcal{A})$, B is eventually exponentially positive, $\gamma(B) \in \{-1, 0, 1\}$, and $B\mathbf{v} = \gamma(B)\mathbf{v}$. Let $C = D^{-1}BD$ for $D = \text{diag}(v_1, \ldots, v_n)$. Then $C \in \mathcal{Q}(\mathcal{A})$ is eventually exponentially positive and $\gamma(C) \in \{-1, 0, 1\}$ with $C\mathbb{1} = \gamma(C)\mathbb{1}$. □

We have only started the study of PEEP sign patterns and there are many open questions. Here we highlight one particular question.

Question 3.2.15. *What is the minimum number of positive entries in an $n \times n$ PEEP sign pattern, or equivalently, what is the minimum number of positive entries in an eventually exponentially positive $n \times n$ matrix?*

This question is motivated by Corollary 4.5 in [1], which states that the minimum number of positive entries in an $n \times n$ PEP sign pattern is $n + 1$ (for $n \geq 2$). An upper bound for the minimum number of + entries in a PEEP sign pattern is given by the following example.

Example 3.2.16. Let $\mathcal{C}_n$ be the $n \times n$ sign pattern

$$\mathcal{C}_n = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & + \\ + & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $\mathcal{C}_n$ is nonnegative and irreducible, it is PEEP; note that $\mathcal{C}_n$ has n positive entries.

Corollary 3.2.17. *The minimum number of positive entries in an $n \times n$ PEEP sign pattern is at most n.*

The sign pattern $\mathcal{B}_{D(-)}$ in (3.3) is a 3×3 pattern that has only 2 positive entries, and from Theorem 3.3.5 in the next section it follows that the minimum number of positive entries in a 3×3 PEEP sign pattern is exactly 2. But we do not have examples of PEEP sign patterns having fewer than n positive entries for $n > 3$.

3.3 Classification of small order PEEP sign patterns

In this section we classify all 2×2 and 3×3 sign patterns as to whether the pattern is PEEP.

Two $n \times n$ sign patterns $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* if $\mathcal{A}' = P^T \mathcal{A} P$ or $\mathcal{A}' = P^T \mathcal{A}^T P$ (where P is a permutation matrix). Throughout this section: ? is one of $0, +, -$; $\oplus$ is one of $0, +$; $\ominus$ is one of $0, -$.

It is clear that every 1×1 sign pattern is PEEP. The classification of 2×2 sign patterns as to whether they are PEEP is immediate from the classification as to whether they are PEP.

Proposition 3.3.1. *A 2×2 sign pattern is PEEP if and only if it is of the form*

$$\begin{bmatrix} ? & + \\ + & ? \end{bmatrix}. \tag{3.4}$$

Proof. Sign patterns of the form (3.4) have $\mathcal{A}^+$ irreducible and so by Theorem 3.2.6, they are PEEP. For the converse, let $\mathcal{A}$ be a 2×2 PEEP sign pattern. Then $\mathcal{A}_{D(+)}$ is PEP. In [1] it was shown that any 2×2 PEP sign pattern has both off-diagonal entries equal to $+$, so $\mathcal{A}$ must also have both off-diagonal entries equal to $+$. □

The classification of 3×3 sign patterns as to whether they are PEEP makes use of the following classification as to whether they are PEP.

Theorem 3.3.2. [1] *A 3×3 sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive or $\mathcal{A}$ is equivalent to a sign pattern of the form*

$$\mathcal{B} = \begin{bmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{bmatrix}. \tag{3.5}$$

Theorem 3.3.3. *Let $B = \begin{bmatrix} x_1 & -b_{12} & -b_{13} \\ b_{21} & x_2 & -b_{23} \\ -b_{31} & b_{32} & x_3 \end{bmatrix}$ with $b_{ij} > 0$ for all $i, j = 1, 2, 3$ be an eventually exponentially positive matrix (note there is no restriction on the signs of x_i, $i = 1, 2, 3$). Then $x_2 < \min\{x_1, x_3\}$.*

Proof. Let γ be the rightmost eigenvalue of B. Observe that $B - \gamma I$ is eventually exponentially positive with rightmost eigenvalue 0. By Proposition 3.1.7, $B - \gamma I$ must have a positive entry in each row and column, so $x_1, x_3 > \gamma$. Since the rightmost eigenvalue of $B - \gamma I$ is simple, $0 > \operatorname{tr}(B - \gamma I) = (x_1 - \gamma) + (x_2 - \gamma) + (x_3 - \gamma)$. The first and third term in this sum are positive, so $\operatorname{tr}(B - \gamma I) < 0$ implies that $x_2 < \gamma$. □

Corollary 3.3.4. *A sign pattern equivalent to one of the the forms*

$$\mathcal{M}_1 = \begin{bmatrix} - & - & - \\ + & + & - \\ - & + & - \end{bmatrix} \text{ or } \mathcal{M}_2 = \begin{bmatrix} - & - & - \\ + & + & - \\ - & + & + \end{bmatrix}$$

is not PEEP.

Theorem 3.3.5. *A 3×3 sign pattern is PEEP if and only if it is equivalent to one of the following four forms:*

$$\mathcal{A}_1 = \begin{bmatrix} ? & + & ? \\ ? & ? & + \\ + & ? & ? \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} ? & + & + \\ + & ? & \ominus \\ + & \ominus & ? \end{bmatrix}, \quad \mathcal{A}_3 = \begin{bmatrix} ? & - & \ominus \\ + & - & - \\ - & + & ? \end{bmatrix}, \quad \mathcal{A}_4 = \begin{bmatrix} + & - & \ominus \\ + & \oplus & - \\ - & + & + \end{bmatrix}.$$

Proof. The sign patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ are PEEP by Theorem 3.2.6. Note that $\mathcal{A}_4$ is of the form $\mathcal{B}$ from Theorem 3.3.2; therefore $\mathcal{A}_4$ is PEP and hence is PEEP. Let

$$A = \begin{bmatrix} 0 & -10 & 0 \\ 22 & -33 & -8 \\ -16 & 22 & 0 \end{bmatrix}.$$

Since the spectrum of A is $\{-5, -14+2i\sqrt{15}, -14-2i\sqrt{15}\}$, $\gamma = -5$ is the rightmost eigenvalue of A, and γ has the right and left eigenvectors $[2,1,2]^T$ and $[18,25,40]^T$ respectively. Thus A is eventually exponentially positive by Proposition 3.1.6. Note that $A \in Q(\mathcal{A}_3(0))$ where $\mathcal{A}_3(0)$ is the form of $\mathcal{A}_3$ with all flexible entries set to zero. Therefore $\mathcal{A}_3(0)$ is PEEP, and by Corollary 3.2.5 every superpattern of $\mathcal{A}_3(0)$ is PEEP. Hence every sign pattern of the form $\mathcal{A}_3$ is PEEP.

Let $\mathcal{A}$ be a 3×3 PEEP sign pattern. Then by Observation 3.1.4, $\mathcal{A}_{D(+)}$ is PEP. By Theorem 3.3.2 either $(\mathcal{A}_{D(+)})^+$ is primitive or $\mathcal{A}_{D(+)}$ is of the form $\mathcal{B}$ in (3.5). If $(\mathcal{A}_{D(+)})^+$ is primitive, then $\mathcal{A}$ is of the form $\mathcal{A}_1$ or $\mathcal{A}_2$. Now suppose that $(\mathcal{A}_{D(+)})^+$ is not primitive. Then we must consider all possible sign patterns $\mathcal{A}$ such that $\mathcal{A}_{D(+)} = \begin{bmatrix} + & - & \ominus \\ + & + & - \\ - & + & + \end{bmatrix}$. Note that the sign patterns $\mathcal{M}_1$ and $\mathcal{M}_2$ in Corollary 3.3.4 and their subpatterns rule out all of the sign patterns that could possibly have this $\mathcal{A}_{D(+)}$ except for those of the form $\mathcal{A}_3$ and $\mathcal{A}_4$. Therefore if $\mathcal{A}$ is a 3×3 PEEP sign pattern, it must be of one of the forms $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ or $\mathcal{A}_4$. □

The symbols $\ominus$ and $\oplus$ are used in Theorem 3.3.5 so that the listed patterns are disjoint classes. For example, if the $(2,2)$-entry of $\mathcal{A}_4$ were changed to ?, then one sign pattern of that form would be equivalent to one sign pattern of the form of $\mathcal{A}_3$.

Bibliography

[1] A. Berman, M. Catral, L.M. DeAlba, A. Elhashash, F.J. Hall, L. Hogben, I.-J. Kim, D.D. Olesky, P. Tarazaga, M.J. Tsatsomeros, P. van den Driessche. Sign patterns that allow eventual positivity. *Electronic Journal of Linear Algebra*, 19 (2010), 108–120.

[2] E.M. Ellison, L. Hogben, and M.J. Tsatsomeros. Sign patterns that require eventual positivity or require eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19 (2010), 98–107.

[3] G.H. Golub and C.F. Van Loan. *Matrix Computations* (third edition). Johns Hopkins University Press, Baltimore, 1996.

[4] D. Handelman. Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains. *Journal of Operator Theory*, 6 (1981), 55-74.

[5] D. Noutsos and M.J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM Journal on Matrix Analysis and Applications*, 30 (2008), 700–712.

CHAPTER 4. GENERAL CONCLUSIONS

4.1 General Discussion

Previously known results related to eventually positive, eventually nonnegative, eventually exponentially positive, and eventually exponentially nonnegative matrices, along with results about sign patterns that require eventual positivity, eventual nonnegativity, or eventual exponential positivity were presented in Section 1.2. In Chapter 2, we discuss sign patterns that require eventual exponential nonnegativity, utilizing several different techniques for evaluating the matrix exponential function. We provide necessary conditions for a sign pattern to require eventual exponential nonnegativity in Section 2.3.1 and in Section 2.3.2 we provide a characterization of sign patterns which are permutationally similar to upper triangular sign patterns that require eventual exponential nonnegativity. In Chapter 3, we discuss potentially eventually exponentially positive sign patterns. We establish some general properties of these sign patterns and provide a classification of $n \times n$ potentially eventually exponentially positive sign patterns for $n \leq 3$.

4.2 Recommendations for Future Research

The most important open question about sign patterns that require eventual exponential nonnegativity is to determine sufficient conditions for sign patterns whose associated digraphs contain a non-trivial connected component, i.e., sign patterns that are not permutationally similar to upper triangular matrices. It is conjectured that the necessary conditions in Theorem 2.3.15 are also sufficient for a sign pattern to require eventual exponential nonnegativity. An open question related to eventually exponentially nonnegative matrices is that of the allows problem; that is, which sign patterns allow eventual exponential nonnegativity?

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Leslie Hogben, for introducing me to many interesting research projects, as well as for her guidance and support over the last five years.

I thank my committee members, Drs. Ryan Martin, Roger Maddux, Justin Peters, and Sung-Yell Song, for their time and advice.

I would also like to thank Dr. Minerva Catral for our many fruitful discussions over the years and for being such a good example of a faculty member at an undergraduate institution being able to remain active in research.